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Lipschitz continuity of spectra of pseudodifferential operators in a weighted Sjöstrand class and Gabor frame bounds

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Abstract. We study one-parameter families of pseudodifferential operators whose Weyl symbols are obtained by dilation and a smooth deformation of a symbol in a weighted Sjöstrand class. We show that their spectral edges are Lipschitz continuous functions of the dilation or deformation parameter. Suitably local estimates hold also for the edges of every spectral gap. These statements extend Bellissard's seminal results on the Lipschitz continuity of spectral edges for families of operators with periodic symbols to a large class of symbols with only mild regularity assumptions.

The abstract results are used to prove that the frame bounds of a family of Gabor systems $\mathcal{G}(g, \alpha\Lambda)$, where Λ is a set of non-uniform time-frequency shifts, $\alpha > 0$, and $g \in M^1_2(\mathbb{R}^d)$, are Lipschitz continuous functions in α. This settles a question about the precise blow-up rate of the condition number of Gabor frames near the critical density.

1. Introduction

We consider a one-parameter family of operators T_δ that depend smoothly on δ. A basic problem of spectral theory is to understand how the spectrum $\sigma(T_\delta)$ depends on the parameter. For self-adjoint operators at least, one would expect that the spectrum depends continuously, e.g., in the Hausdorff metric, on δ. This has been shown in several general settings [2, 3, 18]. As to more quantitative results, one may investigate the behavior of the extreme spectral values, or more generally of spectral edges, and try to understand their smoothness as a function of δ. This is an interesting problem in mathematical physics where δ may be the magnitude of a magnetic field or the value of Planck's constant [2, 5–8, 14, 16, 17].

In this paper we study a general class of pseudodifferential operators with symbols in a weighted Sjöstrand class and δ amounts roughly to a dilation of the symbol. We

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will prove that the spectral edges are Lipschitz continuous in δ. Moreover, we will show the Lipschitz continuity of the spectral gaps.

In our study, we use the Weyl calculus (though other calculi work as well without essential changes). Given a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, its Weyl transform is the operator

$$\sigma^w f(y) = \int_{\mathbb{R}^{2d}} \sigma\Big(\frac{x+y}{2}, \omega\Big) e^{2\pi i (y-x)\cdot\omega} f(x)\,dx\,d\omega, \tag{1.1}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ and a suitable interpretation of the integral. Let D_a denote the dilation $D_a\sigma(z) = \sigma(az)$. Throughout, we will study a one-parameter family of symbols σ_δ that arises by a dilation and a smooth variation of a basic symbol as follows. We will assume that the symbol depends on a parameter $\delta \in (-\delta_0, \delta_0)$ like

$$\sigma_\delta = D_{\sqrt{1+\delta}} G_\delta,$$

and write $T_\delta := \sigma_\delta^w$ for the corresponding operators. While G_δ is allowed to vary with δ, we shall assume that this dependence is moderate, so that σ_δ is roughly a dilation. For the question of spectral perturbation to be meaningful, we will assume that σ_δ is real-valued and that the corresponding operator is bounded on $L^2(\mathbb{R}^d)$, whence T_δ is always self-adjoint.

Our questions are thus the following. How does the spectrum of T_δ depend on δ? Consider in particular the spectral extreme values $\sigma_-(A) := \inf\{\lambda \in \mathbb{R} : \lambda \in \sigma(A)\}$ and $\sigma_+(A) := \sup\{\lambda \in \mathbb{R} : \lambda \in \sigma(A)\}$ of a self-adjoint operator. How does $\sigma_\pm(T_\delta)$ depend on δ? Or more generally, how do the spectral edges – that is, the endpoints of the connected components of $\mathbb{R} \setminus \sigma(T_\delta)$ – depend on δ? What are suitable conditions on the symbols σ_δ so that the spectral edges are Lipschitz continuous?

Our main inspiration comes from J. Bellissard's fundamental paper [8] on the almost-Matthieu operator or Harper operator in a non-commutative torus. He showed that, for certain families of Harper-like operators on the square lattice with constant magnetic field, the spectral edges and in particular the spectral gap boundaries depend Lipschitz continuously on the parameter, improving previous results on (Hölder-)continuity of spectral gaps [18] and spectral edges [3]. Bellissard's work has inspired many authors to extend his results. In [29], for example, Lipschitz continuity for Harper-like operators on crystal lattices is shown, while [14–17] considered continuous magnetic Schrödinger operators with weak magnetic field perturbation, [2] showed spectral continuity of pseudodifferential operators with elliptic symbols in the Hörmander class, and [7] studied dynamically-defined operator families on groups of polynomial growth, and the Lipschitz continuity of their spectra. The methods from [8] have also proved useful to investigate fine properties of rotation algebras [22]. Most notably, Beckus and Bellissard [5] have distilled part of the argument of [8] into a set of powerful abstract principles (see Section 1.3).

1.1. Results

To formulate our results, we use the language and methods of phase-space analysis (time-frequency analysis in applied mathematics) and employ a class of symbols that is tailored to time-frequency analysis. Let $z = (x, \omega) \in \mathbb{R}^{2d}$ be a point in phase-space (time-frequency space), and

$$\rho(z) f(t) = M_{\omega/2} T_x M_{\omega/2} f(t) = e^{-i\pi x\cdot\omega} e^{2\pi i \omega\cdot t} f(t-x)$$

denote the (symmetric) time-frequency shift by z. The associated transform is the short-time Fourier transform

$$V_g f(z) = e^{-i\pi x\cdot\omega} \langle f, \rho(z) g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i\omega\cdot t}\, dt.$$

Let $\varphi(t) = 2^{d/4} e^{-\pi|t|^2}$ denote the standard Gaussian in $\mathbb{R}^d$. The *mixed-norm weighted modulation* space $M^{p,q}_{s,t}(\mathbb{R}^d)$, $1 \le p, q \le \infty$, $s, t \ge 0$, contains all the elements in $\mathcal{S}'(\mathbb{R}^d)$ for which the norm

$$\|f\|_{M^{p,q}_{s,t}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\varphi f(x,\omega)|^p (1+|x|)^{sp}\, dx \right)^{q/p} (1+|\omega|)^{tq}\, d\omega \right)^{1/q} \tag{1.2}$$

is finite, with the usual modification when $p = \infty$ or $q = \infty$. If $p = q$, we write $M^p_{s,t}(\mathbb{R}^d)$, and if $s = t$, we write $M^{p,q}_s(\mathbb{R}^d)$. We omit the subscripts, when $s = t = 0$ and write $M^{p,q}$ for $M^{p,q}_{0,0}$. The use of any nonzero function $g \in \mathcal{S}(\mathbb{R}^d)$ instead of the Gaussian in (1.2) gives an equivalent norm, i.e., $\|f\|_{M^{p,q}_{s,t}} \asymp \|V_g f\|_{L^{p,q}_{s,t}}$. See, e.g., [23] for more details.

Our main result reads as follows.

Theorem 1.1. *Let* $0 < \delta_0 < 1$. *For* $\delta \in (-\delta_0, \delta_0)$, *let* $G_\delta \in M^{\infty,1}_{0,2}(\mathbb{R}^{2d})$ *be real-valued and* $\delta \mapsto G_\delta$ *differentiable*[1] *such that* $\partial_\delta G_\delta \in M^{\infty,1}(\mathbb{R}^{2d})$. *Let* $T_\delta = \sigma^w_\delta$ *be the pseudodifferential operator with Weyl symbol* $\sigma_\delta = D_{\sqrt{1+\delta}} G_\delta$. *Then, for* $\delta_1, \delta_2 \in (-\delta_0, \delta_0)$,

$$\begin{aligned} &|\sigma_\pm(T_{\delta_1}) - \sigma_\pm(T_{\delta_2})| \\ &\quad \le C_d \cdot |\delta_1 - \delta_2| \cdot (1-\delta_0)^{-(d+1)} \cdot \sup_{|t|<\delta_0} \left(\|G_t\|_{M^{\infty,1}_{0,2}} + \|\partial_t G_t\|_{M^{\infty,1}} \right), \end{aligned}$$

where C_d *is a constant that only depends on* d.

[1] Here we mean that the partial derivative of $(z, \delta) \mapsto G_\delta(z)$ with respect to δ exists.

To put this result in perspective, let us compare it with that in [8]. Bellissard studied operators that are linear combinations of phase-space shifts over a lattice, i.e., operators of the form

$$T_\delta = \sum_{k \in \mathbb{Z}^2} a_k(\delta)\rho(\sqrt{1+\delta}k). \tag{1.3}$$

Then the Weyl symbol of T_δ is periodic and given by

$$\sigma_\delta(z) = \sum_{k \in \mathbb{Z}^2} a_k(\delta) e^{2\pi i \sqrt{1+\delta}[k,z]},$$

where $[z, z'] = x' \cdot \omega - x \cdot \omega'$ denotes the standard symplectic form. Bellissard's condition on the coefficients $a_k(\delta)$ explicitly reads

$$\sup_{|t|<\delta_0} \Big(\sum_{k \in \mathbb{Z}^2} |a_k(t)|^2 (1+|k|)^{6+2\varepsilon} + \sum_{k \in \mathbb{Z}^2} |\partial_t a_k(t)|^2 (1+|k|)^{2+2\varepsilon} \Big) < \infty,$$

for some $\varepsilon > 0$. Since

$$\Big\| \sum_{k \in \mathbb{Z}^2} b_k e^{2\pi i [k,\cdot]} \Big\|^2_{M^{\infty,1}_{0,s}} \lesssim \sum_{k \in \mathbb{Z}^2} |b_k|^2 (1+|k|)^{2(1+s+\varepsilon)},$$

it follows that our result extends Bellissard's original conditions.

An important point is that operators of the form (1.3) belong to the non-commutative torus based on $\sqrt{1+\delta}\mathbb{Z}^2$, and thus C^*-algebraic arguments may be applied. By contrast, Theorem 1.1 uses non-periodic symbols within the class $M^{\infty,1}_{0,2}(\mathbb{R}^{2d})$, which roughly corresponds to requiring two bounded derivatives, although the precise membership condition is slightly more subtle. In particular, the class $M^{\infty,1}_{0,2}(\mathbb{R}^{2d})$ contains the Hörmander class $S^0_{0,0}$ of infinitely smooth symbols. The Sjöstrand class $M^{\infty,1}_{0,2}(\mathbb{R}^{2d})$ is perhaps not as known as the Hörmander classes, but it has become a common and very natural class of symbols whenever operators are defined via phase-space shifts. Indeed, since every pseudodifferential operator σ^w can be formally represented as a superposition of phase-space shifts via

$$\sigma^w = \int_{\mathbb{R}^{2d}} \mathcal{U}\hat{\sigma}(z)\rho(z)\, dz,$$

where $\mathcal{U}F(x,\omega) = F(\omega, -x)$, the appearance of the Sjöstrand class is almost inevitable. The Sjöstrand class was studied by Sjöstrand in [31], the time-frequency analysis of $M^{\infty,1}_{s,t}(\mathbb{R}^{2d})$ has its origins in [25]. As it turns out, $M^{\infty,1}_{s,t}(\mathbb{R}^{2d})$ is part of a larger family of function spaces, the *modulation spaces* [19], which have become an indispensable tool in time-frequency analysis and the analysis of pseudodifferential operators. For a survey of this active field we refer, to the recent monographs [9, 13].

To match Bellissard's results for the case of pseudodifferential operators, we will also derive a variant concerning the Lipschitz continuity of spectral gaps. If g is a gap of the spectrum of A, that is, a connected component of the resolvent set $\mathbb{R} \setminus \sigma(A)$, we write $\sigma_+^g(A)$ and $\sigma_-^g(A)$ to denote the edges of g.

Theorem 1.2. *Under the assumptions of Theorem 1.1, let g be a gap of the spectrum of T_0 with length $L(g)$. Then there exist $\varepsilon = \varepsilon(g) > 0$ and functions $E_+^g, E_-^g : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that, for $|\delta| < \varepsilon$,*

(i) $E_+^g(0) = \sigma_+^g(T_0)$, *and* $E_-^g(0) = \sigma_-^g(T_0)$,

(ii) $E_+^g(\delta)$ *(resp.* $E_-^g(\delta)$*) is the right (resp. left) edge of a gap of* $\sigma(T_\delta)$*, and*

(iii) *the* Lipschitz continuity of spectral edges *holds:*

$$|E_\pm^g(\delta) - E_\pm^g(0)| \leq C_d \cdot |\delta| \cdot L(g)^{-1} \cdot \sup_{|t|<\delta_0} \big(\|G_t\|_{M^{\infty,1}} \|\partial_t G_t\|_{M^{\infty,1}} + \|G_t\|^2_{M^{\infty,1}_{0,2}}\big).$$

Note that the Lipschitz estimate in Theorem 1.2 holds only for $|\delta| < \varepsilon$. In fact, for larger δ the gap may disappear, as it occurs for example in certain graphene-like systems submitted to constant magnetic fields [15].[2]

For periodic Weyl symbols Bellissard [8] proved that

$$|E_\pm^g(\delta) - E_\pm^g(0)| \leq C_T \cdot |\delta| \cdot L(g)^{-5}, \quad |\delta| < \varepsilon,$$

and suggested that the correct dependence on the width $L(g)$ of the gap should be $L(g)^{-1}$. This conjecture was confirmed in [5, Theorem 4]; see also [5, Lemma 10]. Along the way, Beckus and Bellissard developed an abstract principle that helps derive such estimates [5], which we shall aptly invoke and combine with Theorem 1.1 to prove Theorem 1.2.

1.2. Gabor frames

Results in the style of Theorems 1.1 and 1.2 are typically investigated in mathematical physics where δ represents Planck's constant or the strength of a magnetic field. While we hope that our results may be useful in such questions, our main motivation comes from an open problem in the theory of Gabor frames. Let $\Lambda \subseteq \mathbb{R}^{2d}$ be a discrete set, not necessarily a lattice, and consider the set of phase-space shifts $\mathcal{G}(g, \Lambda) = \{\rho(\lambda)g\}_{\lambda\in\Lambda}$ for some $g \in L^2(\mathbb{R}^d)$. The main problem is to understand when $\mathcal{G}(g, \Lambda)$ is a frame, i.e., when the *frame inequalities*

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \rho(\lambda)g\rangle|^2 \leq B\|f\|_2^2, \quad \text{for all } f \in L^2(\mathbb{R}^d), \tag{1.4}$$

[2] We thank H. Cornean for pointing this out to us.

hold for some constants $A, B > 0$ independent of f. The optimal constants in (1.4) $B(\Lambda)$ and $A(\Lambda)$ are respectively the largest and the smallest spectral values of the frame operator $S_{g,\Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \rho(\lambda) g \rangle \rho(\lambda) g$. Theorem 1.1 then leads to the following statement about the frame bounds of a non-uniform Gabor frame (where "non-uniform" means that Λ need not be a lattice). Again, the most convenient conditions on g are in terms of a modulation space.

