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A quantitative formula for the imaginary part of a Weyl coefficient

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Abstract. We investigate two-dimensional canonical systems $y' = zJHy$ on an interval, with positive semi-definite Hamiltonian H. Let q_H be the Weyl coefficient of the system. We prove a formula that determines the imaginary part of q_H along the imaginary axis up to multiplicative constants, which are independent of H. We also provide versions of this result for Sturm–Liouville operators and Krein strings.

Using classical Abelian–Tauberian theorems, we deduce characterizations of spectral properties such as integrability of a given comparison function with respect to the spectral measure μ_H, and boundedness of the distribution function of μ_H relative to a given comparison function.

We study in depth Hamiltonians for which $\arg q_H(ir)$ approaches 0 or π (at least on a subsequence). It turns out that this behavior of $q_H(ir)$ imposes a substantial restriction on the growth of $|q_H(ir)|$. Our results in this context are interesting also from a function theoretic point of view.

1. Introduction

We study two-dimensional *canonical systems*

$$y'(t) = zJH(t)y(t), \quad t \in [a,b) \text{ a.e.}, \tag{1.1}$$

where $-\infty < a < b \leq \infty$, $z \in \mathbb{C}$ is a spectral parameter and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The *Hamiltonian* H is assumed to be a locally integrable, $\mathbb{R}^{2\times 2}$-valued function on $[a,b)$ that further satisfies

- $H(t) \geq 0$ and $H(t) \neq 0$, $t \in [a,b)$ a.e.;
- H is definite, i.e., if $v \in \mathbb{C}^2$ is s.t. $H(t)v \equiv 0$ on $[a,b)$, then $v = 0$;
- $\int_a^b \operatorname{tr} H(t)\,\mathrm{d}t = \infty$ (limit point case at b).

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Together with a boundary condition at a, equation (1.1) becomes the eigenvalue equation of a self-adjoint (possibly multi-valued) operator A_H in a Hilbert space $L^2(H)$ associated with H. Throughout this paper, we fix the boundary condition $(1,0)y(a) = 0$, which is no loss of generality.

Many classical second-order differential operators such as Schrödinger and Sturm–Liouville operators, Krein strings, and Jacobi operators can be transformed to the form (1.1), see, e.g., [3, 14, 16, 24, 25]. Canonical systems thus form a unifying framework.

All of the above operators have in common that their spectral theory is centered around the Weyl coefficient q of the operator (also referred to as *Titchmarsh–Weyl m-function*). This function is constructed by Weyl's nested disk method and is a Herglotz function, i.e., it is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and satisfies there $\frac{\operatorname{Im} q(z)}{\operatorname{Im} z} \geq 0$ as well as $q(\overline{z}) = \overline{q(z)}$. It can thus be represented as

$$q(z) = \alpha + \beta z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, \mathrm{d}\mu(t), \quad z \in \mathbb{C} \setminus \mathbb{R} \tag{1.2}$$

with $\alpha \in \mathbb{R}$, $\beta \geq 0$, and μ a positive Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$. The measure μ in the integral representation (1.2) of the Weyl coefficient is a spectral measure of the underlying operator model if $\beta = 0$ (if $\beta > 0$, a one-dimensional component has to be added). The importance of canonical systems in this context lies in the Inverse Spectral Theorem of L. de Branges, stating that each Herglotz function q is the Weyl coefficient of a unique (suitably normalized) canonical system.

Given a Hamiltonian H, we are ultimately interested in the description of properties of its spectral measure μ_H in terms of H. The correspondence between H and μ_H can be best understood using the Weyl coefficient q_H, whose imaginary part $\operatorname{Im} q_H$ determines μ_H via the Stieltjes inversion formula.

In their recent paper [18], Langer, Pruckner, and Woracek gave a two-sided estimate for $\operatorname{Im} q_H(ir)$ in terms of the coefficients of H:

$$L(r) \lesssim \operatorname{Im} q_H(ir) \lesssim A(r), \quad r > 0, \tag{1.3}$$

where L, A are explicit in terms of H, and we used the notation $f(r) \lesssim g(r)$ to state that $f(r) \leq Cg(r)$ for a constant $C > 0$. Moreover, in (1.3) the constants implicit in $\lesssim$ are independent of H. The exact formulation of this result will be recalled in Theorem 2.1.

It may happen that $L(r) = \mathrm{o}(A(r))$, and $\operatorname{Im} q_H(ir)$ is not determined by (1.3). A toy example for this is the Hamiltonian

$$H(t) = t \begin{pmatrix} |\log t| & |\log t|^2 \\ |\log t|^2 & |\log t|^3 \end{pmatrix}, \quad t \in [0, \infty).$$

For $r \to \infty$, a calculation shows that

$$L(r) \asymp (\log r)^{-3}, \quad A(r) \asymp (\log r)^{-1},$$

where $f(r) \asymp g(r)$ means that both $f(r) \lesssim g(r)$ and $g(r) \lesssim f(r)$.

The following theorem, which is our main result, improves the estimate (1.3) by giving a formula for $\operatorname{Im} q_H(ir)$ up to universal multiplicative constants.

1.1 Theorem. *Let H be a Hamiltonian on $[a, b)$, and denote*[1]

$$H(t) = \begin{pmatrix} h_1(t) & h_3(t) \\ h_3(t) & h_2(t) \end{pmatrix}, \quad \Omega_H(t) = \begin{pmatrix} \omega_{H,1}(t) & \omega_{H,3}(t) \\ \omega_{H,3}(t) & \omega_{H,2}(t) \end{pmatrix} := \int_a^t H(s)\, \mathrm{d}s. \tag{1.4}$$

Let $\hat{t}\colon (0, \infty) \to (a, b)$ be a function satisfying[2]

$$\det \Omega_H(\hat{t}(r)) \asymp \frac{1}{r^2}, \quad r \in (0, \infty). \tag{1.5}$$

Then

$$\operatorname{Im} q_H(ir) \asymp \left| q_H(ir) - \frac{\omega_{H,3}(\hat{t}(r))}{\omega_{H,2}(\hat{t}(r))} \right| \asymp \frac{1}{r\,\omega_{H,2}(\hat{t}(r))}, \tag{1.6}$$

$$\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|^2} \asymp \frac{1}{r\,\omega_{H,1}(\hat{t}(r))}, \tag{1.7}$$

for $r \in (0, \infty)$. The constants implicit in $\asymp$ in (1.6) and (1.7) depend on the constants hidden in $\asymp$ in (1.5), but not on H.

If, in addition, $\operatorname{Im} q_H(ir) = \mathrm{o}(|q_H(ir)|)$ for $r \to \infty$ (or $r \to 0$), then[3]

$$q_H(ir) \sim \frac{\omega_{H,3}(\hat{t}(r))}{\omega_{H,2}(\hat{t}(r))}, \qquad r \to \infty\, (r \to 0). \tag{1.8}$$

The two-sided estimate (1.6) has some useful features: its pointwise nature, its applicability for $r \to \infty$ and $r \to 0$, and the universality of the constants hidden in $\asymp$. However, it is rather different from an asymptotic formula: it does not capture small oscillations of $\operatorname{Im} q_H(ir)$ around $\frac{1}{r\omega_{H,2}(\hat{t}(r))}$.

Note also that the first relation in (1.6) can be seen as a statement about the real part of $q_H(ir)$. In fact, $\operatorname{Im} q_H(ir)$ is also obtained if we subtract $\operatorname{Re} q_H(ir)$ from

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[1] When there is no risk of ambiguity, we write Ω and ω_j instead of Ω_H and $\omega_{H,j}$ for short.

[2] We will see later that the equation $\det \Omega_H(t) = \frac{1}{r^2}$ has a unique solution for every $r > 0$. A possible choice of $\hat{t}$ is thus the function that maps $r > 0$ to this solution.

[3] With $f(r) \sim g(r)$ meaning $\lim \frac{f(r)}{g(r)} = 1$.

$q_H(ir)$, then take absolute values. It is an open question whether $\operatorname{Re} q_H(ir)$ can be described more directly in terms of H.

A most important class of operators is that of Sturm–Liouville (in particular, Schrödinger) operators. Let us provide a reformulation of Theorem 1.1 for these operators right away.

Sturm–Liouville operators. We provide a version of Theorem 1.1 for Sturm–Liouville equations

$$-(py')' + qy = zwy \tag{1.9}$$

on (a,b), where $1/p, q, w \in L^1_{loc}(a,b)$, $w > 0$ and p, q are real-valued. Suppose that a is in limit circle case and b is in limit point case. Impose a Dirichlet boundary condition at a, i.e., $y(a) = 0$. The Weyl coefficient for this problem is the unique number $m(z)$ with

$$c(z,\cdot) + m(z)s(z,\cdot) \in L^2((a,b), w(x)\,dx)$$

where $c(z,\cdot)$ and $s(z,\cdot)$ are solutions of (1.9) with initial values

$$\begin{pmatrix} p(a)c'(z,a) \\ c(z,a) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} p(a)s'(z,a) \\ s(z,a) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

1.2 Theorem. *For each $t \in (a,b)$, let $(.,.)_t$ and $\|.\|_t$ denote the scalar product and norm on $L^2((a,t), w(x)\,dx)$, i.e.,*

$$(f,g)_t = \int_a^t f(x)\overline{g(x)}w(x)\,\mathrm{d}x.$$

For $\xi \in \mathbb{R}$, let $\hat{t}_\xi : (0,\infty) \to (a,b)$ be a function satisfying

$$\|c(\xi,\cdot)\|^2_{\hat{t}_\xi(r)} \|s(\xi,\cdot)\|^2_{\hat{t}_\xi(r)} - (c(\xi,\cdot), s(\xi,\cdot))^2_{\hat{t}_\xi(r)} \asymp \frac{1}{r^2}, \quad r \in (0,\infty). \tag{1.10}$$

Then

$$\operatorname{Im} m(\xi + ir) \asymp \frac{1}{r\|s(\xi,\cdot)\|^2_{\hat{t}_\xi(r)}}, \tag{1.11}$$

$$\frac{\operatorname{Im} m(\xi + ir)}{|m(\xi + ir)|^2} \asymp \frac{1}{r\|c(\xi,\cdot)\|^2_{\hat{t}_\xi(r)}}, \tag{1.12}$$

for $r \in (0,\infty)$. The constants implicit in $\asymp$ are independent of p, q, w as well as ξ, but do depend on the constants pertaining to $\asymp$ in (1.10).

In fact, Theorem 1.2 is a direct consequence of Theorem 1.1 upon employing a transformation (cf. [24] for $p = w = 1$ and $\xi = 0$) that maps solutions of (1.9) to solutions of the canonical system $y' = (z - \xi)JH_\xi y$, where

$$H_\xi(t) = w(t) \cdot \begin{pmatrix} c(\xi,t)^2 & -s(\xi,t)c(\xi,t) \\ -s(\xi,t)c(\xi,t) & s(\xi,t)^2 \end{pmatrix}, \quad t \in [a,b).$$

The Weyl coefficients then satisfy $m(z) = q_{H_\xi}(z - \xi)$.

Historical remarks. The origins of the Weyl coefficient in the context of the Sturm–Liouville differential equation are well summarized in Everitt's paper [7]. We give a short account specifically on the history of estimates for the growth of the Weyl coefficient, which date back at least to the 1950s. Particular attention was often given to the deduction of asymptotic formulae for the Weyl coefficient [1, 4, 6, 12, 17, 21]. However, asymptotic results usually depend on rather strong assumptions on the data. When weakening these assumptions, one can still ask for explicit estimates for $q(z)$ as $z \to \infty$ nontangentially in the upper half-plane. There is a number of rather early results that determine $|q(z)|$ up to $\asymp$, e.g., [2, 4, 9], although these still depend on data subject to additional restrictions. Fundamental progress has been made by Jitomirskaya and Last [10], who considered Schrödinger operators with arbitrary (real-valued and locally integrable) potentials. They found a formula up to $\asymp$ for $|q(z)|$, which also covers the case $z \to 0$. An analog of this formula for canonical systems was given in [8].

When it comes to $\operatorname{Im} q(z)$, however, no such formula was available. Only the very recent estimate (1.3) from [18, Theorem 1.1] made it possible to obtain our main result that determines $\operatorname{Im} q(z)$ up to $\asymp$.

Structure of the paper. The proof of Theorem 1.1, together with some immediate corollaries, makes up Section 2. In Section 3, we continue with a first application, a criterion for integrability of a given comparison function with respect to μ_H. We also characterize boundedness of the distribution function of μ_H relative to a given comparison function.

Section 4 is dedicated to the boundary behavior of Herglotz functions. Cauchy integrals and the relative behavior of its imaginary and real part have been intensively studied. For example, for a Herglotz function q it is known [22] that the set of $\xi \in \mathbb{R}$ for which

$$\lim_{r \to 0} \frac{\operatorname{Im} q(\xi + ir)}{|q(\xi + ir)|} = 0 \tag{1.13}$$

is a zero set with respect to μ. In contrast to measure theoretic results like this, we use the de Branges correspondence $H \leftrightarrow q_H$ to investigate this behavior pointwise with respect to ξ. In Theorem 4.3 (a) we show that if ξ is such that (1.13) holds, then

$|q(\xi+ir)|$ is slowly varying (cf. Definition 4.2). Theorem 4.3 (b) is a partial converse of this statement.

