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Trace class properties of resolvents of Callias operators

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Abstract. We present conditions for a family $(A(x))_{x\in\mathbb{R}^d}$ of self-adjoint operators in $H^r = \mathbb{C}^r \otimes H$ for a separable complex Hilbert space H, such that the Callias operator $D = ic\nabla + A(X)$ satisfies that $(D^*D+1)^{-N} - (DD^*+1)^{-N}$ is trace class in $L^2(\mathbb{R}^d, H^r)$. Here, $c\nabla$ is the Dirac operator associated to a Clifford multiplication c of rank r on $\mathbb{R}^d$, and $A(X)$ is fibre-wise multiplication with $A(x)$ in $L^2(\mathbb{R}^d, H^r)$.

1. Introduction

Let $d, r \in \mathbb{N}$ and let c be a Clifford multiplication over $\mathbb{R}^d$ in $\mathbb{C}^r$, and consider the associated Dirac operator $c\nabla$ on $\mathbb{R}^d$,

$$c\nabla f = \sum_{j=1}^{d} c(\mathrm{d}x^j)\partial_{x^j} f, \quad f \in C^1(\mathbb{R}^d, \mathbb{C}^r).$$

Let H be a separable complex Hilbert space. Denote $H^r := \mathbb{C}^r \otimes H$. We consider $c\nabla$ as a self-adjoint linear operator in $L^2(\mathbb{R}^d, H^r)$ with domain $\mathrm{Dom}(c\nabla) := W^{1,2}(\mathbb{R}^d, H^r)$, the L^2-Sobolev space of order 1 with values in H^r.

Now, let $A = (A(x))_{x\in\mathbb{R}^d}$ be a family of self-adjoint operators in H^r. Then we associate to A the fibre wise multiplication operator $A(X)$ in $L^2(\mathbb{R}^d, H^r)$, given by

$$(A(X)f)(x) := A(x)f(x), \quad x \in \mathbb{R}^d.$$

The Callias operator D is then

$$D := ic\nabla + A(X).$$

The goal of this paper is to give conditions on the operator family A, such that the

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resolvent powers of D^*D and DD^* are trace comparable, i.e., such that for some $N \in \mathbb{N}$,

$$(D^*D + 1)^{-N} - (DD^* + 1)^{-N} \in S^1(L^2(\mathbb{R}^d, H^r)), \tag{1}$$

where $S^p(Y)$ denotes the p-th Schatten–von Neumann operators on a Hilbert space Y.

Before we proceed with a proper introduction of the involved operators, let us briefly contextualize why one might be interested in property (1).

The name "Callias operator" reflects the contribution of C. Callias in [5] to the Fredholm index problem of the operator D for a family of matrix potentials A, which arises from Yang–Mills theory. A similar index problem in the special case of dimension $d = 1$ had been considered in the seminal series of articles by M. Atiyah, V. Patodi, and I. Singer in [2, 3], and especially [4], dealing with the index problem on manifolds with boundary. In their setup, the family A is given by first order differential operators on a compact manifold.

A. Pushnitski showed in [10] that the Callias operator D in one dimension $d = 1$ for a family $A(x) = A_- + B(x)$, $x \in \mathbb{R}$, where

$$\int_{\mathbb{R}} \|B'(x)\|_{S^1(H)} \,\mathrm{d}x < \infty, \tag{2}$$

satisfies condition (1) for $N = 1$, and one obtains the trace formula

$$\mathrm{tr}_{L^2(\mathbb{R},H)}((D^*D + z)^{-1} - (DD^* + z)^{-1}) = \frac{1}{2z}\mathrm{tr}_H(g_z(A_+) - g_z(A_-)), \quad z \geq 1,$$

where

$$g_z(\lambda) := \frac{\lambda}{(\lambda^2 + z)^{\frac{1}{2}}},$$

and the limits $A_\pm = \lim_{x\to\pm\infty} A(x)$ exist in an appropriate sense. Furthermore, it was shown that one can calculate the Fredholm index of D in terms of the spectral shift function of the operators A_+ and A_-.

The results by Pushnitski were shown under less restrictive conditions on the family $(A(x))_{x\in\mathbb{R}}$ by F. Gesztesy et al. in [7]. The authors were especially able to weaken the required condition (2) to

$$\int_{\mathbb{R}} \|B'(x)(A_-^2 + 1)^{-\frac{1}{2}}\|_{S^1(H)} \,\mathrm{d}x < \infty, \tag{3}$$

which is the first instance where the family of operators $A(x)$ are generated by possibly unbounded, non-commutative and non-discrete perturbations.

More recently, A. Carey et al. in [6] showed property (1) holds for an appropriate $N \in \mathbb{N}$ in the one-dimensional case $d = 1$, if the family A is itself given by perturbations of a Dirac operator by a matrix potential, and if one imposes weaker trace-class requirements on A than (3), dependent on a parameter (on which the choices for N also depend).

The goal of this paper is to present a collection of conditions on the family A for general $d \in \mathbb{N}$, such that A can be obtained by unbounded, non-commutative and non-discrete perturbations of a general self-adjoint model operator A_0, such that the associated Callias operator D satisfies property (1) for a suitable $N \in \mathbb{N}$. The necessary requirements on the trace-class properties of A will be weaker for larger N. To determine the trace and index formula for D in this instance will however be a task for future work.

Let us present the main result of this paper and review the conditions on the operator family A.

We start with reasonably generic assumptions, which allow us to differentiate the operator family A on $\mathbb{R}^d$. These conditions are chosen such that one obtains favourable domain properties of D, and they do not pertain to the trace-class properties directly.

Let A_0 be a self-adjoint operator in H^r and let $A(x)$, $x \in \mathbb{R}^d$, be symmetric operators, with the dense set

$$\mathcal{D} = \bigcup_{n \in \mathbb{N}} \operatorname{rg}(\mathbb{1}_{[-n,n]}(A_0)) \subseteq H^r,$$

contained in all domains $\operatorname{Dom}(A(x))$, $x \in \mathbb{R}^d$. For compatibility with Clifford multiplication, we require that $c(\mathrm{d}x^j)(\mathcal{D}) \subseteq \operatorname{Dom}(A_0)$, $j \in \{1, \dots, d\}$. We assume that for all $\phi \in \mathcal{D}$ and $\psi \in H^r$ the function

$$x \mapsto \langle A(x)\phi, \psi \rangle_{H^r},$$

is Lebesgue measurable on $\mathbb{R}^d$ (see Hypothesis 2.1). For a yet to choose $N \in \mathbb{N}$, we then require that the function $x \mapsto A(x)\phi$ is $(2N-1)$-times weakly differentiable for $\phi \in \mathcal{D}$, with derivatives in $L^2_{\mathrm{loc}}(\mathbb{R}^d, H^r)$, which we abbreviate with $A \in W^{2N-1,2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r))$ (see Definition 2.2). We impose Kato–Rellich type bounds on the derivatives of A to ensure that $(D^*D)^N$ and $(DD^*)^N$ are self-adjoint operators on the domain $W^{2N,2}(\mathbb{R}^d, H^r) \cap L^2(\mathbb{R}^d, \operatorname{Dom}(A_0^{2N}))$ (see Proposition 3.7).

Let us now discuss the trace-class conditions on A, which are the essential assumptions.

Denote $n := \max(\lfloor \frac{d}{2} - 2 \rfloor + 1, 0) \in \mathbb{N}$, and let $\alpha, \beta \in \mathbb{R}^{\geq 0}$. If $A(x)$ commutes with $c(\mathrm{d}x^j)$ for all $x \in \mathbb{R}^d$ and $j \in \{1, \dots, d\}$, then let $N \in \mathbb{N}$ with $N > \frac{\alpha}{2} + \frac{n}{2} + \frac{d}{4}$, otherwise let $N > \max(\frac{\alpha}{2} + \frac{n}{2} + \frac{d}{4}, \frac{\beta}{2} + \frac{n}{2} + \frac{d}{4} + \frac{1}{2})$.

Denote

$$c\nabla^{\mathrm{End}} A := \sum_{j=1}^{d} c(\mathrm{d}x^j)\partial^{\mathrm{End}}_{x^j} A, \quad \nabla^{\mathrm{End}} Ac := \sum_{j=1}^{d} \partial^{\mathrm{End}}_{x^j} Ac(\mathrm{d}x^j),$$

where $(\partial^{\mathrm{End}}_{x^j} A)f = \partial_{x^j}(Af) - A\partial_{x^j} f$ are appropriately defined partial derivatives of operators (see Definition 2.2). Then we assume

$$\int_{\mathbb{R}^d} \|\partial^{\gamma,\mathrm{End}}(c\nabla^{\mathrm{End}} A\langle A_0\rangle^{-\alpha})(x)\|_{S^1(H^r)}\,\mathrm{d}x < \infty, \tag{4a}$$

$$\int_{\mathbb{R}^d} \|\partial^{\gamma,\mathrm{End}}(\nabla^{\mathrm{End}} Ac\langle A_0\rangle^{-\alpha})(x)\|_{S^1(H^r)}\,\mathrm{d}x < \infty, \tag{4b}$$

$$\int_{\mathbb{R}^d} \|\partial^{\gamma,\mathrm{End}}([c(\mathrm{d}x^j), A]\langle A_0\rangle^{-\beta})(x)\|_{S^1(H^r)}\,\mathrm{d}x < \infty, \quad j \in \{1,\dots,d\}, \tag{4c}$$

for all $\gamma \in \mathbb{N}^d$ with $|\gamma| \le n$.

We should stress that $n = 0$ for dimensions $d \le 3$, which contains the situation discussed in [6, 7, 10]. We note especially that the first line of (4) is analogous to the condition in the one-dimensional case for example like (3) in [7]. In case $d = r = 1$, the second and third line of (4) is trivially satisfied since (scalar) c commutes with A automatically. In dimensions $d \ge 4$, we see that $N > 1$ becomes necessary. In any case, we should note the basic feature of all the conditions in (4): one can increase α and β to decrease the restriction on the family A, in exchange for a possibly larger N, which weakens property (5) of the Callias operator D. This is for example useful if A is a family of (pseudo-)differential operators (cf. [11, Chapter 4]). We will illustrate this last point in Example 4.6 for the concrete case of a Dirac operator perturbed by a potential.

All presented conditions on A together (see Hypothesis 4.5) imply the desired property

$$(D^*D + 1)^{-N} - (DD^* + 1)^{-N} \in S^1(L^2(\mathbb{R}^d, H^r)), \tag{5}$$

for the Callias operator D, which is the principal result of this paper (see Theorem 4.7).

2. Notation and basic definitions

We fix i as the imaginary unit in $\mathbb{C}$. Let H be a separable complex Hilbert space and denote with $H^r := H \otimes \mathbb{C}^r$ for $r \in \mathbb{N}$. Let $L^p(\mathbb{R}^d)$ be the space of p-Lebesgue-integrable elements on $\mathbb{R}^d$, and $W^{k,2}(\mathbb{R}^d)$ the L^2-Sobolev space of order k on $\mathbb{R}^d$.

