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A probabilistic Weyl-law for perturbed Berezin–Toeplitz operators

Izak Oltman

Abstract. This paper proves a probabilistic Weyl-law for the spectrum of randomly perturbed Berezin–Toeplitz operators, generalizing a result proven by Martin Vogel (2020). This is done following Vogel's strategy using the exotic symbol calculus developed by the author (2022).

1. Introduction

This paper generalizes a result of Martin Vogel in [22] which proves a probabilistic Weyl-law for quantizations of functions on tori. Here we do the same, but with the tori replaced by arbitrary Kähler manifolds equipped with positive line bundles.

In [22], Vogel considers Toeplitz quantizations of smooth functions on a real $2d$-dimensional torus, which associates every smooth function f on the torus to a family of $N^d \times N^d$ matrices, f_N, for all $N \in \mathbb{N}$ (here N^{-1} is the semi-classical parameter). A recent physical motivation for such constructions is written by Deleporte in [6, Section 1]. Next, a random matrix with sufficiently small norm is added to f_N, and the spectrum is shown to obey an almost-sure Weyl-law as N goes to infinity. This was conjectured by Christiansen and Zworski in [4] and is a major extension of their work.

This result is most striking when the unperturbed matrix is non-self-adjoint. For example, if $f(x) = \cos(2\pi x) + i\cos(2\pi\xi)$, then the quantization is

$$f_N = \begin{pmatrix} \cos(2\pi/N) & i/2 & 0 & 0 & \cdots & i/2 \\ i/2 & \cos(4\pi/N) & i/2 & 0 & \cdots & 0 \\ 0 & i/2 & \cos(6\pi/N) & i/2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & i/2 & \cos(2(N-1)\pi/N) & i/2 \\ i/2 & 0 & \cdots & 0 & i/2 & \cos(2\pi) \end{pmatrix},$$

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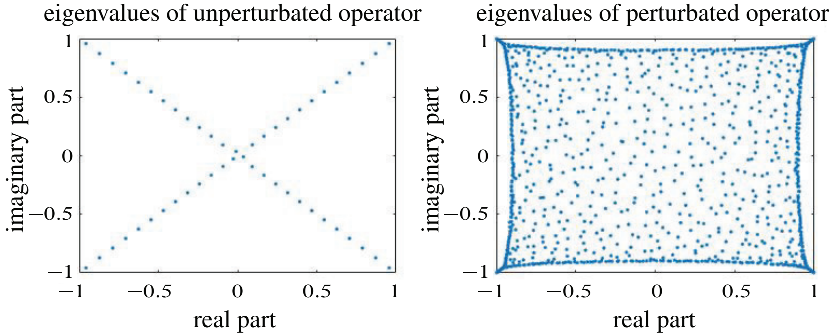

Figure 1. Left: Eigenvalues of the Scottish flag operator with $N = 50$. Right: Eigenvalues of the Scottish flag operator with a small random perturbation with $N = 1000$.

which numerically has spectrum contained on two crossing lines in the complex plane. This operator is aptly named the *Scottish flag operator* and is further described by Embree and Trefethen in [21]. Interestingly, (as far as we are aware) it is unknown analytically where the spectrum of f_N lives. However, if randomly perturbed, the spectrum spreads out with density given by the push-forward of the Lebesgue measure on the torus by f. Figure 1 plots the spectrum of f_N with no perturbation, and with a small perturbation.

The spectral properties of randomly perturbed non-self-adjoint operators were pioneered by Hager in [10], in which the operator $hD_x + g(x)\colon L^2(S^1) \to L^2(S^1)$ was studied. This result, and numerous subsequent results are discussed by Sjöstrand in [15]. There are related results describing spectral properties of randomly perturbed Toeplitz matrices, which can be defined as quantizations of symbols on $\mathbb{T}^2$ with symbol independent of x. See Davies and Hager [5], Guionnet, Wood and Zeitouni [9], Sjöstrand and Vogel [16, 17], and references given there.

This paper is the natural generalization of Vogel's result in [22]. Here we prove a similar result for quantizations of functions on Kähler manifolds (with sufficient structure, as discussed in Section 2). These quantizations, called *Berezin–Toeplitz operators* (or just *Toeplitz operators*) were first described by Berezin in [2] as a particular type of quantization of symplectic manifolds. Following [2], for every smooth function f on a quantizable Kähler manifold X, we get a family of finite rank operators, $T_N f$, indexed by $N \in \mathbb{N}$ (see [13] for a connection between these quantizations, and quantizations on the torus) which have physical interpretations. Deleporte in [6, Appendix A] relates this quantization to spin systems in the large spin limit, and Douglas and Klevtsov in [7] use path integrals for particles in a magnetic field to derive the Bergman kernel (a key ingredient in constructing $T_N f$).

Next, if we add a small Gaussian-type random perturbation $\mathcal{G}_\omega$ to these operators (see Definition 2.3), the empirical measures weakly converge almost surely (see Theorem 2 in Section 2 for a precise statement). Theorem 3 states a result about more general random perturbations $\mathcal{W}_\omega$ (see Definition 2.3) but with a more restrictive coupling constant. A consequence of Theorem 3 is the following probabilistic Weyl-law.

Theorem 1 (A probabilistic Weyl-law). *Given a quantizable Kähler manifold X, $f \in C^\infty(X;\mathbb{C})$ such that there exists $\kappa \in (0,1]$ so that*

$$\mu_d(\{x \in X : |f(x) - z|^2 \leq t\}) = \mathcal{O}(t^\kappa)$$

as $t \to 0$ uniformly for $z \in \mathbb{C}$ (where μ_d is the Liouville volume form on X), $\mathcal{W}_\omega$ a random matrix (see Definition 2.3), and $\Lambda \subset \mathbb{C}$, then almost surely

$$\left(\frac{2\pi}{N}\right)^d \#\{\mathrm{Spec}(T_N f + N^{-d}\mathcal{W}_\omega) \cap \Lambda\} \xrightarrow{N\to\infty} \mu_d(x \in X : f(x) \in \Lambda).$$

Finer results are expected for describing the spectrum of randomly perturbed Toeplitz operators. In [22], precise statements about the number of eigenvalues are obtained using counting functions of holomorphic functions. Here we only show weak convergence of the empirical measures, but achieve this in a relatively simple way using logarithmic potentials as presented in [17].

Here we present numerical examples to motivate the main result of this paper. Consider the Kähler manifold $\mathbb{CP}^1$ (complex protective space of dimension 1) which can be identified with the real 2-sphere with coordinates (x_1, x_2, x_3). In Figure 2, we compute the spectrum of the quantization of the function $f = x_1 + 2x_2^2 + ix_2$. Before perturbation, the spectrum lies on several lines in the complex plane, somewhat analogous to the Scottish flag operator. However, as a perturbation is added, the spectrum fills in. This paper describes the structure of the spectrum of this perturbed operator in the semiclassical limit, as $N \to \infty$.

Numerical verification of this paper's result can be seen if $f = ix_1 + x_2$ (still on $\mathbb{CP}^1$). Figure 3 computes the spectrum of $T_N f$ with a random perturbation added, and plots the number of eigenvalues in circles of increasing radii versus the predicted number of such eigenvalues by Theorem 1. More animations can be found on my website.[1]

Outline of paper. Section 2 reviews background material and states the main result of this paper (Theorem 2). In Section 3, a series of preliminary results about Toeplitz operators are presented. Section 4 reviews logarithmic potentials and reduces Theorem 2 to proving a probabilistic bound involving logarithmic derivatives of Toeplitz

[1] https://math.berkeley.edu/~izak/research/toeplitz/movies.html

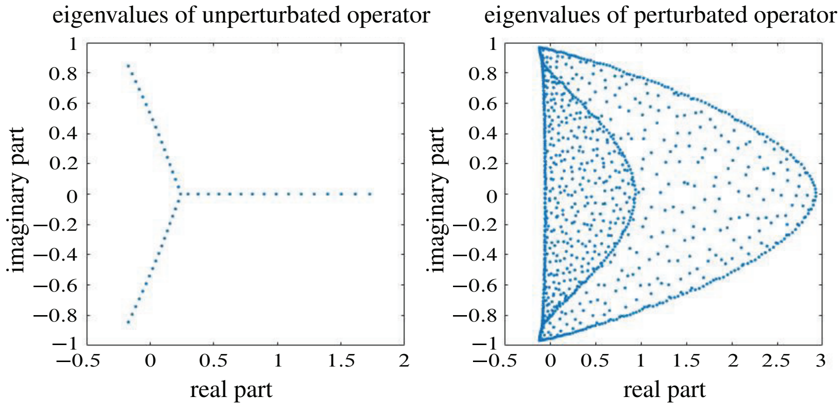

Figure 2. Left: Eigenvalues of the Toeplitz operator on $\mathbb{CP}^1$ identified with the real 2-sphere with symbol $x_1 + 2x_1^2 + ix_2$ and $N = 50$. Right: Eigenvalues of the same operator but with a small random perturbation and $N = 1000$.