In Section 5 we turn to a finer study of $\operatorname{Im} q_H(ir)$ in the context of the geometric origins of (1.3) and (1.6). Namely, the functions L and A describe the imaginary parts of bottom and top of certain Weyl disks containing $q_H(ir)$. We show that there are restrictions on the possible location of $q_H(ir)$ within the disks, and construct a Hamiltonian H for which $q_H(ir)$ oscillates back and forth between the bottoms and tops of the disks. This construction allows us to answer several open problems that were posed in [18].

We conclude our work with a reformulation of Theorem 1.1 for the principal Titchmarsh–Weyl coefficient q_S of a Krein string. This reformulation is the content of Section 6.

Notation associated to Hamiltonians. Let H be a Hamiltonian on $[a,b)$. An interval $(c,d)\subseteq[a,b)$ is called *H-indivisible* if $H(t)$ takes the form $h(t)\binom{\cos\varphi}{\sin\varphi}\binom{\cos\varphi}{\sin\varphi}^*$ a.e. on (c,d), with scalar-valued h and fixed $\varphi\in[0,\pi)$. The angle φ is then called the *type* of the interval.

1.3 Definition. Let

$$\mathring{a}(H):=\inf\Big\{t>a\;\Big|\;(a,t)\text{ is not }H\text{-indivisible of type }0\text{ or }\frac{\pi}{2}\Big\},\tag{1.14}$$

$$\hat{a}(H):=\inf\{t>a\mid(a,t)\text{ is not }H\text{-indivisible}\}.\tag{1.15}$$

Usually, we write $\mathring{a}$ and $\hat{a}$ for short. Since H is assumed to be definite, both of these numbers are smaller than b.

Note that $(\omega_1\omega_2)(t)>0$ if and only if $[a,t)$ is not H-indivisible of type 0 or $\frac{\pi}{2}$, i.e., $t>\mathring{a}$. Using the assumption $\int_a^b \operatorname{tr} H(t)\,dt=\infty$, we infer that $\omega_1\omega_2$ is an increasing bijection from $(\mathring{a},b)$ to $(0,\infty)$.

Similarly, $\det\Omega(t)>0$ is equivalent to $t>\hat{a}$. We have

$$\frac{d}{dt}\Big(\frac{\det\Omega(t)}{\omega_1(t)}\Big)=\omega_1(t)^{-2}\begin{pmatrix}-\omega_3(t)\\ \omega_1(t)\end{pmatrix}^*H(t)\begin{pmatrix}-\omega_3(t)\\ \omega_1(t)\end{pmatrix}\geq 0$$

and (by symmetry) $\frac{d}{dt}(\frac{\det\Omega}{\omega_2})\geq 0$. Since at least one of ω_1 and ω_2 is unbounded, $\det\Omega$ is an increasing bijection from $(\hat{a},b)$ to $(0,\infty)$.

1.4 Definition. For a Hamiltonian H and a number $\eta>0$, set

$$\mathring{r}_{\eta,H}\colon(\mathring{a},b)\to(0,\infty),\quad t\mapsto\frac{\eta}{2\sqrt{(\omega_1\omega_2)(t)}},$$

$$\hat{r}_{\eta,H}\colon(\hat{a},b)\to(0,\infty),\quad t\mapsto\frac{\eta}{2\sqrt{\det\Omega(t)}}.$$

Both of these functions are decreasing and bijective. We define their inverse functions,

$$\mathring{t}_{\eta,H} := \mathring{r}_{\eta,H}^{-1}\colon (0,\infty) \to (\mathring{a}, b), \quad \hat{t}_{\eta,H} := \hat{r}_{\eta,H}^{-1}\colon (0,\infty) \to (\hat{a}, b). \tag{1.16}$$

Note that the functions $\hat{t}_{\eta,H}$, for any $\eta > 0$, satisfy (1.5). Functions of this form will be the default choice of $\hat{t}$ for the sake of Theorem 1.1. We will often fix η and H and write $\mathring{r}$, $\mathring{t}$, $\hat{r}$, $\hat{t}$ for short. If η is fixed but the Hamiltonian is ambiguous, we may write $\mathring{r}_H, \mathring{t}_H, \hat{r}_H, \hat{t}_H$ to indicate dependence on H.

2. On the imaginary part of the Weyl coefficient

We start by providing the details of the estimate (1.3), which is the central result in [18].

2.1 Theorem ([18, Theorem 1.1]). *Let H be a Hamiltonian on $[a, b)$, and let $\eta \in (0, 1 - \frac{1}{\sqrt{2}})$ be fixed. For $r > 0$, let $\mathring{t}(r)$ be the unique number satisfying*

$$(\omega_{H,1}\omega_{H,2})(\mathring{t}(r)) = \frac{\eta^2}{4r^2}, \tag{2.1}$$

cf. Definition 1.4. Set[4]

$$A_{\eta,H}(r) := \frac{\eta}{2r\omega_{H,2}(\mathring{t}(r))}, \qquad L_{\eta,H}(r) := \frac{\det \Omega_H(\mathring{t}(r))}{(\omega_{H,1}\omega_{H,2})(\mathring{t}(r))} \cdot A_{\eta,H}(r).$$

Then the Weyl coefficient q_H associated with the Hamiltonian H satisfies

$$|q_H(ir)| \asymp A_{\eta,H}(r), \tag{2.2}$$

$$L_{\eta,H}(r) \lesssim \operatorname{Im} q_H(ir) \lesssim A_{\eta,H}(r) \tag{2.3}$$

for $r \in (0,\infty)$. The constants implicit in these relations are independent of H. Their dependence on η is continuous.

In the following proof of Theorem 1.1, we will also show that Theorem 2.1 still holds if $\mathring{t}\colon (0,\infty) \to (a, b)$ is a function satisfying $(\omega_{H,1}\omega_{H,2})(\mathring{t}(r)) \asymp \frac{1}{r^2}$, and

$$A(r) := \frac{1}{r\omega_{H,2}(\mathring{t}(r))}, \qquad L(r) := \frac{\det \Omega_H(\mathring{t}(r))}{(\omega_{H,1}\omega_{H,2})(\mathring{t}(r))} \cdot A(r).$$

In particular, we can choose any $\eta > 0$ in (2.1).

[4] If η and H are clear from the context, we may write A and L for short.

Proof of Theorem 1.1. Let $\hat{t}_{\eta,H}$ be defined as in Definition 1.4. We show that for any $\eta > 0$, Theorem 1.1 holds for $\hat{t}_{\eta,H}$ in place of $\hat{t}$, and that the dependence on η of the constants hidden in $\asymp$ in (1.6) and (1.7) is continuous. This then implies that Theorem 1.1 holds for any function $\hat{t}$ satisfying (1.5).

The proof is divided into five steps.

Step 1. We introduce a family of transformations of H that leave the imaginary part of the Weyl coefficient unchanged. If $p \in \mathbb{R}$ and

$$H_p(t) := \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} H(t) \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} = \begin{pmatrix} h_1(t) + 2ph_3(t) + p^2 h_2(t) & h_3(t) + ph_2(t) \\ h_3(t) + ph_2(t) & h_2(t) \end{pmatrix},$$

an easy calculation shows that the Weyl coefficient q_p of H_p is given by $q_p(z) = q_0(z) + p = q_H(z) + p$.

Step 2. We prove (1.6)–(1.8) for fixed $\eta \in (0, 1 - \frac{1}{\sqrt{2}})$. The abbreviations of Table 1 are used only in Step 2.

short form	meaning	short form	meaning	short form	meaning
$\mathring{t}$	$\mathring{t}_{\eta,H}$	$\mathring{t}_p$	$\mathring{t}_{\eta,H_p}$	Ω_p	Ω_{H_p}
$\hat{t}$	$\hat{t}_{\eta,H}$	$\hat{t}_p$	$\hat{t}_{\eta,H_p}$	$\omega_{p,j}$	$\omega_{H_p,j}$
L_p	L_{η,H_p}	A_p	A_{η,H_p}	Ω	Ω_H

Table 1

Let $r > 0$ be fixed (this is important). Our first observation is that $\hat{t}_p(r) = \hat{t}(r)$ for any p since $\det \Omega_p(t) = \det \Omega(t)$ does not depend on p. If we can find p such that $\mathring{t}_p(r) = \hat{t}_p(r) = \hat{t}(r)$, then clearly

$$\frac{L_p(r)}{A_p(r)} = \frac{\det \Omega_p(\mathring{t}_p(r))}{(\omega_{p,1}\omega_{p,2})(\mathring{t}_p(r))} = \frac{\det \Omega_p(\hat{t}_p(r))}{(\omega_{p,1}\omega_{p,2})(\mathring{t}_p(r))} = 1.$$

We apply Theorem 2.1 with η and H_p. The estimate (2.3) then takes the form

$$A_p(r) = L_p(r) \lesssim \operatorname{Im} q_H(ir) \lesssim A_p(r) \tag{2.4}$$

while (2.2) turns into

$$|q_H(ir) + p| \asymp A_p(r), \tag{2.5}$$

where

$$A_p(r) = \frac{\eta}{2r\omega_{p,2}(\mathring{t}_p(r))} = \frac{\eta}{2r\omega_2(\hat{t}(r))}.$$

The right choice of p is

$$p = -\frac{\omega_3(\hat{t}(r))}{\omega_2(\hat{t}(r))},$$

leading to $\omega_{p,3}(\hat{t}(r)) = 0$ and thus

$$(\omega_{p,1}\omega_{p,2})(\hat{t}(r)) = \det \Omega_p(\hat{t}(r)) = \det \Omega(\hat{t}(r)) = \frac{\eta^2}{4r^2}.$$

Consequently, $\mathring{t}_p(r) = \hat{t}(r)$. Observe that the implicit constants in (2.2) and (2.3) are independent of H and r and depend continuously on η. This shows that (1.6) holds, with constants depending continuously on η.

Step 3. (1.7) follows from an application of (1.6) to $\widetilde{H} := J^\top H J = \begin{pmatrix} h_2 & -h_3 \\ -h_3 & h_1 \end{pmatrix}$ and note that $\hat{t}_{\eta,\widetilde{H}} = \hat{t}_{\eta,H}$. Thus

$$\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|^2} = \operatorname{Im}\Big(-\frac{1}{q_H(ir)}\Big) = \operatorname{Im} q_{\widetilde{H}}(ir) \asymp \frac{1}{r\omega_{\widetilde{H},2}(\hat{t}_{\eta,\widetilde{H}}(r))} = \frac{1}{r\omega_{H,1}(\hat{t}_{\eta,H}(r))}.$$

Formula (1.8) follows if we divide (1.6) by $|q_H(ir)|$. Hence, we proved the assertion for $\eta \in (0, 1 - \frac{1}{\sqrt{2}})$.

In the remaining steps we treat the missing case $\eta \geq 1 - \frac{1}{\sqrt{2}}$.

Step 4. Let $k > 0$. For use in Step 5, we show that

$$\operatorname{Im} q_H(ir) \asymp \operatorname{Im} q_H(ikr), \quad |q_H(ir)| \asymp |q_H(ikr)| \tag{2.6}$$

for $r \in (0, \infty)$, where the constants in $\asymp$ depend continuously on k and are independent of H.

For the imaginary part, the statement is easy to see from the integral representation (1.2). For the absolute value, we use the Hamiltonian $\widetilde{H}$ from Step 3 to obtain

$$\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|^2} = \operatorname{Im} q_{\widetilde{H}}(ir) \asymp \operatorname{Im} q_{\widetilde{H}}(ikr) = \frac{\operatorname{Im} q_H(ikr)}{|q_H(ikr)|^2}.$$

This shows that $|q_H(ir)| \asymp |q_H(ikr)|$ as well.

Step 5. Fix a Hamiltonian H, and let $\eta_0 \geq 1 - \frac{1}{\sqrt{2}}$. Then

$$\mathring{t}_{\eta_0,H}(r) = \mathring{t}_{\frac{1}{4},\frac{1}{4\eta_0}H}(r), \qquad \hat{t}_{\eta_0,H}(r) = \hat{t}_{\frac{1}{4},\frac{1}{4\eta_0}H}(r) \tag{2.7}$$

and

$$A_{\eta_0,H}(r) = A_{\frac{1}{4},\frac{1}{4\eta_0}H}(r), \quad L_{\eta_0,H}(r) = L_{\frac{1}{4},\frac{1}{4\eta_0}H}(r). \tag{2.8}$$

Since $\frac{1}{4}$ is less than $1-\frac{1}{\sqrt{2}}$, we can use Theorem 2.1 with $\eta := \frac{1}{4}$ to obtain

$$L_{\eta_0,H}(r) = L_{\frac{1}{4},\frac{1}{4\eta_0}H}(r) \lesssim \operatorname{Im} q_{\frac{1}{4\eta_0}H}(ir) \\ \leq |q_{\frac{1}{4\eta_0}H}(ir)| \asymp A_{\frac{1}{4},\frac{1}{4\eta_0}H}(r) = A_{\eta_0,H}(r) \tag{2.9}$$

for $r \in (0,\infty)$. Since $q_{\frac{1}{4\eta_0}H}(z) = q_H(\frac{z}{4\eta_0})$ and by Step 4, we see that Theorem 2.1 holds for arbitrary $\eta > 0$. It is easy to check that continuous dependence of constants on η is retained. Repeating Steps $1-3$ now shows that also Theorem 1.1 holds for $\hat{t}_{\eta,H}$ for any $\eta > 0$. Moreover, it is not hard to see that everything still works if $\hat{t}$ is a function satisfying (1.5). ∎

2.2 Remark. Theorem 2.1 and Theorem 1.1, in the form we stated them, give information about $q_H(z)$ for $z = ir$. However, if $\vartheta \in (0,\pi)$ is fixed, these theorems also hold

- for $z = re^{i\vartheta}$ uniformly for $r \in (0,\infty)$ and
- for $z = re^{i\varphi}$ uniformly for $r \in (0,\infty)$ and $|\frac{\pi}{2}-\varphi| \leq |\frac{\pi}{2}-\vartheta|$.