For $1 \leq p \leq \infty$, denote with $L^p(\mathbb{R}^d, H^r)$ the H^r-valued (Bochner–Lebesgue-) L^p-elements over $\mathbb{R}^d$, i.e., $f \in L^p(\mathbb{R}^d, H^r)$ if for all $\phi \in H^r$,

$$\mathbb{R}^d \ni x \mapsto \langle f(x), \phi \rangle_{H^r} \text{ is Lebesgue measurable,}$$
$$\|f\|^p_{L^p(\mathbb{R}^d, H^r)} := \int_{\mathbb{R}^d} \|f(x)\|^p_{H^r} \,\mathrm{d}x < \infty, \quad p < \infty,$$
$$\|f\|_{L^\infty(\mathbb{R}^d, H^r)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \|f(x)\|_{H^r} < \infty, \quad p = \infty.$$

Let $\mathcal{F}$ be the (isometric) Fourier transform

$$\mathcal{F}\colon L^2(\mathbb{R}^d, H^r, \mathrm{d}x) \to L^2\Big(\mathbb{R}^d, H^r, \frac{1}{(2\pi)^d}\,\mathrm{d}\xi\Big),$$
$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} e^{-i\langle x, \xi \rangle_{\mathbb{R}^d}} f(x)\,\mathrm{d}x,$$

obtained via continuous extension and Hilbert space tensor product from the classical Fourier transform on Schwartz functions in $\mathbb{R}^d$. Usually we write $\hat{f} := \mathcal{F}(f)$. Note that as in the scalar case the Fourier transform satisfies by complex interpolation for $f \in L^p(\mathbb{R}^d, H^r)$, $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\hat{f}\|_{L^q(\mathbb{R}^d, H^r)} \leq \|f\|_{L^p(\mathbb{R}^d, H^r)}.$$

We also use the abbreviations

$$\langle z \rangle := (1 + |z|^2)^{\frac{1}{2}}, \quad \mathbb{1}_M(z) := \begin{cases} 1, & z \in M, \\ 0, & z \notin M, \end{cases}$$

for $M \subseteq \mathbb{C}$, and $z \in \mathbb{C}$, and $\mathcal{B}(V)$ for the Borel σ-algebra of a topological space V.

Denote with $W^{k,2}(\mathbb{R}^d, H^r)$ the H^r-valued L^2-Sobolev space of order $k \geq 0$ over $\mathbb{R}^d$, i.e., $f \in W^{k,2}(\mathbb{R}^d, H^r)$ if $f \in L^2(\mathbb{R}^d, H^r)$, and

$$\mathbb{R}^d \ni \xi \mapsto \langle \xi \rangle^k \hat{f}(\xi) \in L^2(\mathbb{R}^d, H^r).$$

We abbreviate partial derivatives by

$$(\partial^\gamma f)(x) = \frac{\partial^{|\gamma|}}{\partial^{\gamma_1}_{x^1} \cdot \ldots \cdot \partial^{\gamma_d}_{x^d}} f(x^1, \ldots, x^d), \quad x \in \mathbb{R}^d,\ f \in C^\infty(\mathbb{R}^d),\ \gamma \in \mathbb{N}^d,$$

and denote the extension of ∂^γ to the Sobolev space $W^{|\gamma|,2}(\mathbb{R}^d, H^r)$ with the same letter. We note that for $k \in \mathbb{N}$,

$$f \in W^{k,2}(\mathbb{R}^d, H^r) \iff \partial^\gamma f \in L^2(\mathbb{R}^d, H^r), \quad |\gamma| \leq k.$$

Let c be a Clifford multiplication over $\mathbb{R}^d$ of rank $r \in \mathbb{N}$, and consider the associated Dirac operator $c\nabla$ on $\mathbb{R}^d$, i.e., there are Clifford matrices $(c(\mathrm{d}x^j))_{j=1}^d \in \mathbb{C}^{r\times r}$, such that

$$c(\mathrm{d}x^k)c(\mathrm{d}x^l) + c(\mathrm{d}x^l)c(\mathrm{d}x^k) = -2\delta_{kl}\mathbb{1}_{\mathbb{C}^{r\times r}}, \quad k,l \in \{1,\dots,d\}.$$

Then $c\nabla$ is the operator in $L^2(\mathbb{R}^d, H^r)$ given by

$$c\nabla f := \sum_{j=1}^d c(\mathrm{d}x^j)\partial_{x^j} f, \quad \mathrm{Dom}(c\nabla) := W^{1,2}(\mathbb{R}^d, H^r).$$

It is well known that $c\nabla$ with the above domain is self-adjoint, and that $\mathrm{Dom}(\langle c\nabla\rangle^k) = W^{k,2}(\mathbb{R}^d, H^r)$ as Banach spaces. The square of the Dirac operator is the Laplacian $\Delta = (c\nabla)^2$, with

$$\Delta f = -\sum_{j=1}^d \partial_{x_j}^2 f, \quad \mathrm{Dom}(\Delta) = W^{2,2}(\mathbb{R}^d, H^r).$$

Note that $\Delta + 1 = \langle c\nabla\rangle^2$.

The Fourier transform $\mathcal{F}$ diagonalizes Δ, i.e., for $f\colon\mathbb{R}\to\mathbb{R}$ a Borel function, and if $f(|X|^2)$ denotes the multiplication operator

$$\begin{aligned}(f(|X|^2)g)(\xi) &:= f(|\xi|^2)g(\xi),\ \xi \in \mathbb{R}^d,\\ \mathrm{Dom}(f(|X|^2)) &:= \Big\{g \in L^2\Big(\mathbb{R}^d, H^r, \frac{1}{(2\pi)^d}\,\mathrm{d}\xi\Big)\colon\\ &\qquad \xi \mapsto f(|\xi|^2)g(\xi) \in L^2\Big(\mathbb{R}^d, H^r, \frac{1}{(2\pi)^d}\,\mathrm{d}\xi\Big)\Big\},\end{aligned}$$

we have that

$$\mathcal{F}\colon \mathrm{Dom}(f(\Delta)) \to \mathrm{Dom}(f(|X|^2))$$

is an isometry with respect to the graph norms and

$$f(\Delta) = \mathcal{F}^{-1} f(|X|^2)\mathcal{F}.$$

Throughout this paper, we assume that A_0 is a given self-adjoint operator in H^r with domain $\mathrm{Dom}(A_0)$, which we may consider as a Banach space if equipped with the graph norm of A_0. We introduce the space

$$\mathcal{D} := \bigcup_{n\in\mathbb{N}} \mathrm{rg}(\mathbb{1}_{[-n,n]}(A_0)) \subseteq H^r,$$

which is dense in H^r, a simple consequence of self-adjoint functional calculus.

In $L^2(\mathbb{R}^d, H^r)$ we may define the constant multiplication operator $\widehat{A_0}$ by

$$(\widehat{A_0} f)(x) := A_0 f(x),\ x \in \mathbb{R}^d, \quad \mathrm{Dom}(\widehat{A_0}) := L^2(\mathbb{R}^d, \mathrm{Dom}(A_0)).$$

We introduce the constant coefficient Callias operator D_0 in $L^2(\mathbb{R}^d, \mathbb{C}^r)$, given by

$$D_0 := ic\nabla + \widehat{A_0}, \quad \mathrm{Dom}(D_0) := W^{1,2}(\mathbb{R}^d, H^r) \cap \mathrm{Dom}(\widehat{A_0}).$$

It has been shown in [7, Lemma 4.2] that D_0 is closed and its graph-norm coincides with the sum of the norms of $W^{1,2}(\mathbb{R}^d, H^r)$ and $L^2(\mathbb{R}^d, \mathrm{Dom}(A_0))$. Furthermore, D_0 is normal with

$$D_0^* = -ic\nabla + \widehat{A_0}, \quad \mathrm{Dom}(D_0^*) = \mathrm{Dom}(D_0).$$

The automatically non-negative, self-adjoint operator $H_0 := D_0^* D_0$ satisfies $H_0 = \Delta + \widehat{A_0}^2$ via the commuting functional calculi of $c\nabla$ and $\widehat{A_0}$.

Let us now introduce the first basic conditions on the operator family A.

Hypothesis 2.1. Assume $A = (A(x))_{x\in\mathbb{R}^d}$ is a family of symmetric operators in H^r, with $\mathrm{Dom}(A(x)) \supseteq \mathcal{D}$, $x \in \mathbb{R}^d$. Furthermore, we suppose that for all $\phi, \psi \in \mathcal{D}$,

$$\mathbb{R}^d \ni x \mapsto \langle A(x)\phi, \psi\rangle_{H^r}$$

is (Lebesgue-)measurable.

Under these conditions on A we may define the multiplication operator $A(X)$ in $L^2(\mathbb{R}^d, H^r)$ by

$$\begin{aligned}&(A(X)f)(x) := A(x)f(x),\ x \in \mathbb{R}^d,\\ &\mathrm{Dom}(A(X)) := \{f \in L^2(\mathbb{R}^d, H^r) | x \mapsto A(x)f(x) \in L^2(\mathbb{R}^d, H^r)\}.\end{aligned}$$

Note that so far we have not enough assumptions on the family A to claim that $A(X)$ is densely defined.

However, we may associate the Callias operator D to the operator family A via

$$\begin{gathered}D = ic\nabla + A(X),\\ \mathrm{Dom}(D) = \mathrm{Dom}(c\nabla) \cap \mathrm{Dom}(A(X)).\end{gathered}$$

Again, we cannot claim that D is densely defined. This will be our first goal in the sequel to find additional conditions on the family A to conclude that D is a closed operator with domain $\mathrm{Dom}(D) = \mathrm{Dom}(D_0)$.

Before we proceed with the first results, we also need to introduce a notion of differentiability for the family A.

Definition 2.2. Let $(Y, \|\cdot\|_Y)$ be a Banach space, and assume that $\mathcal{E} \subseteq Y$ is dense. Let $(B(x))_{x\in\mathbb{R}^d}$ be a measurable family of operators in Y, with domains $\mathrm{Dom}(B(x)) \supseteq \mathcal{E}$, $x \in \mathbb{R}^d$. If for $\gamma \in \mathbb{N}^d$ there exists a measurable family of densely defined operators $C_\gamma(x)$, $x \in \mathbb{R}^d$, with $\mathrm{Dom}(C_\gamma(x)) \supseteq \mathcal{E}$, such that for all $f \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{E}$,

$$\int_{\mathbb{R}^d} C_\gamma(x) f(x)\,\mathrm{d}x = (-1)^{|\beta|} \int_{\mathbb{R}^d} B(x)\partial^\beta f(x)\,\mathrm{d}x,$$

where the integral converges in Y as Bochner integrals, we say $(B(x))_{x\in\mathbb{R}^d}$ is γ-*times weakly differentiable* on $\mathcal{E}$, and we set

$$(\partial^{\gamma,\mathrm{End}} B)(x)\psi := C_\gamma(x)\psi, \quad \psi \in \mathcal{E}.$$

If additionally for all $|\gamma| \le n \in \mathbb{N}$, $K \subset \mathbb{R}^d$ compact, and $\psi \in \mathcal{E}$,

$$\int_K \|(\partial^{\gamma,\mathrm{End}} B)(x)\psi\|_Y^2\,\mathrm{d}x < \infty,$$

we write $B \in W^{n,2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{E}, Y))$.

In our setup the spaces are $Y = H^r$ and $\mathcal{E} = \mathcal{D}$.

3. Domain properties of $A(X)$ and D

In this section we will establish conditions on the family A which ensure that the Callias operator D is closed, the operators D^*D and DD^* are self-adjoint, and their powers have the same domain as the powers of $H_0 = D_0^* D_0$.

We begin by establishing a class of dense sets in $L^2(\mathbb{R}^d, H^r)$, which will be cores for some operators in $L^2(\mathbb{R}^d, H^r)$ we will discuss.