Theorem 1.3. *Let $0 < \alpha_0 < 1$, $\alpha_0 < \alpha < 1/\alpha_0$, and $g \in M^1_2(\mathbb{R}^d)$. Let also $\Lambda \subset \mathbb{R}^{2d}$ be relatively separated, i.e.,*

$$\operatorname{rel}(\Lambda) := \sup_{x \in \mathbb{R}^d} \#\{\lambda \in \Lambda \cap x + [0,1]^d\} < \infty.$$

Then

$$|\sigma_\pm(S_{g,\Lambda}) - \sigma_\pm(S_{g,\alpha\Lambda})| \le C_d \cdot \operatorname{rel}(\Lambda) \cdot \alpha_0^{-(4d+2)} \cdot \|g\|^2_{M^1_2} \cdot |1-\alpha|. \tag{1.5}$$

Theorem 1.3 has a rich history. If Λ is a lattice, it was first shown in [20] that the frame bounds depend in a *lower semi-continuous* fashion on α, which implies that the set of lattices that generate a Gabor frame is an open set. For general non-uniform sets Λ, the lower semi-continuity of the frame bounds was proven later in [1].

A particularly important consequence of Theorem 1.3 is the quantitative behavior of the frame bounds near the critical density. By the density theorem for Gabor frames, every frame $\mathcal{G}(g,\Lambda)$ must satisfy the necessary density condition $D^-(\Lambda) \ge 1$, where $D^-(\Lambda)$ is the lower Beurling density and counts the average number of points per unit volume. If $g \in M^1(\mathbb{R}^d)$, then even the strict inequality $D^-(\Lambda) > 1$ holds. In particular, if $g \in M^1(\mathbb{R}^d)$ and $D^-(\Lambda) = 1$, then the lower spectral bound of $S_{g,\Lambda}$ is $A(\Lambda) = 0$. See [27] for a survey of the density theorem and [1, 26] for the relevant result for non-uniform Gabor frames. Since the ratio $B(\Lambda)/A(\Lambda)$ serves as a condition number of the frame and thus measures the stability of various reconstruction procedures, it is important to understand how the lower frame bound $A(\Lambda)$ deteriorates to zero, as the density of Λ decreases to 1. Theorem 1.3 says that for an arbitrary set Λ of density $D^-(\Lambda) = 1$ the lower frame bound of $\mathcal{G}(g,\alpha\Lambda)$ behaves like

$$A(\alpha\Lambda) \lesssim 1 - \alpha,$$

for $\alpha \to 1$, $\alpha < 1$. This amounts to a blow-up of the order $(1-\alpha)^{-1}$ for the condition number $B(\alpha\Lambda)/A(\alpha\Lambda)$.

So far, this behavior of the lower frame bound near the critical density has been proved with special methods only for Gabor frames for square lattices $\alpha\mathbb{Z}^2$ based on the Gaussian $\varphi(t) = e^{-\pi t^2}$, see [10], and the exponential functions $e^{-t}\chi_{[0,\infty)}$ and $e^{-|t|}$, see [28]. In hindsight, these results can be deduced from Bellissard's result [8]. Theorem 1.3 fully settles the question: the asymptotic behavior $A(\alpha\Lambda) \lesssim 1 - \alpha$ near the critical density holds for all Gabor frames $\mathcal{G}(g, \alpha\Lambda)$ with a window in $M^1_2(\mathbb{R}^d)$ and an arbitrary discrete set Λ of density 1, be it a lattice or not, and in arbitrary dimensions. The condition on the window is only slightly more restrictive than the general condition $g \in M^1(\mathbb{R}^d)$, under which the structural results on Gabor frames hold.

Finally, we provide the following companion to Theorem 1.3.

Theorem 1.4. *Under the assumptions of Theorem 1.3, let* $\mathfrak{g}$ *be a gap of the spectrum of* $S_{g,\Lambda}$ *with length* $L(\mathfrak{g})$ *and edges* $\sigma^{\mathfrak{g}}_{\pm}(S_{g,\Lambda})$*. Then there exist* $\varepsilon = \varepsilon(\mathfrak{g}) > 0$ *and gaps of the spectrum of* $S_{g,\alpha\Lambda}$*,* $|1-\alpha| < \varepsilon$*, whose edges* $\sigma^{\mathfrak{g}}_{\pm}(S_{g,\alpha\Lambda})$ *satisfy*

$$|\sigma^{\mathfrak{g}}_{\pm}(S_{g,\Lambda}) - \sigma^{\mathfrak{g}}_{\pm}(S_{g,\alpha\Lambda})| \leq C_d \cdot |1-\alpha| \cdot \mathrm{rel}(\Lambda)^2 \cdot L(\mathfrak{g})^{-1} \cdot \|g\|^4_{M^1_2}.$$

While gaps in the spectrum of Gabor frame operators are comparatively less studied than their spectral edges, general gaps deserve attention. For example, the size of the smallest gap $(0, A)$ of a Gabor frame operator coincides with the so-called *lower Riesz bound* of the Gabor system. Note however that Theorem 1.4 does not describe the size of the smallest gap, as it is in principle possible that a spectral gap $(0, A)$ of $S_{g,\Lambda}$ may evolve into a gap of $S_{g,\alpha\Lambda}$ with $\sigma^{\mathfrak{g}}_{-}(S_{g,\alpha\Lambda}) > 0$ leaving room for a second gap $(0, A')$ in the spectrum of $S_{g,\alpha\Lambda}$ with $A' < \sigma^{\mathfrak{g}}_{-}(S_{g,\alpha\Lambda})$.

1.3. Methods

As our predecessors, the overall structure of our proof of Theorem 1.1 follows Bellissard [8] and consists of three steps: a truncation argument followed by tensorization, and a reverse heat flow estimate.

A key insight of Bellissard was that the C^*-algebra $\mathcal{A}_{1+\delta}$ that is generated by $\{\rho(\sqrt{1+\delta}k)\}_{k\in\mathbb{Z}^2}$ acting on $L^2(\mathbb{R}^d)$ (a non-commutative torus) is isomorphic to a subalgebra of $\mathcal{A}_1 \otimes \mathcal{A}_\delta$. For non-periodic symbols we use a similar tensorization argument. Although we can no longer rely on C^*-algebra techniques, this difficulty is circumvented with the help of the metaplectic representation. See Theorem 3.3 and its comments.

While the overall structure of the proof of Theorem 1.1 is due to Bellissard, the proof techniques are rather different in the case of non-periodic symbols and draw from time-frequency analysis and the theory of modulation spaces. In fact, our main technical contribution is the systematic use of the machinery of time-frequency analysis.

Theorem 1.2 is proved by combining Theorem 1.1 with an abstract principle due to Beckus and Bellissard [5], while Theorems 1.3 and 1.4 follow as a further application of the main results.

After understanding the fundamental principles underlying the continuous dependence of spectra, Beckus and Bellissard with coauthors have axiomatized and greatly expanded the range of their methods [6, 7]. In [6] generalized Schrödinger operators are studied in the context of groupoids and dynamical systems based on C^*-algebraic methods. The final result is a far-reaching general theorem about the continuity of the spectral map that includes, for instance, Schrödinger operators for solids with respect to general point distributions in magnetic fields. The corresponding magnetic translations obey commutation relations similar to those of the time-frequency shifts studied in our work. In [7] the authors study the Lipschitz continuity of spectra of operator families for which the mapping from parameter to operator $\alpha \to T_\alpha$ is driven by a dynamical system. This generalization is motivated by and includes the almost-Mathieu operator, which in our context corresponds to a specific finite sum of time-frequency shifts. Our results are clearly related and fit into this general context, but to the best of our knowledge there is no overlap.

The paper is organized as follows. Section 2 summarizes the tools from time-frequency analysis required for the formulation and for the proof of the main theorem. Section 3 is devoted to the proof of the main theorems. Section 4 discusses Gabor systems and their frame operators.

2. Background and tools

2.1. Notation

Euclidean balls are denoted $B_r(x)$. The dilation operator acts on a function $f:\mathbb{R}^d \to \mathbb{C}$ by $D_a f(x) = f(ax)$, $a > 0$, and for $F:\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, $\mathcal{U}$ denotes the change of variables $\mathcal{U}F(x,\omega) = F(\omega, -x)$. The tensor product of two functions is defined as $f \otimes g(s,t) = f(s)g(t)$. The symbol $\lesssim$ in $f \lesssim g$ means that $f(x) \leq Cg(x)$ for all x with a constant C independent of x.

2.2. Norms and spectral extrema

An important insight in [5, 8] is that the smoothness of *the norms* of (the polynomial calculi of) a family of operators determines the smoothness of their spectra in the Hausdorff metric. We start with the following basic estimate, which we prove for completeness; these and other closely related statements are implicit in the proofs of [4, Theorem 2.3.17 and Theorem 2.8.10].

Lemma 2.1. *Let $\mathcal{H}$ be a Hilbert space, and $A, A_1, A_2 \in B(\mathcal{H})$ be self-adjoint operators. For $\lambda > \|A\|_{B(\mathcal{H})}$, we have*

$$\sigma_+(A) = \|A + \lambda I\|_{B(\mathcal{H})} - \lambda \quad \text{and} \quad \sigma_-(A) = \lambda - \|A - \lambda I\|_{B(\mathcal{H})}.$$

Moreover,

$$|\sigma_\pm(A_1) - \sigma_\pm(A_2)| \le \|A_1 - A_2\|_{B(\mathcal{H})}, \tag{2.1}$$

and

$$\big|\|A_1\|_{B(\mathcal{H})} - \|A_2\|_{B(\mathcal{H})}\big| \le 2 \cdot \max_\pm |\sigma_\pm(A_1) - \sigma_\pm(A_2)|. \tag{2.2}$$

Proof. For a self-adjoint operator A, it holds that $r(A) = \|A\|_{B(\mathcal{H})}$, where $r(A)$ denotes the spectral radius of A. For $\lambda > \|A\|_{B(\mathcal{H})}$ one thus has

$$\sigma_+(A) + \lambda = r(A + \lambda I) = \|A + \lambda I\|_{B(\mathcal{H})}.$$

For σ_- we note that $-(\sigma_-(A) - \lambda) = r(A - \lambda I) = \|A - \lambda I\|_{B(\mathcal{H})}$.

Set $\lambda = 2\max\{\|A_1\|_{B(\mathcal{H})}, \|A_2\|_{B(\mathcal{H})}\}$. By the first part of this lemma and the triangle inequality it follows

$$\begin{aligned} |\sigma_\pm(A_1) - \sigma_\pm(A_2)| &= \big|\|A_1 \pm \lambda I\|_{B(\mathcal{H})} - \|A_2 \pm \lambda I\|_{B(\mathcal{H})}\big| \\ &\le \|A_1 \pm \lambda I - (A_2 \pm \lambda I)\|_{B(\mathcal{H})} = \|A_1 - A_2\|_{B(\mathcal{H})}, \end{aligned}$$

which proves (2.1). To prove (2.2) we first note that

$$\begin{aligned} &\big|\|A_1\|_{B(\mathcal{H})} - \|A_2\|_{B(\mathcal{H})}\big| \\ &\quad = |\max\{|\sigma_+(A_1)|, |\sigma_-(A_1)|\} - \max\{|\sigma_+(A_2)|, |\sigma_-(A_2)|\}|. \end{aligned}$$

For $a, b, c, d \ge 0$, the elementary estimate

$$|\max\{a, b\} - \max\{c, d\}| \le 2\max\{|a - c|, |b - d|\}$$

then yields that

$$\begin{aligned} \big|\|A_1\|_{B(\mathcal{H})} - \|A_2\|_{B(\mathcal{H})}\big| &\le 2\max_\pm \big||\sigma_\pm(A_1)| - |\sigma_\pm(A_2)|\big| \\ &\le 2\max_\pm |\sigma_\pm(A_1) - \sigma_\pm(A_2)|. \end{aligned}$$

■

2.3. The Beckus–Bellissard lemma

Consider a family $\{A_\delta\}_{|\delta|<\delta_0}$ of bounded self-adjoint operators $A_\delta \colon \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$. For a set of polynomials $\mathcal{Q} \subset \mathbb{C}[x]$ let

$$C_{\mathcal{Q}} := \sup_{p \in \mathcal{Q}} \sup_{\delta_1 \neq \delta_2} \frac{\big|\|p(A_{\delta_1})\|_{B(\mathcal{H})} - \|p(A_{\delta_2})\|_{B(\mathcal{H})}\big|}{|\delta_1 - \delta_2|}.$$

We say that $\{A_\delta\}_{\delta \in (-\delta_0, \delta_0)}$ is $(p2)$-*Lipschitz continuous* if for every $M > 0$, we have $C_{\mathcal{P}_M} < \infty$, where $\mathcal{P}_M$ denotes the set of all polynomials of the form $p(x) = \alpha x^2 + \beta x + \gamma$ with $\alpha, \beta, \gamma \in \mathbb{R}$ and $|\alpha| + |\beta| + |\gamma| \le M$.

The following lemma is a minor modification of [5, Lemma 10] (see also [4, Theorem 2.8.10]).

Lemma 2.2. *Let $\{A_\delta\}_{|\delta|<\delta_0}$ be a $(p2)$-Lipschitz continuous family of bounded self-adjoint operators and set*

$$\mathcal{P}(A_0) = \{p(x) = x^2 + \beta x + \gamma : \beta, \gamma \in \mathbb{R}, \\ |\beta| \le 2\|A_0\|_{B(\mathcal{H})}, |\gamma| \le 5\|A_0\|^2_{B(\mathcal{H})}\}. \tag{2.3}$$

Let g be a gap of the spectrum of A_0 with edges $\sigma^g_\pm(A_0)$ and length $L(g)$. Then there exist $\varepsilon = \varepsilon(g) > 0$ and gaps of the spectrum of A_δ, $|\delta| < \varepsilon$, whose edges $\sigma^g_\pm(A_\delta)$ satisfy

$$|\sigma^g_\pm(A_\delta) - \sigma^g_\pm(A_0)| \le 3|\delta| \cdot \frac{C_{\mathcal{P}(A_0)}}{L(g)}, \qquad |\delta| < \varepsilon. \tag{2.4}$$

The statement of [5, Lemma 10] involves a larger class than $\mathcal{P}(A_0)$ defined by imposing the same bound on all polynomial coefficients. The proof in [5], however, readily gives the stronger statement, and will not be repeated.