We restate the explicit constants coming from [18]. Fix $\eta \in (0, 1-\frac{1}{\sqrt{2}})$ and set $\sigma := (1-\eta)^{-2}-1 \in (0,1)$. With

$$c_-(\eta,\vartheta) = \frac{\eta\sin\vartheta}{2(1+|\cos\vartheta|)}\cdot\frac{1-\sigma}{1+\sigma}, \quad c_+(\eta,\vartheta) = \frac{\sigma+\frac{2}{\eta\sin\vartheta}}{1-\sigma},$$

we have[5]

$$c_-(\eta,\vartheta)\cdot\frac{\eta}{2}\cdot\frac{1}{r\omega_2(\hat{t}_{\eta,H}(r))} \leq \operatorname{Im} q_H(re^{i\vartheta}) \\ \leq c_+(\eta,\vartheta)\cdot\frac{\eta}{2}\cdot\frac{1}{r\omega_2(\hat{t}_{\eta,H}(r))}, \tag{2.10}$$

$$c_-(\eta,\vartheta)\cdot\frac{\eta}{2}\cdot\frac{1}{r\omega_1(\hat{t}_{\eta,H}(r))} \leq \frac{\operatorname{Im} q_H(re^{i\vartheta})}{|q_H(re^{i\vartheta})|^2} \\ \leq c_+(\eta,\vartheta)\cdot\frac{\eta}{2}\cdot\frac{1}{r\omega_1(\hat{t}_{\eta,H}(r))}. \tag{2.11}$$

In order to show (2.10), we need to slightly adapt the proof of Theorem 1.1 by replacing ir with $re^{i\vartheta}$ in (2.4) and taking into account the constants provided in [18, Theorem 1.1]. Then (2.11) follows as in Step 3 of the proof.

[5] Since c_- and c_+ are clearly monotonic in ϑ, (2.10) and (2.11) still hold when $q_H(re^{i\vartheta})$ is replaced by $q_H(re^{i\varphi})$, where $|\frac{\pi}{2}-\varphi| \leq |\frac{\pi}{2}-\vartheta|$.

For $\vartheta = \frac{\pi}{2}$, the optimal choice of η is around 0.13833 which gives

$$c_+\Big(0.13833, \frac{\pi}{2}\Big) \approx 1.568,$$
$$c_-\Big(0.13833, \frac{\pi}{2}\Big) \approx 0.002,$$
$$\frac{c_+(0.13833, \frac{\pi}{2})}{c_-(0.13833, \frac{\pi}{2})} \approx 675.772.$$

While it is possible to derive explicit constants also for $\eta \geq 1 - \frac{1}{\sqrt{2}}$, doing so does not result in an improvement of the quotient c_+/c_-.

Immediate consequences of Theorem 1.1. *In order to simplify calculations, unless specified otherwise, we will always assume that $\mathring{t}(r)$ and $\hat{t}(r)$ are defined implicitly by*

$$(\omega_1\omega_2)(\mathring{t}(r)) = \frac{1}{r^2}, \quad \det\Omega(\hat{t}(r)) = \frac{1}{r^2}, \tag{2.12}$$

and similarly for $\mathring{r}$ and $\hat{r}$ (cf. Definition 1.4 with $\eta = 2$).

We revisit the example from the introduction in more generality. The following example was communicated by Matthias Langer. The calculations can be found in the extended preprint [23] of this article.

2.3 Example. Let $\alpha > 0$ and $\beta_1, \beta_2 \in \mathbb{R}$ where $\beta_1 \neq \beta_2$. Set $\beta_3 := \frac{\beta_1+\beta_2}{2}$ and define, for $t \in (0, \infty)$,

$$H(t) = t^{\alpha-1}\begin{pmatrix} |\log t|^{\beta_1} & |\log t|^{\beta_3} \\ |\log t|^{\beta_3} & |\log t|^{\beta_2} \end{pmatrix}.$$

Then for $r \to \infty$, we have

- $L(r) \asymp (\log r)^{\frac{\beta_1-\beta_2}{2}-2}$ and
- $A(r) \asymp |q_H(ir)| \asymp (\log r)^{\frac{\beta_1-\beta_2}{2}}$,

i.e., $L(r) = \mathrm{o}(A(r))$. Using Theorem 1.1, we can now continue the calculations, leading to

$$\operatorname{Im} q_H(ir) \asymp (\log r)^{\frac{\beta_1-\beta_2}{2}-1} \asymp \sqrt{L(r)A(r)}.$$

It is an immediate consequence of Theorem 1.1 that $\operatorname{Im} q_H$ depends monotonically on the off-diagonal of H.

2.4 Corollary. *Let $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ and $\tilde{H} = \begin{pmatrix} h_1 & \tilde{h}_3 \\ \tilde{h}_3 & h_2 \end{pmatrix}$ be two Hamiltonians on $[a, b)$. If $t > \hat{a}(H)$ such that*

$$\Big|\int_a^t h_3(s)\,\mathrm{d}s\Big| \geq \Big|\int_a^t \tilde{h}_3(s)\,\mathrm{d}s\Big|,$$

then

$$\operatorname{Im} q_H(i\hat{r}_H(t)) \lesssim \operatorname{Im} q_{\widetilde{H}}(i\hat{r}_H(t))$$

with a constant independent of t, H, and $\widetilde{H}$.

Proof. Our condition states that $|\omega_{H,3}(t)| \geq |\omega_{\widetilde{H},3}(t)|$. Taking into account that $t > \hat{a}(H)$, this means that $0 < \det\Omega(t) \leq \det\widetilde{\Omega}(t)$. Hence, $\hat{r}_H(t) \geq \hat{r}_{\widetilde{H}}(t)$, and further $\hat{t}_{\widetilde{H}}(\hat{r}_H(t)) \leq t$. Now, by (1.6),

$$\operatorname{Im} q_H(i\hat{r}_H(t)) \asymp \frac{1}{\hat{r}_H(t)\omega_{H,2}(t)} \leq \frac{1}{\hat{r}_H(t)\omega_{\widetilde{H},2}(\hat{t}_{\widetilde{H}}(\hat{r}_H(t)))} \asymp \operatorname{Im} q_{\widetilde{H}}(i\hat{r}_H(t)). \qquad \blacksquare$$

The following result elaborates on the relative behavior of $\operatorname{Im} q_H$ and $|q_H|$. We obtain a quantitative and pointwise relation between $\frac{\operatorname{Im} q_H}{|q_H|}$ and $\frac{\det\Omega}{\omega_1\omega_2}$, leading to the equivalence

$$\lim_{r\to\infty} \frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} = 0 \iff \lim_{t\to\hat{a}} \frac{\det\Omega(t)}{(\omega_1\omega_2)(t)} = 0. \tag{2.13}$$

The relation between $\frac{\det\Omega}{\omega_1\omega_2}$ and $\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|}$ has been investigated also in [19]. Their proof of (2.13),[6] is based on compactness arguments.

Note that our result shows that (2.13) holds true for $r \to 0$ and $t \to b$ as well.

2.5 Proposition. *Let H be a Hamiltonian on $[a, b)$. Then*[7]

$$\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} \asymp \frac{\mathring{r}(\hat{t}(r))}{r} = \sqrt{\frac{\det\Omega(\hat{t}(r))}{(\omega_1\omega_2)(\hat{t}(r))}} \tag{2.14}$$

for $r \in (0,\infty)$. Moreover,

$$|q_H(i\mathring{r}(\hat{t}(r)))| \asymp |q_H(ir)|, \quad r \in (0,\infty). \tag{2.15}$$

All constants implicit in $\asymp$ do not depend on H.

Proof. By definition of $\mathring{r}$ and using (1.6) and (1.7),

$$\mathring{r}(\hat{t}(r)) = \frac{1}{\sqrt{(\omega_1\omega_2)(\hat{t}(r))}} \asymp r\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|}.$$

We also have

$$\sqrt{\frac{\det\Omega(\hat{t}(r))}{(\omega_1\omega_2)(\hat{t}(r))}} = \frac{1}{\sqrt{r^2(\omega_1\omega_2)(\hat{t}(r))}} = \frac{\mathring{r}(\hat{t}(r))}{r},$$

and (2.14) follows.

[6] In [19] $\lim_{t\to a}$ was considered instead of $\lim_{t\to\hat{a}}$.

[7] $\mathring{r}(\hat{t}(r))$ is well defined because of $\hat{t}(r) \in (\hat{a}, b) \subseteq (\mathring{a}, b)$.

For the proof of (2.15), we need the formula

$$\omega_1(\mathring{t}(r)) \asymp \frac{|q_H(ir)|}{r}$$

which we get from Theorem 2.1 applied to $J^\top HJ$. Combine this with (2.2) to get

$$|q_H(ir)|^2 \asymp \frac{\omega_1(\mathring{t}(r))}{\omega_2(\mathring{t}(r))}.$$

On the other hand, (1.6) and (1.7) give

$$|q_H(ir)|^2 \asymp \frac{\omega_1(\hat{t}(r))}{\omega_2(\hat{t}(r))} = \frac{\omega_1(\mathring{t}(\mathring{r}(\hat{t}(r))))}{\omega_2(\mathring{t}(\mathring{r}(\hat{t}(r))))} \asymp |q_H(i\mathring{r}(\hat{t}(r)))|^2. \qquad \blacksquare$$

The freedom in the choice of η leads to the following formula that we will refer to later on.

2.6 Corollary. *Let H be a Hamiltonian on $[a,b)$. Then, for any $k > 0$,*

$$\operatorname{Im} q_H(ikr) \asymp \left| q_H(ikr) - \frac{\omega_3(\hat{t}(r))}{\omega_2(\hat{t}(r))} \right| \asymp \left| q_H(ir) - \frac{\omega_3(\hat{t}(r))}{\omega_2(\hat{t}(r))} \right| \tag{2.16}$$

with constants depending on k, but not on H.

If $\operatorname{Im} q_H(ir) = \mathrm{o}(|q_H(ir)|)$ *for* $r \to \infty$ $[r \to 0]$, *then*

$$q_H(ikr) \sim \frac{\omega_3(\hat{t}(r))}{\omega_2(\hat{t}(r))}, \quad r \to \infty\ [r \to 0]. \tag{2.17}$$

Proof. Apply Theorem 1.1 to H using $\hat{t}_{1,H}$, and to kH using $\hat{t}_{k,kH}$. Then $\hat{t}_{1,H}(r) = \hat{t}_{k,kH}(r)$, and we write $\hat{t}(r)$ for short. Keeping in mind that $q_{kH}(z) = q_H(kz)$, this leads to

$$\operatorname{Im} q_H(ir) \asymp \left| q_H(ir) - \frac{\omega_{H,3}(\hat{t}(r))}{\omega_{H,2}(\hat{t}(r))} \right| \asymp \frac{1}{r\omega_{H,2}(\hat{t}(r))}$$

as well as

$$\operatorname{Im} q_H(ikr) \asymp \left| q_H(ikr) - \frac{k\omega_{H,3}(\hat{t}(r))}{k\omega_{H,2}(\hat{t}(r))} \right| \asymp \frac{1}{kr \cdot \omega_{H,2}(\hat{t}(r))}.$$

(2.16) follows. Now, (2.17) is obtained by dividing (2.16) by $|q_H(ikr)|$. $\blacksquare$

3. Behavior of tails of the spectral measure

Theorem 1.1 that approximately determines the imaginary part of $q_H(ir)$ allows us to determine the growth of the spectral measure μ_H relative to suitable comparison functions. Let us introduce the measure $\tilde{\mu}_H$ on $[0,\infty)$ by

$$\tilde{\mu}_H([0,r)) := \tilde{\mu}_H(r) := \mu_H((-r,r)), \quad r > 0. \tag{3.1}$$

In Section 3.1, equivalent conditions are given for when the function $r \mapsto \tilde{\mu}_H$ is integrable with respect to a given weight function, and also when the measure $\tilde{\mu}_H$ is finite with respect to a rescaling function.

On the other hand, we can view $\tilde{\mu}_H$ as a function of the positive real parameter r, and compare this to a given function $\mathfrak{g}$. This is what we do in Section 3.2.

We note that the content of this section is analogous to [18, Section 4]. The availability of formula (1.6) leads to improved results in the present article, however we provide less detail as was given in [18].

The proofs in this section are based on standard theorems of Abelian-Tauberian type, relating μ_H to its Poisson integral

$$\mathcal{P}[\mu_H](z) := \int_{\mathbb{R}} \operatorname{Im}\Big(\frac{1}{t-z}\Big)\, \mathrm{d}\mu_H(t). \tag{3.2}$$

By (1.2), we have $\mathcal{P}[\mu_H](z) = \operatorname{Im} q_H(z) - \beta \operatorname{Im} z$. If $\beta = 0$, we can proceed with the application of Abelian-Tauberian theorems without problems. The case $\beta > 0$ is equivalent to a being the left endpoint of an H-indivisible interval of type $\frac{\pi}{2}$, i.e., $\mathring{a}(H) > a$ and h_2 vanishes a.e. on $[a, \mathring{a}(H))$. The restricted Hamiltonian $H_- := H|_{[\mathring{a}(H),b)}$ then has the Weyl coefficient $q_{H_-}(z) = q_H(z) - \beta z$ and thus $\operatorname{Im} q_{H_-}(z) = \mathcal{P}[\mu_H](z)$. Hence, we can investigate μ_H by applying the theorems from this section to H_-.