Lemma 3.1. *Let $s \ge 0$. Then*

$$\mathrm{rg}((H_0 + 1)^s|_{C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}})$$

is dense in $L^2(\mathbb{R}^d, H^r)$.

Proof. Let $f \in L^2(\mathbb{R}^d, H^r)$, such that for all $g \in C_c^\infty(\mathbb{R}^d)$ and $\psi \in \mathcal{D}$,

$$\langle (H_0 + 1)^s (g \otimes \psi), f\rangle_{L^2(\mathbb{R}^d, H^r)} = 0.$$

If $\hat{g}$ denotes the Fourier transform of g and $\hat{f} \in L^2(\mathbb{R}^d, H^r)$ the Fourier transform of f, by Plancherel's theorem we find that

$$\int_{\mathbb{R}^d} \hat{g}(\xi)\langle (1 + |\xi|^2 + A_0^2)^s \psi, \hat{f}(\xi)\rangle_{H^r}\,\mathrm{d}\xi = 0.$$

Because $\{\hat{g} | g \in C_c^\infty(\mathbb{R}^d)\}$ is dense in $L^2(\mathbb{R}^d)$, we conclude that for all $\psi \in \mathcal{D}$ exists a nullset $N_\psi \subset \mathbb{R}^d$, such that for $\xi \in \mathbb{R}^d \setminus N_\psi$

$$\langle (1 + |\xi|^2 + A_0^2)^s \psi, \hat{f}(\xi) \rangle_{H^r} = 0.$$

H^r is separable and $\mathcal{D}$ a dense subspace. Therefore, there exists an at most countable subset $\{\psi_n\}_{n \in \mathbb{N}} \subseteq \mathcal{D}$, which is dense in H^r. On the other hand, for all $\xi \in \mathbb{R}^d$ we have $\phi_n := (1 + |\xi|^2 + A_0^2)^{-s} \psi_n \in \mathcal{D}$ for $n \in \mathbb{N}$. Also $N = \bigcup_{n \in \mathbb{N}} N_{\phi_n}$ is still a nullset in $\mathbb{R}^d$. Thus, for $\xi \in \mathbb{R}^d \setminus N$ and all $n \in \mathbb{N}$,

$$\langle \psi_n, \hat{f}(\xi) \rangle_{H^r} = \langle (1 + |\xi|^2 + A_0^2)^s \phi_n, \hat{f}(\xi) \rangle_{H^r} = 0.$$

But $\{\psi_n\}_{n \in \mathbb{N}}$ is dense in H^r, so $\hat{f}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^d$. Thus, $f = 0$ as an element of $L^2(\mathbb{R}^d, H^r)$. Consequently, $\operatorname{rg}((H_0 + 1)^s|_{C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}})$ is dense in $L^2(\mathbb{R}^d, H^r)$. ∎

Let us show that the operator families B we want to consider give rise to densely defined multiplication operators $B(X)$.

Lemma 3.2. *Let*

$$B \in L^{2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r)) := W^{0,2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r)).$$

Then $C_c(\mathbb{R}^d) \otimes \mathcal{D} \in \operatorname{Dom}(B(X))$. *Especially* $B(X)$ *is densely defined.*

Proof. Let $g \in C_c^\infty(\mathbb{R}^d)$ and $\psi \in \mathcal{D}$. We need to show that $x \mapsto g(x)B(x)\psi \in L^2(\mathbb{R}^d, H^r)$, the statement of the lemma then follows by linearity of $B(X)$. To this end let $\operatorname{supp} g \subseteq K$, where $K \subset \mathbb{R}^d$ is compact. Then

$$\int_{\mathbb{R}^d} |g(x)|^2 \|B(x)\psi\|_{H^r}^2 \,\mathrm{d}x \le \sup_{x \in K} |g(x)|^2 \cdot \int_K \|B(x)\psi\|_{H^r}^2 \,\mathrm{d}x < \infty.$$ ∎

The next lemma enables us to decide if a family of bounded operators B gives rise to a bounded multiplication operator $B(X)$ between L^q- and L^2-spaces.

Lemma 3.3. *Let* $B \in L^{2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r))$. *Assume for* $p \ge 2$,

$$x \mapsto \|B(x)\|_{B(H^r)} \in L^p(\mathbb{R}^d),$$

then for $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$,

$$L^q(\mathbb{R}^d, H^r) \subseteq \operatorname{Dom}(B(X)),$$
$$\|B(X)\|_{L^q(\mathbb{R}^d, H^r) \to L^2(\mathbb{R}^d, H^r)} \le \|x \mapsto \|B(x)\|_{B(H^r)}\|_{L^p(\mathbb{R}^d)}.$$

Proof. The statement follows immediately by Hölder's inequality. ∎

Since we want to use Kato–Rellich's theorem for perturbations of the operator $H_0^{\frac{N}{2}}$, we will need to know if a multiplication operator $B(X)$ is relatively bounded with bound strictly less than 1. The next lemma establishes such bounds via Sobolev inequalities.

Lemma 3.4. *Let $B \in L^{2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r))$, and $u \geq 0$. Assume there exists $t \in [0, u]$ and $p \in [2, +\infty]$, such that*

$$\|x \mapsto \|B(x)\langle A_0\rangle^{-2t}\|_{B(H^r)}\|_{L^p(\mathbb{R}^d)} < \infty, \quad \text{for } u - t > \frac{d}{2p},$$

or

$$\|x \mapsto \|B(x)(A_0^2 + z)^{-t}\|_{B(H^r)}\|_{L^p(\mathbb{R}^d)} = o(1), \quad z \to +\infty,$$

for $(u = t, p = 2) \vee (d \geq 3, \frac{d}{p} \in \mathbb{N}, u - t = \frac{d}{2p})$, then

$$\begin{aligned} &\mathrm{Dom}(H_0^u) \subseteq \mathrm{Dom}(B(X)), \\ &\|B(X)(H_0 + z)^{-u}\|_{B(L^2(\mathbb{R}^d, H^r))} = o(1), \quad z \to +\infty. \end{aligned}$$

Proof. Let $z \geq 1$. Then for $p \in (2, +\infty]$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $\frac{1}{q} + \frac{1}{q'} = 1$, and $f \in L^2(\mathbb{R}^d, H^r)$,

$$\begin{aligned} \|(\Delta + z)^{-s} f\|_{L^q(\mathbb{R}^d, H^r)} &\leq \|(|X|^2 + z)^{-s} \hat{f}\|_{L^{q'}(\mathbb{R}^d, H^r)} \\ &\leq \left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (|\xi|^2 + z)^{-sp} \,\mathrm{d}\xi\right)^{\frac{1}{p}} \|f\|_{L^2(\mathbb{R}^d, H^r)} \\ &= (2\pi)^{-pd} z^{\frac{d}{2p} - s} \|\xi \mapsto \langle\xi\rangle^{-2s}\|_{L^p(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d, H^r)}. \end{aligned}$$

Consequently, for $s > \frac{d}{2p}$ there is a constant $C < \infty$, such that

$$\|(\Delta + z)^{-s}\|_{L^2(\mathbb{R}^d, H^r) \to L^q(\mathbb{R}^d, H^r)} \leq C z^{\frac{d}{2t} - s} = o(1), \quad z \to +\infty.$$

For $p = 2$, we obtain

$$\begin{aligned} \|(\Delta + z)^{-s}\|_{B(L^2(\mathbb{R}^d, H^r))} &= \|(|X|^2 + z)^{-s}\|_{B(L^2(\mathbb{R}^d, H^r, \frac{1}{(2\pi)^d} d\xi))} \\ &= \operatorname*{ess\,sup}_{\xi \in \mathbb{R}^d} (|\xi|^2 + z)^{-s} \\ &= z^{-s} = \begin{cases} o(1), \ z \to +\infty, & s > 0, \\ 1, & s = 0. \end{cases} \end{aligned}$$

If $d \geq 3$ and $2s \geq \lceil \frac{d}{p} \rceil$, then the Sobolev inequality ([1, Theorem 4.12, Part III]) yields a constant $C < \infty$, such that for $z \geq 1$,

$$\|(\Delta + z)^{-s}\|_{L^2(\mathbb{R}^d, H^r) \to L^q(\mathbb{R}^d, H^r)} \leq C.$$

This result is only interesting for $\frac{d}{p} \in \mathbb{N}$, otherwise it is superseded by previous estimates.

We also note that by the commuting functional calculi of Δ and $\widehat{A_0}$, that for $s, t \geq 0$, $s + t = u$,

$$\|(\Delta + z)^s(\widehat{A_0}^2 + z)^t(H_0 + z)^{-u}\|_{B(L^2(\mathbb{R}^d, H^r))} \leq 1, \quad z \geq 1.$$

Consequently, we obtain for $s, t \geq 0$, $s + t = u$, by Lemma 3.3 and the assumptions in this lemma,

$$\begin{aligned}
&\|B(X)(H_0 + z)^{-u}\|_{B(L^2(\mathbb{R}^d, H^r))} \\
&\quad \leq \|B(X)(\widehat{A_0}^2 + z)^{-t}\|_{L^q(\mathbb{R}^d, H^r) \to L^2(\mathbb{R}^d, H^r)} \|(\Delta + z)^{-s}\|_{L^2(\mathbb{R}^d, H^r) \to L^q(\mathbb{R}^d, H^r)} \\
&\quad = \|x \mapsto \|B(x)(A_0^2 + z)^{-t}\|_{B(H^r)}\|_{L^p(\mathbb{R}^d)} \\
&\qquad \cdot \begin{cases} o(1), & s > \dfrac{d}{2p}, \\ C, & (s = 0, p = 2) \vee \left(d \geq 3, \dfrac{d}{p} \in \mathbb{N}, s = \dfrac{d}{2p}\right), \end{cases} \qquad z \to +\infty, \\
&\quad = o(1), \quad z \to +\infty. \qquad \blacksquare
\end{aligned}$$

The next ingredient to discuss perturbations of H_0-powers is a Leibniz rule for multiplication operators $B(X)$ in the next statement. The proof is standard, we however give it for self-sufficiency in this operator-valued setup.