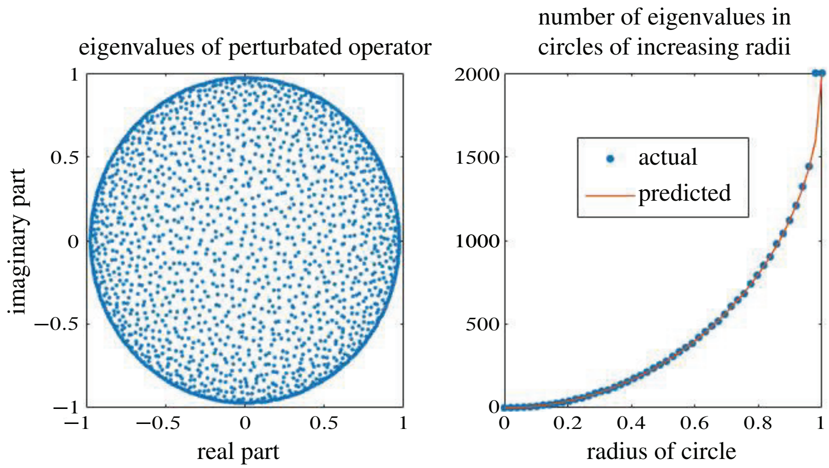

Figure 3. Left: Eigenvalues of the randomly perturbed Toeplitz operator on $\mathbb{CP}^1$ identified with the real 2-sphere with symbol $ix_1 + x_2$ an $N = 2000$. Right: The number of eigenvalues within circles in the complex plane centered at zero with radii ranging from 0 to 1, plotted against the predicted distribution of eigenvalues from Theorem 1.

operators. Section 5 sets up a Grushin problem to further reduce the problem to prove probabilistic bounds on spectral properties of self-adjoint operators. Section 6 proves a deterministic bound involving the logarithmic derivative of Toeplitz operators. The technique involves scaling the symbol by a power of N, and therefore relies on the exotic calculus presented in Section 3. Finally, Section 7 chooses constants to establish the required probabilistic bound for the almost sure convergence in Theorem 2. In Section 8, we describe how to extend this result to the more general random perturbations as stated in Theorem 3.

Notation. We will use the following notation in this paper for functions f and g depending on N. We write $f = \mathcal{O}(g)$ if there exists $C > 0$ independent of N such that $|f| \leq Cg$. We write $f = \mathcal{O}(N^{-\infty})$ if for every $M \in \mathbb{N}$, $f = \mathcal{O}(N^{-M})$. Any subscript in the big-O will denote dependence of C of what is in the subscript. We will write $f \lesssim g$ if there exists a $C > 0$ independent of N such that $f \leq Cg$. We write $f \ll g$ to mean that $Cf \leq g$ for some sufficiently large $C > 0$ independent of N. For a u, v, w elements of a Hilbert space, denote $u \otimes v$ the map that sends w to $u\langle w, v\rangle$.

2. Main result

Let (X, σ) be a compact, connected, d-dimensional Kähler manifold with a holomorphic line bundle L with positively curved Hermitian metric locally given by $h = e^{-\varphi}$. That is, over each fiber $x \in X$, $\|v\|_h := e^{-\varphi(x)}|v|$. Given this, the globally defined symplectic form, σ, is related to the Hermitian metric by $i\partial\bar{\partial}\varphi = \sigma$. Fixing local trivializations, φ can be described as a strictly plurisubharmonic smooth real-valued function (called the *Kähler potential*). This is further outlined by Le Floch in [11].

Let L^N be the Nth tensor power of L, which has Hermitian metric $h_N := e^{-N\varphi}$. Let $\mu_d = \sigma^{\wedge d}/d!$ be the Liouville volume form on X. This provides an L^2 structure on sections of L^N. Indeed, if u and v are smooth sections on L^N, then define

$$\langle u, v\rangle_{L^N} := \int_X h_N(u, v)\, \mathrm{d}\mu_d.$$

Define $L^2(X, L^N)$ to be the space of smooth sections of L^N with finite L^2 norm. In this L^2 space, let $H^0(X, L^N)$ be the space of holomorphic sections.

Proposition 2.1. *The dimension of $H^0(X, L^N)$ is finite, and is asymptotically*

$$\left(\frac{N}{2\pi}\right)^d \mathrm{vol}(X) + \mathcal{O}(N^{d-1}).$$

Proof. See [3, Corollary 2]. ∎

For the remainder of this paper, denote $\dim(H^0, (X, L^N))$ by $\mathcal{N} = \mathcal{N}(N)$. The orthogonal projection from $L^2(X, L^N)$ to $H^0(X, L^N)$ is called the *Bergman projector* and is denoted by Π_N. Finally, given $f \in C^\infty(X; \mathbb{C})$, the Toeplitz operators associated to f, written $T_N f$, are defined for each $N \in \mathbb{N}$ as $T_N f(u) = \Pi_N(fu)$, where

$u \in H^0(X, L^N)$. In this way, $T_N f$ are finite rank operators mapping $H^0(X, L^N)$ to itself. For the remainder of this paper, we will fix a basis for $H^0(X, L^N)$ so that $T_N f$ (and similar operators) can be considered as matrices.

The class of functions to quantize will often depend on N. To define this symbol class requires local control of functions. Fix a finite atlas of neighborhoods $(U_i, \zeta_i)_{i \in \mathcal{I}}$ for the Kähler manifold X.

Definition 2.2 ($S(1)$). $S(1)$ is the set of all smooth functions f on X taking complex values which can be written asymptotically $f \sim \sum N^{-j} f_j$, where $f_j \in C^\infty(X; \mathbb{C})$ do not depend on N. This tilde means that, for all $\alpha \in \mathbb{N}$,

$$\partial_x^\alpha \Big(f \circ \zeta_i(x) - \sum_{j=0}^{M} N^{-j} f_j \circ \zeta_i(x) \Big) = \mathcal{O}_\alpha(N^{-j-1})$$

for all $i \in \mathcal{I}$, and all $\alpha \in \mathbb{N}^d$. By Borel's theorem, given any $f_j \in S(1)$ not depending on N, there exists $f \in S(1)$ such that $f \sim \sum N^{-j} f_j$.

If $f \sim \sum N^{-j} f_j$, we call f_0 the *principal symbol of* f, which is unique modulo $\mathcal{O}(N^{-1})$.

We next add a random perturbation to these Toeplitz operators. For this, we must fix a probability space Ω with probability measure $\mathbb{P}$.

Definition 2.3 ($\mathcal{G}_\omega$ and $\mathcal{W}_\omega$). For each N, let $\{e_i : i = 1, \dots, \mathcal{N}\}$ be an orthonormal basis of $H^0(X, L^N)$. Define

$$\mathcal{G}_\omega = \sum_{i,j=1}^{\mathcal{N}} \alpha_{j,k} e_i \otimes e_j : H^0(X, L^N) \to H^0(X, L^N)$$

where $\alpha_{j,k}$ are independent identically distributed complex Gaussian random variables with mean zero and variance 1.

Similarly define

$$\mathcal{W}_\omega = \sum_{i,j=1}^{\mathcal{N}} \tilde{\alpha}_{j,k} e_i \otimes e_j,$$

with $\tilde{\alpha}_{j,k}$ independent identically distributed copies of a complex random variable with mean zero and bounded second moment.

The ω in the subscript of these objects is to emphasize that these objects are random. That is, for each $\omega \in \Omega$, $\mathcal{G}_\omega$ is a finite rank operator. The majority of this article describes perturbations by $\mathcal{G}_\omega$ (the Gaussian case), while a brief note at the end concerns the more general perturbations by $\mathcal{W}_\omega$.

This paper will prove almost sure weak convergence of the empirical distribution of eigenvalues of randomly perturbed Toeplitz operators. The principal symbol of f must also satisfy the property that there exists $\kappa \in (0,1]$ such that

$$\mu_d(\{x \in X : |f_0(x) - z|^2 \leq t\}) = \mathcal{O}(t^\kappa) \tag{2.1}$$

as $t \to 0$ uniformly for all $z \in \mathbb{C}$. It is observed in [4] that if f is real analytic, then (2.1) holds. See [4], and references presented there, for further discussion of (2.1).

Theorem 2 (Main theorem). *Given $f \in S(1)$ which satisfies (2.1) and $\mathcal{G}_\omega$, a family of random operators on $H^0(X, L^N)$, as defined in Definition 2.3, then for each $\varepsilon > 0$ there exists $\beta = \beta(\varepsilon) \in (0,1)$ and $C > 0$ such that if $\delta = \delta(N)$ satisfies*

$$Ce^{-N^\beta} < \delta < C^{-1} N^{-d/2-\varepsilon}, \tag{2.2}$$

then we have almost sure weak convergence of the empirical measures of $T_N f + \delta \mathcal{G}_\omega$ to $\operatorname{vol}(X)^{-1}(f_0)_* \mu_d$.

More precisely, if $\lambda_i = \lambda_i(N, \omega)$ are the (random) eigenvalues of $T_N f + \delta \mathcal{G}_\omega$, then for all $\varphi \in C_0^\infty(\mathbb{C})$

$$\frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \varphi(\lambda_i) \xrightarrow{N \to \infty} \frac{1}{\operatorname{vol}(X)} \int_{\mathbb{C}} \varphi(z) [(f_0)_* \mu_d](\mathrm{d}z)$$

almost surely, where $(f_0)_* \mu_d$ is the push-forward of the volume form μ_d on X by f_0.

Moreover, for each $\varepsilon > 0$, the constant $\beta(\varepsilon)$ in (2.2) can be chosen at most strictly less than

$$\begin{cases} 2\varepsilon\kappa & \text{if } \varepsilon < \frac{1}{2(\kappa+1)}, \\ \frac{\kappa}{\kappa+1} & \text{if } \varepsilon \geq \frac{1}{2(\kappa+1)}, \end{cases}$$

where κ is defined in (2.1).

We expect Theorem 2 to hold for a much larger class of random perturbations than described in Definition 2.3. Indeed, the only properties of $\mathcal{G}_\omega$ we use is a norm bound (Lemma 4.6) and an anti-concentration bound (Proposition 5.7). See [23] where Vogel and Zeitouni establish similar logarithmic determinant estimates with these classes of random perturbations, and [1, Remark 1.3] where Basak, Paquette, and Zeitouni describe random perturbations satisfying these properties.

Here we present a version of Theorem 2 for the more general random perturbations $\mathcal{W}_\omega$ as described in Definition 2.3.

Theorem 3 (General perturbations). *For $\mathcal{W}_\omega$ defined in Definition 2.3, $f \in S(1)$ satisfying (2.1), $\delta = N^{-d}$, then the empirical measures of $T_N f + \delta \mathcal{W}_\omega$ converge almost surely to $(\operatorname{vol}(X))^{-1}(f_0)_* \mu_d$.*

A proof of this result is presented in Section 8.