3.1. Finiteness of the spectral measure with respect to given weight functions

3.1 Theorem. *Let H be a Hamiltonian defined on $[a,b)$, and assume that h_2 does not vanish identically in a neighborhood of a. Let $\mathfrak{f}$ be a continuous, nondecreasing function, and denote by μ_H the spectral measure of H.*

Then the following statements are equivalent:

(i) *we have*

$$\int_1^\infty \tilde{\mu}_H(r)\frac{\mathfrak{f}(r)}{r^3}\, \mathrm{d}r < \infty; \tag{3.3}$$

(ii) *there is $a' \in (\hat{a}, b)$ such that*

$$\int_{\hat{a}}^{a'} \frac{1}{\omega_2(t)^2} \begin{pmatrix} \omega_2(t) \\ -\omega_3(t) \end{pmatrix}^* H(t) \begin{pmatrix} \omega_2(t) \\ -\omega_3(t) \end{pmatrix} \cdot \ell(\det \Omega(t)^{-\frac{1}{2}})\,\mathrm{d}t < \infty.$$

If, in addition, ℓ is differentiable, then the above condition holds if and only if there is $a' \in (\hat{a}, b)$ such that

$$\int_{\hat{a}}^{a'} \frac{(\det \Omega)'(t)}{\omega_2(t) \det \Omega(t)^{\frac{1}{2}}} \ell'(\det \Omega(t)^{-\frac{1}{2}})\,\mathrm{d}t < \infty.$$

Proof. First note that finiteness of the integrals in the proposition clearly does not depend on $a' \in (\hat{a}, b)$.

Let ξ be the measure on $[1, \infty)$ such that $\ell(r) = \xi([1, r))$, $r \geq 1$. It follows from [13, Lemma 4] that

$$\int_{[1,\infty)} \frac{\mathcal{P}[\mu_H](ir)}{r}\,\mathrm{d}\xi(r) < \infty \iff \int_1^\infty \frac{\tilde{\mu}_H(r)\ell(r)}{r^3}\,\mathrm{d}r < \infty.$$

Since h_2 does not vanish identically in a neighborhood of a, we have $\mathcal{P}[\mu_H] = \operatorname{Im} q_H$. By Theorem 1.1, we have

$$\frac{\mathcal{P}[\mu_H](ir)}{r} \asymp \frac{1}{r^2 \omega_2(\hat{t}(r))} \asymp \frac{\det \Omega(\hat{t}(r))}{\omega_2(\hat{t}(r))}.$$

Hence

$$\int_1^\infty \tilde{\mu}_H(r) \frac{\ell(r)}{r^3}\,\mathrm{d}r < \infty \iff \int_{[1,\infty)} \frac{\det \Omega(\hat{t}(r))}{\omega_2(\hat{t}(r))}\,\mathrm{d}\xi(r) < \infty. \tag{3.4}$$

We define a measure ν on $(0, \infty)$ via $\nu((r, \infty)) = \frac{\det \Omega(\hat{t}(r))}{\omega_2(\hat{t}(r))}$, $r > 0$. Let $\hat{\nu}$ be the measure on $(\hat{a}, b)$ satisfying $\hat{\nu}((\hat{a}, t)) = \nu((\hat{r}(t), \infty)) = \frac{\det \Omega(t)}{\omega_2(t)}$, $t > \hat{a}$. Integrating by parts (see, e.g., [11, Lemma 2]), we can rewrite the first integral in (3.4) as follows:

$$\begin{aligned}
\int_{[1,\infty)} \frac{\det \Omega(\hat{t}(r))}{\omega_2(\hat{t}(r))}\,\mathrm{d}\xi(r) &= \int_{[1,\infty)} \nu((r, \infty))\,\mathrm{d}\xi(r) = \int_{[1,\infty)} \ell(r)\,\mathrm{d}\nu(r) \\
&= \int_{(\hat{a},\hat{t}(1)]} \ell(\hat{r}(t))\,\mathrm{d}\hat{\nu}(t) = \int_{(\hat{a},\hat{t}(1)]} \ell(\hat{r}(t))\,\mathrm{d}\Big(\frac{\det \Omega}{\omega_2}\Big)(t) \\
&= \int_{\hat{a}}^{\hat{t}(1)} \ell(\hat{r}(t)) \cdot \frac{1}{\omega_2(t)^2} \begin{pmatrix} \omega_2(t) \\ -\omega_3(t) \end{pmatrix}^* H(t) \begin{pmatrix} \omega_2(t) \\ -\omega_3(t) \end{pmatrix} \mathrm{d}t.
\end{aligned}$$

To prove the additional statement, let us assume that $\mathscr{f}$ is differentiable. Using a substitution we can rewrite the second integral in (3.4) differently:

$$\int_{[1,\infty)} \frac{\det\Omega(\hat{t}(r))}{\omega_2(\hat{t}(r))}\,\mathrm{d}\,\xi(r) = \int_1^{\infty} \frac{\det\Omega(\hat{t}(r))}{\omega_2(\hat{t}(r))}\mathscr{f}'(r)\,\mathrm{d}\,r = \int_{\hat{t}(r)}^{\hat{a}} \frac{\det\Omega(t)}{\omega_2(t)}\mathscr{f}'(\hat{r}(t))\hat{r}'(t)\,\mathrm{d}\,t$$

$$= \frac{1}{2}\int_{\hat{a}}^{\hat{t}(r)} \frac{\det\Omega(t)}{\omega_2(t)}\mathscr{f}'(\hat{r}(t))\frac{(\det\Omega)'(t)}{\det\Omega(t)^{\frac{3}{2}}}\,\mathrm{d}\,t. \qquad \blacksquare$$

The following result provides, in particular, information on when the measure $\tilde{\mu}_H$ is finite with respect to a regularly varying rescaling function $\mathscr{g}$.

3.2 Corollary. *Let H be a Hamiltonian on $[a,b)$, and assume that h_2 does not vanish identically in a neighborhood of a. Let $\mathscr{g}$ be a continuous function that is regularly varying with index $\alpha \in [0,2]$, and denote by μ_H the spectral measure of H as in* (1.2)*. Then, for $\alpha \in (0,2)$ and every $a' \in (\hat{a},b)$, the following statements are equivalent:*

(i) $$\int_{[1,\infty)} \frac{\mathrm{d}\,\tilde{\mu}_H(r)}{\mathscr{g}(r)} < \infty;$$

(ii) $$\int_{\hat{a}}^{a'} \frac{1}{\omega_2(t)^2}\begin{pmatrix}\omega_2(t)\\ -\omega_3(t)\end{pmatrix}^{*} H(t)\begin{pmatrix}\omega_2(t)\\ -\omega_3(t)\end{pmatrix}\frac{\mathrm{d}\,t}{\det\Omega(t)\,\mathscr{g}(\det\Omega(t)^{-\frac{1}{2}})} < \infty;$$

(iii) $$\int_{\hat{a}}^{a'} \frac{(\det\Omega)'(t)}{\omega_2(t)\det\Omega(t)\,\mathscr{g}(\det\Omega(t)^{-\frac{1}{2}})}\,\mathrm{d}\,t < \infty;$$

If $\alpha = 0$, then (iii) $\Longrightarrow$ (i) *and* (iii) $\Longleftrightarrow$ (ii), *while for $\alpha = 2$ we have* (iii) $\Longrightarrow$ (i) *and* (iii) $\Longrightarrow$ (ii).

Proof. The increasing function $\mathscr{f}(r) := \int_1^r \frac{t}{\mathscr{g}(t)}\,dt$ is regularly varying by Karamata's Theorem [5, Propositions 1.5.8 and 1.5.9a]. Moreover,

$$\mathscr{f}(r)\begin{cases} \asymp \frac{r^2}{\mathscr{g}(r)}, & 0 \le \alpha < 2,\\ \gg \frac{r^2}{\mathscr{g}(r)}, & \alpha = 2. \end{cases} \tag{3.5}$$

Clearly, (iii) is equivalent to

$$\int_{\hat{a}}^{a'} \frac{(\det\Omega)'(t)}{\omega_2(t)\det\Omega(t)^{\frac{1}{2}}}\mathscr{f}'(\det\Omega(t)^{-\frac{1}{2}})\,\mathrm{d}\,t < \infty$$

which is the term appearing in the additional statement of Theorem 3.1. Applying Theorem 3.1 and using (3.5), this is equivalent to (for $\alpha \in [0,2)$) or implies (for $\alpha = 2$) both (ii) and

$$\int_1^\infty \tilde{\mu}_H(r) \frac{\mathrm{d}r}{r g(r)} < \infty.$$

By [18, Proposition 4.5], this is further equivalent to (for $\alpha \in (0,2]$) or implies (for $\alpha = 0$) the first item. ∎

3.2. Comparative growth of the distribution function

In this section we investigate lim sup-conditions for the quotient $\frac{\tilde{\mu}_H(r)}{g(r)}$ instead of integrability conditions. Let us introduce the corresponding classes of measures.

3.3 Definition. Let $g(r)$ be a regularly varying function with index $\alpha \in [0,2]$ and $\lim_{r\to\infty} g(r) = \infty$. Then we set

$$\mathcal{F}_g := \{\mu \mid \tilde{\mu}(r) \lesssim g(r),\ r \to \infty\},$$
$$\mathcal{F}_g^0 := \{\mu \mid \tilde{\mu}(r) = \mathrm{o}(g(r)),\ r \to \infty\},$$

where again $\tilde{\mu}(r) := \mu((-r,r))$.

It should be mentioned that, for nondecreasing g, if

$$\int_{[1,\infty)} \frac{\mathrm{d}\tilde{\mu}(r)}{g(r)} < \infty,$$

then $\mu \in \mathcal{F}_g^0 \subseteq \mathcal{F}_g$. For further discussion of this relation, the reader is referred to [18].

3.4 Theorem. *Let H be a Hamiltonian on $[a,b)$, and assume that h_2 does not vanish identically in a neighborhood of a. Let $g(r)$ be a regularly varying function with index $\alpha \in [0,2]$ and $\lim_{r\to\infty} g(r) = \infty$. Denote by μ_H the spectral measure of H. For $\alpha < 2$, the following statements hold:*

(i) $$\mu_H \in \mathcal{F}_g \iff \limsup_{t\to\hat{a}} \frac{1}{\omega_2(t) g(\det\Omega(t)^{-\frac{1}{2}})} < \infty;$$

(ii) $$\mu_H \in \mathcal{F}_g^0 \iff \lim_{t\to\hat{a}} \frac{1}{\omega_2(t) g(\det\Omega(t)^{-\frac{1}{2}})} = 0.$$

If $\alpha = 2$, then the right-hand side of (i), (ii) *implies the left-hand side, respectively.*

Proof. We use [18, Lemma 4.16] which, adapted to our situation, reads as

$$c_\alpha \limsup_{r\to\infty}\Big(\frac{r}{g(r)}\mathcal{P}[\mu_H](ir)\Big) \le \limsup_{r\to\infty} \frac{\tilde{\mu}_H(r)}{g(r)} \le c'_\alpha \limsup_{r\to\infty}\Big(\frac{r}{g(r)}\mathcal{P}[\mu_H](ir)\Big),$$

and the second inequality holds even for $\alpha = 2$. Since h_2 does not vanish identically in a neighborhood of a, we have $\mathcal{P}[\mu_H] = \operatorname{Im} q_H$. Therefore, the assertion follows from Theorem 1.1 and a substitution $r = \hat{r}(t)$. ∎

4. Weyl coefficients with tangential behavior

In this section, we investigate the scenario

$$\lim_{r\to\infty} \frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} = 0 \quad \text{or} \quad \liminf_{r\to\infty} \frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} = 0. \tag{4.1}$$

This is equivalent to tangential behavior of $q_H(ir)$, i.e.,

$$\lim_{r\to\infty} \arg q_H(ir) \in \{0,\pi\} \quad \text{or} \quad \liminf_{r\to\infty} \min\{\arg q_H(ir), \pi - \arg q_H(ir)\} = 0.$$

From Proposition 2.5 we get that

$$\lim_{n\to\infty} \frac{\operatorname{Im} q_H(ir_n)}{|q_H(ir_n)|} = 0 \iff \lim_{n\to\infty} \frac{\det \Omega(\hat{t}(r_n))}{(\omega_1\omega_2)(\hat{t}(r_n))} = 0. \tag{4.2}$$

for every sequence $r_n \to \infty$. All results in this section can be seen from the canonical systems perspective as well as from the Herglotz functions perspective.

To start with, we observe that the second assertion in (4.1) implies the first unless the limit inferior is assumed only along very sparse sequences. We formulate this fact in the language of Herglotz functions, and prove it within the canonical systems setting. However, we do not know a purely function theoretic proof (which may very well exist in the literature).

4.1 Lemma. *Let q be a Herglotz function. Suppose there is a sequence $(r_n)_{n\in\mathbb{N}}$ with $r_n \to \infty$, $\sup_{n\in\mathbb{N}} \frac{r_{n+1}}{r_n} < \infty$, and*

$$\lim_{n\to\infty} \frac{\operatorname{Im} q(ir_n)}{|q(ir_n)|} = 0.$$

Then $\lim_{r\to\infty} \frac{\operatorname{Im} q(ir)}{|q(ir)|} = 0$.