2.4. Time-frequency representations

We now offer a minimalist account of time-frequency analysis, modulation spaces, and the associated results for pseudodifferential operators. Detailed expositions can be found in the textbook [23] and the two recent monographs [9, 13].

For a point $z = (x, \omega) \in \mathbb{R}^{2d}$, the *phase-space shift* (time-frequency shift) of f is defined as

$$\rho(z) f(t) = e^{-i\pi x\cdot\omega} M_\omega T_x f(t) = e^{-i\pi x\cdot\omega} e^{2\pi i\omega\cdot t} f(t-x),$$

where $T_x f(t) = f(t-x)$ and $M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t)$. In terms of the *symplectic form*

$$[z, z'] = x'\cdot\omega - x\cdot\omega', \quad z = (x,\omega), z' = (x',\omega') \in \mathbb{R}^d \times \mathbb{R}^d, \tag{2.5}$$

the composition of two phase-space shifts gives

$$\rho(z)\rho(z') = e^{i\pi[z,z']}\rho(z+z'), \quad z, z' \in \mathbb{R}^{2d}. \tag{2.6}$$

In particular, $\rho(z)^* = \rho(-z)$ and

$$\rho(z)^*\rho(z')\rho(z) = e^{2\pi i[z',z]}\rho(z'), \quad z, z' \in \mathbb{R}^{2d}. \tag{2.7}$$

The *short-time Fourier transform* of a function or distribution f on $\mathbb{R}^d$ with respect to a window function g is given by

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i\omega\cdot t}\, dt \\ = \langle f, M_\omega T_x g\rangle = e^{-i\pi x\cdot\omega}\langle f, \rho(z) g\rangle.$$

When g is normalized by $\|g\|_2 = 1$, $V_g \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is an isometry [21,23]:

$$\int_{\mathbb{R}^{2d}} |\langle f, \rho(z)g\rangle|^2 = \int_{\mathbb{R}^{2d}} |V_g f(z)|^2 \, dz = \|f\|_2^2, \quad f \in L^2(\mathbb{R}^d). \tag{2.8}$$

In terms of the rank-one projections $q(z) \in B(L^2(\mathbb{R}^d))$,

$$q(z) = \langle \cdot\,, \rho(z)g\rangle \rho(z)g, \tag{2.9}$$

the isometry property of the short-time Fourier transform yields the following continuous resolution of the identity:

$$\int_{\mathbb{R}^{2d}} q(z) \, dz = I, \tag{2.10}$$

where integrals are to be interpreted in the weak sense (2.8).

Identities for the short-time Fourier transform. We will need the following identity for the short-time Fourier transform of a pointwise product of functions:

$$V_g(f \cdot h)(x, \omega) = (\hat{h} *_2 V_g f)(x, \omega) = \int_{\mathbb{R}^d} \hat{h}(\xi) V_g f(x, \omega - \xi) d\xi. \tag{2.11}$$

The *(cross-)Wigner distribution* of $f, g \in L^2(\mathbb{R}^d)$ is defined to be

$$\mathcal{W}(f, g)(x, \omega) = \int_{\mathbb{R}^d} f\Big(x + \frac{t}{2}\Big) \overline{g\Big(x - \frac{t}{2}\Big)} e^{-2\pi i \omega \cdot t} dt. \tag{2.12}$$

If $f = g$, we write $\mathcal{W}(f)$.

Finally, we quote some useful facts about the short-time Fourier transform of a short-time Fourier transform. It is shown in [11, Lemmas 2.1 and 2.2, Proposition 2.5] that[3]

$$\mathcal{F}(V_\varphi f \cdot \overline{V_\varphi g})(z) = \mathcal{U}^{-1}(V_g f \cdot \overline{V_\varphi \varphi})(z), \quad z \in \mathbb{R}^{2d}, \tag{2.13}$$

while with $x, \omega \in \mathbb{R}^{2d}$, $(\tilde{\omega}_1, \tilde{\omega}_2) = (\omega_2, -\omega_1)$, and the window $\Phi = W(\varphi, \varphi)$,

$$|V_\Phi(\mathcal{W}(f, g))(x, \omega)| = |V_\varphi f(x - \tilde{\omega}/2) V_\varphi g(x + \tilde{\omega}/2)|. \tag{2.14}$$

[3] The combination $\mathcal{U}^{-1}\mathcal{F}$ is often called the symplectic Fourier transform and used in addition to $\mathcal{F}$.

2.5. Modulation spaces

The family of modulation spaces $M^{p,q}_{s,t}(\mathbb{R}^d)$ was already defined in (1.2). By changing the order of integration, we obtain the family of *Wiener amalgam spaces*. Let $\varphi(t) = 2^{d/4}e^{-\pi|t|^2}$ denote the standard Gaussian in $\mathbb{R}^d$ and $1 \le p,q \le \infty$, $s,t \ge 0$. Then $W^{p,q}_{s,t}(\mathbb{R}^d)$ consists of all distributions in $\mathcal{S}'(\mathbb{R}^d)$ for which the following norm is finite:

$$\|f\|_{W^{p,q}_{s,t}} = \Big(\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |V_\varphi f(x,\omega)|^p (1+|\omega|)^{sp} \, d\omega \Big)^{q/p} (1+|x|)^{tq} \, dx \Big)^{1/q}, \tag{2.15}$$

with the usual modification when $p = \infty$, or $q = \infty$. If $p = q$, we write $W^p_{s,t}(\mathbb{R}^d)$, if $s = t$, we write $W^{p,q}_s(\mathbb{R}^d)$, and if $s = t = 0$, we write $W^{p,q}(\mathbb{R}^d)$. Any nonzero function $g \in \mathcal{S}(\mathbb{R}^d)$ instead of the Gaussian in (2.15) gives an equivalent norm, i.e., $\|f\|_{W^{p,q}_{s,t}} \asymp \|\mathcal{U} V_g f\|_{L^{p,q}_{s,t}}$; these spaces are often denoted by $W(\mathcal{F}L^p_s, L^q_t)$.

Since $V_g f(x,\omega) = e^{-2\pi i x\cdot\omega} V_{\hat{g}} \hat{f}(\omega,-x)$, a comparison of (1.2) and (2.15) shows that $W^{p,q}_{s,t}(\mathbb{R}^d)$ is the image of the modulation space $M^{p,q}_{s,t}(\mathbb{R}^d)$ under the Fourier transform, in particular

$$\|f\|_{M^{p,q}_{s,t}} \asymp \|\hat{f}\|_{W^{p,q}_{s,t}}. \tag{2.16}$$

Convolution and multiplication in modulation spaces. The space $M^{\infty,1}$ is isometrically translation invariant. As a consequence, it satisfies $L^1 * M^{\infty,1} \to M^{\infty,1}$ together with the estimate

$$\|f * g\|_{M^{\infty,1}} \lesssim \|f\|_{L^1} \|g\|_{M^{\infty,1}}. \tag{2.17}$$

Equivalently, in terms of Wiener amalgam norms,

$$\|f \cdot g\|_{W^{\infty,1}} \lesssim \|f\|_{\mathcal{F}L^1} \|g\|_{W^{\infty,1}}, \tag{2.18}$$

where

$$\|f\|_{\mathcal{F}L^1} := \|\hat{f}\|_{L^1}.$$

We will also use the following estimates for convolution of functions and distributions in modulation spaces, taken from [11, Proposition 2.4]. If $f \in M^\infty(\mathbb{R}^d)$ and $g \in M^1_{0,s}(\mathbb{R}^d)$ with $s \ge 0$, then $f * g \in M^{\infty,1}_{0,s}(\mathbb{R}^d)$ and

$$\|f * g\|_{M^{\infty,1}_{0,s}} \lesssim \|f\|_{M^\infty} \|g\|_{M^1_{0,s}}. \tag{2.19}$$

Let us write X_i to denote the multiplication operator $X_i f(t) = t_i f(t)$, $1 \le i \le d$. The observation

$$X_i(M_\omega T_x f) = M_\omega T_x (X_i f) + x_i M_\omega T_x f,$$

and the fact that different windows generate equivalent norms for the spaces $M^{p,q}_{s,t}(\mathbb{R}^d)$ and $W^{p,q}_{s,t}(\mathbb{R}^d)$ lead to the estimates

$$\|X_i f\|_{M^{p,q}_{s,t}} \lesssim \|f\|_{M^{p,q}_{s+1,t}} \quad \text{and} \quad \|X_i f\|_{W^{p,q}_{s,t}} \lesssim \|f\|_{W^{p,q}_{s,t+1}}. \tag{2.20}$$

Similarly, one sees that

$$\|\partial_i f\|_{M^{p,q}_{s,t}} \lesssim \|f\|_{M^{p,q}_{s,t+1}}.$$

Consequently,

$$\|X_i \partial_i f\|_{M^{p,q}_{s,t}} \lesssim \|f\|_{M^{p,q}_{s+1,t+1}}. \tag{2.21}$$

Next, we show the following variant of [11, Proposition 2.5].

Lemma 2.3. *For $f, g \in M^1_{s+t}(\mathbb{R}^d)$, $s, t \geq 0$, we have $\mathcal{W}(f, g) \in M^1_{s,t}(\mathbb{R}^{2d})$ with the norm estimate*

$$\|\mathcal{W}(f, g)\|_{M^1_{s,t}} \lesssim \|f\|_{M^1_{s+t}} \|g\|_{M^1_{s+t}}.$$

Proof. Using (2.14) with $\Phi = \mathcal{W}(\varphi, \varphi)$ and the substitution $(\tilde{\omega}_1, \tilde{\omega}_2) = (\omega_2, -\omega_1)$,

$$\begin{aligned}
\|\mathcal{W}(f, g)\|_{M^1_{s,t}} &\asymp \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_\Phi(\mathcal{W}(f, g))(x, \omega)|(1 + |x|)^s (1 + |\omega|)^t \, dx d\omega \\
&= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_\varphi f(x - \omega/2) V_\varphi g(x + \omega/2)|(1 + |x|)^s (1 + |\omega|)^t \, dx d\omega \\
&= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_\varphi f(x) V_\varphi g(x + \omega)|(1 + |x + \omega/2|)^s (1 + |\omega|)^t \, dx d\omega \\
&= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_\varphi f(x) V_\varphi g(\omega)|(1 + |(x + \omega)/2|)^s (1 + |\omega - x|)^t \, d\omega dx \\
&\lesssim \|f\|_{M^1_{s+t}} \|g\|_{M^1_{s+t}},
\end{aligned}$$

where we used the submultiplicativity of the polynomial weights. ■

As we note below, the space $M^\infty(\mathbb{R}^d)$ contains atomic measures supported on relatively separated sets.

Lemma 2.4. *Let $\Lambda \subset \mathbb{R}^d$ be relatively separated. Then $\mu := \sum_{\lambda \in \Lambda} \delta_\lambda \in M^\infty(\mathbb{R}^d)$. Moreover, if* $\operatorname{rel}(\Lambda) := \sup_{x \in \mathbb{R}^d} \#\{\lambda \in \Lambda \cap x + [0, 1]^d\}$ *then*

$$\|\mu\|_{M^\infty} \lesssim \operatorname{rel}(\Lambda).$$

The proof of Lemma 2.4 follows from a direct calculation (which is easily carried out by taking a window function $g \in M^1(\mathbb{R}^d)$ supported on $[0, 1]^d$) and is therefore omitted.

Finally, we will need the embeddings

$$\|f\|_{M^\infty} \lesssim \|f\|_{L^\infty} \lesssim \|f\|_{M^{\infty,1}}, \tag{2.22}$$

and the following norm estimates for dilations on modulation spaces:

$$\|D_a f\|_{M^{\infty,1}_{0,s}} \le C_{d,s} \max\{1, a^{d+s}\} \|f\|_{M^{\infty,1}_{0,s}}, \quad a > 0,\ s \ge 0, \tag{2.23}$$

$$\|D_a f\|_{M^{1}_{0,s}} \le C_{d,s} \max\{a^{-d}, a^{s}\} \|f\|_{M^{1}_{0,s}}, \quad a > 0,\ s \ge 0. \tag{2.24}$$

See [32, Theorem 1.1] and [12, Theorem 3.2] for the weighted versions.

2.6. Pseudodifferential operators

The modulation spaces $M^{\infty,1}_{0,s}(\mathbb{R}^{2d})$, $s \ge 0$, are important symbol classes in the theory of pseudodifferential operators. In particular, the space $M^{\infty,1}(\mathbb{R}^{2d})$ was first used by Sjöstrand [30] as a class of non-smooth ("rough") symbols that contains the Hörmander class $S^0_{0,0}$. See also the early papers [24,25] for a detailed time-frequency analysis of this symbol class.

The standard definition of the Weyl calculus (1.1) does not reveal how modulation spaces and phase-space methods enter the analysis. This becomes more plausible when we write a pseudodifferential operator as

$$\langle \sigma^w f, g\rangle = \langle \sigma, \mathcal{W}(g, f)\rangle$$

or as a superposition of phase-space shifts

$$\sigma^w = \int_{\mathbb{R}^{2d}} \mathcal{U}\hat{\sigma}(z)\rho(z)\, dz$$

with $\mathcal{U}F(x,\omega) = F(\omega, -x)$. Taking these formulas for the Weyl calculus as the starting point, the appearance of modulation spaces is natural and ultimately led to the following results, which we will use in an essential way.