Remark 2.1. We expect a wider range of δ's and more general random perturbations in Theorem 3 should lead to the same conclusion.

3. Review of an exotic calculus of Toeplitz operators

In proving Theorem 2, non-negative symbols are scaled by powers of N^{-1}. These functions belong to a more exotic symbol class than smooth functions uniformly bounded in N. Toeplitz operators of functions in this symbol class still have natural composition formulas. A summary of these results is contained in this section. For proofs see [12].

Definition 3.1 (Order function). For $\rho \in [0, 1/2)$, a ρ-order function m on X is a function $m \in C^\infty(X; \mathbb{R}_{>0})$, depending on N, such that there exists $M_0 \in \mathbb{N}$ such that, for all $x, y \in X$,

$$m(x)/m(y) \lesssim (1 + \operatorname{dist}(x, y)N^\rho)^{M_0},$$

where $\operatorname{dist}(x, y)$ is the distance between x and y with respect to the Riemannian metric on X induced by the symplectic form σ.

Definition 3.2 ($S_\rho(m)$). Given $\rho \in [0, 1/2)$ and a ρ-order function m on X. $S_\rho(m)$ is defined as the set of smooth functions on X depending on N such that, for all $i \in \mathcal{I}$, $\alpha \in \mathbb{N}^d$,

$$|\partial^\alpha (f \circ \zeta_i^{-1}(x))| \lesssim_\alpha N^{\delta|\rho|} m \circ \zeta_i^{-1}(x)$$

for all $x \in \zeta_i(U_i)$ (recall $\{(U_i, \zeta_i) : i \in \mathcal{I}\}$ is a finite atlas on X).

Proposition 3.3 (Composition). *Given $\rho \in [0, 1/2)$, ρ-order functions m_1, m_2 on X, $f \in S_\rho(m_1)$ and $g \in S_\rho(m_2)$, then there exists $h \in S_\rho(m_1 m_2)$ such that*

$$T_N f \circ T_N g = T_N h + \mathcal{O}(N^{-\infty}),$$

where $\mathcal{O}$ is in terms of the norm from $L^2(X, L^N) \to L^2(X, L^N)$. Moreover, the principal symbol of h is $f_0 g_0$.

Claim 3.1. *Given $f \in S(1)$ with $f_0 \geq 0$, if $\rho \in [0, 1/2)$, then $m(x) = f_0 N^{2\rho} + 1$ is a ρ-order function on X and $f N^{2\rho} \in S_\rho(m)$.*

Proposition 3.4 (Parametrix construction). *Given $\rho \in [0, 1/2)$, a ρ-order function m on X, $\rho \in [0, 1/2)$, and $f \in S_\rho(m)$ such that there exists $C > 0$ so that $f > Cm$,* *then there exists* $g \in S_\rho(m^{-1})$ *such that*

$$T_N f \circ T_N g = 1 + \mathcal{O}(N^{-\infty}), \quad T_N g \circ T_N f = 1 + \mathcal{O}(N^{-\infty}).$$

Proposition 3.5 (Functional calculus). *Given a* ρ*-order function* $m \geq 1$ *on* X *(for a fixed* $\rho \in [0, 1/2)$*), a family of operators* $\{R_N\}_{N\in\mathbb{N}}$ *mapping* $H^0(X, L^N)$ *to itself such that* $\|R_N\| = \mathcal{O}(N^{-\infty})$ *and* $T_N f + R_N$ *is self-adjoint for all* N*, and* $f \in S_\rho(m)$ *taking real non-negative values such that there exists* $C > 0$ *with* $|f| \geq mC^{-1} - C$*, then for any* $\chi \in C^\infty(\mathbb{R}; \mathbb{C})$*, there exists* $g \in S_\rho(m^{-1})$ *such that*

$$\chi(T_N f + R_N) = T_N g + \mathcal{O}(N^{-\infty})$$

and g *has principal symbol* $\chi(f_0)$*.*

Typically, Proposition 3.5 will be applied with $R_N = 0$ for all N.

Proposition 3.6 (Trace formula). *If* m *is a* ρ*-order function on* X *(for fixed* $\rho \in [0, 1/2)$*), and* $f \in S_\rho(m)$*, then*

$$\begin{aligned} \operatorname{Tr} T_N f &= \Big(\frac{N}{2\pi}\Big)^d \int_X f(x) \,\mathrm{d}\mu_d(x) + \mathcal{O}(N^{d-(1-2\rho)}) \max_{x\in X} m(x) \\ &= \Big(\frac{N}{2\pi}\Big)^d \int_X f_0(x) \,\mathrm{d}\mu_d(x) + \mathcal{O}(N^{d-(1-2\rho)}) \max_{x\in X} m(x), \end{aligned}$$

where f_0 *is the principal symbol of* f*.*

Note that if $f = 1$, then $\operatorname{Tr} T_N 1 = \operatorname{Tr}(\Pi_N) = \dim(H^0(X, L^N)) = \mathcal{N}$ which is an alternative way of proving that $\mathcal{N} = \operatorname{vol}(X)(N/2\pi)^d + \mathcal{O}(N^{d-1})$.

4. Probabilistic preliminaries

This paper uses the probabilistic machinery of logarithmic potentials. An overview is presented in this section.

Definition 4.1 ($\mathcal{P}(\mathbb{C})$). Let $\mathcal{P}(\mathbb{C})$ be the collection of probability measures μ on $\mathbb{C}$ such that $\int \log(1 + |z|) \,\mathrm{d}\mu(z) < \infty$.

Definition 4.2 (Logarithmic potential). For $\nu \in \mathcal{P}(\mathbb{C})$, define the logarithmic potential as $U_\nu(z) := \int_{\mathbb{C}} \log|z - w| \,\mathrm{d}\nu(w)$.

Using the fact that $\log|z|$ is the fundamental solution of the Laplacian, it can be shown that, in the sense of distributions, $\Delta U_\nu = 2\pi\nu$, which is the key ingredient in proving the following theorem.

Proposition 4.3 (Convergence of random measures by logarithmic potentials). *Given* $\{\nu_N\} \subset \mathcal{P}(\mathbb{C})$ *random measures such that almost surely* $\operatorname{supp} \nu_N \subset \Lambda$ *for* $N \gg 1$ *(with* $\Lambda \Subset \bar{\Lambda} \Subset \Lambda' \Subset \mathbb{C}$*) and for almost all* $z \in \Lambda'$*, one has* $U_{\nu_N}(z) \to U_\nu(z)$ *almost surely for some* $\nu \in \mathcal{P}(C)$ *with* $\operatorname{supp} \nu \subset \Lambda$*, then almost surely* $\nu_N \to \nu$ *weakly.*

Proof. See [17, Theorem 7.1]. ■

We wish to use Proposition 4.3 to prove almost sure weak convergence of the empirical measures of $T_N f + \delta \mathcal{G}_\omega$.

Definition 4.4 (ν_N). Let σ_N be the spectrum of $T_N f + \delta \mathcal{G}_\omega$. Let

$$\nu_N = \mathcal{N}^{-1} \sum_{\lambda \in \sigma_N} \hat{\delta}_\lambda$$

where $\delta > 0$ depends on N, and $\hat{\delta}_\lambda$ is the Dirac distribution centered at λ. The logarithmic potentials for these random measures are

$$U_{\nu_N}(z) = \frac{1}{\mathcal{N}} \sum_{\lambda \in \sigma_N} \log|z - \lambda| = \frac{1}{\mathcal{N}} \log|\det(T_N f + \delta \mathcal{G}_\omega - z)|.$$

Definition 4.5 (ν). Let $\nu = \operatorname{vol}(X)^{-1} (f_0)_* \mu_d$ (recall μ_d is the volume measure on X) which has logarithmic potential

$$U_\nu(z) = \fint_X \log|z - f_0(x)| \, d\mu_d(x).$$

Where $\fint_X f \, d\mu_d$ is defined as $\operatorname{vol}(X)^{-1} \int f \, d\mu_d$.

Claim 4.1. *For all* N*,* $\nu_N, \nu \in \mathcal{P}(\mathbb{C})$*.*

Proof. For each $N \in \mathbb{N}$

$$\begin{aligned}\int_{\mathbb{C}} \log(1 + |z|) \, d\nu_N(z) &= \frac{1}{\mathcal{N}} \sum_{\lambda \in \sigma_N} \log(1 + |\lambda|) \le \max_{\lambda \in \sigma_N} \log(1 + |\lambda|) \\ &\le \log(1 + \|T_N f + \delta \mathcal{G}_\omega\|) < \infty.\end{aligned}$$

And similarly,

$$\begin{aligned}\int_{\mathbb{C}} \log(1 + |z|) \, d\nu(z) &= \frac{1}{\operatorname{vol}(X)} \int_{\mathbb{C}} \log(1 + |z|)[(f_0)_* \mu_d](dz) \\ &\le \max_{x \in X} \log(1 + |f(x)|) < \infty.\end{aligned}$$

■

Let Λ be a neighborhood of $f(X)$. Clearly, $\operatorname{supp} \nu \subset \Lambda$, the same is true with probability 1 for ν_N, for sufficiently large N. A standard random matrix lemma is required to show this.

Lemma 4.6 (Norm of Gaussian matrix). *There exists $C > 0$ such that*

$$\mathbb{P}(\|\mathcal{G}_\omega\| \leq C\mathcal{N}^{1/2}) \geq 1 - \exp(-\mathcal{N}).$$

If an event has this lower bound of probability, it is said to occur with overwhelming probability.

Proof. See [19, Exercise 2.3.3]. ∎

For a fixed $\varepsilon > 0$, we will choose $\delta = \delta(N)$ such that

$$0 < \delta = \mathcal{O}(\mathcal{N}^{-1/2-\varepsilon}). \tag{4.1}$$

Lemma 4.7 (Borel–Cantelli). *If A_n are events such that $\sum_1^\infty \mathbb{P}(A_n) < \infty$, then the probability that A_n occurs infinitely often is* 0.