Proof. Let H be a Hamiltonian (on $[0,\infty)$), such that $q = q_H$. Let $d(t) := \frac{\det \Omega(t)}{(\omega_1\omega_2)(t)}$. Set $t_n := \hat{t}(r_n)$, then by (2.14),

$$d(t_n) \asymp \Big(\frac{\operatorname{Im} q(ir_n)}{|q(ir_n)|}\Big)^2 \xrightarrow{n\to\infty} 0.$$

Suppose that the assertion was not true, i.e., there is a sequence $\xi_1 > \xi_2 > \cdots$ converging to 0, such that $d(\xi_k) \geq C > 0$ for all k. For $k \in \mathbb{N}$, set

$$n(k) := \max\{n \in \mathbb{N} \mid t_n > \xi_k\}.$$

We obtain

$$\Big(\frac{r_{n(k)+1}}{r_{n(k)}}\Big)^2 = \frac{\det \Omega(t_{n(k)})}{\det \Omega(t_{n(k)+1})} \geq \frac{\det \Omega(\xi_k)}{\det \Omega(t_{n(k)+1})}$$
$$= \frac{d(\xi_k)}{d(t_{n(k)+1})} \cdot \frac{(\omega_1\omega_2)(\xi_k)}{(\omega_1\omega_2)(t_{n(k)+1})} \geq \frac{C}{d(t_{n(k)+1})} \xrightarrow{k\to\infty} \infty$$

which contradicts our assumption. ∎

Recall formulae (2.15) and (2.14). On an intuitive level, they tell us that in the case that $\operatorname{Im} q_H(ir) \not\asymp |q_H(ir)|$, the growth of $|q_H(ir)|$ is restricted since $\mathring{r}(\hat{t}(r))$ is then far away from r. If read in the other direction, this means that if $|q_H(ir)|$ grows quickly and without oscillating too much, then $\mathring{r}(\hat{t}(r))$ and r should be close to each other, and hence the quotient $\frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|}$ should not decay.

The following definition introduces the notions needed in Theorems 4.3 (a) and 4.3 (b), which confirm this intuition.

4.2 Definition. A measurable function $f\colon(0,\infty)\to(0,\infty)$ is called *regularly varying (at infinity) with index* $\alpha \in \mathbb{R}$ if, for any $\lambda > 0$,

$$\lim_{r\to\infty} \frac{f(\lambda r)}{f(r)} = \lambda^\alpha. \tag{4.3}$$

If $\alpha = 0$, then f is also called *slowly varying (at infinity)*.

A measurable function $f\colon(0,\infty)\to(0,\infty)$ is *positively increasing (at infinity)* if there is $\lambda \in (0,1)$ such that

$$\limsup_{r\to\infty} \frac{f(\lambda r)}{f(r)} < 1. \tag{4.4}$$

Let us say explicitly that we do not require f to be monotone.

4.3 (a) Theorem. *Let $q \neq 0$ be a Herglotz function. If $|q(ir)|$ or $\frac{1}{|q(ir)|}$ is positively increasing at infinity (in particular, if $|q(ir)|$ is regularly varying with index $\alpha \neq 0$), then* $\operatorname{Im} q(ir) \asymp |q(ir)|$ *as $r \to \infty$.*

4.3 (b) Theorem. *Let $q \neq 0$ be a Herglotz function. If* $\operatorname{Im} q(ir) = \mathrm{o}(|q(ir)|)$ *as $r \to \infty$, then, for every $\delta \in [0,1)$,*

$$\lim_{r\to\infty} \frac{q\Big(ir\big[\frac{\operatorname{Im} q(ir)}{|q(ir)|}\big]^\delta\Big)}{q(ir)} = 1. \tag{4.5}$$

For $k > 0$, we also have $\lim_{r\to\infty} \frac{q(ikr)}{q(ir)} = 1$*, in particular, $|q(ir)|$ is slowly varying at infinity.*

4.4 Remark. In Theorem 4.3 (a), the requirement that $|q(ir)|$ should be positively increasing is meaningful. It is not enough that $|q(ir)|$ grows sufficiently fast, say, $|q(ir)| \gtrsim r^\delta$ for $r \to \infty$ and some $\delta > 0$.

In fact, for any given $\delta \in (0,1)$, we construct in Definition 5.2 a Hamiltonian[8] H whose Weyl coefficient q_H satisfies (see Lemma 5.6) $|q_H(ir)| \gtrsim r^\delta$ as $r \to \infty$, but

$$\liminf_{r\to\infty} \frac{\operatorname{Im} q_H(ir)}{|q_H(ir)|} = 0.$$

In other words, $\operatorname{Im} q_H(ir) \not\asymp |q_H(ir)|$.

Note also that for the above-mentioned H, certainly $|q_H(ir)|$ is not slowly varying [5, Proposition 1.3.6]. Hence, in Theorem 4.3 (b) it is not enough to require $\operatorname{Im} q(ir_n) = \mathrm{o}(|q(ir_n)|)$ on some sequence $r_n \to \infty$.

4.5 Example. Let $q(z) = \log z$, satisfying $|q(ir)| = [(\log r)^2 + \frac{\pi^2}{4}]^{1/2}$ which is increasing. However, $\operatorname{Im} q(ir)$ is constant and hence $\operatorname{Im} q(ir) = \mathrm{o}(|q(ir)|)$ as $r \to \infty$. Theorem 4.3 (a) fails because $|q(ir)|$ is not positively increasing.

Proof of Theorem 4.3 (a). Assume first that $|q(ir)|$ is positively increasing. Then there are $\lambda, \sigma \in (0,1)$ and $R > 0$ such that

$$\frac{|q(i\lambda r)|}{|q(ir)|} \le \sigma, \qquad r \ge R. \tag{4.6}$$

Let H be a Hamiltonian with Weyl coefficient $q_H = q$, allowing us to use (2.15).

Suppose that the assertion was not true. Then there is a (without loss of generality, monotone) sequence $r_n \to \infty$ with $\lim_{n\to\infty} \frac{\operatorname{Im} q(ir_n)}{|q(ir_n)|} = 0$. Let $m(n)$ be such that

$$\lambda^{m(n)+1} \le \frac{\mathring{r}(\hat{t}(r_n))}{r_n} < \lambda^{m(n)}.$$

Note that $m(n) \to \infty$ because of (2.14).

Furthermore, (2.15) ensures that there is $\beta > 0$ with

$$\beta \le \frac{|q(i\mathring{r}(\hat{t}(r)))|}{|q(ir)|}, \qquad r \in (0,\infty).$$

We will also need that for $0 < r < r'$,

$$\frac{|q(ir)|}{|q(ir')|} \asymp \frac{r'\omega_2(\mathring{t}(r'))}{r\omega_2(\mathring{t}(r))} \le \frac{r'}{r}$$

because ω_2 is nondecreasing.

---

[8] Choose suitable parameters $p, l \in (0,1)$, such that $\delta = \frac{\log l}{\log(pl)}$, i.e., $p = l^{\delta^{-1}-1}$.

Choosing n so big that $\mathring{r}(\hat{t}(r_n)) \geq R$, we get the contradiction

$$\beta \leq \frac{|q(i\mathring{r}(\hat{t}(r_n)))|}{|q(ir_n)|} = \frac{|q(i\mathring{r}(\hat{t}(r_n)))|}{|q(i\lambda^{m(n)}r_n)|} \cdot \prod_{j=0}^{m(n)-1} \frac{|q(i\lambda^{j+1}r_n)|}{|q(i\lambda^j r_n)|}$$
$$\lesssim \frac{\lambda^{m(n)}r_n}{\mathring{r}(\hat{t}(r_n))}\sigma^{m(n)} \leq \frac{\sigma^{m(n)}}{\lambda} \xrightarrow{n\to\infty} 0.$$

This proves the theorem in the case that $|q(ir)|$ is positively increasing.

If, on the other hand, $\frac{1}{|q(ir)|}$ is positively increasing, we may set $\tilde{q} := -\frac{1}{q}$, for which $|\tilde{q}(ir)|$ is positively increasing. We obtain

$$\frac{\operatorname{Im} q(ir)}{|q(ir)|} = \frac{\operatorname{Im} \tilde{q}(ir)}{|\tilde{q}(ir)|} \asymp 1.$$

Finally, we note that if $|q(ir)|$ is regularly varying with index $\alpha > 0$, then it is also positively increasing. If $|q(ir)|$ is regularly varying with index $\alpha < 0$, then $\frac{1}{|q(ir)|}$ is regularly varying with index $-\alpha > 0$ and thus positively increasing. ■

Our proof of Theorem 4.3 (b) is elementary - only folklore facts that follow from the Herglotz integral representation (1.2) are needed. We would be interested in an elementary proof of Theorem 4.3 (a) as well, which so far we have not found.

One fact needed in the following proof is the following: For any Herglotz function q and any $z \in \mathbb{C}_+$, we have

$$|q'(z)| \leq \frac{\operatorname{Im} q(z)}{\operatorname{Im} z}. \tag{4.7}$$

This can be seen using the representation (1.2): We write

$$q'(z) = b + \int_{\mathbb{R}} \frac{d\sigma(t)}{(t-z)^2}$$

and obtain

$$|q'(z)| \leq b + \int_{\mathbb{R}} \frac{d\sigma(t)}{|t-z|^2} = \frac{\operatorname{Im} q(z)}{\operatorname{Im} z}.$$

Proof of Theorem 4.3 (b). Let $k \in (0,1)$. Then

$$|\log q(ikr) - \log q(ir)| = \left|\int_{kr}^{r} i(\log q)'(is)\, ds\right| \leq \int_{kr}^{r} |(\log q)'(is)|\, ds. \tag{4.8}$$

Apply (4.7) to $\log q$ and to $i\pi - \log q$ to obtain

$$|(\log q)'(is)| \leq \frac{1}{s} \min\{\operatorname{Im}[\log q(is)], \pi - \operatorname{Im}[\log q(is)]\}$$
$$= \frac{1}{s} \min\{\arg q(is), \pi - \arg q(is)\} \asymp \frac{\operatorname{Im} q(is)}{s|q(is)|}$$

for all $s > 0$. We will also need monotonicity in s of $s\frac{\operatorname{Im} q(is)}{|q(is)|}$. In fact, it is easy to see from (1.2) that $s \operatorname{Im} q(is)$ is nondecreasing in s. Now, we can write

$$s\frac{\operatorname{Im} q(is)}{|q(is)|} = \sqrt{s \operatorname{Im} q(is) \cdot s \operatorname{Im}\Big(-\frac{1}{q(is)}\Big)}$$

and hence $s\frac{\operatorname{Im} q(is)}{|q(is)|}$ is nondecreasing in s. Putting together and continuing the estimation in (4.8), we obtain

$$\begin{aligned} |\log q\big(ikr\big) - \log q(ir)| &\lesssim \int_{kr}^{r} \frac{\operatorname{Im} q(is)}{s|q(is)|}\, \mathrm{d}s \leq r\frac{\operatorname{Im} q(ir)}{|q(ir)|} \cdot \int_{kr}^{r} \frac{\mathrm{d}s}{s^2} \\ &= r\frac{\operatorname{Im} q(ir)}{|q(ir)|}\Big(\frac{1}{kr} - \frac{1}{r}\Big) \asymp \frac{\operatorname{Im} q(ir)}{|q(ir)|} \xrightarrow{r\to\infty} 0. \end{aligned} \tag{4.9}$$

This shows $\lim_{r\to\infty} \frac{q(ikr)}{q(ir)} = 1$. To prove (4.5), set $k(r) := \frac{\operatorname{Im} q(ir)}{|q(ir)|}$ and repeat the calculations up to the second to last term in (4.9), but with k replaced by $k(r)^\delta$, where $\delta \in [0, 1)$. Since

$$rk(r)\Big(\frac{1}{rk(r)^\delta} - \frac{1}{r}\Big) \asymp k(r)^{1-\delta} \xrightarrow{r\to\infty} 0,$$

we arrive at (4.5). ∎

Note that $\lim_{r\to\infty} \frac{q(ikr)}{q(ir)} = 1$ is also a consequence of (2.17). The preceding proof, in addition to being elementary, is needed to show (4.5) which, upon taking absolute values, can be seen as slow variation with a rate.

5. Maximal oscillation within Weyl disks

In order to explain the aim of this section, let us first recall the notion of Weyl disks. Let $W(t, z) \in \mathbb{C}^{2\times 2}$ be the fundamental solution of

$$\frac{d}{dt}W(t, z)J = zW(t, z)H(t), \tag{5.1}$$

with initial condition $W(a, z) = I$, solving the transpose of equation (1.1). We define the *Weyl disks*

$$D_{t,z} := \Big\{\frac{w_{11}(t, z)\tau + w_{12}(t, z)}{w_{21}(t, z)\tau + w_{22}(t, z)}\Big|\tau \in \overline{\mathbb{C}_+}\Big\} \subseteq \overline{\mathbb{C}_+}, \tag{5.2}$$

where $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, and the closure is taken in the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For fixed $z \in \mathbb{C}_+$ and $t_1 \leq t_2$, we have $D_{t_1,z} \supseteq D_{t_2,z}$, and the disks shrink down to a single point which is $q_H(z)$:

$$\bigcap_{t \in [a,b)} D_{t,z} = \{q_H(z)\}.$$

Now, we review the estimate (1.3) which has a geometric interpretation. Namely, the functions $L(r)$ and $A(r)$ give, up to $\asymp$, the imaginary part of the bottom and top point of $D_{\mathring{t}(r),ir}$, respectively. The size of $\operatorname{Im} q_H(ir)$ relative to $L(r)$ and $A(r)$ thus corresponds to the vertical position of $q_H(ir)$ within the disk $D_{\mathring{t}(r),ir}$.

