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Scattering theory with both regular and singular perturbations

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Abstract. We provide an asymptotic completeness criterion and a representation formula for the scattering matrix of the scattering couple (A_{B}, A), where both A and A_{B} are self-adjoint operator and A_{B} formally corresponds to adding to A two terms, one regular and the other singular. In particular, our abstract results apply to the couple $(\Delta_{\mathsf{B}}, \Delta)$, where Δ is the free self-adjoint Laplacian in $L^2(\mathbb{R}^3)$ and Δ_{B} is a self-adjoint operator in a class of Laplacians with both a regular perturbation, given by a short-range potential, and a singular one describing boundary conditions (like Dirichlet, Neumann and semi-transparent δ and δ' ones) at the boundary of an open, bounded Lipschitz domain. The results hinge upon a limiting absorption principle for A_{B} and a Kreĭn-like formula for the resolvent difference $(-A_{\mathsf{B}}+z)^{-1}-(-A+z)^{-1}$ which puts on an equal footing the regular (here, in the case of the Laplacian, a Kato–Rellich potential suffices) and the singular perturbations.

1. Introduction

The mathematical scattering theory for short-range potential is a well-developed subject; the existence and completeness of the wave operators can be obtained by two essentially different approaches: the trace-class method and the smooth method (see, e.g., [21]). An important object defined in terms of the wave operators is the scattering operator and, even more important from the point of view of its physical applications, the scattering matrix, which is its reduction to a multiplication operator in the spectral representation of the self-adjoint free Laplacian.

The scattering problem for singular perturbations of self-adjoint operators, which is outside the original scope of these methods, is connected with scattering from obstacles with impenetrable or semi-transparent boundary conditions (see, e.g., [3, 4, 11–14]). On this side, a general scheme has been developed in [11] by combining the construction in [16] with an abstract version of the Limiting Absorption Principle (simply LAP in the following) due to W. Renger (see [18]) and a variant of the smooth method due to M. Schechter (see [19]). In particular, the results in [11] apply to obstacle scattering with a large class of interface conditions on Lipschitz hypersurfaces in

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any dimension. Let us recall that in [4] boundary triple theory and properties of the associated operator-valued Weyl functions were used to obtain a similar representation of the scattering matrix for singularly coupled self-adjoint extensions. It is worth to remark that, while the approach in [11] avoids any trace-class condition, these are needed in [4] and so the applications there are limited to the case of smooth obstacles in two dimensions.

The target of the present paper is to provide a general framework for the scattering with both potential type and singular perturbations. Since our concern is the scattering theory with respect to the free Laplacian, we regard the regular and the singular parts of the perturbation as a single object; this constitutes the main novelty of our approach. In particular, we give an abstract resolvent formula, generalizing the one in [16], which puts on an equal footing the two components of the perturbation. Such a representation is a key ingredient in the derivation of LAP which leads then to the main results of the first part: the asymptotic completeness and an explicit formula for the scattering matrix. These results rely on a certain number of assumptions whose validity is carefully analyzed in the second part where we consider the specific case of a short range potential plus a distributional term, supported on a closed surface and describing self-adjoint interface conditions. In this way, we obtain new representation formulae for the scattering matrix which are expected to be relevant in different physical applications involving wave propagation in inhomogeneous media with impenetrable or semi-transparent obstacles.

Here, in more details, the contents of the paper. In Section 2, following the scheme proposed in [16], we provide an abstract resolvent formula for a perturbation A_{B} of the self-adjoint A by a linear combination of the adjoint of two bounded trace-like maps $\tau_1\colon \mathrm{dom}(A) \to \mathfrak{h}_1$ and $\tau_2\colon \mathrm{dom}(A) \to \mathfrak{h}_2$; while the kernel of τ_2 is required to be dense, so τ_2^* plays the role of a singular perturbation, no further hypothesis is required for τ_1 and in applications that allows τ_1^* to represent a regular perturbation by a short-range potential. In Section 2.3, by block operator matrices and the Schur complement, we re-write the obtained resolvent formula in terms of the resolvent of the operator corresponding to the non-singular part of the perturbations; that plays an important role in the subsequent part regarding LAP and the scattering theory.