Lemma 3.5. *Let $B \in W_{\mathrm{loc}}^{N,2,\mathrm{End}}(\mathbb{R}^d, (\mathcal{D}, H^r))$. For some $N \in \mathbb{N}$. If $f \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$, then $B(X)f \in \mathrm{Dom}(\partial^\gamma) = W^{|\gamma|,2}(\mathbb{R}^d, H^r)$ for $\gamma \in \mathbb{N}^d$, $|\gamma| \leq N$, and*

$$\partial^\gamma B(X)f = \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} (\partial^{\delta,\mathrm{End}} B)(X) \partial^{\gamma - \delta} f.$$

Proof. We proceed by induction on N. The statements of the lemma hold for $|\gamma| = 0$, by Lemma 3.2. Let us assume the statements are true for $n \leq N$. Let $|\hat{\gamma}| = N + 1$. Then there is $\gamma \in \mathbb{N}^d$, and $j \in \{1, \ldots, d\}$, such that $\hat{\gamma} = \gamma + \delta_j$. Since by assumption $B(X)f \in W^{N,2}(\mathbb{R}^d, H^r)$, it suffices to check that $\partial^\gamma B(X)f \in W^{1,2}(\mathbb{R}^d, H^r)$ to conclude $B(X)f \in W^{N+1,2}(\mathbb{R}^d, H^r)$. Also, by assumption, we know that

$$\partial^\gamma B(X)f = \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} (\partial^{\delta,\mathrm{End}} B)(X) \partial^{\gamma - \delta} f. \tag{6}$$

Since, by Lemma 3.2, we have $(\partial^{\delta,\mathrm{End}} B)(X)\partial^{\gamma-\delta} f \in L^2(\mathbb{R}^d, H^r)$, we may consider $(\partial^{\delta,\mathrm{End}} B)(X)\partial^{\gamma-\delta} f$ as an H^r-valued distribution. As such, for any $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$
\begin{aligned}
&(\partial_{x_j}(\partial^{\delta,\mathrm{End}} B)(X)\partial^{\gamma-\delta} f)[\phi] \\
&\quad = -\int_{\mathbb{R}^d} (\partial^{\delta,\mathrm{End}} B)(x)(\partial^{\gamma-\delta} f)(x)(\partial_{x_i}\phi)(x)\,\mathrm{d}x \\
&\quad = ((\partial^{\delta,\mathrm{End}} B)(X)\partial^{\gamma-\delta+\delta_j} f)[\phi] - \int_{\mathbb{R}^d} (\partial^{\delta,\mathrm{End}} B)(x)(\partial_{x_i}((\partial^{\gamma-\delta} f)\phi))(x)\,\mathrm{d}x \\
&\quad = ((\partial^{\delta,\mathrm{End}} B)(X)\partial^{\gamma-\delta+\delta_j} f)[\phi] + ((\partial^{\delta+\delta_j,\mathrm{End}} B)(X)\partial^{\gamma-\delta} f)[\phi]. \qquad (7)
\end{aligned}
$$

The above steps are justified by Lemma 3.2 and the fact that for any $\eta \in \mathbb{N}^d$ the derivative $\partial^\eta f \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$, and $\partial^{\delta,\mathrm{End}} B$, $\partial^{\delta+\delta_j,\mathrm{End}} B \in L^{2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r))$. This argument also yields

$$
(\partial^{\delta,\mathrm{End}} B)(X)\partial^{\gamma-\delta+\delta_j} + (\partial^{\delta+\delta_j,\mathrm{End}} B)(X)\partial^{\gamma-\delta} f \in L^2(\mathbb{R}^d, H^r),
$$

which implies by (7) that

$$
(\partial^{\delta,\mathrm{End}} B)(X)\partial^{\gamma-\delta} f \in W^{1,2}(\mathbb{R}^d, H^r),
$$

and thus by (6), $\partial^\gamma B(X) f \in W^{1,2}(\mathbb{R}^d, H^r)$. Finally, we combine (6) and (7) to conclude

$$
\begin{aligned}
\partial^{\hat{\gamma}} B(X) f &= \sum_{\delta \le \gamma} \binom{\gamma}{\delta} ((\partial^{\delta+\delta_j,\mathrm{End}} B)(X)\partial^{\gamma-\delta} f + (\partial^{\delta,\mathrm{End}} B)(X)\partial^{\gamma-\delta+\delta_j} f) \\
&= \sum_{\beta \le \hat{\gamma}} \binom{\hat{\gamma}}{\delta} (\partial^{\delta,\mathrm{End}} B)(X)\partial^{\hat{\gamma}-\delta} f.
\end{aligned}
$$

∎

The next lemma is closely related with the Leibniz rule in Lemma 3.5, however its usefulness will become only apparent in the next section, when we discuss trace-class properties of a certain class of operators in $L^2(\mathbb{R}^d, H^r)$. Its rough content is that differentiability of an operator family B enables us to extract smoothing operators in $L^2(\mathbb{R}^d, H^r)$ from the left of the multiplication operator $B(X)$.

Lemma 3.6. *Let $B \in W^{2l+k,2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r))$ for $l \in \mathbb{N}$ and $k \in \{0, 1\}$. Then there are constant matrices*

$$
C^{k,l}_{\gamma,\delta} \in \Big\{\lambda_0 1_{H^r} + \sum_{j=1}^{d} \lambda_j c(\mathrm{d}x^j),\ \lambda_j \in \mathbb{Z} + i\mathbb{Z},\ j \in \{0, \ldots, d\}\Big\}, \quad \gamma, \delta \in \mathbb{N}^d,
$$

such that for $f \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$*,*

$$B(X)f = (ic\nabla + 1)^{-k}(\Delta + 1)^{-l} \sum_{\substack{\gamma,\delta \in \mathbb{N}^d \\ |\gamma+\delta| \le 2l+k}} C_{\gamma,\delta}^{k,l} (\partial^{\gamma,\mathrm{End}} B)(X) \partial^\delta f.$$

Proof. The conditions on B imply by Lemma 3.5 that for any $\gamma \in \mathbb{N}^d$, $|\gamma| \le 2l + k$,

$$B(X)f \in \mathrm{Dom}(\partial^\gamma).$$

On the other hand, we may expand $(c\nabla + i)^k(\Delta + 1)^l$ on $C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$ to find matrices

$$c_{\gamma,\delta}^{k,l} \in \Big\{\lambda_0 1_{H^r} + \sum_{j=1}^d \lambda_j c(\mathrm{d}x^j),\ \lambda_j \in \mathbb{Z} + i\mathbb{Z},\ j \in \{0, \dots, d\}\Big\},$$

such that

$$(c\nabla + i)^k(\Delta + 1)^l g = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \le 2l+k}} c_\alpha^{k,l} \partial^\alpha g,\ g \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}.$$

Thus, $B(X)f \in \mathrm{Dom}((c\nabla + i)^k(\Delta + 1)^l)$, and by Lemma 3.5,

$$((c\nabla + i)^k(\Delta + 1)^l)B(X)f = \sum_{\substack{\gamma \in \mathbb{N}^d \\ |\gamma| \le 2l+k}} c_\alpha^{k,l} \sum_{\delta \le \gamma} \binom{\gamma}{\delta} (\partial^{\delta,\mathrm{End}} B)(X) \partial^{\gamma-\delta} f.$$

By applying $(c\nabla + i)^{-k}(\Delta + 1)^{-l}$ to the left, and using that the resolvents of $c\nabla$ and Δ commute, we obtain the claimed statement with $C_{\gamma,\delta}^{k,l} = \binom{\gamma+\delta}{\gamma} c_\gamma^{k,l}$. ■

We arrive at the fundamental result of this section which establishes conditions on the operator family A such that the Callias operator D is closed and

$$\mathrm{Dom}((D^*D)^{\frac{N}{2}}) = \mathrm{Dom}((DD^*)^{\frac{N}{2}}) = \mathrm{Dom}(H_0^{\frac{N}{2}}),$$

for given $N \in \mathbb{N}$.

Proposition 3.7. *Let* $N \in \mathbb{N}$, $N \ge 1$, *and* $A \in W_{\mathrm{loc}}^{N-1,2,\mathrm{End}}(\mathbb{R}^d, (\mathcal{D}, H^r))$. *Assume for* $\gamma \in \mathbb{N}^d$, $1 \le |\gamma| \le N - 1$, *there exist* $t \in [0, \frac{|\gamma|}{2}]$ *and* $p \in [2, +\infty]$ *such that*

$$\|x \mapsto \|(\partial^{\gamma,\mathrm{End}} A)(x)\langle A_0\rangle^{-2t}\|_{B(H^r)}\|_{L^p(\mathbb{R}^d)} < \infty, \quad \text{for } \frac{|\gamma|}{2} - t > \frac{d}{2p},$$

or

$$\|x \mapsto \|(\partial^{\gamma,\mathrm{End}} A)(x)(A_0^2 + z)^{-t}\|_{B(H^r)}\|_{L^p(\mathbb{R}^d)} = o(1),\ z \to +\infty,$$

for $(\frac{|\gamma|}{2} = t, p = 2) \vee (d \geq 3, \frac{d}{p} \in \mathbb{N}, \frac{|\gamma|}{2} - t = \frac{d}{2p})$, *and*

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \| A_0^n (A(x) - A_0)(A_0^2 + z)^{-\frac{n+1}{2}} \|_{B(H^r)} = o(1), \tag{8}$$

for $z \to +\infty$ *and* $0 \leq n \leq N-1$. *Denote for* $j \in \{0, 1\}$ *the operators*

$$D_j = (-1)^j i c \nabla + A(X) \quad \text{and} \quad D_{j,0} = (-1)^j i c \nabla + \widehat{A_0}.$$

Then for any $\eta \in \{0, 1\}^N$, *the operator* $\prod_{k=1}^N D_{\eta_k}$ *is closed with*

$$\begin{aligned} \operatorname{Dom}\Big(\prod_{k=1}^N D_{\eta_k}\Big) &= \operatorname{Dom}\Big(\prod_{k=1}^N D_{\eta_k,0}\Big) = \operatorname{Dom}(H_0^{\frac{N}{2}}) \\ &= W^{N,2}(\mathbb{R}^d, H^r) \cap L^2(\mathbb{R}^d, \operatorname{Dom}(A_0^N)). \end{aligned}$$

Additionally, $\prod_{k=1}^N D_{\eta_k}$ *is self-adjoint if it is symmetric.*

Proof. We first note that by Lemma 3.4 we have

$$\operatorname{Dom}((\partial^{\gamma,\mathrm{End}} A)(X)) \supseteq \operatorname{Dom}(H_0^{\frac{|\gamma|+1}{2}})$$

for $\gamma \in \mathbb{N}^d$, $1 \leq |\gamma| \leq N-1$, and

$$\|(\partial^{\gamma,\mathrm{End}} A)(X)(H_0 + z)^{-\frac{|\gamma|+1}{2}}\|_{B(L^2(\mathbb{R}^d, H^r))} = o(1), \ z \to +\infty.$$

Let $n \in \mathbb{N}$ and let $\phi \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$. Then, if $\hat{\phi}$ denotes the H^r-valued Fourier transform of ϕ,

$$\|H_0^{\frac{n}{2}} \phi\|_{L^2(\mathbb{R}^d, H^r)}^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\sigma(A_0)} (|\xi|^2 + \lambda^2)^n \langle \mathrm{d} E_{A_0}(\lambda) \hat{\phi}(\xi), \hat{\phi}(\xi) \rangle_{H^r} \, \mathrm{d}\xi. \tag{9}$$

Then after multiplying out the polynomial $(|\xi|^2 + \lambda^2)^n$, by Young's inequality, there are constants C_n such that

$$|\xi|^{2n} + \lambda^{2n} \leq (|\xi|^2 + \lambda^2)^n \leq C_n (|\xi|^{2n} + \lambda^{2n}), \quad \xi \in \mathbb{R}^d, \lambda \in \mathbb{R}.$$

Consequently, by (9), we have

$$\begin{aligned} \operatorname{Dom}(H_0^{\frac{n}{2}}) &= \operatorname{Dom}(\langle c\nabla \rangle^n) \cap \operatorname{Dom}(\widehat{A_0}^n) \\ &= W^{n,2}(\mathbb{R}^d, H^r) \cap L^2(\mathbb{R}^d, \operatorname{Dom}(A_0^n)), \end{aligned} \tag{10}$$

where the equality also holds with respect to the topologies induced by graph norms. A similar argument, using the commuting Fourier transform and spectral resolution of A_0, shows that

$$\|(\widehat{A_0}^2 + z)^{\frac{n}{2}}\phi\|_{L^2(\mathbb{R}^d,H^r)} \leq \|(H_0 + z)^{\frac{n}{2}}\phi\|_{L^2(\mathbb{R}^d,H^r)},\ z \geq 1.$$