In this section we give answers to several questions from [18]. For instance, the question was raised whether there is a Hamiltonian H for which $L(r) \asymp \operatorname{Im} q_H(ir) \not\asymp A(r)$ for $r \to \infty$. The answer to this particular question is no, cf. Proposition 5.1. However[9], $L(r_n) \asymp \operatorname{Im} q_H(ir_n) \ll A(r_n)$ on a subsequence $r_n \to \infty$ is possible, and we provide examples for this in Definition 5.2 and in Example 5.7. The Weyl coefficient of the Hamiltonian constructed in Definition 5.2 exhibits “maximal” oscillatory behavior in the sense that it goes back and forth between the bottoms and tops of the disks $D_{\mathring{t}(r),ir}$.

5.1 Proposition. *Let H be a Hamiltonian on (a,b). The following statements hold.*

(i) *Suppose that $L(r) \not\asymp A(r)$ as $r \to \infty$. Then there exists a sequence $(r_n)_{n\in\mathbb{N}}$ such that $r_n \to \infty$, $L(r_n) \ll A(r_n)$, and*

$$\operatorname{Im} q_H(ir_n) \gtrsim \sqrt{L(r_n)A(r_n)}.$$

(ii) *Suppose that $L(r) \not\asymp A(r)$, but not $L(r) \ll A(r)$ as $r \to \infty$. Then there is also $(r'_n)_{n\in\mathbb{N}}$ with $r'_n \to \infty$, $L(r'_n) \ll A(r'_n)$, and*

$$\operatorname{Im} q_H(ir'_n) \asymp \sqrt{L(r'_n)A(r'_n)}. \tag{5.3}$$

Proof. We shorten notation by setting

$$d(t) := \frac{\det \Omega(t)}{(\omega_1\omega_2)(t)}.$$

By assumption, $\liminf_{t\to\hat{a}} d(t) = 0$. Let $c \in (\hat{a}, b)$ be fixed and set

$$t_n := \max\Big\{t \leq c \;\Big|\; d(t) \leq \frac{1}{n}\Big\}.$$

[9] In this section, we use the more transparent notation $f(r) \ll g(r)$ instead of $f(r) = o(g(r))$.

With $t_n^+ := \hat{t}(\mathring{r}(t_n)) \geq t_n$, we have $d(t_n^+) \geq \frac{1}{n} = d(t_n)$ if n is large enough for $t_n^+ \leq c$ to hold. Using (2.14), we obtain

$$\Big(\frac{\operatorname{Im} q_H(i\mathring{r}(t_n))}{A(\mathring{r}(t_n))}\Big)^2 \asymp d(t_n^+) \geq d(t_n) = \frac{L(\mathring{r}(t_n))}{A(\mathring{r}(t_n))}.$$

Note that $L(\mathring{r}(t_n)) \ll A(\mathring{r}(t_n))$ because of $d(t_n) \to 0$.

Suppose now that $s := \limsup_{r\to\infty} \frac{L(r)}{A(r)} > 0$. Set $\xi_n := \max\{t \leq t_n \mid d(\xi_n) = \frac{s}{2}\}$ and find τ_n between ξ_n and t_n such that $d(\tau_n) = \min\{d(t) \mid t \in [\xi_n, t_n]\}$. Certainly, $d(\tau_n) \leq d(t_n) = \frac{1}{n}$ and $d(\tau_n) \leq d(t)$ for all $t \in [\xi_n, c]$. Also note that by the same arguments as above,

$$\operatorname{Im} q_H(i\mathring{r}(\tau_n)) \gtrsim \sqrt{L(\mathring{r}(\tau_n)A(\mathring{r}(\tau_n)))}. \tag{5.4}$$

We prove next that $\hat{r}(\tau_n) \ll \mathring{r}(\xi_n)$. Note that by passing to a subsequence and possibly switching signs of ω_3 by looking at $J^\top HJ$ instead of H, we can assume that

$$\lim_{n\to\infty} \frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}} = 1.$$

A calculation shows that $\sqrt{(\omega_1\omega_2)(t)} - \omega_3(t)$ is increasing. Hence

$$\begin{aligned}
\Big(\frac{\hat{r}(\tau_n)}{\mathring{r}(\xi_n)}\Big)^2 &= \frac{(\omega_1\omega_2)(\xi_n)}{\det\Omega(\tau_n)} \\
&= \frac{(\omega_1\omega_2)(\xi_n)\big(1 - \frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\big)}{(\sqrt{(\omega_1\omega_2)(\tau_n)} - \omega_3(\tau_n))^2\big(1 + \frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\big)} \\
&\leq \frac{(\omega_1\omega_2)(\xi_n)\big(1 - \frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\big)}{(\sqrt{(\omega_1\omega_2)(\xi_n)} - \omega_3(\xi_n))^2\big(1 + \frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\big)} \\
&= \frac{\big(1 - \frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\big)}{\big(1 - \frac{\omega_3(\xi_n)}{\sqrt{(\omega_1\omega_2)(\xi_n)}}\big)^2\big(1 + \frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}}\big)} \lesssim 1 - \frac{\omega_3(\tau_n)}{\sqrt{(\omega_1\omega_2)(\tau_n)}} \to 0.
\end{aligned} \tag{5.5}$$

Let $\tau_n^- := \mathring{t}(\hat{r}(\tau_n))$. By the calculation above, $\mathring{r}(\tau_n^-) = \hat{r}(\tau_n) < \mathring{r}(\xi_n)$ for large enough n, implying $\tau_n^- > \xi_n$ and hence $d(\tau_n^-) \geq d(\tau_n)$. Consequently,

$$\frac{L(\hat{r}(\tau_n))}{A(\hat{r}(\tau_n))} = d(\tau_n^-) \geq d(\tau_n) \asymp \Big(\frac{\operatorname{Im} q_H(i\hat{r}(\tau_n))}{A(\hat{r}(\tau_n))}\Big)^2.$$

This means that

$$\operatorname{Im} q_H(i\hat{r}(\tau_n)) \leq C\sqrt{L(\hat{r}(\tau_n))A(\hat{r}(\tau_n))}$$

for some $C > 0$ and all large n. Recall (5.4) and choose $C' > 0$, w.l.o.g. $C' < C$, such that

$$\operatorname{Im} q_H(i\mathring{r}(\tau_n)) \geq C' \sqrt{L(\mathring{r}(\tau_n))A(\mathring{r}(\tau_n))}$$

for large n. By continuity, we find, for each large n, an $r'_n \in [\mathring{r}(\tau_n), \hat{r}(\tau_n)]$ with

$$\frac{\operatorname{Im} q_H(ir'_n)}{\sqrt{L(r'_n)A(r'_n)}} \in [C', C],$$

such that $(r'_n)_{n\in\mathbb{N}}$ satisfies (5.3). The only thing left to prove is that $L(r'_n) \ll A(r'_n)$.

Suppose not, then on a subsequence we would have $L(r'_n) \asymp A(r'_n)$. Consider $\xi'_n := \mathring{t}(r'_n) \leq \tau_n$ which would then satisfy $d(\xi'_n) \gtrsim 1$ and hence

$$1 - \frac{\omega_3(\xi'_n)}{\sqrt{(\omega_1\omega_2)(\xi'_n)}} \gtrsim 1.$$

Now, look at (5.5), but with ξ_n replaced by ξ'_n. It follows that, for large n, $\hat{r}(\tau_n) < r'_n$, contradicting the choice of r'_n. ∎

In the following definition, we construct a Hamiltonian by prescribing $f := \frac{\omega_3}{\sqrt{\omega_1\omega_2}}$ and choosing f to be a highly oscillating function. It should be mentioned that the method we use for prescription works on a general basis: any locally absolutely continuous function with values in $(-1, 1)$ occurs as $\frac{\omega_3}{\sqrt{\omega_1\omega_2}}$ for some Hamiltonian. Details can be found in the appendix of the extended preprint [23] of this article.

5.2 Definition. Let $(t_n)_{n\in\mathbb{N}}, (\xi_n)_{n\in\mathbb{N}}$ be sequences of positive numbers converging to zero, where $\xi_{n+1} < t_n < \xi_n$ for all $n \in \mathbb{N}$. Choose $p, l \in (0, 1)$ and set

$$f(t_n) = 1 - p^n, \quad f(\xi_n) = l^n$$

and interpolate between those points using monotone and absolutely continuous functions (e.g., linear interpolation). Set

$$\alpha_1(t) := \begin{cases} \dfrac{f'(t)}{1 - f(t)}, & t \in (\xi_{n+1}, t_n), \\ 0, & t \in (t_n, \xi_n) \end{cases}$$

and

$$\alpha_2(t) := \begin{cases} \dfrac{f'(t)}{1 - f(t)}, & t \in (\xi_{n+1}, t_n), \\ -2\dfrac{f'(t)}{f(t)}, & t \in (t_n, \xi_n) \end{cases}$$

For $t \in [0, t_1]$, let $\omega_i(t) := \exp(-\int_t^{t_1} \alpha_i(s)\,\mathrm{d}s)$, $i = 1, 2$, and $\omega_3(t) := \sqrt{(\omega_1\omega_2)(t)} \cdot f(t)$. Set $h_i(t) = \omega_i'(t)$, $i = 1, 2, 3$, $t \in [0, t_1]$. For $t \in (t_1, \infty)$, let $h_1(t) := 1$ and $h_2(t) := h_3(t) := 0$. Finally, define

$$H_{p,l} := \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}.$$

5.3 Lemma. *$H_{p,l}$ is a Hamiltonian on $[0, \infty)$, and $\omega_i(t) = \int_0^t h_i(s)\,ds$ for $i = 1, 2, 3$ and $t \in [0, t_1]$. Moreover, 0 is not the left endpoint of an $H_{p,l}$-indivisible interval.*

Proof. We write H instead of $H_{p,l}$ for short. First we show that $H(t) \geq 0$ for all $t \in [0, t_1]$. Start by noting that, for $i = 1, 2$,

$$\frac{h_i(t)}{\omega_i(t)} = (\log \omega_i)'(t) = \alpha_i(t),$$

and calculate

$$\begin{aligned} \frac{h_3(t)^2}{(\omega_1\omega_2)(t)} = \frac{[(\sqrt{\omega_1\omega_2} f)'(t)]^2}{(\omega_1\omega_2)(t)} &= \Big(f'(t) + \frac{1}{2}\Big[\frac{h_1(t)}{\omega_1(t)} + \frac{h_2(t)}{\omega_2(t)}\Big] f(t)\Big)^2 \\ &= \Big(f'(t) + \frac{\alpha_1(t) + \alpha_2(t)}{2} f(t)\Big)^2. \end{aligned}$$

If $t \in (t_n, \xi_n)$, then this equates to 0, as does

$$\frac{(h_1h_2)(t)}{(\omega_1\omega_2)(t)} = \alpha_1(t)\alpha_2(t) = 0.$$

For $t \in (\xi_{n+1}, t_n)$,

$$\Big(f'(t) + \frac{\alpha_1(t) + \alpha_2(t)}{2} f(t)\Big)^2 = \Big(\frac{f'(t)}{1 - f(t)}\Big)^2 = \alpha_1(t)\alpha_2(t) = \frac{(h_1h_2)(t)}{(\omega_1\omega_2)(t)}.$$

In both cases, $\det H(t) = 0$. For $i = 1, 2$, as $\alpha_i(t) \geq 0$, $t \in [0, t_1]$, certainly $\omega_i(t)$ is increasing and thus $h_i(t) \geq 0$. This suffices to show that $H(t) \geq 0$.

H is in limit point case since, for $t > t_1$, the trace of $H(t)$ equals 1. To show that $\omega_i(t) = \int_0^t h_i(s)\,ds$, $i = 1, 2, 3$, $t \in [0, t_1]$, we need to check that $\lim_{t\to 0} \omega_i(t) = 0$. For $i = 1$, this follows from

$$\int_0^{t_1} \alpha_1(s)\,\mathrm{d}s = \sum_{n=1}^{\infty} \int_{\xi_{n+1}}^{t_n} \frac{f'(s)}{1 - f(s)}\,\mathrm{d}s = \sum_{n=1}^{\infty} [\log(1 - l^{n+1}) - \log(p^n)] = \infty. \quad (5.6)$$

For $i = 2$, it follows from the fact that $\alpha_2(t) \geq \alpha_1(t)$ for all $t \in [0, t_1]$, and for $i = 3$ it follows from the definition of ω_3 and the fact that $f(t) < 1$, $t \in [0, t_1]$.