The composition of Weyl transforms defines a bilinear form on the space of symbols (*twisted product*)

$$\sigma^w \tau^w = (\sigma \sharp \tau)^w.$$

Using the symplectic form (2.5), the twisted product of Schwartz class symbols can be written explicitly as

$$\sigma \sharp \tau(z) = 4^d \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \sigma(z')\tau(z'') e^{4\pi i [z-z', z-z'']}\, dz'\, dz'', \tag{2.25}$$

while for general σ and τ this formula holds in the distributional sense. The *twisted convolution*

$$\sigma \natural \tau(z) = \int_{\mathbb{R}^{2d}} \sigma(z')\tau(z-z')e^{-\pi i[z-z',z']}\, dz'$$

is related to the twisted product by

$$\mathcal{F}(\sigma \sharp \tau) = (\mathcal{F}\sigma)\natural(\mathcal{F}\tau). \tag{2.26}$$

We now quote some basic properties of weighted Sjöstrand classes.

Theorem 2.5. *The following facts hold.*

(i) *If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, then σ is a bounded operator on $L^2(\mathbb{R}^d)$ and*

$$\|\sigma^w\|_{B(L^2(\mathbb{R}^d))} \lesssim \|\sigma\|_{M^{\infty,1}}.$$

(ii) *If $F \in W^{\infty,1}(\mathbb{R}^{2d})$ and $\rho(F) = \int_{\mathbb{R}^{2d}} F(z)\rho(z)\, dz$, then $\rho(F)$ is bounded on $L^2(\mathbb{R}^d)$ with operator norm*

$$\|\rho(F)\|_{B(L^2(\mathbb{R}^d))} \lesssim \|F\|_{W^{\infty,1}}.$$

(iii) *For $s \geq 0$, $M^{\infty,1}_{0,s}(\mathbb{R}^{2d})$ is a Banach $*$-algebra with respect to the twisted product $\sharp$ and the involution $\sigma \mapsto \overline{\sigma}$. In particular,*

$$\|\sigma \sharp \tau\|_{M^{\infty,1}_{0,s}} \lesssim \|\sigma\|_{M^{\infty,1}_{0,s}} \|\tau\|_{M^{\infty,1}_{0,s}}.$$

(iv) *Let $|\delta| < \delta_0 < 1$, $G_\delta \in M^{\infty,1}(\mathbb{R}^{2d})$ be real-valued, and set*

$$T_\delta = (D_{\sqrt{1+\delta}} G_\delta)^w.$$

Then

$$\begin{aligned} \|T_\delta\|_{B(L^2(\mathbb{R}^d))} &\lesssim \max\{1, (1+\delta)^d\} \|G_\delta\|_{M^{\infty,1}} \\ &\leq (1+\delta_0)^d \|G_\delta\|_{M^{\infty,1}}. \end{aligned} \tag{2.27}$$

Proof. For (i) and (iii), see [24,25,30]. Item (ii) is just a reformulation when the operator is written as a superposition of phase-space shifts. (iv) follows from the invariance of $M^{\infty,1}(\mathbb{R}^{2d})$ under dilations expressed by (2.23). Note that symbols are functions on $\mathbb{R}^{2d}$ and the correct norm of the dilation is therefore $\sqrt{1+\delta}^{2d}$. ∎

3. Proof of the main results

In our proof of Theorems 1.1 we follow Bellissard's strategy [8] consisting of three main ingredients: (i) truncate the symbol of T_δ to define an operator $\mathcal{T}_R(T_\delta)$ and compare its spectral extreme values to the ones of T_δ (Lemma 3.1); (ii) introduce a tensorization $\mathcal{T}_R(T_\delta)^\otimes$ acting on $L^2(\mathbb{R}^{2d})$ which preserves the spectrum of $\mathcal{T}_R(T_\delta)$ (Theorem 3.3); and (iii) rewrite $\mathcal{T}_R(T_\delta)^\otimes$ adequately (Lemma 3.5) so that reverse heat-flow estimates can be used to compare the spectral extreme values of $\mathcal{T}_R(T_\delta)^\otimes$ and $\mathcal{T}_R(T_0)$ (Lemmas 3.7 and 3.8).

The challenge in our case is to find suitable alternative arguments to treat non-periodic symbols. As a first step, we write the pseudodifferential operator T_δ with symbol $\sigma_\delta = D_{\sqrt{1+\delta}} G_\delta$ as a superposition of phase-space shifts (spreading representation in engineering language)

$$T_\delta = \int_{\mathbb{R}^{2d}} \mathcal{U}\widehat{\sigma_\delta}(z)\rho(z)\,dz = \int_{\mathbb{R}^{2d}} \mathcal{U}\widehat{G_\delta}(z)\rho(\sqrt{1+\delta}z)\,dz.$$

Although in general $\widehat{G_\delta}$ is a distribution, the analysis of T_δ becomes feasible in this representation. A main tool is the boundedness estimate from Theorem 2.5, which we will use several times. We will at first assume that δ is positive and use reflection arguments to cover negative values.

3.1. Truncation error

Fix a real-valued, even, radial function $\theta \in C^\infty(\mathbb{R}^{2d})$ such that $\theta(z) = 1$, for $|z| \le 1$, $\theta(z) = 0$, for $|z| \ge 2$, and $0 \le \theta(z) \le 1$ else, and set $\theta_R(z) = \theta(z/R)$, with $R > 0$.

We define the truncation of T_δ by

$$\mathcal{T}_R(T_\delta) = \int_{\mathbb{R}^{2d}} \mathcal{U}(\theta_R\widehat{G_\delta})(z)\rho(\sqrt{1+\delta}z)\,dz. \tag{3.1}$$

Since θ is real-valued and even, it follows that $\mathcal{T}_R(T_\delta)$ is self-adjoint. By Theorem 2.5, it is enough to bound $\|G_\delta - \widehat{\theta_R} * G_\delta\|_{M^{\infty,1}}$ in order to derive an estimate of the norm of $T_\delta - \mathcal{T}_R(T_\delta)$.

Lemma 3.1. *Let $0 \le \delta < \delta_0$ and $G_\delta \in M^{\infty,1}_{0,2}(\mathbb{R}^{2d})$ be real-valued. If we assume that $\sup_{|t|<\delta_0} \|G_t\|_{M^{\infty,1}_{0,2}} < \infty$, then*

$$\|G_\delta - \widehat{\theta_R} * G_\delta\|_{M^{\infty,1}} \lesssim R^{-2} \cdot \sup_{|t|<\delta_0} \|G_t\|_{M^{\infty,1}_{0,2}}.$$

In particular, by (2.27)*,*

$$\|T_\delta - \mathcal{T}_R(T_\delta)\|_{B(L^2(\mathbb{R}^d))} \lesssim R^{-2} \cdot (1+\delta_0)^d \cdot \sup_{|t|<\delta_0} \|G_t\|_{M^{\infty,1}_{0,2}}. \tag{3.2}$$

Proof. Fix a constant $K > 0$ and choose $\Phi \in M^1(\mathbb{R}^{2d})$ to be compactly supported in $B_K(0)$. Let $R \geq 2K$. If $|x| \leq R-K$, then $\operatorname{supp} T_x\Phi \subseteq B_R(0)$ and consequently $V_\Phi \widehat{G_\delta}(x,\omega) - V_\Phi(\theta_R \widehat{G_\delta})(x,\omega) = 0$ for $|x| \leq R-K$ and all $\omega \in \mathbb{R}^{2d}$. Therefore,

$$\begin{aligned}
&\|G_\delta - \widehat{\theta_R} * G_\delta\|_{M^{\infty,1}} \\
&\quad = \|\widehat{G_\delta} - \theta_R \widehat{G_\delta}\|_{W^{\infty,1}} \\
&\quad \asymp \int_{\mathbb{R}^{2d}} \sup_{\omega \in \mathbb{R}^{2d}} |V_\Phi \widehat{G_\delta}(x,\omega) - V_\Phi(\theta_R \widehat{G_\delta})(x,\omega)| \, dx \\
&\quad = \int_{\mathbb{R}^{2d} \setminus B_{R-K}(0)} \sup_{\omega \in \mathbb{R}^{2d}} |V_\Phi \widehat{G_\delta}(x,\omega) - V_\Phi(\theta_R \widehat{G_\delta})(x,\omega)| \, dx \\
&\quad \lesssim R^{-2} \int_{\mathbb{R}^{2d}} \sup_{\omega \in \mathbb{R}^{2d}} (|V_\Phi \widehat{G_\delta}(x,\omega)| + |\widehat{\theta_R} *_2 V_\Phi \widehat{G_\delta}(x,\omega)|)(1+|x|)^2 dx \\
&\quad \leq R^{-2}(1 + \|\widehat{\theta_R}\|_1) \int_{\mathbb{R}^{2d}} \sup_{\omega \in \mathbb{R}^{2d}} |V_\Phi \widehat{G_\delta}(x,\omega)|(1+|x|)^2 dx \\
&\quad \asymp R^{-2}(1 + \|\widehat{\theta_R}\|_1)\|\widehat{G_\delta}\|_{W^{\infty,1}_{0,2}} \\
&\quad \asymp R^{-2}(1 + \|\widehat{\theta}\|_1)\|G_\delta\|_{M^{\infty,1}_{0,2}}.
\end{aligned}$$

Here, we have used that $R \leq 2(R-K)$, (2.11), and Young's inequality to show that $\sup_\omega |\widehat{\theta_R} *_2 V_\Phi \widehat{G_\delta}(x,\omega)| \leq \|\theta_R\|_1 \sup_\omega |V_\Phi \widehat{G_\delta}(x,\omega)|$, and

$$\|\widehat{\theta_R}\|_1 = R^{2d} \int_{\mathbb{R}^{2d}} |\widehat{\theta}(R\omega)| \, d\omega = \|\widehat{\theta}\|_1.$$

Finally, for $0 \leq R \leq 2K$,

$$\begin{aligned}
\|G_\delta - \widehat{\theta_R} * G_\delta\|_{M^{\infty,1}} &= \|\widehat{G_\delta} - \theta_R \widehat{G_\delta}\|_{W^{\infty,1}} \\
&\asymp \int_{\mathbb{R}^{2d}} \sup_{\omega \in \mathbb{R}^{2d}} |V_\Phi \widehat{G_\delta}(x,\omega) - V_\Phi(\theta_R \widehat{G_\delta})(x,\omega)| \, dx \\
&\lesssim \int_{\mathbb{R}^{2d}} \sup_{\omega \in \mathbb{R}^{2d}} (|V_\Phi \widehat{G_\delta}(x,\omega)| + |\widehat{\theta_R} *_2 V_\Phi \widehat{G_\delta}(x,\omega)|) dx \\
&\leq (1 + \|\widehat{\theta_R}\|_1) \int_{\mathbb{R}^{2d}} \sup_{\omega \in \mathbb{R}^{2d}} |V_\Phi \widehat{G_\delta}(x,\omega)| dx
\end{aligned}$$

$$\asymp (1 + \|\hat{\theta}\|_1)\|G_\delta\|_{M^{\infty,1}}$$
$$\leq \frac{4K^2}{R^2}(1 + \|\hat{\theta}\|_1)\|G_\delta\|_{M^{\infty,1}_{0,2}},$$

which concludes the proof after adjusting the implied constants. ∎

The following estimate will be helpful to analyze truncations.

Lemma 3.2. *For* $0 < R \leq \delta^{-1/2}$, *we have* $\|e^{\frac{\pi\delta}{2}|\cdot|^2}\theta_R\|_{\mathcal{F}L^1} \lesssim 1$.

Proof. To prove the estimate we apply the dilation invariance of $\mathcal{F}L^1$ and the Sobolev-type embedding $M^2_{0,d+1}(\mathbb{R}^{2d}) \hookrightarrow \mathcal{F}L^1(\mathbb{R}^{2d})$. Using multi index notation for derivatives, we obtain

$$\begin{aligned}
\|e^{\frac{\pi\delta}{2}|\cdot|^2}\theta(\cdot/R)\|_{\mathcal{F}L^1} &= \|e^{\frac{\pi\delta R^2}{2}|\cdot|^2}\theta\|_{\mathcal{F}L^1} \\
&\lesssim \sum_{|\alpha|,|\beta|\leq d+1} \|\partial^\alpha[e^{\frac{\pi\delta R^2}{2}|\cdot|^2}]\cdot\partial^\beta\theta\|_{L^2} \\
&\lesssim \sum_{|\alpha|\leq d+1} \|\partial^\alpha[e^{\frac{\pi\delta R^2}{2}|\cdot|^2}]\|_{L^\infty(B_2(0))} \lesssim 1,
\end{aligned}$$

because $\delta R^2 \leq 1$. ∎

3.2. Tensorization

Let $\delta > 0$ and define $T_\delta^{\otimes}\colon L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ as follows:

$$\begin{aligned}
T_\delta^{\otimes} &= \int_{\mathbb{R}^{2d}} \mathcal{U}\widehat{G_\delta}(z)\rho(z)\otimes\rho(\sqrt{\delta}z)\,dz \\
&= \int_{\mathbb{R}^{2d}} \mathcal{U}\widehat{G_\delta}(x,\omega)\rho(x,\sqrt{\delta}x,\omega,\sqrt{\delta}\omega)\,dx\,d\omega.
\end{aligned}$$

Note that if G_δ is real-valued, then both T_δ and $T_\delta^{\otimes}$ are self-adjoint. We emphasize that the tensorized operator $T_\delta^{\otimes}$ acts on $L^2(\mathbb{R}^{2d})$, whereas the original operator T_δ acts on $L^2(\mathbb{R}^d)$. We similarly define a tensorized operator associated with the truncation (3.1):

$$\mathcal{T}_R(T_\delta)^{\otimes} = \int_{\mathbb{R}^{2d}} \mathcal{U}(\theta_R\widehat{G_\delta})(z)\rho(z)\otimes\rho(\sqrt{\delta}z)\,dz.$$

Let $\mathrm{Sp}(d)$ denote the *symplectic group* of all matrices $\mathcal{R} \in \mathrm{GL}(2d,\mathbb{R})$ that satisfy $\mathcal{R}^* J\mathcal{R} = J$, with $J = \left(\begin{smallmatrix} 0 & I_d \\ -I_d & 0\end{smallmatrix}\right)$. For each symplectic matrix, there exists a unitary operator $\mu(\mathcal{R})$, called *metaplectic operator*, such that

$$\rho(\mathcal{R}(x,\omega)) = \mu(\mathcal{R})\rho(x,\omega)\mu(\mathcal{R})^{-1}, \tag{3.3}$$

see for example [23, Lemma 9.4.3]. We are now ready to prove that T_δ and $T_\delta^{\otimes}$ are isospectral.