With these preparations we find for $0 \leq n \leq N - 1$

$$\begin{aligned}
&\|\widehat{A_0}^n(A(X) - \widehat{A_0})\phi\| \\
&\quad \leq \|\widehat{A_0}^n(A(X) - \widehat{A_0})(\widehat{A_0}^2 + z)^{-\frac{n+1}{2}}\|_{B(L^2(\mathbb{R}^d,H^r))}\|(H_0 + z)^{-\frac{n+1}{2}}\phi\|_{L^2(\mathbb{R}^d,H^r)} \\
&\quad = \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \|A_0^n(A(x) - A_0)(A_0^2 + z)^{-\frac{n+1}{2}}\|_{B(H^r)}\|(H_0 + z)^{-\frac{n+1}{2}}\phi\|_{L^2(\mathbb{R}^d,H^r)}.
\end{aligned} \tag{11}$$

Furthermore, there are constant matrices $C^{\gamma,\delta}$ in $\mathbb{C}^{r\times r}$, such that

$$\begin{aligned}
&(c\nabla)^n(A(X) - \widehat{A_0})\phi \\
&\quad = \sum_{|\gamma+\delta|\leq n} C^{\gamma,\delta}(\partial^{\gamma,\mathrm{End}}(A - A_0))(X)\partial^\delta\phi \\
&\quad = \sum_{|\gamma+\delta|\leq n} C^{\gamma,\delta}(\partial^{\gamma,\mathrm{End}}(A-A_0))(X)(H_0 + z)^{-\frac{|\gamma|+1}{2}}\partial^\delta(H_0+z)^{-\frac{|\delta|}{2}}(H_0+z)^{\frac{|\gamma+\delta|+1}{2}}\phi.
\end{aligned}$$

We thus find a constant $C < \infty$, such that

$$\begin{aligned}
&\|(c\nabla)^n(A(X) - \widehat{A_0})\phi\|_{L^2(\mathbb{R}^d,H^r)} \\
&\quad \leq C \sum_{1\leq|\gamma|\leq n} \|(\partial^{\gamma,\mathrm{End}}(A - A_0))(X)(H_0 + z)^{-\frac{|\gamma|+1}{2}}\|_{B(L^2(\mathbb{R}^d,H^r))} \\
&\qquad\qquad \cdot \|(H_0 + z)^{\frac{n+1}{2}}\phi\|_{L^2(\mathbb{R}^d,H^r)} \\
&\qquad + \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \|(A(x) - \widehat{A_0})(A_0^2 + z)^{-\frac{1}{2}}\|_{B(H^r)}\|(H_0 + z)^{\frac{1}{2}}\phi\|_{L^2(\mathbb{R}^d,H^r)}.
\end{aligned} \tag{12}$$

We combine (10), (11), (12), and the prerequisites of this proposition to state, for $0 \leq n \leq N - 1$,

$$\|H_0^{\frac{n}{2}}(A(X) - \widehat{A_0})(H_0 + z)^{-\frac{n+1}{2}}\|_{B(L^2(\mathbb{R}^d,H^r))} = o(1), \quad z \to +\infty,$$

and since

$$\|(H_0 + z)^s(H_0 + 1)^{-s}\|_{B(L^2(\mathbb{R}^d,H^r))} \leq 1, \quad \text{for } z \geq 1,\ s \geq 0,$$

which follows by the functional calculi of $\widehat{A_0}$ and Δ, we may conclude, for $0 \leq n \leq N - 1$,

$$\|(H_0 + z)^{\frac{n}{2}}(A(X) - \widehat{A_0})(H_0 + z)^{-\frac{n+1}{2}}\|_{B(L^2(\mathbb{R}^d,H^r))} = o(1), \quad z \to +\infty. \tag{13}$$

We want to apply the well-known Kato–Rellich theorems ([9, Theorem 4.1.1 and Theorem 5.4.3]) to the operator

$$T_1 := \prod_{k=1}^{N} D_{\eta_k}$$

as a perturbation of $T_0 = \prod_{k=1}^{N} D_{\eta_k,0}$. If we show that $T_1 - T_0$ is relatively bounded by T_0 with a bound strictly less than 1, then the domains of T_0 and T_1 must coincide, T_1 is closed and additionally self-adjoint if it is symmetric. The spectral resolution of $\widehat{A_0}$ and the Fourier transform commute, so we have that

$$\mathrm{Dom}(T_0) = \mathrm{Dom}(H_0^{\frac{N}{2}}),$$

which implies that it suffices to show the relative bound for $H_0^{\frac{N}{2}}$ instead of T_0. Thus, we need to show that

$$\sup_{\phi \in C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}, \phi\neq 0} \frac{\|(T_1 - T_0)\phi\|_{L^2(\mathbb{R}^d, H^r)}}{\|(H_0+z)^{\frac{N}{2}}\phi\|_{L^2(\mathbb{R}^d, H^r)}} = o(1), \quad z \to +\infty. \tag{14}$$

We note that for $\phi \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$,

$$(T_1 - T_0)\phi = \sum_{\alpha\in\{0,1\}^N, |\alpha|\geq 1} \prod_{k=1}^{N} S_{k,\alpha_k}\phi,$$

where $S_{k,0} := D_{\eta_k,0}$, and $S_{k,1} = A(X) - \widehat{A_0}$. We treat each summand separately.

$$\Big\| \prod_{k=1}^{N} S_{k,\alpha_k}\phi \Big\|_{L^2(\mathbb{R}^d, H^r)} \leq \prod_{k=1}^{N} \|(H_0+z)^{\frac{k-1}{2}} S_{k,\alpha_k}(H_0+z)^{-\frac{k}{2}}\|_{B(L^2(\mathbb{R}^d,H^r))} \|(H_0+z)^{\frac{N}{2}}\phi\|_{L^2(\mathbb{R}^d,H^r)}.$$

Because

$$\|(H_0+z)^{\frac{k-1}{2}} D_{\eta_k,0}(H_0+z)^{-\frac{k}{2}}\|_{B(L^2(\mathbb{R}^d,H^r))} \leq 1 \quad \text{for } z \geq 1,$$

and at least one of the factors S_{k,α_k} equals $A(X) - \widehat{A_0}$, we conclude by (13),

$$\Big\| \prod_{k=1}^{N} S_{k,\alpha_k}\phi \Big\|_{L^2(\mathbb{R}^d, H^r)} = \|(H_0+z)^{\frac{N}{2}}\phi\|_{L^2(\mathbb{R}^d,H^r)} o(1), \ z \to +\infty,$$

which implies the required asymptotic (14). ∎

Remark 3.8. We note that condition (8) from Proposition 3.7 implies for a.e. $x \in \mathbb{R}^d$,

$$\|(A(x) - A_0)(A_0^2 + z)^{-\frac{1}{2}}\|_{B(H^r)} = o(1), \quad z \to +\infty,$$

which implies by the self-adjoint Kato–Rellich theorem ([9, Theorem 5.4.3]) that $\mathrm{Dom}(A(x)) = \mathrm{Dom}(A_0)$, for a.e. $x \in \mathbb{R}^d$.

4. Main results

In this chapter we will discuss trace class properties of operators arising from multiplication operators, which then leads to the principal result of this work, Theorem 4.7. We begin therefore with a lemma giving conditions on an operator family B, such that a class of operators associated with the multiplication operator $B(X)$ and the Dirac operator $c\nabla$ are Hilbert–Schmidt operators in $L^2(\mathbb{R}^d, H^r)$.

Lemma 4.1. *Let $(B(x))_{x\in\mathbb{R}^d}$ be a measurable family of operators with*

$$\mathcal{D} \subseteq \mathrm{Dom}(B(x)), \quad x \in \mathbb{R}^d,$$

such that

$$\int_{\mathbb{R}^d} \|B(x)\|_{S^1(H^r)}\,\mathrm{d}x < \infty.$$

Let $s \geq 0$, $u \geq \frac{s}{2}$, $t < 2u - s - \frac{d}{2}$, and $f : \mathbb{R}^d \to \mathbb{C}$ measurable with

$$\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^d} |f(\xi)|\langle\xi\rangle^{-t} < \infty,$$

then the operator,

$$Q := \langle\widehat{A_0}\rangle^s f(c\nabla)(H_0 + 1)^{-u}|B(X)|^{\frac{1}{2}},$$

is densely defined and admits a Hilbert–Schmidt extension in $L^2(\mathbb{R}^d, H^r)$.

Proof. Let $K \subset \mathbb{R}^d$ be compact and $\psi \in \mathcal{D}$. Then

$$\begin{aligned}\int_K \||B(x)|^{\frac{1}{2}}\psi\|_{H^r}^2\,\mathrm{d}x &\leq \|\psi\|_{H^r}^2 \int_{\mathbb{R}^d} \||B(x)|^{\frac{1}{2}}\|_{S^2(H^r)}^2\,\mathrm{d}x \\ &= \|\psi\|_{H^r}^2 \int_{\mathbb{R}^d} \|B(x)\|_{S^1(H^r)}\,\mathrm{d}x < \infty.\end{aligned}$$

So, $|B|^{\frac{1}{2}} \in L^{2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r))$. Lemma 3.2 then implies

$$C_c(\mathbb{R}^d) \otimes \mathcal{D} \subseteq \mathrm{Dom}(|B(X)|^{\frac{1}{2}}),$$

thus Q is densely defined.

For $\varepsilon \searrow 0$ and $n \to \infty$ the operators

$$\kappa_{\varepsilon,n} := e^{-\varepsilon\Delta} \mathbb{1}_{[-n,n]}(\widehat{A_0}),$$

jointly converge strongly to 1 in $L^2(\mathbb{R}^d, H^r)$. Moreover, for $\varepsilon > 0$ and $n \in \mathbb{N}$

$$\kappa_{\varepsilon,n}(x, y) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle x-y,\xi\rangle} e^{-\varepsilon|\xi|^2} \,\mathrm{d}\xi \; \mathbb{1}_{[-n,n]}(A_0)$$

is a smooth $B(H^r)$-valued kernel of $\kappa_{\varepsilon,n}$. Let

$$Q_{\varepsilon,n} := \kappa_{\varepsilon,n} Q.$$

For $\varepsilon > 0$ and $n \in \mathbb{N}$ its $B(H^r)$-valued integral kernel $k_{Q_{\varepsilon,n}}$ at $x, y \in \mathbb{R}^d$ is given by

$$\begin{aligned} k_{Q_{\varepsilon,n}}(x, y) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle x-y,\xi\rangle} e^{-\varepsilon|\xi|^2} f(\xi)(1 + |\xi|^2 + A_0^2)^{-u} \,\mathrm{d}\xi \\ &\qquad \cdot \langle A_0\rangle^s \mathbb{1}_{[-n,n]}(A_0)|B(y)|^{\frac{1}{2}} \\ &= (2\pi)^{-d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{-i\langle x-y,\xi\rangle} e^{-\varepsilon|\xi|^2} f(\xi)(1 + |\xi|^2 + \lambda^2)^{-u} \,\mathrm{d}\xi \\ &\qquad \cdot \langle\lambda\rangle^s \mathbb{1}_{[-n,n]}(\lambda) \,\mathrm{d}E_{A_0}(\lambda)\, |B(y)|^{\frac{1}{2}}. \end{aligned}$$