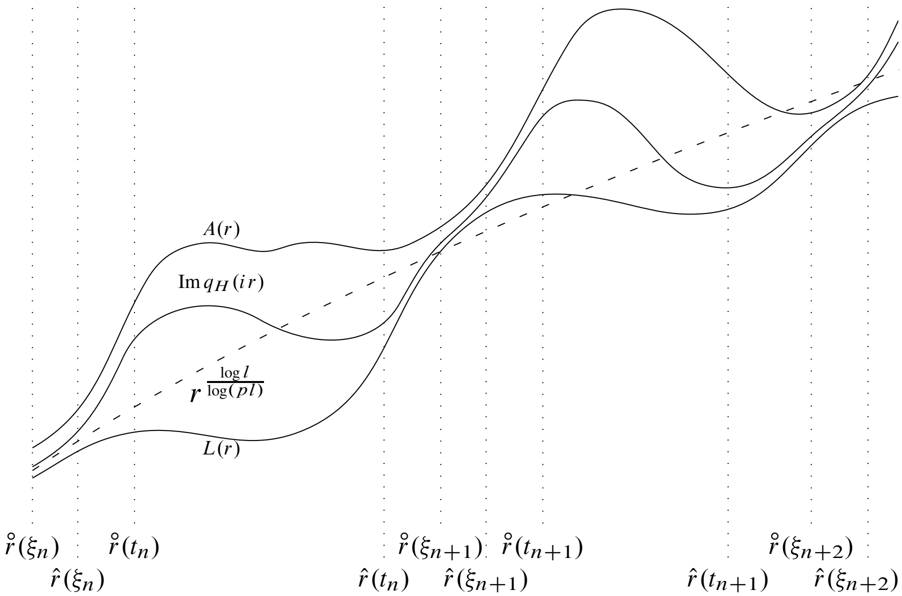

Figure 1. A sketch of the behavior of $q_{H_{p,l}}$.

Finally, 0 is not the left endpoint of an H-indivisible interval because

$$\det \Omega(t) = (\omega_1\omega_2)(t)(1 - f(t)^2) > 0$$

for all $t \in (0, t_1]$. ∎

We investigate the behavior for $r \to \infty$ of $\operatorname{Im} q_{H_{p,l}}(ir)$ as well as $L(r)$ and $A(r)$. A rough description of the situation is given in Figure 1 Formal details are given in the following theorem as well as in Lemma 5.6.

5.4 Theorem. *Let $p, l \in (0, 1)$. For the Hamiltonian $H = H_{p,l}$ from Definition 5.2 and for all sufficiently large $n \in \mathbb{N}$, we have*

$$\mathring{r}(\xi_n) < \hat{r}(\xi_n) < \mathring{r}(t_n) < \hat{r}(t_n) < \mathring{r}(\xi_{n+1}). \tag{5.7}$$

On the intervals delimited by the terms in (5.7), the functions $L(r)$, $\operatorname{Im} q_H(ir)$, and $A(r)$ behave in the following way.

(i) $\operatorname{Im} q_H(ir) \asymp A(r)$ *uniformly for* $r \in [\mathring{r}(\xi_n), \hat{r}(\xi_n)]$, $n \in \mathbb{N}$.

(ii) $\operatorname{Im} q_H(ir) \asymp A(r)$ *uniformly for* $r \in [\hat{r}(\xi_n), \mathring{r}(t_n)]$, $n \in \mathbb{N}$. *Moreover,* $L(\mathring{r}(t_n)) \ll A(\mathring{r}(t_n))$.

(iii) $L(r) \ll A(r)$ *uniformly for* $r \in [\mathring{r}(t_n), \hat{r}(t_n)]$, $n \in \mathbb{N}$. *In addition,* $L(\mathring{r}(t_n)) \ll \operatorname{Im} q_H(i\mathring{r}(t_n)) \asymp A(\mathring{r}(t_n))$ *as well as* $L(\hat{r}(t_n)) \asymp \operatorname{Im} q_H(i\hat{r}(t_n)) \ll A(\hat{r}(t_n))$.

(iv) $L(r) \asymp \operatorname{Im} q_H(ir) \ll A(r)$ *uniformly for* $r \in [\hat{r}(t_n), \mathring{r}(\xi_{n+1})]$, $n \in \mathbb{N}$.

The proof of this theorem involves some (partly tedious) computations that are partly contained in the forthcoming lemma. The symbol $\approx$ should mean equality up to an additive term that is bounded in n and t.

5.5 Lemma. *For the Hamiltonian $H_{p,l}$, the following formulae hold:*

$$\log \mathring{r}(t_n) \approx -n^2\frac{\log(pl)}{2} - n\frac{\log(\frac{l}{p})}{2}, \qquad \log \hat{r}(t_n) \approx -n^2\frac{\log(pl)}{2} - n\frac{\log l}{2},$$
$$\log \mathring{r}(\xi_n) \approx -n^2\frac{\log(pl)}{2} + n\frac{\log(pl)}{2}, \quad \log \hat{r}(\xi_n) \approx -n^2\frac{\log(pl)}{2} + n\frac{\log(pl)}{2},$$
$$\log \mathring{r}(t) \approx -n^2\frac{\log(pl)}{2} - n\frac{\log(\frac{l}{p})}{2} + \log f(t), \qquad t \in [t_n, \xi_n],$$
$$\log \mathring{r}(t) \approx -n^2\frac{\log(pl)}{2} - n\frac{\log(pl)}{2} + \log(1 - f(t)), \qquad t \in [\xi_{n+1}, t_n],$$
$$\log \hat{r}(t) \approx -n^2\frac{\log(pl)}{2} - n\frac{\log(\frac{l}{p})}{2} + \log f(t) - \frac{\log(1 - f(t))}{2}, \quad t \in [t_n, \xi_n],$$
$$\log \hat{r}(t) \approx -n^2\frac{\log(pl)}{2} - n\frac{\log(pl)}{2} + \frac{\log(1 - f(t))}{2}, \qquad t \in [\xi_{n+1}, t_n].$$

Proof. First we calculate

$$\begin{aligned}
\log(\mathring{r}(t_n)) &= -\frac{1}{2}\log[(\omega_1\omega_2)(t_n)] = \frac{1}{2}\int_{t_n}^{t_1} (\alpha_1(s) + \alpha_2(s))\,\mathrm{d}s \\
&= \sum_{k=1}^{n-1}\left(\int_{t_{k+1}}^{\xi_{k+1}} \frac{-f'(s)}{f(s)}\,\mathrm{d}s + \int_{\xi_{k+1}}^{t_k} \frac{f'(s)}{1 - f(s)}\,\mathrm{d}s\right) \\
&= \sum_{k=1}^{n-1}\big(\log(1 - p^{k+1}) - (k+1)\log l + \log(1 - l^{k+1}) - k\log p\big) \\
&\approx -n^2\frac{\log(pl)}{2} - n\frac{\log(\frac{l}{p})}{2}.
\end{aligned} \tag{5.8}$$

This also leads to

$$\begin{aligned}
\log \hat{r}(t_n) &= -\frac{1}{2}\log(1 - f(t_n)^2) + \log \mathring{r}(t_n) \\
&\approx -\frac{1}{2}\log(1 - f(t_n)) + \log \mathring{r}(t_n) \\
&\approx -n^2\frac{\log(pl)}{2} - n\frac{\log l}{2}.
\end{aligned}$$

If $t \in [t_n, \xi_n]$, then

$$\log \mathring{r}(t) = \log \mathring{r}(t_n) - \int_{t_n}^{t} \frac{-f'(s)}{f(s)}\,\mathrm{d}s \approx -n^2 \frac{\log(pl)}{2} - n \frac{\log(\frac{l}{p})}{2} + \log f(t).$$

If $t \in [\xi_{n+1}, t_n]$, then

$$\begin{aligned}\log \mathring{r}(t) &= \log \mathring{r}(t_n) + \int_{t}^{t_n} \frac{f'(s)}{1 - f(s)}\,\mathrm{d}s \\ &\approx -n^2 \frac{\log(pl)}{2} - n \frac{\log(pl)}{2} + \log(1 - f(t)).\end{aligned}$$

By adding $-\frac{1}{2}\log(1 - f(t)^2) \approx -\frac{1}{2}\log(1 - f(t))$, the analogous formula for $\hat{r}(t)$ follows. Lastly,

$$\begin{aligned}\log \mathring{r}(\xi_n) &\approx -n^2 \frac{\log(pl)}{2} - n \frac{\log\left(\frac{l}{p}\right)}{2} + \log f(\xi_n) \\ &\approx -n^2 \frac{\log(pl)}{2} + n \frac{\log(pl)}{2}.\end{aligned}$$

and

$$\log \hat{r}(\xi_n) = -\frac{1}{2}\log(1 - f(\xi_n)^2) + \log \mathring{r}(\xi_n) \approx \log \mathring{r}(\xi_n). \qquad \blacksquare$$

Proof of Theorem 5.4. It follows from Lemma 5.5 that $\hat{r}(\xi_n) < \mathring{r}(t_n)$ and $\hat{r}(t_n) < \mathring{r}(\xi_{n+1})$ for large enough n. The remaining two inequalities in (5.7) follow from the basic fact that $\mathring{r}(t) < \hat{r}(t)$ for all $t \in (0, \infty)$.

We will now prove (i)–(iv) in reverse order.

(iv) $\xi_{n+1} \le \mathring{t}(r) \le t_n$ and $\xi_{n+1} \le \hat{t}(r) \le t_n$. By Lemma 5.5,

$$\begin{aligned}&-n^2 \frac{\log(pl)}{2} - n \frac{\log(pl)}{2} + \frac{1}{2}\log(1 - f(\hat{t}(r))) \approx \log \hat{r}(\hat{t}(r)) = \log r \\ &= \log \mathring{r}(\mathring{t}(r)) \approx -n^2 \frac{\log(pl)}{2} - n \frac{\log(pl)}{2} + \log(1 - f(\mathring{t}(r))).\end{aligned}$$

Hence,

$$\frac{\operatorname{Im} q_H(ir)}{A(r)} \asymp \sqrt{1 - f(\hat{t}(r))^2} \asymp 1 - f(\mathring{t}(r))^2 = \frac{L(r)}{A(r)}.$$

In addition,

$$\frac{L(\mathring{r}(\xi_{n+1}))}{A(\mathring{r}(\xi_{n+1}))} \asymp 1 - f(\xi_{n+1})^2 \asymp 1,$$

while
$$\frac{L(\hat{r}(t_n))}{A(\hat{r}(t_n))} \asymp 1 - f(\mathring{t}(\hat{r}(t_n)))^2 \asymp \sqrt{1 - f(t_n)^2} = p^{\frac{n}{2}} \ll 1.$$

(iii) $\xi_{n+1} \le \mathring{t}(r) \le t_n$ and $t_n \le \hat{t}(r) \le \xi_n$. Thus

$$-n^2\frac{\log(pl)}{2} - n\frac{\log(\frac{l}{p})}{2} + \log f(\hat{t}(r)) - \frac{1}{2}\log(1 - f(\hat{t}(r)))$$
$$\approx \log \hat{r}(\hat{t}(r)) = \log \mathring{r}(\mathring{t}(r)) \approx -n^2\frac{\log(pl)}{2} - n\frac{\log(pl)}{2} + \log(1 - f(\mathring{t}(r))).$$

Consequently,

$$\frac{1}{2}\log\big(1 - f(\hat{t}(r))\big) \approx n\log p + \log f(\hat{t}(r)) - \log\big(1 - f(\mathring{t}(r))\big),$$

which implies
$$\sqrt{1 - f(\hat{t}(r))} \asymp p^n \frac{f(\hat{t}(r))}{1 - f(\mathring{t}(r))}.$$

Let us check that the term $f(\hat{t}(r))$ can be neglected. Using that $f(\mathring{t}(r)) \le 1 - p^n$, we get
$$\sqrt{1 - f(\hat{t}(r))} \lesssim f(\hat{t}(r))$$
which is only possible if $f(\hat{t}(r))$ stays away from 0. As $f(\hat{t}(r)) < 1$, this means that $f(\hat{t}(r)) \asymp 1$, leading to
$$\frac{\operatorname{Im} q_H(ir)}{A(r)} \asymp \sqrt{1 - f(\hat{t}(r))} \asymp \frac{p^n}{1 - f(\mathring{t}(r))}.$$

Hence, $\operatorname{Im} q_H(i\mathring{r}(t_n)) \asymp A(\mathring{r}(t_n))$. Looking back at case (iv), we know that
$$\operatorname{Im} q_H(i\hat{r}(t_n)) \asymp L(\hat{r}(t_n)) \ll A(\hat{r}(t_n)).$$

In particular, since $\frac{L(r)}{A(r)} = 1 - f(\mathring{t}(r))^2$ is increasing for r in $[\mathring{r}(t_n), \hat{r}(t_n)]$, we have $L(r) \ll A(r)$ uniformly on this interval.

(ii) $t_n \le \mathring{t}(r) \le \xi_n$ and $t_n \le \hat{t}(r) \le \xi_n$, leading to

$$-n^2\frac{\log(pl)}{2} - n\frac{\log(\frac{l}{p})}{2} + \log f(\hat{t}(r)) - \frac{1}{2}\log(1 - f(\hat{t}(r)))$$
$$\approx \log \hat{r}(\hat{t}(r)) = \log \mathring{r}(\mathring{t}(r)) \approx -n^2\frac{\log(pl)}{2} - n\frac{\log(\frac{l}{p})}{2} + \log f(\mathring{t}(r)).$$

Hence

$$\sqrt{1 - f(\hat{t}(r))} \asymp \frac{f(\hat{t}(r))}{f(\mathring{t}(r))} > f(\hat{t}(r)).$$

In particular, $1 - f(\hat{t}(r))$ stays away from 0, which means that

$$\frac{\operatorname{Im} q_H(ir)}{A(r)} \asymp \sqrt{1 - f(\hat{t}(r))} \asymp 1.$$

In other words, $\operatorname{Im} q_H(ir) \asymp A(r)$ uniformly for $r \in [\hat{r}(\xi_n), \mathring{r}(t_n)]$. As we already know, $L(\mathring{r}(t_n)) \ll \operatorname{Im} q_H(i\mathring{r}(t_n)) \asymp A(\mathring{r}(t_n))$.