Proof. See [8]. ∎

Lemma 4.8 (Bound of $T_N f$). *Given $f \in S(1)$, then $\|T_N f\|_{L^N \to L^N} \leq \sup|f|$.*

Proof. This follows immediately by writing $T_N f = \Pi_N \circ M_f \circ \Pi_N$ and recalling that Π_N is unitary. ∎

Claim 4.2. *Almost surely,* $\operatorname{supp} \nu_N \subset \Lambda$ *for* $N \gg 1$.

Proof. First note that $\|T_N f + \delta\mathcal{G}_\omega\| \leq \|T_N f\| + \delta\|\mathcal{G}_\omega\| \leq \sup f + \mathcal{N}^{-\varepsilon}$ with overwhelming probability (by Lemma 4.6, (4.1), and Lemma 4.8). Let σ_N be the spectrum of $T_N f + \delta\mathcal{G}_\omega$. In this event, for sufficiently large N, $\sigma_N \subset \Lambda$. So, if A_N^c is the event that $\sigma_N \subset \Lambda$, then $\mathbb{P}(A_N^c) \geq 1 - e^{-\mathcal{N}}$. Therefore, $\sum \mathbb{P}(A_N) < \infty$ and so by Lemma 4.7, almost surely $P(A_N^c) = 1$ for $N \gg 1$. ∎

Lemma 4.9 (Almost sure convergence). *If $\{Y_N\}_{N\in\mathbb{N}}$ and Y are random variables on a probability space* $(\Omega, \mathbb{P})$ *and ε_N is a sequence of numbers converging to* 0 *such that*

$$\sum_{N=1}^{\infty} \mathbb{P}(|Y_N - Y| > \varepsilon_N) < \infty,$$

then $Y_N \to Y$ almost surely.

Proof. See [8]. ∎

Therefore, ν_N and ν satisfy the conditions of Proposition 4.3. So, it suffices to show that $U_{\nu_N}(z) \to U_\nu(z)$ for almost all z in the bounded set containing Λ. To prove this almost sure convergence, it suffices to apply Lemma 4.9 with $Y_N = \mathcal{N}^{-1} \log|\det(T_N f + \delta \mathcal{G}_\omega - z)|$ and $Y = \fint \log|z - f_0(x)| \, d\mu_d(x)$ for suitably chosen ε_N.

5. Setting up a Grushin problem

To control $\log|\det(T_N f + \delta \mathcal{G}_\omega - z)|$ we follow the now standard method of setting up a Grushin problem. This approach was used in [10, 22], and is comprehensively reviewed in [18].

Let $P = T_N f$ and $\mathcal{H}_N = H^0(X, L^N)$. Define the z-dependent self-adjoint operators $Q = (P - z)^*(P - z)$ and $\widetilde{Q} = (P - z)(P - z)^*$. These operators share the same eigenvalues $0 \le t_1^2 \le \cdots \le t_{\mathcal{N}}^2$. We can find an orthonormal basis of eigenvectors of Q for these eigenvalues, denoted by e_i, and similarly, an orthonormal basis of eigenvectors of $\widetilde{Q}$ denoted by f_i. These eigenvectors can be chosen such that

$$(P - z)^* f_i = t_i e_i, \quad (P - z) e_i = t_i f_i, \quad i = 1, \ldots, \mathcal{N}.$$

Next we fix $\rho \in (0, \min(1/2, \varepsilon))$, and define

$$\alpha := N^{-2\rho}, \quad A := \max\{i \in \mathbb{Z} : t_i^2 \le \alpha\}.$$

Definition 5.1 ($\mathcal{P}^\delta$). Let δ_j be the standard basis of $\mathbb{C}^A$, and define the operators $R_+(z) = \sum_1^A \delta_i \otimes e_i : \mathcal{H}_N \to \mathbb{C}^A$ and $R_-(z) = \sum_1^A f_i \otimes \delta_i : \mathbb{C}^A \to \mathcal{H}_N$, where we use the notation $(u \otimes v)(w) = \langle w, v \rangle u$. For each $z \in \mathbb{C}$ and $\delta \ge 0$, define

$$\mathcal{P}^\delta(z) := \begin{pmatrix} P + \delta \mathcal{G}_\omega - z & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H}_N \\ \mathbb{C}^A \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_N \\ \mathbb{C}^A \end{pmatrix}. \tag{5.1}$$

Lemma 5.2. *If $\delta = 0$, then $\mathcal{P}^\delta$, as defined in (5.1), is bijective with inverse*

$$\begin{aligned} \mathcal{E}^0(z) &= \begin{pmatrix} \sum_{A+1}^{\mathcal{N}} \frac{1}{t_i} e_i \otimes f_i & \sum_1^A e_i \otimes \delta_i \\ \sum_1^A \delta_i \otimes f_i & -\sum_1^A t_i \delta_i \otimes \delta_i \end{pmatrix} \\ &:= \begin{pmatrix} E^0(z) & E_+^0(z) \\ E_-^0(z) & E_{-+}^0(z) \end{pmatrix}. \end{aligned} \tag{5.2}$$

Proof. See [22, Section 5.1]. ∎

To ease notation, the z in the argument for these operators will often be dropped. Unless specified, all estimates are uniform in z.

Claim 5.1 (Invertibility of $\mathcal{P}^\delta$). *$\mathcal{P}^\delta$ is invertible if $\delta\|\mathcal{G}_\omega E^0\| \ll 1$.*

Proof. By computation,

$$\mathcal{P}^\delta \mathcal{E}^0 = 1 + \begin{pmatrix} \delta\mathcal{G}_\omega E^0 & \delta\mathcal{G}_\omega E_+^0 \\ 0 & 0 \end{pmatrix} := 1 + K.$$

If $\|K\| < 1$ (which is true given the hypothesis), then $(I + K)^{-1}$ exists as a Neumann series, and we get $\mathcal{P}^\delta \mathcal{E}^0 (I + K)^{-1} = I$ (a similar argument shows this is a left inverse as well). ∎

Lemma 5.3 (Norm of E^0). *In the notation of* (5.2)*, $\|E^0\| \le \alpha^{-1/2}$.*

Proof. By construction, $E^0 = \sum_{M+1}^{\mathcal{N}} (t_i)^{-1} e_i \otimes f_i$, so that

$$\|E^0\| = \|E^0 f_{M+1}\| = (t_{M+1})^{-1} \le \alpha^{-1/2}. \qquad ∎$$

Lemma 5.4 (Norm of E_+^0). *In the notation of* (5.2)*, $\|E_+^0\| = 1$.*

Proof. By construction $E_+^0(z) = \sum_1^M e_i \otimes \delta_i$ which has norm 1. ∎

These lemmas, along with Lemma 4.6, guarantee that if $\delta = \mathcal{O}(\alpha^{1/2}\mathcal{N}^{-1/2})$, then $\mathcal{P}^\delta$ is invertible with overwhelming probability. Denote the inverse of $\mathcal{P}^\delta$ by $\mathcal{E}^\delta$ with the same notation for its components as in (5.2).

Define $P^\delta = P + \delta\mathcal{G}_\omega$. By Schur's complement formula, if $P^\delta - z$ is invertible,

$$\det \begin{pmatrix} P^\delta - z & R_- \\ R_+ & 0 \end{pmatrix} = \det(P^\delta - z)\det(-R_+(P^\delta - z)^{-1} R_-).$$

Writing $\mathcal{P}^\delta \mathcal{E}^\delta = 1$, we get that $-R_- = (P^\delta - z)E_+^\delta (E_{-+}^\delta)^{-1}$ and $R_+ E_+^\delta = 1$. Therefore, $-R_+(P^\delta - z)^{-1} R_- = (E_{-+}^\delta)^{-1}$, so that

$$\log|\det(P^\delta - z)| = \log|\det \mathcal{P}^\delta(z)| + \log|\det E_{-+}^\delta(z)|. \tag{5.3}$$

Note that $P^\delta - z$ is invertible if and only if E_{-+}^δ is invertible. Therefore, (5.3) holds even when $P^\delta - z$ is not invertible.

Therefore, to prove Theorem 2, it suffices to show summability of the probability of the events:

$$\mathcal{A}_N := \Bigg\{ \underbrace{\Bigg| (\mathcal{N})^{-1}(\log|\det \mathcal{P}^\delta| + \log|\det E_{-+}^\delta(z)|) - \fint_X \log|z - f_0(x)|\, \mathrm{d}\mu \Bigg|}_{:=B} > \varepsilon_N \Bigg\}.$$

We let $\varepsilon_N = N^{-\gamma}$ for a suitably chosen $\gamma = \gamma(d, \kappa) > 0$. Expand

$$B = B_1 + B_2 + B_3,$$

where

$$B_1 = \mathcal{N}^{-1} \log|\det \mathcal{P}^0| - \fint_X \log|z - f_0(x)| \, \mathrm{d}\mu(x), \tag{5.4}$$

$$B_2 = \mathcal{N}^{-1}(\log|\det \mathcal{P}^\delta| - \log|\det \mathcal{P}^0|), \tag{5.5}$$

$$B_3 = \mathcal{N}^{-1} \log|\det E_{-+}^\delta|. \tag{5.6}$$

Controlling B_1 requires the most work as it requires utilizing the calculus of Toeplitz operators. However, it is completely deterministic, and remains true for unperturbed operators. B_2 will be easily shown to be negligible. Proving a lower bound on B_3 is the key ingredient in proving Theorem 2, as it will force the events $\mathcal{A}_N$ to sufficiently small probability. Without a perturbation, B_3 will have no lower bound.

Proving bounds on B_2 and B_3 closely follows [22].