Theorem 3.3. *Let* $0 \le \delta < \delta_0$*, and* $G_\delta \in M^{\infty,1}(\mathbb{R}^{2d})$ *be real-valued. Then* $\sigma(T_\delta) = \sigma(T_\delta^{\otimes})$*, and* $\sigma(\mathcal{T}_R(T_\delta)) = \sigma(\mathcal{T}_R(T_\delta)^{\otimes})$*, for all* $R > 0$.

Proof. Let $R_\delta \in \mathcal{O}(2d)$ be an orthogonal transformation that satisfies

$$R_\delta(x, \sqrt{\delta}x) = (\sqrt{1+\delta}x, 0), \quad \text{for every } x \in \mathbb{R}^d,$$

and set $\mathcal{R}_\delta = \begin{pmatrix} R_\delta & 0 \\ 0 & R_\delta \end{pmatrix} \in \mathbb{R}^{4d\times 4d}$. Then $\mathcal{R}_\delta \in \mathrm{Sp}(2d)$ and

$$\begin{aligned}\mu(\mathcal{R}_\delta)\rho(x, \sqrt{\delta}x, \omega, \sqrt{\delta}\omega)\mu(\mathcal{R}_\delta)^{-1} &= \rho(R_\delta(x, \sqrt{\delta}x), R_\delta(\omega, \sqrt{\delta}\omega)) \\ &= \rho(\sqrt{1+\delta}x, 0, \sqrt{1+\delta}\omega, 0).\end{aligned}$$

Since

$$\rho(\sqrt{1+\delta}x, 0, \sqrt{1+\delta}\omega, 0) = \rho(\sqrt{1+\delta}x, \sqrt{1+\delta}\omega) \otimes I,$$

it follows that

$$\begin{aligned}T_\delta \otimes I &= \int_{\mathbb{R}^{2d}} \mathcal{U}\widehat{G_\delta}(z)\rho(\sqrt{1+\delta}x, 0, \sqrt{1+\delta}\omega, 0)dz \\ &= \int_{\mathbb{R}^{2d}} \mathcal{U}\widehat{G_\delta}(z)\mu(\mathcal{R}_\delta)\rho(x, \sqrt{\delta}x, \omega, \sqrt{\delta}\omega)\mu(\mathcal{R}_\delta)^{-1}dz \\ &= \mu(\mathcal{R}_\delta)T_\delta^{\otimes}\mu(\mathcal{R}_\delta)^{-1}.\end{aligned}$$

Hence, $T_\delta \otimes I$ and $T_\delta^{\otimes}$ are unitarily equivalent and therefore $\sigma(T_\delta) = \sigma(T_\delta \otimes I) = \sigma(T_\delta^{\otimes})$. The same argument applies to $\mathcal{T}_R(T_\delta)$. ■

Bellissard [8] proved a special case of Theorem 3.3 for periodic symbols with C^*-algebra arguments. Our main insight is that the metaplectic representation allows one to treat also non-periodic symbols (and perhaps provides a more direct argument even for periodic ones).

We now extend the resolution of the identity (2.10) to obtain the following expansion of phase-space shifts. Recall that φ is always the normalized Gaussian

$$\varphi(t) = 2^{d/4}e^{-\pi|t|^2}, \quad t \in \mathbb{R}^d. \tag{3.4}$$

The following proposition, which can be found in [8, Proposition 2 (vi)], is the core of the argument leading to Theorem 1.1. We provide a short proof for the reader's convenience.

Lemma 3.4. *Let φ be the normalized Gaussian (3.4), $[\,\cdot\,,\,\cdot\,]$ the symplectic form (2.5), and q the rank-one projection (2.9) associated with φ. Then*

$$\rho(z) = e^{\frac{\pi}{2}|z|^2} \int_{\mathbb{R}^{2d}} e^{2\pi i [z,z']} q(z') dz', \quad z \in \mathbb{R}^{2d}, \tag{3.5}$$

where integral converges in the weak sense.

Proof. Let $f, g \in L^2(\mathbb{R}^d)$. Using (2.13), we find that

$$\begin{aligned} \int_{\mathbb{R}^{2d}} e^{2\pi i [z,z']} \langle q(z') f, g\rangle dz' &= \mathcal{F}(V_\varphi f \overline{V_\varphi g})(-\omega, x) \\ &= \mathcal{U}\mathcal{F}(V_\varphi f \overline{V_\varphi g})(-x, -\omega) \\ &= V_g f(-z)\overline{V_\varphi \varphi(-z)} = \langle \rho(z) f, g\rangle e^{-\pi |z|^2/2}. \end{aligned}$$

Since f and g were arbitrary, this implies (3.5). ∎

Next, we apply Lemma 3.4 to inspect the tensorized operator $T_\delta^{\otimes}$.

Lemma 3.5. *Fix $G_\delta \in M^{\infty,1}(\mathbb{R}^{2d})$ for $\delta > 0$ and set*

$$Q_R(T_\delta)(z') := \int_{\mathbb{R}^{2d}} e^{\frac{\pi\delta}{2}|z|^2} e^{2\pi i [z,z']} \mathcal{U}(\theta_R \widehat{G_\delta})(z) \rho(z) dz. \tag{3.6}$$

Then

$$\mathcal{T}_R(T_\delta)^{\otimes} = \frac{1}{\delta^d} \int_{\mathbb{R}^{2d}} Q_R(T_\delta)(z') \otimes q(z'/\sqrt{\delta}) dz'. \tag{3.7}$$

Proof. Let us first observe that if $h(z) = z \cdot z'$, then $\mathcal{U}h(z) = [z, z']$ which implies that for $f, g \in L^2(\mathbb{R}^d)$

$$\langle Q_R(T_\delta)(z') f, g\rangle = \mathcal{U}\mathcal{F}(e^{\frac{\pi\delta}{2}|\cdot|^2} \mathcal{U}(\theta_R \widehat{G_\delta}) \langle \rho(\cdot) f, g\rangle)(z'), \tag{3.8}$$

as well as

$$\int_{\mathbb{R}^{2d}} e^{2\pi i \sqrt{\delta}[z,z']} \langle q(z') f, g\rangle dz' = D_{\sqrt{\delta}} \mathcal{U}\mathcal{F}^{-1}(\langle q(\cdot) f, g\rangle)(z).$$

Using (3.5) for the second factor in $\rho(z) \otimes \rho(\sqrt{\delta} z)$, we may formally write $\mathcal{T}_R(T_\delta)^{\otimes}$ as

$$\mathcal{T}_R(T_\delta)^{\otimes} = \int_{\mathbb{R}^{2d}} \mathcal{U}(\theta_R \widehat{G_\delta})(z) \rho(z) \otimes \rho(\sqrt{\delta} z) dz$$

$$= \int_{\mathbb{R}^{2d}} \mathcal{U}(\theta_R \widehat{G_\delta})(z)\rho(z) \otimes \left(e^{\frac{\pi\delta}{2}|z|^2} \int_{\mathbb{R}^{2d}} e^{2\pi i \sqrt{\delta}[z,z']} q(z')dz' \right) dz$$

$$= \int_{\mathbb{R}^{2d}} e^{\frac{\pi\delta}{2}|z|^2} \mathcal{U}(\theta_R \widehat{G_\delta})(z)\rho(z) \otimes \int_{\mathbb{R}^{2d}} e^{2\pi i \sqrt{\delta}[z,z']} q(z')dz'dz.$$

For $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$, we therefore get

$$\begin{aligned}
&\langle \mathcal{T}_R(T_\delta)^{\otimes}(f_1 \otimes f_2), (g_1 \otimes g_2)\rangle \\
&= \langle \mathcal{U}(e^{\frac{\pi\delta}{2}|\cdot|^2}\theta_R \widehat{G_\delta})\langle \rho(\cdot) f_1, g_1\rangle, \overline{D_{\sqrt{\delta}} \mathcal{U}\mathcal{F}^{-1}(\langle q(\cdot) f_2, g_2\rangle)}\rangle \\
&= \frac{1}{\delta^d}\langle \mathcal{F}^{-1}\mathcal{U}^*(\mathcal{U}(e^{\frac{\pi\delta}{2}|\cdot|^2}\theta_R \widehat{G_\delta})\langle \rho(\cdot) f_1, g_1\rangle), \overline{D_{1/\sqrt{\delta}}(\langle q(\cdot) f_2, g_2\rangle)}\rangle \\
&= \frac{1}{\delta^d}\langle \mathcal{U}\mathcal{F}(\mathcal{U}(e^{\frac{\pi\delta}{2}|\cdot|^2}\theta_R \widehat{G_\delta})\langle \rho(\cdot) f_1, g_1\rangle), \overline{D_{1/\sqrt{\delta}}(\langle q(\cdot) f_2, g_2\rangle)}\rangle \\
&= \frac{1}{\delta^d}\langle \langle Q_R(T_\delta)(\cdot) f_1, g_1\rangle, \overline{\langle q(\cdot/\sqrt{\delta}) f_2, g_2\rangle}\rangle \\
&= \left\langle \frac{1}{\delta^d} \int_{\mathbb{R}^{2d}} Q_R(T_\delta)(z') \otimes q(z'/\sqrt{\delta})dz'(f_1 \otimes f_2), (g_1 \otimes g_2)\right\rangle.
\end{aligned}$$

To justify these calculations, we show that $\langle Q_R(T_\delta)(\cdot) f_1, g_1\rangle \in L^\infty(\mathbb{R}^{2d})$ and that $\langle q(\cdot/\sqrt{\delta}) f_2, g_2\rangle \in L^1(\mathbb{R}^{2d})$ (with norms that may depend on δ and R). To this end note, first

$$\|\langle q(\cdot) f, g\rangle\|_{L^1} = \|V_\varphi f \cdot \overline{V_\varphi g}\|_{L^1} \le \|V_\varphi f\|_{L^2}\|V_\varphi g\|_{L^2} = \|f\|_2\|g\|_2.$$

Second, letting for simplicity $\|f\|_2 = \|g\|_2 = 1$,

$$\begin{aligned}
|\langle Q_R(T_\delta)(z) f, g\rangle| &\le \|Q_R(T_\delta)(z)\|_{B(L^2(\mathbb{R}^d))} \lesssim \|\mathcal{U}(M_{-z} e^{\frac{\pi\delta}{2}|\cdot|^2}\theta_R \widehat{G_\delta})\|_{W^{\infty,1}} \\
&\lesssim \|e^{\frac{\pi\delta}{2}|\cdot|^2}\theta_R \widehat{G_\delta}\|_{W^{\infty,1}} \lesssim \|e^{\frac{\pi\delta}{2}|\cdot|^2}\theta_R\|_{\mathcal{F}L^1}\|\widehat{G_\delta}\|_{W^{\infty,1}} < \infty,
\end{aligned}$$

where we used (2.18) and $e^{\frac{\pi\delta}{2}|\cdot|^2}\theta_R \in \mathcal{S}(\mathbb{R}^{2d})$. Taking the supremum over $z \in \mathbb{R}^{2d}$ shows that $\langle Q_R(T_\delta)(\cdot) f, g\rangle \in L^\infty(\mathbb{R}^{2d})$. ∎

We need the following estimate on the reversal of the heat-flow, which follows from a Taylor expansion; see, e.g., [8, Lemma 6].

Lemma 3.6. *Let $F \in C_b^2(\mathbb{R}^{2d})$ and $\Phi_\delta(z) = \frac{1}{\delta^d} e^{-\pi|z|^2/\delta}$ with $z \in \mathbb{R}^{2d}$ and $\delta > 0$. Then*

$$\|F - \Phi_\delta * F\|_\infty \lesssim \delta\|\partial^2 F\|_\infty := \delta \sum_{|\alpha|=2} \|\partial^\alpha F\|_\infty. \tag{3.9}$$

Finally, we compare the spectral extreme values of the tensorization $\mathcal{T}_R(T_\delta)^{\otimes}$ and $Q_R(T_\delta)(0)$.

Lemma 3.7. *Let* $0 < \delta < \delta_0$ *and* $G_\delta \in M^{\infty,1}_{0,2}(\mathbb{R}^{2d})$ *be real-valued. If we assume that* $\sup_{|t|<\delta_0} \|G_t\|_{M^{\infty,1}_{0,2}} < \infty$, *and* $0 < R \le \delta^{-1/2}$, *then*

$$|\sigma_\pm(Q_R(T_\delta)(0)) - \sigma_\pm(\mathcal{T}_R(T_\delta)^\otimes)| \lesssim \delta \cdot \sup_{|t|<\delta_0} \|G_t\|_{M^{\infty,1}_{0,2}}.$$

Proof. We proceed in four steps.

Step 1. Note that $Q_R(T_0)(0) = \mathcal{T}_R(T_0)$. In addition, by (2.7),

$$Q_R(T_\delta)(z') = \rho(z')^* Q_R(T_\delta)(0)\rho(z'). \tag{3.10}$$

Consequently, $\sigma(Q_R(T_\delta)(z)) = \sigma(Q_R(T_\delta)(0))$, as $Q_R(T_\delta)(z)$ and $Q_R(T_\delta)(0)$ are unitarily equivalent.

Step 2. We note that $Q_R(T_\delta)(z)$ is self-adjoint for every $z \in \mathbb{R}^{2d}$ because $Q_R(T_\delta)(0)$ is. Let $f \in L^2(\mathbb{R}^d)$ with $\|f\|_2 = 1$ and fix $\lambda \in \mathbb{R}$. Then, (3.10) shows that

$$\begin{aligned}\langle (Q_R(T_\delta)(z) - \lambda I) f, f\rangle &= \langle (Q_R(T_\delta)(0) - \lambda I)\rho(z) f, \rho(z) f\rangle \\ &\le \|Q_R(T_\delta)(0) - \lambda I\|_{B(L^2(\mathbb{R}^d))}.\end{aligned}$$

Hence, $(Q_R(T_\delta)(z) - \lambda I) \otimes q(z/\sqrt{\delta}) \le \|Q_R(T_\delta)(0) - \lambda I\|_{B(L^2(\mathbb{R}^d))} I \otimes q(z/\sqrt{\delta})$.