(i) $t_n \leq \mathring{t}(r) \leq \xi_n$ and $\xi_n \leq \hat{t}(r) \leq t_{n-1}$. In this case

$$\begin{aligned} &-n^2 \frac{\log(pl)}{2} + n\frac{\log(pl)}{2} + \frac{1}{2}\log(1 - f(\hat{t}(r))) \\ &\quad \approx \log \hat{r}(\hat{t}(r)) = \log \mathring{r}(\mathring{t}(r)) \approx -n^2 \frac{\log(pl)}{2} - n\frac{\log(\frac{l}{p})}{2} + \log f(\mathring{t}(r)). \end{aligned}$$

Taking into account that $f(\mathring{t}(r)) \geq l^n$ by definition, it follows that

$$\frac{\operatorname{Im} q_H(ir)}{A(r)} \asymp \sqrt{1 - f(\hat{t}(r))} \asymp \frac{f(\mathring{t}(r))}{l^n} \asymp 1.$$

Therefore, $\operatorname{Im} q_H(ir) \asymp A(r)$ uniformly for $r \in [\mathring{r}(\xi_n), \hat{r}(\xi_n)]$. At the left end of this interval, we even have $L(\mathring{r}(\xi_n)) \asymp A(\mathring{r}(\xi_n))$ by case (iv). ■

Before we state our next result, we note that by definition of $H_{p,l}$,

$$\liminf_{t \to 0} \frac{\det \Omega(t)}{(\omega_1 \omega_2)(t)} = \liminf_{t \to 0} (1 - f(t)^2) = 0. \tag{5.9}$$

In view of (2.14), we have

$$\liminf_{r \to \infty} \frac{\operatorname{Im} q_{H_{p,l}}(ir)}{|q_{H_{p,l}}(ir)|} = 0$$

and hence $\operatorname{Im} q_{H_{p,l}}(ir) \not\asymp |q_{H_{p,l}}(ir)|$.

Nevertheless, the following lemma shows that $|q_{H_{p,l}}(ir)|$ grows faster than a power. Recalling Theorem 4.3 (a), this means that $|q(ir)| \gtrsim r^\delta$ for $r \to \infty$ is not a sufficient condition for $\operatorname{Im} q(ir) \asymp |q(ir)|$ as $r \to \infty$. Instead, we see that $|q(ir)|$ being positively increasing really means that not only does $|q(ir)|$ grow sufficiently fast, but also without oscillating too much.

5.6 Lemma. *Let* $\delta := \frac{\log l}{\log(pl)} \in (0, 1)$. *Then*

- $|q_{H_{p,l}}(ir)| \gtrsim r^\delta$, $r \to \infty$,
- $|q_{H_{p,l}}(i\mathring{r}(\xi_n))| \asymp \mathring{r}(\xi_n)^\delta$.

Proof. We start the proof with calculating, for $t \in [t_n, \xi_n]$,

$$
\begin{aligned}
\log \sqrt{\frac{\omega_1(t)}{\omega_2(t)}} &= \frac{1}{2}\log\Big(\frac{\omega_1(t)}{\omega_2(t)}\Big) = \sum_{k=1}^{n-1} \int_{t_{k+1}}^{\xi_{k+1}} \frac{-f'(s)}{f(s)}\,\mathrm{d}s + \int_{t_n}^{t} \frac{f'(s)}{f(s)}\,\mathrm{d}s \\
&= \sum_{k=1}^{n-1} (\log(1-p^{k+1}) - (k+1)\log l) + \log f(t) - \log(1-p^n) \\
&\approx -(n^2+n)\frac{\log l}{2} + \log f(t).
\end{aligned}
$$

Now, we use our formula for $\log \mathring{r}(t)$:

$$
\begin{aligned}
\log \sqrt{\frac{\omega_1(t)}{\omega_2(t)}} &\approx \frac{\log l}{\log(pl)} \log \mathring{r}(t) + \frac{1}{2}\Big(\frac{\log(l)\log(\frac{l}{p})}{\log(pl)} - \log l\Big)n \\
&\quad + \Big(1 - \frac{\log l}{\log(pl)}\Big)\log f(t) \\
&= \frac{\log l}{\log(pl)} \log \mathring{r}(t) + \frac{\log p}{\log(pl)}\big(\log f(t) - n\log l\big), \quad t \in [t_n, \xi_n].
\end{aligned}
\tag{5.10}
$$

Since f was assumed to be monotone decreasing on $[t_n, \xi_n]$, and $\log f(\xi_n) = n\log l$,

$$
\log \sqrt{\frac{\omega_1(t)}{\omega_2(t)}} \gtrapprox \frac{\log l}{\log(pl)} \log \mathring{r}(t) = \delta \log \mathring{r}(t),
$$

where $\gtrapprox$ indicates that the inequality holds up to an additive term that is bounded in n and t. Therefore

$$
|q_{H_{p,l}}(i\mathring{r}(t))| \asymp \sqrt{\frac{\omega_1(t)}{\omega_2(t)}} \gtrsim \mathring{r}(t)^{\delta}, \quad t \in [t_n, \xi_n].
$$

Observing that $\frac{\omega_1}{\omega_2}$ is constant on $[\xi_{n+1}, t_n]$ (since $\alpha_1 - \alpha_2 = 0$ there), we obtain this estimate also for $t \in [\xi_{n+1}, t_n]$:

$$
|q_{H_{p,l}}(i\mathring{r}(t))| \asymp \sqrt{\frac{\omega_1(t)}{\omega_2(t)}} = \sqrt{\frac{\omega_1(\xi_{n+1})}{\omega_2(\xi_{n+1})}} \gtrsim \mathring{r}(\xi_{n+1})^{\delta} \geq \mathring{r}(t)^{\delta}.
$$

Finally, setting $t = \xi_n$ in (5.10) yields $|q_{H_{p,l}}(i\mathring{r}(\xi_n))| \asymp \mathring{r}(\xi_n)^{\delta}$. ∎

5.7 Example. Let H be as in Definition 5.2, but $f(\xi_n) = 1 - l^{n-1}$ instead, where $l > \sqrt{p}$. Similarly to Theorem 5.4, one can show that

$$L(\hat{r}(t_n)) \asymp \operatorname{Im} q_H(i\hat{r}(t_n)) \ll A(\hat{r}(t_n)).$$

However, for our new Hamiltonian,

$$\lim_{t\to 0} \frac{\det \Omega(t)}{(\omega_1\omega_2)(t)} = \limsup_{t\to 0} \frac{\det \Omega(t)}{(\omega_1\omega_2)(t)} = \limsup_{t\to 0} \big(1 - f(t)^2\big) = 0$$

as opposed to (5.9).

6. Reformulation for Krein strings

Recall that a *Krein string* is a pair $S[L, \mathfrak{m}]$ consisting of a number $L \in (0,\infty]$ and a nonnegative Borel measure $\mathfrak{m}$ on $[0, L]$, such that $\mathfrak{m}([0,t])$ is finite for every $t \in [0, L)$, and $\mathfrak{m}(\{L\}) = 0$. To this pair we associate the equation

$$y'_+(x) + z \int_{[0,x]} y(t)\, \mathrm{d}\,\mathfrak{m}(t) = 0, \quad x \in [0, L), \tag{6.1}$$

where y'_+ denotes the right-hand derivative of y, and z is a complex spectral parameter.

For each string, we can construct a function q_S called the *principal Titchmarsh–Weyl coefficient* of the string ([20] following [15]). This function belongs to the Stieltjes class, i.e., it is analytic on $\mathbb{C} \setminus [0,\infty)$, its imaginary part is nonnegative on $\mathbb{C}_+$, and its values on $(-\infty, 0)$ are positive. The correspondence between Krein strings and functions of Stieltjes class is bijective, as was shown by M. G. Krein.

Theorem 6.1 below is the reformulation of Theorem 1.1 for the Krein string case.

6.1 Theorem. *Let $S[L, \mathfrak{m}]$ be a Krein string and set*

$$\delta(t) := \Big(\int_{[0,t)} \xi^2\, \mathrm{d}\,\mathfrak{m}(\xi)\Big) \cdot \Big(\int_{[0,t)} \mathrm{d}\,\mathfrak{m}(\xi)\Big) - \Big(\int_{[0,t)} \xi\, \mathrm{d}\,\mathfrak{m}(\xi)\Big)^2 \tag{6.2}$$

for $t \in [0, L)$. Let

$$\hat{\tau}(r) := \inf\Big\{t > 0 \;\Big|\; \frac{1}{r^2} \le \delta(t)\Big\}, \quad r \in (0,\infty).$$

We set

$$f(r) := \mathfrak{m}([0, \hat{\tau}(r))) + \mathfrak{m}(\{\hat{\tau}(r)\}) \frac{\frac{1}{r^2} - \delta(\hat{\tau}(r))}{\delta(\hat{\tau}(r)+) - \delta(\hat{\tau}(r))} \tag{6.3}$$

if δ is discontinuous at $\hat{\tau}(r)$, and $f(r) := \mathfrak{m}([0, \hat{\tau}(r)))$ otherwise. Then

$$\operatorname{Im} q_S(ir) \asymp \frac{1}{rf(r)}, \quad r \in (0, \infty), \tag{6.4}$$

with constants independent of the string.

Before proving Theorem 6.1, we need to introduce the concept of dual strings as well as a Hamiltonian associated to a string. Writing

$$m(t) := \mathfrak{m}([0, t)), \quad t \in [0, L)$$

we can define the dual string $S[\hat{L}, \hat{\mathfrak{m}}]$ of $S[L, \mathfrak{m}]$ by setting

$$\hat{L} := \begin{cases} m(L) & \text{if } L + m(L) = \infty, \\ \infty & \text{else} \end{cases}$$

and

$$\hat{m}(\xi) := \inf\{t > 0 \mid \xi \leq m(t)\}.$$

The function $\hat{m}$ is increasing and left-continuous and thus gives rise to a nonnegative Borel measure $\hat{\mathfrak{m}}$.

The Hamiltonian defined by

$$H(t) := \begin{cases} \begin{pmatrix} \hat{m}(t)^2 & \hat{m}(t) \\ \hat{m}(t) & 1 \end{pmatrix} & \text{if } t \in [0, \hat{L}], \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \hat{L} + \int_0^{\hat{L}} \hat{m}(t)^2 \, dt < \infty, \ \hat{L} < t < \infty \end{cases} \tag{6.5}$$

then satisfies $q_S = q_H$, see, e.g., [16].

Proof of Theorem 6.1. In view of Theorem 1.1 and the fact that $q_S = q_H$ for the Hamiltonian H defined in (6.5), our task is to express $\hat{t}_H(r)$ in terms of the string. If $\delta(\hat{\tau}(r)) = \frac{1}{r^2}$, this is easy because of [16, Corollary 3.4] giving

$$\det \Omega_H(m(\hat{\tau}(r))) = \delta(\hat{\tau}(r)) = \frac{1}{r^2}$$

and hence $\hat{t}_H(r) = m(\hat{\tau}(r))$.

Otherwise, we have $\delta(\hat{\tau}(r)) < \frac{1}{r^2}$ and $\delta(\hat{\tau}(r)+) \geq \frac{1}{r^2}$. Using again [16, Corollary 3.4], we have

$$\det \Omega_H(m(\hat{\tau}(r))) = \delta(\hat{\tau}(r)) < \frac{1}{r^2}, \quad \det \Omega_H(m(\hat{\tau}(r)+)) = \delta(\hat{\tau}(r)+) \geq \frac{1}{r^2} \tag{6.6}$$

which tells us that $\hat{t}_H(r) \in (m(\hat{\tau}(r)), m(\hat{\tau}(r)+)]$. By [16, Lemma 3.1], $\widehat{m}$ is constant on this interval. Therefore, for $t \in (m(\hat{\tau}(r)), m(\hat{\tau}(r)+)]$,

$$\begin{aligned}\det \Omega_H(t) &= \Big(\int_0^{m(\hat{\tau}(r))} \widehat{m}(x)^2 \, \mathrm{d}x + (t - m(\hat{\tau}(r)))\widehat{m}(t)^2 \Big) \cdot t \\ &\quad - \Big(\int_0^{m(\hat{\tau}(r))} \widehat{m}(x) \, \mathrm{d}x + (t - m(\hat{\tau}(r)))\widehat{m}(t) \Big)^2 \\ &= c_1(r)t + c_2(r)\end{aligned}$$

for some constants $c_1(r), c_2(r)$. Using (6.6), this leads to

$$\det \Omega_H(t) = \delta(\hat{\tau}(r)) + \frac{t - m(\hat{\tau}(r))}{m(\hat{\tau}(r)+) - m(\hat{\tau}(r))}\big(\delta(\hat{\tau}(r)+) - \delta(\hat{\tau}(r))\big).$$

If we equate this to $\frac{1}{r^2}$, we find that

$$\hat{t}_H(r) = m(\hat{\tau}(r)) + \big(m(\hat{\tau}(r)+) - m(\hat{\tau}(r))\big)\frac{\frac{1}{r^2} - \delta(\hat{\tau}(r))}{\delta(\hat{\tau}(r)+) - \delta(\hat{\tau}(r))} = f(r).$$

Now, we have $\omega_{H;2}(t) = \int_0^t h_2(s)\,ds = t$, and Theorem 1.1 now shows

$$\operatorname{Im} q_S(ir) = \operatorname{Im} q_H(ir) \asymp \frac{1}{r\hat{t}_H(r)} = \frac{1}{rf(r)}. \qquad \blacksquare$$

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