Lemma 5.5 (Bound on E_{-+}). *In the notation of (5.2), $\|E_{-+}^0\| \le \sqrt{\alpha}$.*

Proof. By construction, $E_{-+}^0 = -\sum_1^A t_j \delta_j \otimes \delta_j$, so

$$\|E_{-+}^0\| = |E_{-+}^0(\delta_A)| = t_A \le \sqrt{\alpha}.$$

■

Lemma 5.6 (Bound on E^δ). *In the notation of (5.2), $\|E^\delta\| \le 2\alpha^{-1/2}$ with overwhelming probability.*

Proof. By the Neumann construction, $\|E^\delta\| = \|E^0(1 + \delta \mathcal{G}_\omega E^0)^{-1}\| \le 2\|E^0\|$ which is bounded by $2\alpha^{-1/2}$ by Lemma 5.3. ■

Claim 5.2 (Bound on B_2). *In the notation of (5.5), $B_2 = \mathcal{O}(\delta\alpha^{-1/2}\mathcal{N}^{1/2})$ with overwhelming probability.*

Proof. Using Jacobi's formula, $(\log\det A)' = \mathrm{Tr}(A^{-1}A')$, we have that

$$\begin{aligned} \mathcal{N} B_2 &= \log|\det \mathcal{P}^\delta| - \log|\det \mathcal{P}| \\ &= \int_0^\delta \frac{d}{d\tau} \log|\det \mathcal{P}^\tau| \, \mathrm{d}\tau \\ &= \int_0^\delta \mathrm{Re}(\mathrm{Tr}(\mathcal{E}^\tau \frac{d}{d\tau} \mathcal{P}^\tau)) \, \mathrm{d}\tau \\ &= \int_0^\delta \mathrm{Re}(\mathrm{Tr}(E^\tau \mathcal{G}_\omega)) \, \mathrm{d}\tau. \end{aligned}$$

Taking absolute values and using properties of trace norms,

$$\begin{aligned} |\log|\det \mathcal{P}^\delta| - \log|\det \mathcal{P}^0|| &\leq \delta \sup_{\tau\in[0,\delta]} \|E^\tau\| \|\mathcal{G}_\omega\|_{\mathrm{tr}} \\ &\leq \mathcal{O}(\delta\alpha^{-1/2}\mathcal{N}\|\mathcal{G}_\omega\|), \end{aligned} \tag{5.7}$$

where we used Lemma 5.6, and Hölder's inequality for the Schatten norm. Recalling the bound on $\mathcal{G}_\omega$, (5.7) is $\mathcal{O}(\delta\alpha^{-1/2}\mathcal{N}^{3/2})$ with overwhelming probability. ∎

The following theorem about singular values of randomly perturbed matrices is required for proving a lower bound of B_3. Given a matrix B, let $s_1(B) \geq s_2(B) \geq \cdots \geq s_N(B)$ be its singular values.

Proposition 5.7. *If B is an $N \times N$ complex matrix and $\mathcal{G}_\omega$ is a random matrix with independent identically distributed complex Gaussian entries of mean 0 and variance 1, then there exists $C > 0$ such that, for all $\delta > 0$, $t > 0$,*

$$\mathbb{P}(s_N(B + \delta\mathcal{G}_\omega) < \delta t) \leq CNt^2.$$

Proof. See [22, Theorem 23], which is a complex version proven by Sankar, Spielman, and Teng in [14, Lemma 3.2]. ∎

Claim 5.3 (Bound on B_3). *In the notation of (5.6), B_3 obeys the probabilistic upper bound*

$$\mathbb{P}(\mathcal{N}^{-1}\log|\det E^\delta_{-+}| < 0) > 1 - e^{-\mathcal{N}}, \tag{5.8}$$

for $N \gg 1$. And B_3 obeys the probabilistic lower bound: there exists there exists $C > 0$ such that for all $\delta > 0$

$$\mathbb{P}(\mathcal{N}^{-1}\log|\det E^\delta_{-+}| \geq A\mathcal{N}^{-1}\log(\delta t)) > 1 - C\mathcal{N}t^2 - e^{-\mathcal{N}}.$$

Proof. First, by the Neumann series construction and choice of δ, with overwhelming probability,

$$\begin{aligned} \|E^\delta_{-+}\| &\leq \|E^\delta_{-+} - E^0_{-+}\| + \|E^0_{-+}\| \\ &= \|E^0_-(1 - \delta\mathcal{G}_\omega E^0)^{-1}\delta\mathcal{G}_\omega E^0_+\| + \|E^0_{-+}\| \\ &\leq 2\|\delta\mathcal{G}_\omega\| + \alpha^{1/2} \leq C\alpha^{1/2}. \end{aligned}$$

So, in this event, $\|E^\delta_{-+}\| \leq C\alpha^{1/2} < 1$ for $N \gg 1$, and therefore $\log|\det E^\delta_{-+}| < 0$ proving (5.8).

For the lower bound, first note that

$$\log|\det E^\delta_{-+}| = \sum_1^A \log s_j(E^\delta_{-+}) \geq A\log s_A(E^\delta_{-+}).$$

For a matrix B, let $t_1(B)$ be the smallest eigenvalue of $\sqrt{B^*B}$, so one has $s_A(E^\delta_{-+}) = t_1(E^\delta_{-+})$. Assume that $P - z$ is invertible. Using that $(E^0_{-+})^{-1} = -R_+(P-z)^{-1}R_-$ and properties of singular values of sums and products of trace class operators, we get

$$\begin{aligned}(t_1(E^0_{-+}))^{-1} &= s_1((E^0_{-+})^{-1}) \\ &\leq s_1(R_-)s_1(R_+)s_1((P-z)^{-1}) \\ &= \|R_+\|\|R_-\|s_1((P-z)^{-1}) \\ &= s_1((P-z)^{-1}) = (t_1(P-z))^{-1} \\ &= s_{\mathcal{N}}((P-z)^{-1}).\end{aligned}$$

For $\delta = \mathcal{O}(\mathcal{N}^{-1/2}\alpha^{1/2})$, this holds for E^δ_{-+} (the event of a singular matrix has probability zero and the singular values depend continuously on δ) so

$$s_A(E^\delta_{-+}) = t_1(E^\delta_{-+}) \geq s_{\mathcal{N}}(P + \delta\mathcal{G}_\omega - z)$$

with overwhelming probability.

Using Proposition 5.7, in the event that $\|\mathcal{G}_\omega\| \leq C\mathcal{N}^{1/2}$ (overwhelming probability) and $s_{\mathcal{N}}(P - z + \delta\mathcal{G}_\omega) > \delta t$ (probability at least $1 - C\mathcal{N}t^2$), we have that $s_A(E^\delta_{-+}) > \delta t$ with probability greater than $1 - C\mathcal{N}t^2 - e^{-\mathcal{N}}$. Therefore

$$\log|\det E^\delta_{-+}| \geq A \log s_A(E^\delta_{-+}) \geq A\log(\delta t)$$

with probability $\geq 1 - e^{-\mathcal{N}} - C\mathcal{N}t^2$. ∎

6. Bound on B_1

This section is devoted to estimating B_1 (as in (5.4)) which involves computing the trace of a function of a Toeplitz operator belonging to an exotic symbol class. This closely follows [22], however several simplifications arise partially due to requiring weaker bounds, and several modifications are required as we are working with Toeplitz operators.

Claim 6.1 (Bound on B_1). *For $\mathcal{P}$ defined in (5.1),*

$$\log|\det \mathcal{P}^0| = N^d \fint_X \log|f_0(x) - z|^2 \,\mathrm{d}\mu + \mathcal{O}(N^{d-\min(2\rho\kappa,(1-2\rho))}\log(N)).$$

Proof. Let us first consider some preliminary reductions in computing $\log|\det \mathcal{P}^0|$. By Schur's complement formula, $|\det \mathcal{P}^0|^2 = |\det(P-z)|^2|\det E^0_{-+}|^{-2}$. The first term is

$$|\det(P-z)|^2 = \det Q = \prod_{i=1}^{\mathcal{N}} t_i^2.$$

Because $E^0_{-+} = -\sum_1^A t_j \delta_j \otimes \delta_j$ (recall A is the largest integer such that $t_A^2 \leq \alpha$), the second term is

$$|\det E^0_{-+}|^{-2} = \Big(\prod_{i=1}^{A} t_i^2\Big)^{-2},$$

therefore

$$|\det \mathcal{P}^0|^2 = \prod_{i=A+1}^{\mathcal{N}} t_i^2 = \alpha^{-A} \prod_{i=1}^{\mathcal{N}} 1_\alpha(t_i^2) = \alpha^{-A} \det 1_\alpha(Q)$$

where $1_\alpha = \max(x, \alpha)$. If χ is a cut-off function identically 1 on [0, 1], and supported in $[-1/2, 2]$, then $x + (\alpha/4)\chi(4x/\alpha) \leq 1_\alpha(x) \leq x + \alpha\chi(x/\alpha)$ for $x \geq 0$. Therefore,

$$\det(Q + 4^{-1}\alpha\chi(Q/(4^{-1}\alpha))) \leq \det(1_\alpha(Q)) \leq \det(Q + \alpha\chi(Q/\alpha)). \tag{6.1}$$

Now, fix $1 \gg \alpha_1 > \alpha$, so that $\log\det(Q + \alpha\chi(Q/\alpha))$ can be written

$$-\int_\alpha^{\alpha_1} \frac{d}{dt} \log\det(Q + t\chi(Q/t))\, \mathrm{d}t + \log\det(Q + \alpha_1\chi(Q/\alpha_1)). \tag{6.2}$$

First, the integrand is estimated. Let

$$\psi(t) = (t - t\chi'(t))(1 + \chi(t))^{-1}$$

so that

$$\frac{d}{dt} \log(x + t\chi(x/t)) = t^{-1}\psi(x/t)$$

for $t > 0$ and $\psi \in C_0^\infty(\mathbb{R}_{\geq 0})$. Therefore, by Jacobi's identity,

$$\frac{d}{dt} \log\det(Q + t\chi(Q/t)) = \mathrm{Tr}(t^{-1}\psi(Q/t)).$$

While morally the same, here we diverge from Vogel's proof [22] to handle this trace term, and must rely on Section 3. The main issues are that Q is the composition of Toeplitz operators, which may no longer be a Toeplitz operator (but is modulo $\mathcal{O}(N^{-\infty})$ error), Q/t belongs to an exotic symbol class so to compute $\psi(Q/t)$ requires an exotic calculus, and the trace formula (Proposition 3.6) has weaker remainder than for quantizations of tori.