Next, we invoke Lemmas 3.4 and (3.7) of Lemma 3.5 to obtain

$$\begin{aligned}\mathcal{T}_R(T_\delta)^\otimes - \lambda I \otimes I &= \frac{1}{\delta^d}\int_{\mathbb{R}^{2d}} Q_R(T_\delta)(z) \otimes q(z/\sqrt{\delta}) dz - \lambda I \otimes I \\ &= \frac{1}{\delta^d}\int_{\mathbb{R}^{2d}} (Q_R(T_\delta)(z) - \lambda I) \otimes q(z/\sqrt{\delta}) dz \\ &\le \|Q_R(T_\delta)(0) - \lambda I\|_{B(L^2(\mathbb{R}^d))} \frac{1}{\delta^d}\int_{\mathbb{R}^{2d}} I \otimes q(z/\sqrt{\delta}) dz \\ &= \|Q_R(T_\delta)(0) - \lambda I\|_{B(L^2(\mathbb{R}^d))} I \otimes I.\end{aligned} \tag{3.11}$$

Repeating this argument for $\lambda I \otimes I - \mathcal{T}_R(T_\delta)^\otimes$ shows that

$$\|\mathcal{T}_R(T_\delta)^\otimes - \lambda I \otimes I\|_{B(L^2(\mathbb{R}^{2d}))} \le \|Q_R(T_\delta)(0) - \lambda I\|_{B(L^2(\mathbb{R}^d))}. \tag{3.12}$$

Step 3. For a lower estimate let $f \in L^2(\mathbb{R}^d)$, $\|f\|_2 = 1$, and apply Lemma 3.5:

$$\begin{aligned}&\|\mathcal{T}_R(T_\delta)^\otimes - \lambda I \otimes I\|_{B(L^2(\mathbb{R}^{2d}))} \\ &\quad \ge |\langle (\mathcal{T}_R(T_\delta)^\otimes - \lambda I \otimes I)(f \otimes \varphi), (f \otimes \varphi)\rangle| \\ &\quad = \left| \frac{1}{\delta^d}\int_{\mathbb{R}^{2d}} \langle Q_R(T_\delta)(z) f, f\rangle \langle q(z/\sqrt{\delta})\varphi, \varphi\rangle \, dz - \lambda \|f\|_2^2 \|\varphi\|_2^2 \right| = (*).\end{aligned}$$

Since the short-time Fourier transform of a Gaussian is again a Gaussian, we find that

$$\langle q(z/\sqrt{\delta})\varphi, \varphi\rangle = |\langle \varphi, \rho(z/\sqrt{\delta})\varphi\rangle|^2 = e^{-\pi|z|^2/\delta} = \delta^d \Phi_\delta(z),$$

and that the last expression involves a convolution with the scaled Gaussian Φ_δ. We can continue as follows:

$$\begin{aligned}
(*) &= \Big| \int_{\mathbb{R}^{2d}} \langle Q_R(T_\delta)(z) f, f\rangle \Phi_\delta(0-z)\, dz - \lambda \Big| \\
&= |(\langle Q_R(T_\delta)(\cdot) f, f\rangle * \Phi_\delta)(0) - \lambda| \\
&\geq |\langle Q_R(T_\delta)(0) f, f\rangle - \lambda| - \|\langle Q_R(T_\delta)(\cdot) f, f\rangle - \langle Q_R(T_\delta)(\cdot) f, f\rangle * \Phi_\delta\|_\infty \\
&\geq |\langle Q_R(T_\delta)(0) f, f\rangle - \lambda| \\
&\quad - \sup_{\|h\|_2=1} \|\langle Q_R(T_\delta)(\cdot)h, h\rangle - \langle Q_R(T_\delta)(\cdot)h, h\rangle * \Phi_\delta\|_\infty.
\end{aligned}$$

In the first term we take the supremum over $f \in L^2(\mathbb{R}^d)$, $\|f\|_2 = 1$; to the second term we apply Lemma 3.6. In view of (3.12), this leads to

$$\begin{aligned}
&|\|\mathcal{T}_R(T_\delta)^{\otimes} - \lambda I \otimes I\|_{B(L^2(\mathbb{R}^{2d}))} - \|Q_R(T_\delta)(0) - \lambda I\|_{B(L^2(\mathbb{R}^d))}| \\
&\qquad \leq \sup_{\|h\|_2=1} \|\langle Q_R(T_\delta)(\cdot)h, h\rangle - \langle Q_R(T_\delta)(\cdot)h, h\rangle * \Phi_\delta\|_\infty \\
&\qquad \lesssim \delta \sup_{\|h\|_2=1} \|\partial^2 \langle Q_R(T_\delta)(\cdot)h, h\rangle\|_\infty.
\end{aligned}$$

The deductions above together with the first part of Lemma 2.1 (for an appropriate choice of λ) then show

$$|\sigma_\pm(Q_R(T_\delta)(0)) - \sigma_\pm(\mathcal{T}_R(T_\delta)^{\otimes})| \lesssim \delta \sup_{\|h\|_2=1} \|\partial^2 \langle Q_R(T_\delta)(\cdot)h, h\rangle\|_\infty. \tag{3.13}$$

Step 4. It remains to further estimate the right-hand side of (3.13). Let $\|h\|_2 = 1$. Using (3.8), the partial derivatives of $\langle Q_R(T_\delta)(z')h, h\rangle$ are given as follows: with the notation $z'_i = (x'_i, \omega'_i) \in \mathbb{R}^2$ and $i = 1, \ldots, d$, let u'_i be either x'_i or ω'_i. Then

$$\begin{aligned}
&|\partial_{u'_i} \partial_{u'_j} \langle Q_R(T_\delta))(z')h, h\rangle| \\
&\quad = |\partial_{u'_i} \partial_{u'_j} \mathcal{U}\mathcal{F}(e^{\frac{\pi\delta}{2}|\cdot|^2} \mathcal{U}(\theta_R \widehat{G_\delta}) \langle \rho(\cdot)h, h\rangle)(z')| \\
&\quad = 4\pi^2 |\mathcal{U}\mathcal{F}(X_{i'} X_{j'} e^{\frac{\pi\delta}{2}|\cdot|^2} \mathcal{U}(\theta_R \widehat{G_\delta}) \langle \rho(\cdot)h, h\rangle)(z')|,
\end{aligned}$$

for suitable indices $i', j' \in \{1, \ldots, 2d\}$. We apply Theorem 2.5 (ii) and obtain

$$
\begin{aligned}
&\|\partial^2 \langle Q_R(T_\delta)(\cdot)h, h\rangle\|_\infty \\
&\qquad \lesssim \sup_{z' \in \mathbb{R}^{2d}} \sum_{i,j=1,\ldots,2d} \left| \int_{\mathbb{R}^{2d}} \mathcal{U}(M_{-z'} e^{\frac{\pi\delta}{2}|\cdot|^2} X_i X_j \theta_R \widehat{G_\delta})(z) \langle \rho(z)h, h\rangle dz \right| \\
&\qquad \lesssim \sup_{z' \in \mathbb{R}^{2d}} \sum_{i,j=1,\ldots,2d} \|\mathcal{U}(M_{-z'} e^{\frac{\pi\delta}{2}|\cdot|^2} X_i X_j \theta_R \widehat{G_\delta})\|_{W^{\infty,1}} \\
&\qquad \lesssim \sum_{i,j=1,\ldots,2d} \|e^{\frac{\pi\delta}{2}|\cdot|^2} X_i X_j \theta_R \widehat{G_\delta}\|_{W^{\infty,1}}.
\end{aligned}
$$

For each of the terms, we use the product property (2.18) and obtain

$$
\|\partial^2 \langle Q_R(T_\delta)(\cdot)h, h\rangle\|_\infty \lesssim \sum_{i,j=1,\ldots,2d} \|e^{\frac{\pi\delta}{2}|\cdot|^2} \theta_R\|_{\mathcal{F}L^1} \|X_i X_j \widehat{G_\delta}\|_{W^{\infty,1}}.
$$

By Lemma 3.2, $\|e^{\frac{\pi\delta}{2}|\cdot|^2} \theta_R\|_{\mathcal{F}L^1} \lesssim 1$ for $R \le \delta^{-1/2}$, whereas, by (2.20),

$$
\|X_i X_j \widehat{G_\delta}\|_{W^{\infty,1}} \lesssim \|\widehat{G_\delta}\|_{W^{\infty,1}_{0,2}} \asymp \|G_\delta\|_{M^{\infty,1}_{0,2}}. \tag{3.14}
$$

In conclusion, we have shown that

$$
\|\partial^2 \langle Q_R(T_\delta)(\cdot)h, h\rangle\|_\infty \lesssim \|G_\delta\|_{M^{\infty,1}_{0,2}},
$$

which, combined with (3.13), completes the proof. ■

3.3. Differentiation of the symbol

Lemma 3.8. *Assume that $G_\delta \in M^{\infty,1}_{0,2}(\mathbb{R}^{2d})$ is real-valued, $\delta \mapsto G_\delta$ is differentiable, $\partial_\delta G_\delta \in M^{\infty,1}(\mathbb{R}^{2d})$ for $0 < \delta < \delta_0 < 1$, and $0 < R \le \delta^{-1/2}$. Then*

$$
\|Q_R(T_\delta)(0) - \mathcal{T}_R(T_0)\|_{B(L^2(\mathbb{R}^d))} \lesssim \delta \cdot \sup_{|t|<\delta_0} (\|G_t\|_{M^{\infty,1}_{0,2}} + \|\partial_t G_t\|_{M^{\infty,1}}). \tag{3.15}
$$

Proof. Recall that

$$
\begin{aligned}
&Q_R(T_\delta)(0) - \mathcal{T}_R(T_0) \\
&\quad = \int_{\mathbb{R}^{2d}} e^{\pi\delta|z|^2/2} \mathcal{U}(\theta_R \widehat{G_\delta})(z) \rho(z) dz - \int_{\mathbb{R}^{2d}} \mathcal{U}(\theta_R \widehat{G_0})(z) \rho(z) dz.
\end{aligned}
$$

Using Theorem 2.5 (ii), we estimate the operator norm by

$$
\|Q_R(T_\delta)(0) - \mathcal{T}_R(T_0)\|_{B(L^2(\mathbb{R}^d))} \lesssim \|e^{\frac{\pi\delta}{2}|\cdot|^2} \theta_R \widehat{G_\delta} - \theta_R \widehat{G_0}\|_{W^{\infty,1}}.
$$

Although $\widehat{G_\delta}$ is a tempered distribution, the short-time Fourier transform

$$H(\delta, x, \omega) = V_\varphi(e^{\frac{\pi\delta}{2}|\cdot|^2}\theta_R\widehat{G_\delta})(x,\omega), \quad x,\omega \in \mathbb{R}^{2d},$$

is a smooth function and therefore we may express it as

$$H(\delta, x, \omega) - H(0, x, \omega) = \int_0^\delta \partial_t H(t, x, \omega) dt.$$

Then by the definition of the $W^{\infty,1}$-norm we have

$$\begin{aligned}
\|e^{\frac{\pi\delta}{2}|\cdot|^2}\theta_R\widehat{G_\delta} - \theta_R\widehat{G_0}\|_{W^{\infty,1}} &= \int_{\mathbb{R}^{2d}} \sup_{\omega\in\mathbb{R}^{2d}} |H(\delta,x,\omega) - H(0,x,\omega)|\, dx \\
&\le \int_0^\delta \int_{\mathbb{R}^{2d}} \sup_{\omega\in\mathbb{R}^{2d}} |\partial_t H(t,x,\omega)|\, dx\, dt \\
&\le \delta \sup_{|t|\le\delta} \int_{\mathbb{R}^{2d}} \sup_{\omega\in\mathbb{R}^{2d}} |\partial_t H(t,x,\omega)|\, dx.
\end{aligned}$$

Spelling out $\partial_t H(t,\cdot)$ explicitly and using the identity

$$V_{\hat{g}}\hat{f}(x,\omega) = e^{-2\pi i x\cdot\omega} V_g f(-\omega, x)$$

with $\hat{f} = e^{\pi\delta|\cdot|^2/2}\theta_R\widehat{G_\delta}$ and $\hat{g} = \hat{\varphi} = \varphi$ gives

$$\begin{aligned}
\partial_t H(t,x,\omega) &= e^{-2\pi i x\cdot\omega}\partial_t V_\varphi(\mathcal{F}^{-1}(e^{\frac{\pi t}{2}|\cdot|^2}\theta_R) * G_t)(-\omega, x) \\
&= e^{-2\pi i x\cdot\omega} V_\varphi(\mathcal{F}^{-1}(\partial_t e^{\frac{\pi t}{2}|\cdot|^2}\theta_R) * G_t)(-\omega, x) \\
&\quad + e^{-2\pi i x\cdot\omega} V_\varphi(\mathcal{F}^{-1}(e^{\frac{\pi t}{2}|\cdot|^2}\theta_R) * \partial_t G_t)(-\omega, x) \\
&= \frac{\pi}{2} V_\varphi(|\cdot|^2 e^{\frac{\pi t}{2}|\cdot|^2}\theta_R\widehat{G_t})(x,\omega) + V_\varphi(e^{\frac{\pi t}{2}|\cdot|^2}\theta_R\widehat{\partial_t G_t})(x,\omega). \qquad (3.16)
\end{aligned}$$

In the calculations above, we interchanged integration (hidden in V_φ) and differentiation twice to obtain the second equality. To justify this, we construct two integrable majorants and apply the Leibniz integral rule. We note that, by regularity and support assumptions on θ,

$$\max\{|\mathcal{F}^{-1}(e^{\frac{\pi t}{2}|\cdot|^2}\theta_R)(z)|, |\mathcal{F}^{-1}(|\cdot|^2 e^{\frac{\pi t}{2}|\cdot|^2}\theta_R)(z)|\} \le C_R(1+|z|)^{-(2d+1)},$$

for a constant C_R independent of t, as long as $|t| \leq 1$, but dependent on R. Therefore, using the embedding $M^{\infty,1}(\mathbb{R}^{2d}) \subseteq L^\infty(\mathbb{R}^{2d})$ from (2.22),

$$\begin{aligned}
&|\mathcal{F}^{-1}(|\cdot|^2 e^{\frac{\pi t}{2}|\cdot|^2}\theta_R)(z)G_t(w-z) + \mathcal{F}^{-1}(e^{\frac{\pi t}{2}|\cdot|^2}\theta_R)(z)\partial_t G_t(w-z)| \\
&\quad \lesssim C_R(1+|z|)^{-(2d+1)} \sup_{|t|<\delta_0} (\|G_t\|_\infty + \|\partial_t G_t\|_\infty) \\
&\quad \lesssim C_R(1+|z|)^{-(2d+1)} \sup_{|t|<\delta_0} (\|G_t\|_{M^{\infty,1}} + \|\partial_t G_t\|_{M^{\infty,1}}),
\end{aligned}$$

as well as

$$\begin{aligned}
&|(\mathcal{F}^{-1}(|\cdot|^2 e^{\frac{\pi t}{2}|\cdot|^2}\theta_R) * G_t(y) + \mathcal{F}^{-1}(e^{\frac{\pi t}{2}|\cdot|^2}\theta_R) * \partial_t G_t(y))\rho(z)\varphi(y)| \\
&\quad \lesssim |(\|\mathcal{F}^{-1}(|\cdot|^2 e^{\frac{\pi t}{2}|\cdot|^2}\theta_R) * G_t\|_\infty + \|\mathcal{F}^{-1}(e^{\frac{\pi t}{2}|\cdot|^2}\theta_R) * \partial_t G_t\|_\infty) T_x\varphi(y)| \\
&\quad \lesssim C_R \sup_{|t|<\delta_0} (\|G_t\|_{M^{\infty,1}} + \|\partial_t G_t\|_{M^{\infty,1}})|T_x\varphi(y)|.
\end{aligned}$$

This provides the required majorants.