In Section 3, following the scheme proposed in [13] and further generalized in [11], at first we provide, under suitable hypothesis, a Limiting Absorption Principle for A_{B} (see Theorem 3.1) and then an asymptotic completeness criterion for the scattering couple (A_{B}, A) (see Theorem 3.5). Then, by a combination of LAP with stationary scattering theory in the Birman–Yafaev scheme and the invariance principle, we obtain a representation formula for the scattering matrix of the couple (A_{B}, A) (see Theorem 3.10). Whenever A is the free Laplacian in $L^2(\mathbb{R}^3)$, such a formula contains, as subcases, both the usual formula for the perturbation given by a short-range potential as given, e.g., in [21] and the formula for the case of a singular perturbation describing self-adjoint boundary conditions on a hypersurface as given in [11].

In Section 4, in order to apply our abstract results to the case in which A is the free 3D Laplacian and the regular part represents a perturbation by a potential, we give various regularity results for the boundary layer operators associated to $\Delta + \mathsf{v}$, where v is a potential of Kato–Rellich type.

In Sections 5 and 6, we present various applications, where the free Laplacian is perturbed both by a regular term, given by a short range potential v decaying as $|x|^{-\kappa(1+\epsilon)}$, and by a singular one describing either separating boundary conditions (as Dirichlet and Neumann ones) or semi-transparent (as δ and δ'-type ones). In order to satisfy all our hypotheses, we need $\kappa = 2$. However, all our hypotheses but a single one (see Lemma 5.6) hold with $\kappa = 1$; we conjecture that the requirement $\kappa = 2$ is merely of technical nature and that our results are true for a short range potential decaying as $|x|^{-(1+\epsilon)}$. Finally, let us remark that whenever one is only interested in the construction of the operators and not in the scattering theory, then it is sufficient to assume that v is a Kato–Rellich potential (see Section 5.1).

Schrödinger operators with a Kato–Rellich potential plus a δ-like perturbation with a p-summable strength ($p > 2$) have been already considered in [14], while for a different construction with a bounded potential and a δ- or a δ'-like perturbation with bounded strength we refer to [3]. None of such references considered the scattering matrix (however, [14] provided a limiting absorption principle). Whenever the singular part of the perturbations is absent, our framework extends from compactly supported potentials in one dimension to short range potentials in three dimensions the kind of results provided in [5, Section 5].

Let us notice that, building on the results in [1,11], the abstract models introduced in Section 2 and the related scattering theory presented in Section 3 apply to perturbations of the Laplacian in $\mathbb{R}^n$, $n \geq 2$, with a suitable short-range potential plus a singular term supported on a bounded hypersurface of codimension one.

1.1. Some notation and definition

We introduce the following notation.

- $\|\cdot\|_X$ denotes the norm on the complex Banach space X; in case X is a Hilbert space, $\langle\cdot,\cdot\rangle_X$ denotes the (conjugate-linear with respect to the first argument) scalar product.
- $\langle\cdot,\cdot\rangle_{X^*,X}$ denotes the duality (assumed to be conjugate-linear with respect to the first argument) between the dual couple (X^*, X).
- $L^*\colon \mathrm{dom}(L^*) \subseteq Y^* \to X^*$ denotes the dual of the densely defined linear operator $L\colon \mathrm{dom}(L) \subseteq X \to Y$; in a Hilbert spaces setting L^* denotes the adjoint operator.