Let ρ_t be such that $t = N^{-2\rho_t}$. By Proposition 3.3, one has $Q = T_N q + \mathcal{O}(N^{-\infty})$, where the principal symbol of q is $|f_0 - z|^2$. For each t, Q/t is (modulo $\mathcal{O}(N^{-\infty})$) a Toeplitz operator with symbol in $S_{\rho_t}(m_t)$ where $m_t = q_0/t + 1$, by Claim 3.1. And so, by Proposition 3.5, there exists $q_t \in S_{\rho_t}(m_t^{-1})$, such that one has

$\psi(Q/t) = T_N(q_t) + E_N(t)$. Where q_t has principal symbol $\psi(q/t)$ and $E_N(t) = \mathcal{O}(N^{-\infty})$ (with estimates uniform over t). Therefore,

$$\int_{\alpha}^{\alpha_1} \frac{d}{dt} \log\det(Q + t\chi(Q/t))\, \mathrm{d}t = \int_{\alpha}^{\alpha_1} \mathrm{Tr}(t^{-1}\psi(Q/t))\, \mathrm{d}t$$
$$= \int_{\alpha}^{\alpha_1} t^{-1}\, \mathrm{Tr}(T_N(q_t) + E_N(t))\, \mathrm{d}t.$$

The error term is

$$\int_{\alpha}^{\alpha_1} t^{-1}\, \mathrm{Tr}(E_N(t)) dt = \mathcal{O}(N^{-\infty})$$

because $E_N(t)$ is uniformly $\mathcal{O}(N^{-\infty})$. While for each t, Proposition 3.6 shows that

$$\mathrm{Tr}(T_N(q_t)) = \Big(\frac{N}{2\pi}\Big)^d \int_X \psi(q_0/t)\, \mathrm{d}\mu_d(x) + t^{-1}\mathcal{O}(N^{d-1})$$

because m^{-1} is bounded. Therefore,

$$\int_{\alpha}^{\alpha_1} \frac{d}{dt} \log\det(Q + t\chi(Q/t))\, \mathrm{d}t$$
$$= \int_{\alpha}^{\alpha_1} \Big(\int_X \Big(\frac{N}{2\pi}\Big)^d t^{-1}\psi(q_0/t)\, \mathrm{d}\mu_d(x) + t^{-2}\mathcal{O}(N^{d-1}) \Big) \mathrm{d}t$$
$$= \Big(\frac{N}{2\pi}\Big)^d \int_X \log(q_0 + t\chi(q_0/t))\Big|_{t=\alpha}^{t=\alpha_1} \mathrm{d}\mu(x) + \mathcal{O}(N^{d-1}\alpha).$$

Next, the second term of (6.2) is computed. Because α_1 is fixed, Q/α_1 has symbol in $S(1)$. Therefore, by Proposition 3.5, $Q + \alpha_1\chi(Q/\alpha_1) = T_N r + E_N$ (with $\|E_N\| = \mathcal{O}(N^{-\infty})$) where $r \in S(1)$ with principal symbol $q_0 + \alpha_1\chi(q_0/\alpha_1)$. Let $r^t = tr + (1-t) \in S(1)$, so that

$$\log\det(Q + \alpha_1\chi(Q/\alpha_1)) = \int_0^1 \frac{d}{dt} \log\det(T_N r^t + tE_N)\, \mathrm{d}t$$
$$= \int_0^1 \mathrm{Tr}\Big((T_N r^t + tE_N)^{-1}\Big(\frac{d}{dt}T_N r^t + E_N\Big)\Big)\, \mathrm{d}t.$$

The principal symbol of r^t is $r_0^1 = t(q_0 + \alpha_1\chi(q_0/\alpha_1)) + (1-t)$. Note that when $x \geq 0$, then $x + \alpha_1\chi(x/\alpha_1) \geq \alpha_1 > 0$. Therefore, $(r_0^t) \geq \alpha_1$.

Lemma 6.1. *There exists $s(t) \in S(1)$ (with bounds uniform in t) such that*

$$(T_N r^t + tE_N)^{-1} = T_N s(t) + \mathcal{O}(N^{-\infty}),$$

and the principal symbol of $s(t)$ is $(r_0^t)^{-1}$.

Proof. By Proposition 3.4, there exists a symbol $\ell = \ell(t) \in S(1)$ which inverts (modulo $\mathcal{O}(N^{-\infty})$ error) $T_N r^t$, and has principal symbol $(r_0^t)^{-1}$. But then

$$(T_N r^t + tE_N)T_N\ell = 1 + K$$

with $K = \mathcal{O}(N^{-\infty})$, using that $tE_N = \mathcal{O}(N^{-\infty})$ and $T_N\ell$ has norm bounded independent of N. By Neumann series, for $N \gg 1$, $(1+K)$ is invertible, so that

$$(T_N r^t + tE_N)(T_N\ell)(1+K)^{-1} = 1.$$

$(T_N\ell)(1+K)^{-1}$ will be a Toeplitz operator, modulo a $\mathcal{O}(N^{-\infty})$ term, with symbol ℓ which has principal symbol $(r_0^t)^{-1}$. By repeating this argument, but left-composing by $T_N\ell$, we get the lemma. ∎

Clearly, $\frac{d}{dt}T_N r^t = T_N(r-1)$ so using Lemma 6.1, we get that

$$(T_N r^t + tE_N)^{-1}\Big(\frac{d}{dt}T_N r^t + E_N\Big)$$

is (modulo $\mathcal{O}(N^{-\infty})$) a Toeplitz operator with principal symbol $(r_0^t)^{-1}(\frac{d}{dt}r_0^t)$. So, by Proposition 3.6,

$$\begin{aligned}&\mathrm{Tr}\Big((T_N r^t + tE_N)^{-1}\Big(\frac{d}{dt}T_N r^t + E_N\Big)\Big)\\ &\quad= \Big(\frac{N}{2\pi}\Big)^d \int_X (r_0^t)^{-1}\Big(\frac{d}{dt}r_0^t\Big)\,\mathrm{d}\mu_d(x) + \mathcal{O}(N^{d-1}),\end{aligned}$$

which when integrated from $t = 0$ to $t = 1$ becomes

$$\begin{aligned}&\Big(\frac{N}{2\pi}\Big)^d \int_X \log(r_0^1)dx + \mathcal{O}(N^{d-1})\\ &\quad= \Big(\frac{N}{2\pi}\Big)^d \int_X \log(q_0 + \alpha_1\chi(q_0/\alpha_1))\,\mathrm{d}\mu_d(x) + \mathcal{O}(N^{d-1}).\end{aligned}$$

Therefore, (6.2) becomes

$$\left(\frac{N}{2\pi}\right)^d \int\limits_X \log(q_0 + \alpha\chi(q_0/\alpha))\,\mathrm{d}\,\mu_d + \mathcal{O}(N^{d-1}\alpha^{-1}).$$

A calculus lemma is required to estimate $\int_X \log(q_0 + \alpha\chi(q_0/\alpha))\,\mathrm{d}\,x$.

Lemma 6.2. *Given $q \in C^\infty(X;\mathbb{R}_{\geq 0})$ such that $\mu_d(\{x \in X : q(x) \leq t\}) = \mathcal{O}(t^\kappa)$ as $t \to 0$ for $\kappa \in (0,1]$, and $\chi \in C_0^\infty((-1/2,2);[0,1])$ identically 1 on $[0,1]$, then*

$$\int\limits_X \log(q + \alpha\chi(q/\alpha))\,\mathrm{d}\,\mu_d = \int\limits_X \log(q)\,\mathrm{d}\,\mu_d + \mathcal{O}(\alpha^\kappa).$$

Proof. Let $g(t) = \log(t + \alpha\chi(t/\alpha))$ and $m(t) = \mu_d(\{x \in X : q(x) \leq t\})$. Then, letting $q_1 = \max q + 2\alpha$,

$$\begin{aligned}
&\int\limits_X \log(q + \alpha\chi(q/\alpha)) - \log(\alpha)\,\mathrm{d}\,\mu_d \\
&\quad= \int\limits_X g(q(x)) - g(0)\,\mathrm{d}\,\mu_d = \int\limits_X \int\limits_0^{q(x)} g'(t)\,\mathrm{d}\,t\,\mathrm{d}\,\mu_d \\
&\quad= \int\limits_0^{q_1} g'(t) \int\limits_{q(x)>t} \mathrm{d}\,\mu_d\,\mathrm{d}\,t \\
&\quad= \int\limits_0^{q_1} g'(t)(\mathrm{vol}(X) - m(t))\,\mathrm{d}\,t \\
&\quad= \mathrm{vol}(X)(g(q_1) - \log(\alpha)) - \int\limits_0^{q_1} g'(t)m(t)\,\mathrm{d}\,t.
\end{aligned}$$

So, that

$$\int\limits_X \log(q + \alpha\chi(q/\alpha))\,\mathrm{d}\,\mu_d = \mathrm{vol}(X)g(q_1) - \int\limits_0^{q_1} g'(t)m(t)\,\mathrm{d}\,t. \tag{6.3}$$

Similarly, if $\tilde{g}(t) = \log(t)$, we get an analogous expression as (6.3), that is,

$$\int\limits_X \log(q)\,\mathrm{d}\,\mu_d = \mathrm{vol}(X)\tilde{g}(q_1) - \int\limits_0^{q_1} \tilde{g}'(t)m(t)\,\mathrm{d}\,t.$$