Finally, applying (2.18) to the expression (3.16) and using Lemma 3.2 and (2.20), we obtain

$$\begin{aligned}
&\|Q_R(T_\delta)(0) - \mathcal{T}_R(T_0)\|_{B(L^2(\mathbb{R}^d))} \\
&\quad \lesssim \delta \sup_{|t|\leq\delta} (\|e^{\frac{\pi t}{2}|\cdot|^2}\theta_R|\cdot|^2 \widehat{G_t}\|_{W^{\infty,1}} + \|e^{\frac{\pi t}{2}|\cdot|^2}\theta_R \widehat{\partial_t G_t}\|_{W^{\infty,1}}) \\
&\quad \lesssim \delta \sup_{|t|<\delta_0} (\||\cdot|^2 \widehat{G_t}\|_{W^{\infty,1}} + \|\widehat{\partial_t G_t}\|_{W^{\infty,1}}) \\
&\quad = \delta \sup_{|t|<\delta_0} (\|G_t\|_{M^{\infty,1}_{0,2}} + \|\partial_t G_t\|_{M^{\infty,1}}). \qquad \blacksquare
\end{aligned}$$

3.4. Proof of Theorem 1.1

We only consider the right spectral extreme value; the proof for left spectral extreme value works exactly in the same way.

Let us first assume that $\delta = \delta_1 > 0$ and $\delta_2 = 0$, and let $R = \delta^{-1/2}$. We estimate

$$\begin{aligned}
|\sigma_+(T_\delta) - \sigma_+(T_0)| \leq\; & |\sigma_+(T_\delta) - \sigma_+(\mathcal{T}_R(T_\delta))| + |\sigma_+(\mathcal{T}_R(T_\delta)) - \sigma_+(\mathcal{T}_R(T_\delta)^{\otimes})| \\
&+ |\sigma_+(\mathcal{T}_R(T_\delta)^{\otimes}) - \sigma_+(Q_R(T_\delta)(0))| \\
&+ |\sigma_+(Q_R(T_\delta)(0)) - \sigma_+(\mathcal{T}_R(T_0))| \\
&+ |\sigma_+(\mathcal{T}_R(T_0)) - \sigma_+(T_0)|.
\end{aligned}$$

The first and last terms can be bounded using Lemma 3.1. The second term is zero by Theorem 3.3. Lemma 3.7 bounds the third term, and Lemma 3.8 bounds the fourth term. Altogether, we arrive at

$$\begin{aligned} |\sigma_+(T_\delta) - \sigma_+(T_0)| &\lesssim \Big(\frac{1}{R^2} + \delta\Big)(1+\delta_0)^d \sup_{|t|<\delta_0} (\|G_t\|_{M^{\infty,1}_{0,2}} + \|\partial_t G_t\|_{M^{\infty,1}}) \\ &= 2\delta(1+\delta_0)^d \sup_{|t|<\delta_0} (\|G_t\|_{M^{\infty,1}_{0,2}} + \|\partial_t G_t\|_{M^{\infty,1}}). \end{aligned} \tag{3.17}$$

Hence, Lipschitz continuity of the spectral extreme values at 0 holds for $\delta > 0$.

The general case $-\delta_0 < \delta_1 \le \delta_2 < \delta_0$, $\delta_0 < 1$, needs an additional argument for which we introduce a new parameter θ. Fix δ_1, δ_2 and define $\widetilde{T}_\theta$, $0 \le \theta < \theta_0 = \frac{\delta_0-\delta_1}{1+\delta_1}$, via its Weyl symbol $D_{\sqrt{1+\theta}}\widetilde{G}_\theta$, where $\widetilde{G}_\theta = D_{\sqrt{1+\delta_1}} G_{(1+\delta_1)\theta+\delta_1}$. For this choice, we have $\widetilde{T}_0 = T_{\delta_1}$, and $\widetilde{T}_{(\delta_2-\delta_1)/(1+\delta_1)} = T_{\delta_2}$. By the dilation property (2.23) and $0 < 1+\delta_1 < 2$ we get

$$\begin{aligned} \|\widetilde{G}_\theta\|_{M^{\infty,1}_{0,2}} &= \|D_{\sqrt{1+\delta_1}} G_{(1+\delta_1)\theta+\delta_1}\|_{M^{\infty,1}_{0,2}} \\ &\lesssim \|G_{(1+\delta_1)\theta+\delta_1}\|_{M^{\infty,1}_{0,2}} \le \sup_{|t|<\delta_0} \|G_t\|_{M^{\infty,1}_{0,2}}, \end{aligned}$$

as well as

$$\|\partial_\theta \widetilde{G}_\theta\|_{M^{\infty,1}} = \|D_{\sqrt{1+\delta_1}} \partial_\theta [G_{(1+\delta_1)\theta+\delta_1}]\|_{M^{\infty,1}} \lesssim \sup_{|t|<\delta_0} \|\partial_t G_t\|_{M^{\infty,1}}.$$

Applying the estimate from (3.17) to $\widetilde{T}_\theta$ and choosing in particular $\theta = \frac{\delta_2-\delta_1}{1+\delta_1}$ shows

$$\begin{aligned} |\sigma_+(T_{\delta_1}) - \sigma_+(T_{\delta_2})| &= |\sigma_+(\widetilde{T}_0) - \sigma_+(\widetilde{T}_{(\delta_2-\delta_1)/(1+\delta_1)})| \\ &\lesssim |\theta|(1+\theta_0)^d \sup_{|t|<\delta_0} (\|G_t\|_{M^{\infty,1}_{0,2}} + \|\partial_t G_t\|_{M^{\infty,1}}). \end{aligned}$$

The Lipschitz dependence follows from

$$|\theta|(1+\theta_0)^d = \frac{|\delta_2-\delta_1|}{1+\delta_1}\Big(1 + \frac{\delta_0-\delta_1}{1+\delta_1}\Big)^d \le |\delta_1-\delta_2| \frac{2^d}{(1-\delta_0)^{d+1}},$$

since $\delta_0 < 1$ and $1+\delta_1 > 1-\delta_0$. All in all,

$$|\sigma_+(T_{\delta_1}) - \sigma_+(T_{\delta_2})| \lesssim |\delta_1-\delta_2|(1-\delta_0)^{-(d+1)} \sup_{|t|<\delta_0} (\|G_t\|_{M^{\infty,1}_{0,2}} + \|\partial_t G_t\|_{M^{\infty,1}}),$$

as claimed.

3.5. Proof of Theorem 1.2

Since the desired conclusion should hold for sufficiently small δ, we may assume that $\delta_0 < 1/2$. We shall apply the Beckus–Bellissard lemma (Lemma 2.2). Consider a polynomial $p(x) = x^2 + \beta x + \gamma$, with $\beta, \gamma \in \mathbb{R}$. The Weyl symbol of $p(T_\delta)$ is given by $D_{\sqrt{1+\delta}} \widetilde{G}_\delta$, where

$$\widetilde{G}_\delta = D_{1/\sqrt{1+\delta}}((D_{\sqrt{1+\delta}} G_\delta) \sharp (D_{\sqrt{1+\delta}} G_\delta)) + \beta \cdot G_\delta + \gamma.$$

Since $M^{\infty,1}_{0,2}(\mathbb{R}^{2d})$ is a Banach algebra with unit element 1 (Theorem 2.5 (iii)), it follows from (2.23) and $\delta_0 < 1/2$ that

$$\|\widetilde{G}_\delta\|_{M^{\infty,1}_{0,2}} \lesssim (\|G_\delta\|^2_{M^{\infty,1}_{0,2}} + |\beta| \|G_\delta\|_{M^{\infty,1}_{0,2}} + |\gamma|). \tag{3.18}$$

Let us use the notation $G \sharp_\delta H := D_{1/\sqrt{1+\delta}}((D_{\sqrt{1+\delta}} G) \sharp (D_{\sqrt{1+\delta}} H))$. If G and H are Schwartz functions, a computation with (2.26) gives

$$\mathcal{F}(G \sharp_\delta H)(z) = \int_{\mathbb{R}^{2d}} \widehat{G}(z') \widehat{H}(z - z') e^{-\pi(1+\delta) i [z-z', z']} \, dz'.$$

Hence, assuming for a moment that G_δ is a Schwartz function,

$$\begin{aligned}
\mathcal{F}(\partial_\delta(G_\delta \sharp_\delta G_\delta))(z) &= \partial_\delta(\mathcal{F}(G_\delta \sharp_\delta G_\delta))(z) \\
&= \int_{\mathbb{R}^{2d}} \widehat{\partial_\delta G_\delta}(z') \widehat{G_\delta}(z - z') e^{-\pi(1+\delta) i [z-z', z']} \, dz' \\
&\quad + \int_{\mathbb{R}^{2d}} \widehat{G_\delta}(z') \widehat{\partial_\delta G_\delta}(z - z') e^{-\pi(1+\delta) i [z-z', z']} \, dz' \\
&\quad + \pi i \sum_{j=1}^{d} \int_{\mathbb{R}^{2d}} z'_{j+d} \cdot \widehat{G_\delta}(z') \cdot (z - z')_j \cdot \widehat{G_\delta}(z - z') e^{-\pi(1+\delta) i [z-z', z']} \, dz' \\
&\quad - \pi i \sum_{j=1}^{d} \int_{\mathbb{R}^{2d}} z'_j \cdot \widehat{G_\delta}(z') \cdot (z - z')_{j+d} \cdot \widehat{G_\delta}(z - z') e^{-\pi(1+\delta) i [z-z', z']} \, dz'.
\end{aligned}$$

Consequently,

$$\partial_\delta(G_\delta \sharp_\delta G_\delta) = ((\partial_\delta G_\delta) \sharp_\delta G_\delta) + (G_\delta \sharp_\delta (\partial_\delta G_\delta)) + \sum_{k=1}^{2d} \lambda_k (\partial_{z_{j_k}} G_\delta) \sharp_\delta (\partial_{z_{j'_k}} G_\delta), \tag{3.19}$$

for suitable indices $j'_k, j_k \in \{0, \dots, 2d\}$ and $\lambda_k \in \mathbb{C}$, with $|\lambda_k| = 1/2$. To see that (3.19) is valid in general, we take a sequence $G^k_\delta \in \mathcal{S}(\mathbb{R}^{2d})$ such that $\partial_\delta G^k_\delta \in \mathcal{S}(\mathbb{R}^{2d})$,

$$G^k_\delta \xrightarrow{w^*} G_\delta \qquad \text{in } M^{\infty,1}_{0,2}(\mathbb{R}^{2d}), \tag{3.20}$$

and

$$\partial_\delta G_\delta^k \xrightarrow{w^*} \partial_\delta G_\delta \quad \text{in } M^{\infty,1}(\mathbb{R}^{2d}). \tag{3.21}$$

For example, define $G_\delta^k(z) := \psi(z/k) \cdot (G_\delta * \phi_k)(z)$, where $\psi \in \mathcal{S}(\mathbb{R}^{2d})$ is chosen such that $0 \leq \psi \leq 1$, and $\psi(z) = 1$ for $z \in B_1(0)$, and $\phi_k(z) = k^{2d}\phi(kz)$ for a mollifier ϕ.