- $\varrho(A)$ and $\sigma(A)$ denote the resolvent set and the spectrum of the self-adjoint operator A; $\sigma_{\rm p}(A)$, $\sigma_{\rm ess}(A)$, $\sigma_{\rm pp}(A)$, $\sigma_{\rm cont}(A)$, $\sigma_{\rm ac}(A)$, $\sigma_{\rm sc}(A)$, denote the point, discrete, essential, pure point, continuous, absolutely continuous and singular continuous spectra.
- $\mathcal{B}(X,Y)$, $\mathcal{B}(X)\equiv\mathcal{B}(X,X)$, denote the Banach space of bounded linear operator on the Banach space X to the Banach space Y; $\|\cdot\|_{X,Y}$ denotes the corresponding norm.
- $\mathfrak{S}_\infty(X,Y)$ denotes the space of compact operators on X to Y.
- $X\hookrightarrow Y$ means that X is continuously embedded into Y.
- $\Omega\equiv\Omega_{\rm in}\subset\mathbb{R}^3$ denotes an open and bounded subset with a Lipschitz boundary Γ; $\Omega_{\rm ex}:=\mathbb{R}^3\backslash\bar\Omega$.
- $H^s(\Omega)$ and $H^s(\Omega_{\rm ex})$ denote the scales of Sobolev spaces.
- $H^s(\mathbb{R}^3\backslash\Gamma):=H^s(\Omega)\oplus H^s(\Omega_{\rm ex})$.
- $|x|$ denotes the norm of $x\in\mathbb{R}^n$. $\langle x\rangle$ denotes the function $x\mapsto(1+|x|^2)^{1/2}$.
- $L^2_w(\mathbb{R}^3)$, $w\in\mathbb{R}$, denotes the set of complex-valued functions f such that $\langle x\rangle^w f\in L^2(\mathbb{R}^3)$.
- $H^s_w(\mathbb{R}^3\backslash\Gamma):=H^s(\Omega)\oplus H^s_w(\Omega_{\rm ex})$, where $H^s_w(\Omega_{\rm ex})$ denotes the weighted Sobolev space relative to the weight $\langle x\rangle^w$.
- $\gamma_0^{\rm in/ex}$ and $\gamma_1^{\rm in/ex}$ denote the interior/exterior Dirichlet and Neumann traces on the boundary Γ.
- $\gamma_0:=\frac12(\gamma_0^{\rm in}+\gamma_0^{\rm ex})$, $\gamma_1:=\frac12(\gamma_1^{\rm in}+\gamma_1^{\rm ex})$.
- $[\gamma_0]:=\gamma_0^{\rm in}-\gamma_0^{\rm ex}$, $[\gamma_1]:=\gamma_1^{\rm in}-\gamma_1^{\rm ex}$.
- SL_z and DL_z denote the single- and double-layer operators.
- $S_z:=\gamma_0\,\mathrm{SL}_z$, $D_z:=\gamma_1\,\mathrm{DL}_z$.
- $D\subset\mathbb{R}$ is said to be discrete in the open set $E\supset D$ whenever the (possibly empty) set of its accumulations point is contained in $\mathbb{R}\backslash E$; D is said to be discrete whenever $E=\mathbb{R}$.
- $\mathring{D}$ denotes the open part of the set $D\subseteq\mathbb{R}$; ∂D denotes its boundary; $D^-:=D\cap(-\infty,0]$.
- Given $x\ge 0$ and $y\ge 0$, $x\lesssim y$ means that there exists $c\ge 0$ such that $x\le c\,y$.

2. An abstract Kreĭn-type resolvent formula

2.1. The resolvent formula

Let $A\colon \mathrm{dom}(A) \subseteq \mathsf{H} \to \mathsf{H}$ be a self-adjoint operator in the Hilbert space H. We denote by $R_z := (-A+z)^{-1}$, $z \in \varrho(A)$, its resolvent; one has $R_z \in \mathcal{B}(\mathsf{H}, \mathsf{H}_A)$, where H_A is the Hilbert space given by $\mathrm{dom}(A)$ equipped with the scalar product

$$\langle u, u\rangle_{\mathsf{H}_A} := \langle (A^2+1)^{1/2}u, (A^2+1)^{1/2}v\rangle_{\mathsf{H}}.$$

Let

$$\mathfrak{h}_k \hookrightarrow \mathfrak{h}_k^\circ \hookrightarrow \mathfrak{h}_k^*, \quad k = 1, 2,$$

be auxiliary Hilbert spaces with dense continuous embedding; we do not identify $\mathfrak{h}_k$ with its dual $\mathfrak{h}_k^*$ (however, we use $\mathfrak{h}_k \equiv \mathfrak{h}_k^{**}$) and we work with the $\mathfrak{h}_k^*$-$\mathfrak{h}_k$ duality $\langle\cdot,\cdot\rangle_{\mathfrak{h}_k^*,\mathfrak{h}_k}$ defined in terms of the scalar product of the intermediate Hilbert space $\mathfrak{h}_k^\circ$. The scalar product and hence the duality are supposed to be conjugate linear with respect to the first variable; notice that $\langle\varphi,\phi\rangle_{\mathfrak{h}_k,\mathfrak{h}_k^*} = \langle\phi,\varphi\rangle_{\mathfrak{h}_k^*,\mathfrak{h}_k}^*$.