Note that $g(q_1) = \tilde{g}(q_1)$. Therefore,

$$\begin{aligned}\left|\int_X \log(q + \alpha\chi(q/\alpha)) - \log(q)\,\mathrm{d}\mu_d\right| &= \left|\int_0^{q_1} (\tilde{g}'(t) - g'(t))m(t)\,\mathrm{d}t\right| \\ &= \left|\int_0^{q_1} \Big(\frac{1}{t} - \frac{1+\chi'(t/\alpha)}{t+\alpha\chi(t/\alpha)}\Big)m(t)\,\mathrm{d}t\right| \\ &= \left|\int_0^{q_1/\alpha} \Big(\frac{1}{s} - \frac{1+\chi'(s)}{s+\chi(s)}\Big)m(s\alpha)\,\mathrm{d}s\right| \\ &\lesssim \int_0^2 s^{-1} m(s\alpha)\,\mathrm{d}s \\ &\lesssim \alpha^\kappa \int_0^2 s^{\kappa-1}\,\mathrm{d}s \lesssim \alpha^\kappa.\end{aligned}$$

Here we use that $\chi(0) = 1$ to get a lower bound on $|s + \chi(s)|$, and the fact that $\chi(s) - s\chi'(s)$ is supported in $(0, 2)$. ∎

Applying this lemma, we get

$$\log\det(Q + \alpha\chi(Q/\alpha)) = \Big(\frac{N}{2\pi}\Big)^d \int_X \log(q)\,\mathrm{d}\mu_d(x) + \mathcal{O}(\alpha^\kappa) + \mathcal{O}(N^{d-(1-2\rho)}).$$

Recalling that $(N/2\pi)^d \mathcal{N}^{-1} = \mathrm{vol}(X)^{-1} + \mathcal{O}(N^{-1})$, we get that

$$\begin{aligned}&\log\det(Q + \alpha\chi(Q/\alpha)) \\ &\quad = (\mathcal{N} + \mathcal{O}(N^{-1})) ⨍ \log(q)\,\mathrm{d}\mu_d + \mathcal{O}(N^{d-(1-2\rho)}).\end{aligned} \tag{6.4}$$

$\int_X \log(q)\,\mathrm{d}\mu_d$ can be uniformly bounded in z, so that the $\mathcal{O}(N^{-1})$ term can be absorbed into $\mathcal{O}(N^{d-(1-2\rho)})$. By (6.1), we get the following lower bound by replacing α by $\alpha/4$:

$$\log\det(Q + \alpha\chi(Q/\alpha)) \geq \mathcal{N} ⨍ \log(q)\,\mathrm{d}\mu_d + \mathcal{O}(N^{d-(1-2\rho)}). \tag{6.5}$$

Lemma 6.3 (Bound on A). *The number of eigenvalues of Q that are less than α is* $\mathcal{O}(N^d N^{-\min(2\rho\kappa,(1-2\rho))})$.

Proof. Let $\psi \in C_0^\infty([-1/2, 3/2]; [0, 1])$ be identically 1 on $[0, 1]$. It then suffices to estimate $\mathrm{Tr}(\psi(Q/\alpha))$. By Proposition 3.5, $\psi(Q/\alpha) = T_{N,q_2} + \mathcal{O}(N^{-\infty})$, where $q_2 \in S_\rho(1)$ with principal symbol $\psi(q/\alpha)$. Then, by Proposition 3.6,

$$\begin{aligned}\mathrm{Tr}(\psi(Q/\alpha)) &= \mathrm{Tr}(T_{N,q_2} + \mathcal{O}(N^{-\infty}))\\ &= (N/2\pi)^d \int_X \psi(q/\alpha)\,\mathrm{d}\mu_d(x) + \mathcal{O}(N^{d-(1-2\rho)})\\ &\lesssim N^d\alpha^\kappa + N^{d-(1-2\rho)}\\ &= \mathcal{O}(N^d N^{-\min(2\rho\kappa,1-2\rho)}).\end{aligned}$$

■

Therefore, putting everything together, we get that

$$\begin{aligned}\log|\det \mathcal{P}^0| &= \frac{1}{2}\log(|\det \mathcal{P}^0|^2)\\ &= \frac{1}{2}\log(\alpha^{-A}\det 1_\alpha(Q))\\ &= \frac{A}{2}\log(1/\alpha) + \frac{1}{2}\log\det(1_\alpha Q)).\end{aligned}$$

Formulas (6.4) and (6.5) provide upper and lower bounds of $2^{-1}\log\det(1_\alpha(Q))$. Then, using that $2^{-1}\log q_0 = |f_0 - z|$ and Lemma 6.3, we get

$$\begin{aligned}&\Big|\log|\det \mathcal{P}^0| - \mathcal{N} \fint_X \log|f_0 - z| d\mu_d\Big|\\ &\lesssim A\log(1/\alpha) + \alpha^\kappa + N^{d-(1-2\rho)}\\ &\lesssim N^{d-\min(2\rho\kappa,(1-2\rho))}\log(N) + N^{-2\rho\kappa} + N^{d-(1-2\rho)}\\ &\lesssim N^{d-\min(2\rho\kappa,(1-2\rho))}\log(N).\end{aligned}$$

Recall $\mathcal{N} B_1 = \log|\det \mathcal{P}^0| - \mathcal{N} \fint \log|z - f_0(x)|\,\mathrm{d}\mu_d$, so that

$$B_1 = \mathcal{O}(N^{-\min(2\rho\kappa,(1-2\rho))}\log(N)).$$

■

7. Summability of $\mathcal{A}_N$

Recall that $\mathcal{A}_N = \{|B(N)| > \varepsilon_N\}$, where $B(N) = B_1 + B_2 + B_3$ with

$$\begin{aligned}B_1 &= \mathcal{N}^{-1}\log|\det \mathcal{P}^0| - \fint \log|z - f_0(x)|\,\mathrm{d}\mu_d(x),\\ B_2 &= \mathcal{N}^{-1}(\log|\det \mathcal{P}^\delta| - \log|\det \mathcal{P}^0|),\\ B_3 &= \mathcal{N}^{-1}\log|\det E_{-+}^\delta|.\end{aligned}$$

The following table summarizes the bounds on B_1, B_2, and B_3:

Bound	Probability of Bound	Reference
$B_1 = \mathcal{O}(N^{-\min(2\rho\kappa,(1-2\rho))}\log(N))$	1	Claim 6.1
$B_2 = \mathcal{O}(\delta\alpha^{-1/2}\mathcal{N}^{1/2})$	$> 1 - \exp(-\mathcal{N})$	Claim 5.2
$B_3 \geq \mathcal{N}^{-1} A \log(t\delta)$	$> 1 - C\mathcal{N}t^2 - \exp(-\mathcal{N})$	Claim 5.3
$B_3 < 0$	$> 1 - \exp(-\mathcal{N})$	Claim 5.3

Recall that $\rho \in (0, \min(1/2, \varepsilon))$ and that $\alpha = N^{-2\rho}$. Theorem 2 will follow if $\sum \mathbb{P}(\mathcal{A}_N) < \infty$ for $\varepsilon_N = N^{-\gamma}$. Recall that $\delta = \mathcal{O}(N^{-d/2-\varepsilon}) = \mathcal{O}(N^{-d/2}\alpha^{1/2})$. Fix $0 < \gamma < \min(\varepsilon - \rho, 2\rho\kappa, 1 - 2\rho)$. Then $\mathbb{P}(\mathcal{A}_N) = \mathbb{P}(B > N^{-\gamma}) + \mathbb{P}(B < -N^{-\gamma})$. The first term is

$$\mathbb{P}(B > N^{-\gamma}) = \mathbb{P}(B_3 > N^{-\gamma} - B_2 - B_1).$$

Because $\gamma < \varepsilon - \rho$ and $B_2 = \mathcal{O}(N^{\rho-\varepsilon})$ (with overwhelming probability), we see that $B_2 = \mathcal{O}(N^{-\gamma})$ (with overwhelming probability). Similarly, because of the bound on B_1 and the choice of γ, $B_1 = \mathcal{O}(N^{-\gamma})$. So, if N is sufficiently large, $N^{-\gamma} - B_2 - B_1 \geq CN^{-\gamma} > 0$. But then by Claim 5.3, $\mathbb{P}(B > N^{-\gamma}) \leq e^{-N^d}$ for $N \gg 1$.

Similarly, for N sufficiently large, there exists $C_0 \in (0, 1/2)$ such that, $|B_1| + |B_2| < C_0 N^{-\gamma}$, so

$$\begin{aligned}\mathbb{P}(B < -N^{-\gamma}) &\leq \mathbb{P}(B_3 < -(1 - C_0)N^{-\gamma}) \\ &= 1 - \mathbb{P}(B_3 \geq -(1 - C_0)N^{-\gamma}).\end{aligned}$$

By the choice of γ, bound on A from Lemma 6.3, and selecting $t = \mathcal{N}^{-2/d-1/2}$, we get, for large enough N, $-(1 - C_0)N^{-\gamma} \leq \mathcal{N}^{-1} A \log(\delta t)$ as long as

$$-N^{-\gamma}(1 - C_0) \leq \mathcal{N}^{-1} A \log(\delta).$$

This requires that $\delta \gg e^{-N^{\beta}}$ for $\beta = \min(2\rho\kappa, 1 - 2\rho) - \gamma \in (0, 1)$. In this case, by Claim 5.3,

$$\begin{aligned}\mathbb{P}(B_3 > -N^{-\gamma}) &\geq \mathbb{P}(B_3 > A\mathcal{N}^{-1}\log(\delta t)) \\ &\geq 1 - C\mathcal{N}t^2 - e^{-\mathcal{N}} \\ &= 1 - C\mathcal{N}^{-2/d} + e^{-\mathcal{N}}.\end{aligned}$$

Therefore, $\mathbb{P}(B < -N^{-\gamma}) \leq CN^{-2} + e^{-N^d}$ for $N \gg 1$.

With this,

$$\sum_{N=1}^{\infty} \mathbb{P}(\mathcal{A}_N) = C + \sum_{N \gg 1} \mathbb{P}(A_N) \leq C + \sum_{N \gg 1} (N^{-2} + 2e^{-N^d}) < \infty,$$

which proves Theorem 2.