Applying consecutively (2.23), (2.20), and Theorem 2.5 (iii), it thus follows

$$\begin{aligned}
&\|\partial_\delta(G_\delta \sharp_\delta G_\delta)\|_{M^{\infty,1}} \\
&\quad\lesssim \|D_{\sqrt{1+\delta}} G_\delta \sharp D_{\sqrt{1+\delta}}(\partial_\delta G_\delta)\|_{M^{\infty,1}} \\
&\qquad + \max_{j,j'=1,\dots,2d} \|(D_{\sqrt{1+\delta}}(\partial_{z_j} G_\delta))\sharp(D_{\sqrt{1+\delta}}(\partial_{z_{j'}} G_\delta))\|_{M^{\infty,1}} \\
&\quad\lesssim \|G_\delta\|_{M^{\infty,1}} \|\partial_\delta G_\delta\|_{M^{\infty,1}} + \|G_\delta\|^2_{M^{\infty,1}_{0,1}} \\
&\quad\lesssim \|G_\delta\|_{M^{\infty,1}} \|\partial_\delta G_\delta\|_{M^{\infty,1}} + \|G_\delta\|^2_{M^{\infty,1}_{0,2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\partial_\delta \widetilde{G}_\delta\|_{M^{\infty,1}} &\leq \|\partial_\delta(G_\delta \sharp_\delta G_\delta)\|_{M^{\infty,1}} + |\beta| \|\partial_\delta G_\delta\|_{M^{\infty,1}} \\
&\lesssim \|G_\delta\|_{M^{\infty,1}} \|\partial_\delta G_\delta\|_{M^{\infty,1}} + \|G_\delta\|^2_{M^{\infty,1}_{0,2}} + |\beta| \|\partial_\delta G_\delta\|_{M^{\infty,1}}. \qquad (3.22)
\end{aligned}$$

Let $\delta_1, \delta_2 \in (-\delta_0, \delta_0) \subset (-1/2, 1/2)$. Combining Theorem 1.1 with (3.18) and (3.22),

$$\begin{aligned}
&|\sigma_\pm(p(T_{\delta_1})) - \sigma_\pm(p(T_{\delta_2}))| \\
&\quad\lesssim |\delta_1 - \delta_2| \sup_{|\delta|<\delta_0} (\|\widetilde{G}_\delta\|_{M^{\infty,1}_{0,2}} + \|\partial_\delta \widetilde{G}_\delta\|_{M^{\infty,1}}) \\
&\quad\lesssim |\delta_1 - \delta_2| \sup_{|\delta|<\delta_0} [(\|G_\delta\|^2_{M^{\infty,1}_{0,2}} + \|G_\delta\|_{M^{\infty,1}} \|\partial_\delta G_\delta\|_{M^{\infty,1}}) \\
&\qquad + |\beta|(\|G_\delta\|_{M^{\infty,1}_{0,2}} + \|\partial_\delta G_\delta\|_{M^{\infty,1}}) + |\gamma|].
\end{aligned}$$

In particular, if $|\beta| \leq 2\|T_0\|_{B(L^2(\mathbb{R}^d))}$, $|\gamma| \leq 5\|T_0\|^2_{B(L^2(\mathbb{R}^d))}$, then, by Theorem 2.5, $|\beta| \lesssim \|G_\delta\|_{M^{\infty,1}}$ and $|\gamma| \lesssim \|G_\delta\|^2_{M^{\infty,1}} \leq \|G_\delta\|^2_{M^{\infty,1}_{0,2}}$. Hence,

$$\begin{aligned}
&|\sigma_\pm(p(T_{\delta_1})) - \sigma_\pm(p(T_{\delta_2}))| \\
&\quad\lesssim |\delta_1 - \delta_2| \sup_{|\delta|<\delta_0} (\|G_\delta\|^2_{M^{\infty,1}_{0,2}} + \|G_\delta\|_{M^{\infty,1}} \|\partial_\delta G_\delta\|_{M^{\infty,1}}).
\end{aligned}$$

Therefore, by Lemma 2.1, for $\delta_1 \neq \delta_2$,

$$\frac{|\|p(T_{\delta_1})\|_{B(L^2)} - \|p(T_{\delta_2})\|_{B(L^2)}|}{|\delta_1 - \delta_2|} \lesssim \sup_{|\delta|<\delta_0} (\|G_\delta\|^2_{M^{\infty,1}_{0,2}} + \|G_\delta\|_{M^{\infty,1}} \|\partial_\delta G_\delta\|_{M^{\infty,1}}),$$

holds uniformly for all polynomials $p \in \mathcal{P}(T_0)$. This shows that the number $C_{\mathcal{P}(T_0)}$ defined in Lemma 2.2 satisfies

$$C_{\mathcal{P}(T_0)} \lesssim \sup_{|t|<\delta_0} \left(\|G_t\|^2_{M^{\infty,1}_{0,2}} + \|G_t\|_{M^{\infty,1}} \|\partial_t G_t\|_{M^{\infty,1}}\right). \tag{3.23}$$

In addition, reinspection of the previous estimates shows that $\{T_\delta\}_{|\delta|<\delta_0}$ is $(p2)$-Lipschitz continuous. Hence, we can invoke Lemma 2.2 and the conclusion follows from (3.23).

4. Gabor frames

We now apply the results on the Lipschitz continuity of the spectral edges to the Gabor frame operator.

Let $g \in M^1(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^{2d}$ be a relatively separated set. The frame operator of the associated set of phase-space shifts is

$$S_{g,\Lambda} f = \sum_{\lambda\in\Lambda} \langle f, \rho(\lambda)g\rangle \rho(\lambda)g = \sum_{\lambda\in\Lambda} q(\lambda) f, \quad f \in L^2(\mathbb{R}^d),$$

where q is the rank-one projection (2.9). The Weyl symbol of $q(z)$ is just the shift $T_z \mathcal{W}(g)$ of the Wigner distribution of g (2.12). Hence, the Weyl symbol of $S_{g,\Lambda}$ is

$$\sigma_{g,\Lambda} = \sum_{\lambda\in\Lambda} T_\lambda \mathcal{W}(g).$$

The spectral extreme values of $S_{g,\Lambda}$ and $S_{g,\alpha\Lambda}$ are equal to the optimal frame bounds (1.4) of $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(g,\alpha\Lambda)$ respectively. We set

$$1/\alpha = \sqrt{1+\delta}.$$

The Weyl symbol corresponding to $S_{g,\alpha\Lambda}$ is

$$\sigma_{g,\alpha\Lambda} = \sum_{\lambda\in\Lambda} T_{\alpha\lambda} \mathcal{W}(g) = D_{\sqrt{1+\delta}}\Big(\sum_{\lambda\in\Lambda} T_\lambda D_{1/\sqrt{1+\delta}} \mathcal{W}(g)\Big).$$

Thus, $\sigma_{g,\alpha\Lambda} = D_{\sqrt{1+\delta}} G_\delta$ with

$$G_\delta = \sum_{\lambda\in\Lambda} T_\lambda D_{1/\sqrt{1+\delta}} \mathcal{W}(g).$$

In order to apply Theorem 1.1, we need to calculate the norms of G_δ and $\partial_\delta G_\delta$ in the corresponding weighted Sjöstrand classes.

Lemma 4.1. *Let* $\Lambda \subset \mathbb{R}^{2d}$ *be relatively separated and* $0 < \delta_0 < 1$. *If* $g \in M_2^1(\mathbb{R}^d)$, *then*

(i) $\|G_\delta\|_{M^{\infty,1}} \lesssim \mathrm{rel}(\Lambda) \cdot (1+\delta)^d \cdot \|g\|_{M^1}^2$, $\delta \in [0, \infty)$,

(ii) $\|G_\delta\|_{M_{0,2}^{\infty,1}} \lesssim \mathrm{rel}(\Lambda) \cdot (1-\delta_0)^{-1} \cdot \|g\|_{M_2^1}^2$, $\delta \in (-\delta_0, \delta_0)$,

(iii) $\|\partial_\delta G_\delta\|_{M^{\infty,1}} \lesssim \mathrm{rel}(\Lambda) \cdot (1-\delta_0)^{-1} \cdot \|g\|_{M_2^1}^2$, $\delta \in (-\delta_0, \delta_0)$.

Proof. Let $\mu = \sum_{\lambda \in \Lambda} \delta_\lambda$. Then $G_\delta = \mu * D_{1/\sqrt{1+\delta}} \mathcal{W}(g)$ and $\|\mu\|_{M^\infty} \lesssim \mathrm{rel}(\Lambda)$ by Lemma 2.4.

Furthermore, since $g \in M_2^1(\mathbb{R}^d)$, its Wigner distribution satisfies

$$\mathcal{W}(g) \in M_{0,2}^1(\mathbb{R}^{2d}),$$

as a consequence of Lemma 2.3. The convolution relation (2.19) and the dilation property (2.24) on $\mathbb{R}^{2d}$ show that

$$\|G_\delta\|_{M^{\infty,1}} \lesssim \|\mu\|_{M^\infty} \|D_{1/\sqrt{1+\delta}} \mathcal{W}(g)\|_{M^1} \lesssim \mathrm{rel}(\Lambda) \cdot (1+\delta)^d \cdot \|g\|_{M^1}^2,$$

as claimed in (i). For (ii) we argue similarly

$$\begin{aligned}
\|G_\delta\|_{M_{0,2}^{\infty,1}} &\lesssim \|\mu\|_{M^\infty} \|D_{1/\sqrt{1+\delta}} \mathcal{W}(g)\|_{M_{0,2}^1} \\
&\lesssim \max\{1, (1+\delta)^{-1}\} \|\mu\|_{M^\infty} \|g\|_{M_2^1}^2 \\
&\lesssim (1-\delta_0)^{-1} \cdot \mathrm{rel}(\Lambda) \cdot \|g\|_{M_2^1}^2.
\end{aligned}$$

It remains to determine $\partial_\delta G_\delta$ and estimate its norm. First, we note

$$\begin{aligned}
\partial_\delta G_\delta(z) &= \partial_\delta \Big(\sum_{\lambda \in \Lambda} \mathcal{W}(g) \Big(\frac{z-\lambda}{\sqrt{1+\delta}} \Big) \Big) \\
&= \sum_{\lambda \in \Lambda} \sum_{i=1}^{2d} -\frac{z_i - \lambda_i}{2(1+\delta)^{3/2}} \partial_i \mathcal{W}(g) \Big(\frac{z-\lambda}{\sqrt{1+\delta}} \Big) \\
&= -\frac{1}{2(1+\delta)} \, \mu * D_{1/\sqrt{1+\delta}} \Big(\sum_{i=1}^{2d} X_i \partial_i \mathcal{W}(g) \Big)(z).
\end{aligned}$$

Using (2.19) and (2.24) as above, we prove (iii):

$$\begin{aligned}
\|\partial_\delta G_\delta\|_{M^{\infty,1}} &\lesssim (1-\delta_0)^{-1} \cdot \|\mu\|_{M^\infty} \sum_{i=1}^{2d} \|X_i \partial_i \mathcal{W}(g)\|_{M^1} \\
&\lesssim (1-\delta_0)^{-1} \cdot \mathrm{rel}(\Lambda) \cdot \|\mathcal{W}(g)\|_{M_{1,1}^1} \\
&\lesssim (1-\delta_0)^{-1} \cdot \mathrm{rel}(\Lambda) \cdot \|g\|_{M_2^1}^2.
\end{aligned}$$

In the last step we have applied (2.21) and Lemma 2.3. ∎

An application of Theorem 1.1 now allows us to show the Lipschitz continuity of the frame bounds of $\mathscr{G}(g, \alpha\Lambda)$.

Proof of Theorem 1.3. Recall that $\alpha^{-1} = \sqrt{1+\delta}$ with $\delta \in (-1, +\infty)$. Suppose first that $\delta \le 1/2$ and set $\delta_0 = \max\{1/2, 1-\alpha_0^2\}$. Then $0 < \delta_0 < 1$. Let us check that $\delta \in (-\delta_0, \delta_0)$. By assumption, $\delta \le 1/2 \le \delta_0$. In addition, $\sqrt{1+\delta} = 1/\alpha > \alpha_0$, and consequently $\delta > \alpha_0^2 - 1$, which shows that $-\delta < 1 - \alpha_0^2 \le \delta_0$. We now invoke Theorem 1.1 and Lemma 4.1 to conclude that

$$
\begin{aligned}
|\sigma_\pm(S_{g,\Lambda}) - \sigma_\pm(S_{g,\alpha\Lambda})| &\lesssim |\delta| \cdot (1-\delta_0)^{-(d+1)} \cdot \sup_{|t|<\delta_0} (\|G_t\|_{M^{\infty,1}_{0,2}} + \|\partial_t G_t\|_{M^{\infty,1}}) \\
&\lesssim |\delta| \cdot \mathrm{rel}(\Lambda) \cdot (1-\delta_0)^{-(d+2)} \cdot \|g\|^2_{M^1_2} \\
&\le |\delta| \cdot \mathrm{rel}(\Lambda) \cdot \alpha_0^{-2(d+2)} \cdot \|g\|^2_{M^1_2}.
\end{aligned}
$$

On the other hand, if $\delta \ge 1/2$, we use the following crude estimate based on Theorem 2.5, (2.23) and Lemma 4.1:

$$
\begin{aligned}
|\sigma_\pm(S_{g,\Lambda}) - \sigma_\pm(S_{g,\alpha\Lambda})| &\le |\sigma_\pm(S_{g,\Lambda})| + |\sigma_\pm(S_{g,\alpha\Lambda})| \\
&\le \|S_{g,\Lambda}\|_{B(L^2)} + \|S_{g,\alpha\Lambda}\|_{B(L^2)} \\
&\lesssim \|G_0\|_{M^{\infty,1}} + \|D_{\sqrt{1+\delta}} G_\delta\|_{M^{\infty,1}} \\
&\lesssim (1+\delta)^d (\|G_0\|_{M^{\infty,1}} + \|G_\delta\|_{M^{\infty,1}}) \\
&\lesssim (1+\delta)^{2d} \cdot \mathrm{rel}(\Lambda) \cdot \|g\|^2_{M^1} \\
&< |\delta| \cdot \alpha_0^{-4d} \cdot \mathrm{rel}(\Lambda) \cdot \|g\|^2_{M^1}.
\end{aligned}
$$

Hence, for all $\delta \in (-1, \infty)$,

$$
|\sigma_\pm(S_{g,\Lambda}) - \sigma_\pm(S_{g,\alpha\Lambda})| \lesssim |\delta| \cdot \mathrm{rel}(\Lambda) \cdot \alpha_0^{-4d} \cdot \|g\|^2_{M^1_2}. \tag{4.1}
$$

Finally, observe that since $\alpha_0 < \alpha < 1/\alpha_0$ and $\alpha_0 < 1$,

$$
\begin{aligned}
|\delta| = \frac{|1-\alpha^2|}{\alpha^2} &= \frac{1+\alpha}{\alpha^2}|1-\alpha| \\
&\le (\alpha_0^{-1} + \alpha_0{}^{-2})|1-\alpha| \\
&\le 2\alpha_0{}^{-2}|1-\alpha|, \qquad (4.2)
\end{aligned}
$$

which in combination with (4.1) yields (1.5). ∎

Proof of Theorem 1.4. We let again $\alpha^{-1} = \sqrt{1+\delta}$ and take $|\alpha - 1| < \varepsilon$ with ε sufficiently small so that $\delta \in (-\delta_0, \delta_0)$ and $\alpha_1 < \alpha < 1/\alpha_1$ with $\delta_0 \le 1/2$ and $1/2 < \alpha_1 < 1$.

We invoke Theorem 1.2, Lemma 4.1 and (4.2) to obtain, with possibly a smaller value of ε,

$$
\begin{aligned}
|\sigma_{\pm}^{g}(S_{g,\Lambda}) - \sigma_{\pm}^{g}(S_{g,\alpha\Lambda})| &\lesssim \frac{|\alpha-1|}{L(g)} \cdot \sup_{|t|<\delta_0} (\|G_t\|_{M^{\infty,1}} \|\partial_t G_t\|_{M^{\infty,1}} + \|G_t\|^2_{M^{\infty,1}_{0,2}}) \\
&\lesssim \frac{|\alpha-1|}{L(g)} \cdot \mathrm{rel}(\Lambda)^2 \cdot \|g\|^4_{M^1_2},
\end{aligned}
$$

as claimed. ∎

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