Given the bounded linear maps

$$\tau_k\colon \mathsf{H}_A \to \mathfrak{h}_k, \quad k = 1, 2,$$

such that

$$\ker(\tau_2) \text{ is dense in } \mathsf{H} \text{ and } \mathrm{ran}(\tau_2) \text{ is dense in } \mathfrak{h}_2, \tag{2.1}$$

we introduce the bounded operators

$$\tau\colon \mathsf{H}_A \to \mathfrak{h}_1 \oplus \mathfrak{h}_2, \quad \tau u := \tau_1 u \oplus \tau_2 u,$$

and

$$G_z\colon \mathfrak{h}_1^* \oplus \mathfrak{h}_2^* \to \mathsf{H}, \quad G_z := (\tau R_{\bar z})^*, \quad z \in \varrho(A).$$

We further suppose that there exist reflexive Banach spaces $\mathfrak{b}_k$, $k = 1, 2$, with dense continuous embeddings $\mathfrak{h}_k \hookrightarrow \mathfrak{b}_k$ (hence $\mathfrak{b}_k^* \hookrightarrow \mathfrak{h}_k^*$), such that $\mathrm{ran}(G_z|\mathfrak{b}_1^* \oplus \mathfrak{b}_2^*)$ is contained in the domain of definition of some (supposed to exist) $(\mathfrak{b}_1 \oplus \mathfrak{b}_2)$-valued extension of τ (which we denote by the same symbol) in such a way that

$$\tau G_z|\mathfrak{b}_1^* \oplus \mathfrak{b}_2^* \in \mathcal{B}(\mathfrak{b}_1^* \oplus \mathfrak{b}_2^*, \mathfrak{b}_1 \oplus \mathfrak{b}_2). \tag{2.2}$$

Given these hypotheses, we set $\mathsf{B} = (B_0, B_1, B_2)$, with

$$B_0 \in \mathcal{B}(\mathfrak{b}_2^*, \mathfrak{b}_{2,2}^*), \quad B_1 \in \mathcal{B}(\mathfrak{b}_1, \mathfrak{b}_1^*), \quad B_2 \in \mathcal{B}(\mathfrak{b}_2, \mathfrak{b}_{2,2}^*), \tag{2.3}$$

where $\mathfrak{b}_{2,2}$ is a reflexive Banach space,

$$B_1 = B_1^*, \quad B_0 B_2^* = B_2 B_0^*, \tag{2.4}$$

and introduce the map

$$Z_{\mathsf{B}} \ni z \mapsto \Lambda_z^{\mathsf{B}} \in \mathcal{B}(\mathfrak{b}_1 \oplus \mathfrak{b}_2, \mathfrak{b}_1^* \oplus \mathfrak{b}_2^*), \quad \Lambda_z^{\mathsf{B}} := (M_z^{\mathsf{B}})^{-1}(B_1 \oplus B_2), \tag{2.5}$$

where

$$\begin{aligned} Z_{\mathsf{B}} &:= \{z \in \varrho(A) \colon (M_w^{\mathsf{B}})^{-1} \in \mathcal{B}(\mathfrak{b}_1^* \oplus \mathfrak{b}_{2,2}^*, \mathfrak{b}_1^* \oplus \mathfrak{b}_2^*),\ w = z, \bar{z}\} \\ M_z^{\mathsf{B}} &:= (1 \oplus B_0) - (B_1 \oplus B_2)\tau G_z \in \mathcal{B}(\mathfrak{b}_1^* \oplus \mathfrak{b}_2^*, \mathfrak{b}_1^* \oplus \mathfrak{b}_{2,2}^*). \end{aligned} \tag{2.6}$$

Theorem 2.1. *Suppose hypotheses* (2.1)*,* (2.2)*,* (2.3)*, and* (2.4) *hold and that* Z_{B} *defined in* (2.6) *is not empty. Then, defined* Λ_z^{B} *as in* (2.5)*,*

$$R_z^{\mathsf{B}} := R_z + G_z \Lambda_z^{\mathsf{B}} G_{\bar{z}}^*, \quad z \in Z_{\mathsf{B}}, \tag{2.7}$$

is the resolvent of a self-adjoint operator A_{B} *and* $Z_{\mathsf{B}} = \varrho(A_{\mathsf{B}}) \cap \varrho(A)$*.*