Note that if $\varepsilon > (2(\kappa+1))^{-1}$, then we can select $\rho = (2(\kappa+1))^{-1}$ and choose γ arbitrarily small, so that $\beta = \kappa(\kappa+1)^{-1} - \gamma$. While if $\varepsilon < (2(\kappa+1))^{-1}$, then the maximum β can be is $2\varepsilon\kappa$. Therefore, we have

$$\beta < \begin{cases} 2\varepsilon\kappa & \text{if } \varepsilon < \frac{1}{2(\kappa+1)}, \\ \frac{\kappa}{\kappa+1} & \text{if } \varepsilon \geq \frac{1}{2(\kappa+1)}. \end{cases}$$

8. General random perturbations

In this section we provide a discussion about how to modify the proof of Theorem 2 (Gaussian random perturbations) to prove Theorem 3 (more general random perturbations). We also deduce Theorem 1 (stated in the introduction) from Theorem 3.

Proof. Under the assumptions of $\mathcal{W}_\omega$ (see Definition 2.3), we have the following probabilistic norm bound:

$$\mathbb{E}[\|\mathcal{W}_\omega\|^2] = \sum_{i,j=1}^{\mathcal{N}} \mathbb{E}[|(\mathcal{W}_\omega)_{i,j}|^2] = \mathcal{O}(\mathcal{N}^2), \tag{8.1}$$

as well as the following anti-concentration bound (from [20, Theorem 3.2]): for $\gamma_0 \geq 1/2$, $A_0 \geq 0$, there exists a $c > 0$ such that if M is a deterministic matrix with $\|M\| \leq \mathcal{N}^{\gamma_0}$ then

$$\begin{aligned} &\mathbb{P}(s_{\mathcal{N}}(M + \mathcal{W}_\omega) \leq \mathcal{N}^{-(2A_0+1)\gamma_0}) \\ &\quad \leq c(\mathcal{N}^{-A_0+o(1)} + \mathbb{P}(\|\mathcal{W}_\omega\| \geq \mathcal{N}^{\gamma_0})). \end{aligned} \tag{8.2}$$

Recall, for an $N \times N$ matrix A, we denote $s_1 \geq s_2 \geq \cdots \geq s_N(A)$ the singular values of A.

From (8.1), and Markov's inequality, we get

$$\mathbb{P}(\|\mathcal{W}_\omega\| \geq N^{d-1}) = \mathcal{O}(N^{-2})$$

therefore if $\delta = N^{-d}$ then $\delta\|\mathcal{W}_\omega\| = \mathcal{O}(N^{-1})$ with probability at least $1 - CN^{-2}$. From this, Claim 4.2 (the supports of the random empirical measures being contained in a bounded set for $N \gg 1$) will follow by an identical argument.

Next, with probability at least $1 - CN^{-2}$, we have $\delta\|\mathcal{W}_\omega\|_{\mathrm{L}}\alpha^{1/2} \ll 1$. In this event, we can build our perturbed Grushin problem the same way as in Section 5.

Next, we have to modify the estimate of B_2 which was estimated in Claim 5.2. For this, we simply modify (5.7) with a weaker estimate on the probability $\|\mathcal{W}_\omega\|$ is small. Specifically, we see there exists $C > 0$ such that

$$\mathbb{P}(B_2 = \mathcal{O}(\alpha^{-1/2}N^{-1})) > 1 - CN^{-2}.$$

The final modification is in estimating $B_3 = \mathcal{N}^{-1} \log|\det E^{\delta}_{-+}|$. We see, by the same argument presented in Section 5, that

$$\mathbb{P}(B_3 < 0) \geq 1 - CN^{-2}.$$

To prove a lower bound, we go through the same argument, to get that

$$\log|\det E^{\delta}_{-+}| \geq A \log|s_{\mathcal{N}}(T_N f - z + \delta \mathcal{W}_\omega)|.$$

Next, let

$$K_0 := \sup_{z\in\Lambda} \|T_N f - z\| = \mathcal{O}(1)$$

(recall Λ is a neighborhood of $f(X)$). By (8.2) (with $\gamma_0 = 1$ and $A_0 = 2$), we have (for $N \gg 1$)

$$\begin{aligned}
&\mathbb{P}(s_{\mathcal{N}}(T_N f - z + \delta \mathcal{W}_\omega) \leq N^{-7d}) \\
&\quad = \mathbb{P}\big(s_{\mathcal{N}}(\delta^{-1} K_0^{-1}(T_N f - z) + K_0^{-1}\mathcal{W}_\omega) \leq (N^d)^{-(2A_0+1)\gamma_0}\big) \\
&\quad \leq c\big(N^{-2d+o(1)} + \mathbb{P}(\|K_0^{-1}\mathcal{W}_\omega\| \geq N^{-d})\big) \\
&\quad \leq cN^{-2}.
\end{aligned}$$

Here we use that $\|\delta^{-1} K_0^{-1}(T_N f - z)\| \leq N^d$. With this, we can proceed as in Section 7, with weaker probabilistic estimates. We choose $\rho \in (0, 1/2)$, and $0 < \gamma < \min(2\rho\kappa, 1 - 2\rho)$. Writing $\mathbb{P}(\mathcal{A}_N) = \mathbb{P}(B > N^{-\gamma}) + \mathbb{P}(B < -N^{-\gamma})$, we see that

$$\mathbb{P}(B > N^{-\gamma}) \leq CN^{-2}$$

for $N \gg 1$. Similarly, in the event $s_{\mathcal{N}}(T_N f - z + \delta \mathcal{W}_\omega) \geq N^{-7d}$, we have (for $N \gg 1$)

$$A \log|s_{\mathcal{N}}(T_N f - z + \delta \mathcal{W}_\omega)| \leq N^{d-\gamma}$$

so that

$$\mathbb{P}(B_3 > -N^{-\gamma}) \geq \mathbb{P}(B_3 > A\mathcal{N}^{-1} \log|s_{\mathcal{N}}(T_N f - z + \delta \mathcal{W}_\omega)|) \geq 1 - CN^{-2}.$$

Therefore, $\mathbb{P}(B < -N^{-\gamma}) \leq CN^{-2}$ for $N \gg 1$. With this, $\sum_1^\infty \mathbb{P}(\mathcal{A}_N) < \infty$, and we have almost sure weak convergence of the empirical measures of $T_N f + \delta \mathcal{W}_\omega$ to $\mathrm{vol}(X)^{-1}(f_0)_*\mu_d$. ∎

Proposition 8.1. *Theorem 3 implies the probabilistic Weyl law (Theorem 1) stated in the introduction.*

Proof. For $\Lambda \subset \mathbb{C}$ given in the hypothesis, let

$$A_N = (\mathrm{vol}(X)/\mathcal{N})\#\{\mathrm{Spec}(T_N f + N^{-d}\mathcal{W}_\omega) \cap \Lambda\}.$$

It suffices to show that for each $\varepsilon > 0$

$$\mathbb{P}(\limsup_{N\to\infty} |A_N - \mu_d(f \in \Lambda)| > \varepsilon) = 0.$$

We may assume Λ is bounded. If not, let $\widetilde{\Lambda}$ be an open, bounded neighborhood of $f(X)$. Recall that almost surely $\mathrm{Spec}(T_N f + \delta \mathcal{W}_\omega) \subset \widetilde{\Lambda}$ for $N \gg 1$. Therefore, if

$$\widetilde{A}_N = (\mathrm{vol}(X)/\mathcal{N})\#\{\mathrm{Spec}(T_N f + N^{-d}\mathcal{W}_\omega) \cap \Lambda \cap \widetilde{\Lambda}\},$$

then

$$\mathbb{P}(\limsup_{N\to\infty} |A_N - \mu_d(f \in \Lambda)| > \varepsilon) = \mathbb{P}(\limsup_{N\to\infty} |\widetilde{A}_N - \mu_d(f \in \Lambda)| > \varepsilon).$$

Now, relabel $\Lambda \cap \widetilde{\Lambda}$ as Λ. Let $\varphi, \psi \in C_0^\infty(\mathbb{C}; [0,1])$ be such that $\mathrm{supp}\,\varphi \subset \Lambda$, $\varphi(x) \equiv 1$ for $\mathrm{dist}(x, \partial\Lambda) > \varepsilon$, $\psi(x) \equiv 1$ for $x \in \Lambda$, and $\psi(x) = 0$ for $\mathrm{dist}(x, \partial\Lambda) > \varepsilon$ (here $\partial\Lambda$ is the boundary of Λ). Therefore, we have

$$\frac{\mathrm{vol}(X)}{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \varphi(\lambda_i) \le A_N \le \frac{\mathrm{vol}(X)}{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \psi(\lambda_i). \tag{8.3}$$

By Theorem 3, the lower bound of (8.3) convergences almost surely to

$$\int_{\mathbb{C}} \varphi(z)(f_*\mu_d)(\mathrm{d}z) = \mu_d(f \in \Lambda) + \mathcal{O}(\varepsilon^\kappa).$$

And similarly the upper bound of (8.3) converges almost surely to $\mu_d(f \in \Lambda) + \mathcal{O}(\varepsilon^\kappa)$ (where the constant in $\mathcal{O}(\varepsilon^\kappa)$ is deterministic). Therefore, there exists $C > 0$ such that

$$\mathbb{P}(\limsup_{N\to\infty} |A_N - \mu_d(f \in \Lambda)| > C\varepsilon^\kappa) = 0.$$

Because $\varepsilon > 0$ is arbitrary, this implies A_N converges almost surely to $\mu_d(f \in \Lambda)$. Then, because $\mathcal{N} = \mathrm{vol}(X)(N/2\pi)^d + \mathcal{O}(N^{d-1})$, $(N/2\pi)^d\,\mathrm{vol}(X)\mathcal{N}^{-1}A_N$ converges almost surely to $\mu_d(f \in \Lambda)$. ∎

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Izak Oltman
Department of Mathematics, University of California, 970 Evans Hall, Berkeley, CA 94720, USA; izak@math.berkeley.edu