Let us introduce the finite Borel measure μ_B on $\mathbb{R}$ given by

$$\mu_B(I) := \int_{\mathbb{R}^d} \|\mathbb{1}_I(A_0)|B(y)|^{\frac{1}{2}}\|^2_{S^2(H^r)} \,\mathrm{d}y, \quad I \in \mathcal{B}(\mathbb{R}),$$

with

$$\begin{aligned} \mu_B(\mathbb{R}) &= \int_{\mathbb{R}^d} \||B(y)|^{\frac{1}{2}}\|^2_{S^2(H^r)} \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \|B(y)\|_{S^1(H^r)} \,\mathrm{d}y < \infty. \end{aligned}$$

If $g\colon \mathbb{R} \to \mathbb{C}$ is an essentially bounded Borel function, then for any orthonormal basis[1] $(\phi_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ of H^r,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left\| \int_{\mathbb{R}} g(\lambda)\,\mathrm{d}\,E_{A_0}(\lambda)|B(y)|^{\frac{1}{2}} \right\|^2_{S^2(H^r)} \mathrm{d}\,y \\
&\quad = \int_{\mathbb{R}^d} \sum_{n\in\mathbb{N}} \int_{\mathbb{R}} |g(\lambda)|^2\,\mathrm{d}\langle E_{A_0}(\lambda)|B(y)|^{\frac{1}{2}}\phi_n, |B(y)|^{\frac{1}{2}}\phi_n\rangle_{H^r}\,\mathrm{d}\,y \\
&\quad = \int_{\mathbb{R}} |g(\lambda)|^2\,\mathrm{d}\Big(\int_{\mathbb{R}^d} \sum_{n\in\mathbb{N}} \langle E_{A_0}(\lambda)|B(y)|^{\frac{1}{2}}\phi_n, |B(y)|^{\frac{1}{2}}\phi_n\rangle_{H^r}\,\mathrm{d}\,y \Big) \\
&\quad = \int_{\mathbb{R}} |g(\lambda)|^2\,\mathrm{d}\,\mu_B(\lambda).
\end{aligned} \tag{15}$$

We return to the integral kernel of $Q_{\varepsilon,n}$. For $\varepsilon, \delta > 0$, $n, m \in \mathbb{N}$, the function

$$\begin{aligned}
&f_{n,m,\varepsilon,\delta}\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}, \\
&f_{n,m,\varepsilon,\delta}(z,\lambda) := (2\pi)^{-d}\langle\lambda\rangle^s \int_{\mathbb{R}^d} e^{-i\langle z,\xi\rangle}(e^{-\varepsilon|\xi|^2}\mathbb{1}_{[-n,n]}(\lambda) - e^{-\delta|\eta|^2}\mathbb{1}_{[-m,m]}(\lambda)) \\
&\qquad\qquad \cdot f(\xi)(1+|\xi|^2+\lambda^2)^{-u}\,\mathrm{d}\,\xi,
\end{aligned}$$

is a bounded Borel function in λ for every $z \in \mathbb{R}^d$. Additionally, $f_{n,m,\varepsilon,\delta}$ is given as a partial Fourier transform in ξ, thus we may utilize Plancherel's theorem for the L^2-norm in z. Let us consider the operator $Q_{\varepsilon,n} - Q_{\delta,m}$

$$\begin{aligned}
&\|Q_{\varepsilon,n} - Q_{\delta,m}\|^2_{S^2(L^2(\mathbb{R}^d,H^r))} \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}d} \|k_{Q_{\varepsilon,n}}(x,y) - k_{Q_{\delta,m}}(x,y)\|^2_{S^2(H^r)}\,\mathrm{d}\,x\,\mathrm{d}\,y \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left\| \int_{\mathbb{R}} f_{n,m,\varepsilon,\delta}((x-y),\lambda)\,\mathrm{d}\,E_{A_0}(\lambda)|B(y)|^{\frac{1}{2}} \right\|^2_{S^2(H^r)} \mathrm{d}\,x\,\mathrm{d}\,y \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left\| \int_{\mathbb{R}} f_{n,m,\varepsilon,\delta}(z,\lambda)\,\mathrm{d}\,E_{A_0}(\lambda)|B(y)^*|^{\frac{1}{2}} \right\|^2_{S^2(H^r)} \mathrm{d}\,y\,\mathrm{d}\,z \\
&\quad \overset{(15)}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}} |f_{n,m,\varepsilon,\delta}(z,\lambda)|^2\,\mathrm{d}\,\mu_B(\lambda)\,\mathrm{d}\,z.
\end{aligned} \tag{16}$$

[1] $\mathcal{D}$ is a dense linear subspace of a separable Hilbert space, thus such an orthonormal basis always exists.

We continue by applying Plancherel's theorem,

$$= (2\pi)^{-d} \int_{\mathbb{R}} \langle\lambda\rangle^{2s} \int_{\mathbb{R}^d} |e^{-\varepsilon|\xi|^2} \mathbb{1}_{[-n,n]}(\lambda) - e^{-\delta|\xi|^2} \mathbb{1}_{[-m,m]}(\lambda)|^2 \cdot |f(\xi)|^2 (1 + |\xi|^2 + \lambda^2)^{-2u} \,\mathrm{d}\xi \;\mathrm{d}\mu_B(\lambda)$$

$$\leq (2\pi)^{-d} \operatorname*{ess\,sup}_{\eta\in\mathbb{R}^d} |f(\eta)|^2 \langle\eta\rangle^{-2t} \cdot \int_{\mathbb{R}} \int_{\mathbb{R}^d} |e^{-\varepsilon|\xi|^2} \mathbb{1}_{[-n,n]}(\lambda) - e^{-\delta|\xi|^2} \mathbb{1}_{[-m,m]}(\lambda)|^2 \langle\xi\rangle^{-4u+2s+2t} \,\mathrm{d}\xi \,\mathrm{d}\mu_B(\lambda).$$

The integrand in the last line of (16) possesses the dominant $(\xi, \lambda) \mapsto \langle\xi\rangle^{-4u+2s+2t}$, which is integrable against $\mathrm{d}\xi \otimes \mathrm{d}\mu_B$, because $-4u + 2s + 2t < -d$ and μ_B is finite. Therefore, we may interchange limits in $\varepsilon, \delta \searrow 0$ and $n, m \to \infty$ with both integrals. Thus

$$\| Q_{\varepsilon,n} - Q_{\delta,m} \|^2_{S^2(L^2(\mathbb{R}^d, H^r))} \xrightarrow{\varepsilon,\delta\searrow 0,\; n,m\to\infty} 0.$$

Since for $\varepsilon > 0$ and $n \in \mathbb{N}$ we may similarly show that

$$\begin{aligned} \| Q_{\varepsilon,n} \|^2_{S^2(L^2(\mathbb{R}^d, H^r))} &\leq (2\pi)^{-d} \operatorname*{ess\,sup}_{\eta\in\mathbb{R}^d} |f(\eta)|^2 \langle\eta\rangle^{-2t} \cdot \int_{\mathbb{R}} \int_{\mathbb{R}^d} |e^{-\varepsilon|\xi|^2} \mathbb{1}_{[-n,n]}(\lambda)|^2 \langle\xi\rangle^{-4u+2s+2t} \,\mathrm{d}\xi \;\mathrm{d}\mu_B(\lambda) \\ &\leq (2\pi)^{-d} \operatorname*{ess\,sup}_{\eta\in\mathbb{R}^d} |f(\eta)|^2 \langle\eta\rangle^{-2t} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \langle\xi\rangle^{-4u+2s+2t} \,\mathrm{d}\xi \;\mathrm{d}\mu_B(\lambda) < \infty, \end{aligned}$$

we conclude that the operators $Q_{\varepsilon,n}$ admit Hilbert–Schmidt extensions which form a Cauchy sequence of Hilbert–Schmidt operators, and thus converge to some $\widetilde{Q} \in S^2(L^2(\mathbb{R}^d, H^r))$. On the other hand, the strong limit of $Q_{\varepsilon,n}$ on their common domain containing $C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$ is Q. Thus, Q admits the extension $\widetilde{Q}$. ∎

An immediate consequence of the previous lemma is that we can give conditions on an operator family B, such that a class of operators associated with the multiplication operator $B(X)$ and the Dirac operator $c\nabla$ become trace-class operators in $L^2(\mathbb{R}^d, H^r)$, by splitting into a product of Hilbert–Schmidt operators.

Corollary 4.2. *Let $B \in L^{2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r))$, such that*

$$\int_{\mathbb{R}^d} \|B(x)\|_{S^1(H^r)} \,\mathrm{d}x < \infty.$$

Let $s_j \geq 0$, $u_j \geq \frac{s_j}{2}$, *and* $t_j < 2u_j - s_j - \frac{d}{2}$ *for* $j \in \{1, 2\}$, *then the operator*

$$S := \langle \widehat{A_0} \rangle^{s_1} \langle c\nabla \rangle^{t_1} (H_0 + 1)^{-u_1} B(X) \langle \widehat{A_0} \rangle^{s_2} \partial^\delta (H_0 + 1)^{-u_2}, \quad \delta \in \mathbb{N}^d, |\delta| \leq t_2,$$

is densely defined on $\operatorname{rg}((H_0 + 1)^{u_2}|_{C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}})$ *and admits a trace-class extension in* $L^2(\mathbb{R}^d, H^r)$.

Proof. The space $\operatorname{rg}((H_0 + 1)^{u_2}|_{C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}})$ is dense in $L^2(\mathbb{R}^d, H^r)$ by Lemma 3.1. Let $f \in \operatorname{rg}((H_0 + 1)^{u_2}|_{C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}})$. Then there is $g \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$, such that $(H_0 + 1)^{-u_2} f = g$. Since $\langle \widehat{A_0} \rangle^{s_2}$ and ∂^δ map $C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$ to itself, we obtain that $\langle \widehat{A_0} \rangle^{s_2} \partial^\delta (H_0 + 1)^{-u_2} f \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$. Since $B \in L^{2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r))$, Lemma 3.2 implies that

$$\langle \widehat{A_0} \rangle^{s_2} \partial^\delta (H_0 + 1)^{-u_2} f \in \operatorname{Dom}(B(X)).$$

Therefore, $\operatorname{rg}((H_0 + 1)^{u_2}|_{C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}}) \subseteq \operatorname{Dom}(S)$, which shows that S is densely defined.

By polar decomposition there exists a measurable family $(U(x))_{x \in \mathbb{R}^d}$ of unitary operators in H^r, such that for a.e. $x \in \mathbb{R}^d$,

$$B(x) = |B(x)^*|^{\frac{1}{2}} U(x) |B(x)|^{\frac{1}{2}}.$$

Accordingly we may decompose

$$\begin{aligned} S &= Q_1 Q_2, \\ Q_1 &= \langle \widehat{A_0} \rangle^{s_1} \langle c\nabla \rangle^{t_2} (H_0 + 1)^{-u_1} |B(X)^*|^{\frac{1}{2}}, \\ Q_2 &= U(X) |B(X)|^{\frac{1}{2}} \langle \widehat{A_0} \rangle^{s_2} \partial^\delta (H_0 + 1)^{-u_2}. \end{aligned}$$

For Q_2 we obtain, using the commutativity of $\widehat{A_0}$, ∂ and H_0 on adequate domains and the $*$-invariance of Hilbert–Schmidt operators,

$$\begin{aligned} \|Q_2\|_{S^2(L^2(\mathbb{R}^d, H^r))} &= \| |B(X)|^{\frac{1}{2}} \langle \widehat{A_0} \rangle^{s_2} \partial^\delta (H_0 + 1)^{-u_2} \|_{S^2(L^2(\mathbb{R}^d, H^r))} \\ &= \| \langle \widehat{A_0} \rangle^{s_2} \partial^\delta (H_0 + 1)^{-u_2} |B(X)|^{\frac{1}{2}} \|_{S^2(L^2(\mathbb{R}^d, H^r))}, \end{aligned}$$

so Q_2 possesses a Hilbert–Schmidt extension by Lemma 4.1. Moreover, Lemma 4.1 also implies that Q_1 admits a Hilbert–Schmidt extension, because $*$-invariance of trace-class operators yields,

$$\int_{\mathbb{R}^d} \|B(x)^*\|_{S^1(H^r)} \, \mathrm{d}x = \int_{\mathbb{R}^d} \|B(x)\|_{S^1(H^r)} \, \mathrm{d}x < \infty.$$

Consequently, S admits a trace-class extension. ■

The next result is concerned with giving conditions on a family B, such that the operator $(H_0+1)^{-1}B(X)(H_0+1)^{-N}$ admits a trace-class extension in $L^2(\mathbb{R}^d, H^r)$. To that end, we will use Lemma 3.6 to rewrite $B(X)$, and obtain a decomposition of

$$(H_0+1)^{-1}B(X)(H_0+1)^{-N}$$

into operators of the type discussed in Corollary 4.2.

Proposition 4.3. *Denote* $n := \max(\lfloor \frac{d}{2} - 2 \rfloor + 1, 0)$. *Let* $\alpha \in \mathbb{R}^{\geq 0}$ *be fixed. Let* $B \in W^{n,2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r))$, *and*

$$\int_{\mathbb{R}^d} \|\partial^{\gamma,\mathrm{End}}(B\langle A_0\rangle^{-\alpha})(x)\|_{S^1(H^r)}\,\mathrm{d}x < \infty,$$

for all $\gamma \in \mathbb{N}^d$ *with* $|\gamma| \leq n$. *Then for* $N > \frac{\alpha}{2} + \frac{n}{2} + \frac{d}{4}$, *the operator*

$$(H_0+1)^{-1}B(X)(H_0+1)^{-N}$$

is densely defined on $\mathrm{rg}((H_0+1)^N|_{C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}})$, *and admits a trace-class extension in* $L^2(\mathbb{R}^d, H^r)$.

Proof. Let $f \in \mathrm{rg}((H_0+1)^N|_{C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}})$, which is dense in $L^2(\mathbb{R}^d, H^r)$ by Lemma 3.1. Then $g = (H_0+1)^{-N}f \in C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}$. Lemma 3.2 then implies that

$$(H_0+1)^{-1}B(X)(H_0+1)^{-N}$$

is densely defined. Let $2l + k = n$ with $k \in \{0,1\}$ and $l \in \mathbb{N}$, and denote

$$B^{k,l}_{\gamma,\delta}(x) := C^{k,l}_{\gamma,\delta}(\partial^{\gamma,\mathrm{End}}C)(x)\langle A_0\rangle^{-\alpha}$$

for $\gamma, \delta \in \mathbb{N}^d$. Then Lemma 3.6 implies

$$\begin{aligned}
(H_0+1)^{-1}B(X)(H_0+1)^{-N}f &= (H_0+1)^{-1}B(X)g \\
&= (H_0+1)^{-1}(ic\nabla+1)^{-k}(\Delta+1)^{-l}\sum_{\substack{\gamma,\delta\in\mathbb{N}^d\\|\gamma+\delta|\leq n}} C^{k,l}_{\gamma,\delta}(\partial^{\gamma,\mathrm{End}}B)(X)\partial^\delta g \\
&= \sum_{\substack{\gamma,\delta\in\mathbb{N}^d\\|\gamma+\delta|\leq n}} (H_0+1)^{-1}(ic\nabla+1)^{-k}(\Delta+1)^{-l}C^{k,l}_{\gamma,\delta}(\partial^{\gamma,\mathrm{End}}B)(X)\partial^\delta(H_0+1)^{-N}f \\
&= \langle c\nabla\rangle^k(1+ic\nabla)^{-k}\sum_{\substack{\gamma,\delta\in\mathbb{N}^d\\|\gamma+\delta|\leq n}}\langle c\nabla\rangle^{-n}(H_0+1)^{-1}B^{k,l}_{\gamma,\delta}(X)\langle\widehat{A_0}\rangle^\alpha\partial^\delta(H_0+1)^{-N}f.
\end{aligned}$$

The operator $\langle c\nabla\rangle^k(1+ic\nabla)^{-k}$ is bounded. For all $\gamma, \delta \in \mathbb{N}^d$ with $|\gamma+\delta| \leq n$, the operators

$$\langle c\nabla\rangle^{-n}(H_0+1)^{-1}B^{k,l}_{\gamma,\delta}(X)\langle\widehat{A_0}\rangle^\alpha\partial^\delta(H_0+1)^{-N}$$

admit trace-class extensions, which follows by Corollary 4.2, because

$$B_{\gamma,\delta}^{k,l} \in L_{\mathrm{loc}}^{2,\mathrm{End}}(\mathbb{R}^d, (\mathcal{D}, H^r)),$$

and

$$\int_{\mathbb{R}^d} \|B_{\gamma,\delta}^{k,l}(x)\|_{S^1(H^r)} < \infty,$$

by assumption on B, and choosing $u_1 = 1$, $s_1 = 0$, $t_1 = -n$, $u_2 = N$, $s_2 = \alpha$, and $t_2 = n$. Consequently, $(H_0 + 1)^{-1} B(X)(H_0 + 1)^{-N}$ admits a trace-class extension. ∎

By a complex interpolation argument, we may extend the trace-class membership of $(H_0 + 1)^{-1} B(X)(H_0 + 1)^{-N}$ to the operator

$$(H_0 + 1)^{-m-1} B(X)(H_0 + 1)^{-N+m}.$$

Corollary 4.4. *Let B, α, n, and N satisfy the same prerequisites as in Proposition 4.3. Additionally, assume that $B^* \in W_{\mathrm{loc}}^{n,2,\mathrm{End}}(\mathbb{R}^d, (\mathcal{D}, H^r))$, and*

$$\int_{\mathbb{R}^d} \|\partial^{\beta,\mathrm{End}}(B^* \langle A_0 \rangle^{-\alpha})(x)\|_{S^1(H^r)} \, \mathrm{d}x < \infty.$$

Then for all $m \in \mathbb{N}$ with $0 \le m \le N - 1$ the operators

$$(H_0 + 1)^{-m-1} B(X)(H_0 + 1)^{-N+m}$$

are densely defined on $\mathrm{rg}((H_0 + 1)^N|_{C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}})$, and admit trace-class extensions in $L^2(\mathbb{R}^d, H^r)$.

Proof. First we show that

$$(H_0 + 1)^{-m-1} B(X)(H_0 + 1)^{-N+m}|_{\mathrm{rg}((H_0+1)^N|_{C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}})}$$

is densely defined. This follows by Lemma 3.1 and Lemma 3.2 and if we note that $(H_0 + 1)^m$ maps $C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}$ to itself.

Now, consider the operator $S_0 := (H_0 + 1)^{-1} B(X)(H_0 + 1)^{-1}$. S_0 is densely defined on

$$\mathrm{rg}((H_0 + 1)|_{C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}}) \subseteq \mathrm{Dom}(S_0),$$

by Lemma 3.1 and Lemma 3.2. We also find

$$\begin{aligned} S_0^* &\supseteq (H_0 + 1)^{-1}(B(X))^*(H_0 + 1)^{-1} \\ &\supseteq (H_0 + 1)^{-1} B^*(X)(H_0 + 1)^{-1}|_{\mathrm{rg}((H_0+1)|_{C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}})}, \end{aligned} \tag{17}$$

which shows that S_0^* is densely defined. Since S_0, S_0^* are both densely defined, S_0 is closable and admits therefore a closure S. Denote $T = (H_0 + 1)^{N-1}$. Then by Proposition 4.3 the operator $S_0 T^{-1}$ admits a trace-class extension R, which is thus bounded, therefore $R = \overline{S_0 T^{-1}}$. Since ST^{-1} is also a closed extension of $S_0 T^{-1}$, we conclude $R = ST^{-1}$. Because $S_0^* = S^*$, and if we consider (17), the operator $S^* T^{-1}$ is a closed extension of

$$(H_0 + 1)^{-1} B^*(X)(H_0 + 1)^{-N}|_{\mathrm{rg}((H_0+1)^N|_{C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}})},$$

which is densely defined and admits a trace-class extension R' by Corollary 4.2, which must be its closure. Therefore, $S^* T^{-1} = R'$. We thus meet the prerequisites of an interpolation theorem ([8, Theorem 3.2]), ensuring that for $x \in [0, 1]$ the operators $T^{-x} S T^{-1+x}$ defined on $\mathrm{Dom}(T)$ are closable and $R_x = \overline{T^{-x} S T^{-1+x}}$ are trace-class operators. Now, choose $x = \frac{m}{N-1}$, then

$$\begin{aligned} &R_{\frac{m}{N-1}}|_{\mathrm{rg}((H_0+1)^N|_{C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}})} \\ &\quad = (H_0 + 1)^{-m-1} B(X)(H_0 + 1)^{-N+m}|_{\mathrm{rg}((H_0+1)^N|_{C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}})}, \end{aligned}$$

which finishes the proof. ∎

We have now all ingredients prepared to show the principal result of this work. Before stating the main theorem, let us summarize all accumulated conditions on the operator family A in the following Hypothesis for convenience. Let us introduce the notation

$$c\nabla^{\mathrm{End}} A := \sum_{j=1}^{d} c(\mathrm{d}x^j)\partial_{x^j}^{\mathrm{End}} A, \quad \nabla^{\mathrm{End}} Ac := \sum_{j=1}^{d} \partial_{x^j}^{\mathrm{End}} Ac(\mathrm{d}x^j).$$

Hypothesis 4.5. Denote $n := \max(\lfloor \frac{d}{2} - 2 \rfloor + 1, 0)$. Let $\alpha, \beta \in \mathbb{R}^{\geq 0}$ be fixed.

Let A_0 be a self-adjoint operator in H^r, and let $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathrm{rg}(\mathbb{1}_{[-n,n]}(A_0))$ and assume that $c(\mathrm{d}x^j)(\mathcal{D}) \subseteq \mathrm{Dom}(A_0)$, $j \in \{1, \ldots, d\}$.

Let $A = (A(x))_{x\in\mathbb{R}^d}$ be a family of symmetric operators in H^r such that $\mathcal{D} \subseteq \mathrm{Dom}(A(x))$. If $A(x)$ commutes with $c(\mathrm{d}x^j)$ for all $x \in \mathbb{R}^d$ and $j \in \{1, \ldots, d\}$, then let $N \in \mathbb{N}$ with $N > \frac{\alpha}{2} + \frac{n}{2} + \frac{d}{4}$, otherwise let $N > \max(\frac{\alpha}{2} + \frac{n}{2} + \frac{d}{4}, \frac{\beta}{2} + \frac{n}{2} + \frac{d}{4} + \frac{1}{2})$.

Assume that $A \in W_{\mathrm{loc}}^{2N-1,2,\mathrm{End}}(\mathbb{R}^d, (\mathcal{D}, H^r))$, and

$$\int_{\mathbb{R}^d} \|\partial^{\gamma,\mathrm{End}}(c\nabla^{\mathrm{End}} A\langle A_0\rangle^{-\alpha})(x)\|_{S^1(H^r)}\, \mathrm{d}x < \infty,$$

$$\int_{\mathbb{R}^d} \|\partial^{\gamma,\mathrm{End}}(\nabla^{\mathrm{End}} Ac\langle A_0\rangle^{-\alpha})(x)\|_{S^1(H^r)}\, \mathrm{d}x < \infty,$$

$$\int_{\mathbb{R}^d} \|\partial^{\gamma,\mathrm{End}}([c(\mathrm{d}x^j), A]\langle A_0\rangle^{-\beta})(x)\|_{S^1(H^r)}\,\mathrm{d}x < \infty, \qquad j \in \{1,\dots,d\}, \tag{18a}$$

for all $\gamma \in \mathbb{N}^d$ with $|\gamma| \le n$. Assume that for $\gamma \in \mathbb{N}^d$, $1 \le |\gamma| \le 2N-1$, there exist $t \in [0, \frac{|\gamma|}{2}]$ and $p \in [2, +\infty]$ such that, for $\frac{|\gamma|}{2} - t > \frac{d}{2p}$,

$$\|x \mapsto \|(\partial^{\gamma,\mathrm{End}} A)(x)\langle A_0\rangle^{-2t}\|_{B(H^r)}\|_{L^p(\mathbb{R}^d)} < \infty \tag{19}$$

or, for $(\frac{|\gamma|}{2} = t, p = 2) \vee (d \ge 3, \frac{d}{p} \in \mathbb{N}, \frac{|\gamma|}{2} - t = \frac{d}{2p})$,

$$\|x \mapsto \|(\partial^{\gamma,\mathrm{End}} A)(x)(A_0^2 + z)^{-t}\|_{B(H^r)}\|_{L^p(\mathbb{R}^d)} = o(1), \qquad z \to +\infty. \tag{20}$$

Finally, also assume, for $0 \le k \le 2N-1$,

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \|A_0^k(A(x) - A_0)(A_0^2 + z)^{-\frac{k+1}{2}}\|_{B(H^r)} = o(1), \qquad z \to +\infty. \tag{21}$$

Since Hypothesis 4.5 comprises of several technical conditions on the family A, we shall discuss their necessity and differentiate their importance. Let us first consider the set of conditions (19), (20), and (21), which are chosen such that

$$\mathrm{Dom}((D^*D)^N) = \mathrm{Dom}((DD^*)^N) = W^{2N,2}(\mathbb{R}^d, H^r) \cap L^2(\mathbb{R}^d, \mathrm{Dom}(A_0^{2N})),$$

see Proposition 3.7. We may categorize these conditions as “minor” since they do not pertain to trace-class properties of D but only of its domain. The essential conditions on A, central to trace-class properties of D, are given by (18), and they are in analogy to the conditions (2) in [10] and (3) in [7]. We should note that the additional derivatives $\partial^{\gamma,\mathrm{End}}$ for $\gamma \le n$ appear only in dimension $d \ge 4$, while the entire second line of (18) is not present for $d = 1$, since without loss of generality Clifford multiplication is scalar and therefore commutes with any operator family. We also note that the requirement on the Clifford multiplication, $c(\mathrm{d}x^j)(\mathcal{D}) \subseteq \mathrm{Dom}(A_0)$, $j \in \{1,\dots,d\}$, is always satisfied if A_0 and $c(\mathrm{d}x^j)$ commute.

Let us illustrate the essential conditions by giving a simple example concerned with operator families generated by a perturbed Dirac operator on $\mathbb{R}^m$, which is similar to the discussed case in [6] for a diagonal matrix potential.

Example 4.6. Denote $n := \max(\lfloor \frac{d}{2} - 2\rfloor + 1, 0)$. Let $m, s \in \mathbb{N}$ and let $\hat{c}$ be a Clifford multiplication over $\mathbb{R}^m$ in $\mathbb{C}^s$, and consider the self-adjoint Dirac operator $A_0 := \hat{c}\hat{\nabla}$ in H^r for $H = L^2(\mathbb{R}^m, \mathbb{C}^s)$. Let $\delta > \frac{m}{2}$, $\alpha > \frac{m}{2} + \delta$, and let $N \in \mathbb{N}$ with $N > \frac{\alpha}{2} + \frac{n}{2} + \frac{d}{4}$. Let $v \colon \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be a C^{2N-1}-function such that its derivatives are bounded, and that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^m} |\partial_x^{\gamma'} v(x,y)|\langle y\rangle^{\delta}\,\mathrm{d}y\ \mathrm{d}x < \infty, \qquad 1 \le |\gamma'| \le n+1. \tag{22}$$

Define

$$(V(x)f)(y) := v(x,y)f(y), \qquad x \in \mathbb{R}^d,\ y \in \mathbb{R}^m,$$
$$\operatorname{Dom}(V(x)) = H^r = L^2(\mathbb{R}^m, \mathbb{C}^s) \otimes \mathbb{C}^r, \quad x \in \mathbb{R}^d,$$

and

$$A(x) := A_0 + V(x), \qquad x \in \mathbb{R}^d,$$
$$\operatorname{Dom}(A(x)) = \operatorname{Dom}(A_0) = W^{1,2}(\mathbb{R}^m, \mathbb{C}^s) \otimes \mathbb{C}^r, \quad x \in \mathbb{R}^d.$$

Because v is smooth, we have $A \in W^{2N-1,2,\mathrm{End}}_{\mathrm{loc}}(\mathbb{R}^d, (\mathcal{D}, H^r))$. Furthermore, we note that for all $\gamma \in \mathbb{N}^d$ with $1 \leq |\gamma| \leq 2N-1$ one obtains

$$\sup_{x\in\mathbb{R}^d, y\in\mathbb{R}^m} |\partial_x^\gamma v(x,y)| =: C_\gamma < \infty,$$

which yields condition (19). Because v is smooth with bounded derivatives, we also find

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d, y\in\mathbb{R}^m} |\partial_x^\gamma v(x,y)| \|(A_0^2 + z)^{-\frac{k+1}{2}}\|_{B(H^r)} \\ &= C_\gamma z^{-\frac{k+1}{2}} \\ &= o(1), \quad z \to +\infty, 0 \leq |\gamma| = k \leq 2N-1,\end{aligned}$$

which yields condition (21). The essential condition (18) reads

$$\int_{\mathbb{R}^d} \|(\partial^{\gamma'} V)(x)\langle \hat{c}\widehat{\nabla}\rangle^{-\alpha}\|_{S^1(H^r)}\,\mathrm{d}x < \infty,\ 1 \leq |\gamma'| \leq n+1,$$

which is satisfied according to [11, Corollary 4.8] and 22. The second and third line of condition (18) is voided since V commutes with $\hat{c}$.

The operator family A therefore satisfies Hypothesis 4.5 and is admissible for the following Theorem 4.7. It states that the associated Callias operator $D = ic\nabla + A(X)$ in this example satisfies

$$(D^*D+1)^{-N} - (DD^*+1)^{-N} \in S^1(L^2(\mathbb{R}^d, H^r)) = S^1(L^2(\mathbb{R}^{d+m}, \mathbb{C}^{rs})).$$

We state the principal result of this work.

Theorem 4.7. *Assume Hypothesis 4.5. Then*

$$(D^*D+1)^{-N} - (DD^*+1)^{-N} \in S^1(L^2(\mathbb{R}^d, H^r)).$$

Proof. A direct consequence of Proposition 3.7 is that

$$\mathrm{Dom}((D^*D)^k) = \mathrm{Dom}((DD^*)^k) = \mathrm{Dom}(H_0^k),$$

for $0 \le k \le N$. Thus, by the resolvent identity, we find

$$\begin{aligned}
&(D^*D+1)^{-N} - (DD^*+1)^{-N} \\
&\quad = \sum_{k=0}^{N-1} (DD^*+1)^{-k-1}[D, D^*](D^*D+1)^{-N+k} \\
&\quad = \sum_{k=0}^{N-1} B_k (H_0+1)^{-k-1}[D, D^*](H_0+1)^{-N+k} \widetilde{B}_{N-k},
\end{aligned}$$

where the operators

$$\begin{aligned}
B_k &:= \overline{(DD^*+1)^{-k-1}(H_0+1)^{k+1}}, \\
\widetilde{B}_k &:= \overline{(H_0+1)^k (D^*D+1)^{-k}},
\end{aligned}$$

are bounded. Thus, it suffices to show that

$$(H_0+1)^{-k-1}[D, D^*](H_0+1)^{-N+k} \in S^1(L^2(\mathbb{R}^d, H^r)), \quad 0 \le k \le N-1.$$

For $0 \le k \le N-1$ and $\phi \in \mathrm{rg}((H_0+1)^N|_{C_c^\infty(\mathbb{R}^d)\otimes\mathcal{D}})$, one obtains

$$\begin{aligned}
&(H_0+1)^{-k-1}[D, D^*](H_0+1)^{-N+k}\phi \\
&\quad = 2i(H_0+1)^{-k-1}[c\nabla, A(X)](H_0+1)^{-N+k}\phi \\
&\quad = 2i \sum_{j=1}^{d} (H_0+1)^{-k-1}[c(\mathrm{d}x^j), A](X)\partial_{x^j}(H_0+1)^{-N+k}\phi \\
&\qquad + 2i \sum_{j=1}^{d} (H_0+1)^{-k-1} c(\mathrm{d}x^j)(\partial_{x^j}^{\mathrm{End}} A)(X)(H_0+1)^{-N+k}\phi \\
&\quad = 2i(H_0+1)^{-k-1} \sum_{j=1}^{d} [c(\mathrm{d}x^j), A](X)\partial_{x^j}(H_0+1)^{-N+k}\phi \\
&\qquad + 2i(H_0+1)^{-k-1}(c\nabla^{\mathrm{End}} A)(X)(H_0+1)^{-N+k}\phi,
\end{aligned}$$

where we use that the conditions posed on A imply that $\mathrm{Dom}(A(x)) = \mathrm{Dom}(A_0)$, for a.e. $x \in \mathbb{R}^d$ (see Remark 3.8), and that $c(\mathrm{d}x^j)(\mathcal{D}) \subseteq \mathrm{Dom}(A_0)$, $j \in \{1, \ldots, d\}$. Since $\partial_{x^j}(H_0+1)^{-\frac{1}{2}}$ is bounded for $j \in \{1, \ldots, d\}$, it suffices to show that

$$\begin{aligned}
&(H_0+1)^{-k-1}[c(\mathrm{d}x^j), A](X)(H_0+1)^{-N+k+\frac{1}{2}}, \quad j \in \{1, \ldots, d\}, \\
&(H_0+1)^{-k-1}(c\nabla^{\mathrm{End}} A)(X)(H_0+1)^{-N+k}
\end{aligned}$$

admit trace-class extensions for $0 \leq k \leq N-1$, which is a direct consequence of Corollary 4.4. ∎

The above presented Theorem 4.7 should provide a large enough store of operator families A such that one may investigate trace formulas and spectral shift functions of the pair (D^*D, DD^*), which might lead to new results for the index of the Callias operator D in this abstract setup.

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