Proof. By (2.4), one gets

$$\begin{aligned} &((1 \oplus B_0) - (B_1 \oplus B_2)\tau G_{\bar{z}})(B_1 \oplus B_2^*) \\ &\quad = (B_1 \oplus B_2)((1 \oplus B_0^*) - \tau G_{\bar{z}}(B_1 \oplus B_2^*)) \\ &\quad = (B_1 \oplus B_2)((1 \oplus B_0) - (B_1 \oplus B_2)\tau G_z)^*. \end{aligned}$$

This entails, by the definitions (2.5) and (2.6),

$$(\Lambda_z^{\mathsf{B}})^* = \Lambda_{\bar{z}}^{\mathsf{B}}. \tag{2.8}$$

By the resolvent identity, there follows

$$\begin{aligned} &((1 \oplus B_0) - (B_1 \oplus B_2)\tau G_z) - ((1 \oplus B_0) - (B_1 \oplus B_2)\tau G_w) \\ &\quad = (B_1 \oplus B_2)\tau(G_w - G_z) = (z - w)(B_1 \oplus B_2)\tau R_w G_z \\ &\quad = (z - w)(B_1 \oplus B_2)G_{\bar{w}}^* G_z, \end{aligned}$$

which entails

$$\begin{aligned} &((1 \oplus B_0) - (B_1 \oplus B_2)\tau G_w)^{-1} - ((1 \oplus B_0) - (B_1 \oplus B_2)\tau G_z)^{-1} \\ &\quad = (z - w)((1 \oplus B_0) - (B_1 \oplus B_2)\tau G_w)^{-1}(B_1 \oplus B_2)G_{\bar{w}}^* G_z \\ &\qquad \times ((1 \oplus B_0) - (B_1 \oplus B_2)\tau G_z)^{-1}, \end{aligned}$$

and hence

$$\Lambda_w^{\mathsf{B}} - \Lambda_z^{\mathsf{B}} = (z - w)\Lambda_w^{\mathsf{B}} G_{\bar{w}}^* G_z \Lambda_z^{\mathsf{B}}. \tag{2.9}$$

By (2.8) and (2.9),

$$(R_z^{\mathsf{B}})^* = R_{\bar{z}}^{\mathsf{B}}, \quad R_z^{\mathsf{B}} = R_w^{\mathsf{B}} + (w - z)R_z^{\mathsf{B}} R_w^{\mathsf{B}}$$

(see [16, p. 113]). Hence, R_z^{B} is the resolvent of a self-adjoint operator whenever it is injective (see, e.g., [20, Theorems 4.10 and 4.19]). By (2.7),

$$\begin{aligned}(B_1 \oplus B_2)\tau R_z^{\mathsf{B}} &= (B_1 \oplus B_2)(1 + \tau G_z \Lambda_z^{\mathsf{B}})G_{\bar z}^* \\ &= ((B_1 \oplus B_2) + (B_1 \oplus B_2)\tau G_z \Lambda_z^{\mathsf{B}})G_{\bar z}^* \\ &= ((B_1 \oplus B_2) + ((1 \oplus B_0) - ((1 \oplus B_0) - (B_1 \oplus B_2)\tau G_z))\Lambda_z^{\mathsf{B}})G_{\bar z}^* \\ &= (1 \oplus B_0)\Lambda_z^{\mathsf{B}} G_{\bar z}^*.\end{aligned}$$

Thus, if $R_z^{\mathsf{B}} u = 0$ then

$$0 \oplus 0 = (1 \oplus B_0)\Lambda_z^{\mathsf{B}} G_{\bar z}^* u = (\Lambda_z^{\mathsf{B}} G_{\bar z}^* u)_1 \oplus B_0(\Lambda_z^{\mathsf{B}} G_{\bar z}^* u)_2$$

By

$$G_z(\phi_1 \oplus \phi_2) = G_z^1\phi_1 + G_z^2\phi_2, \quad G_z^k := (\tau_k R_{\bar z})^*,$$

there follows

$$0 = R_z^{\mathsf{B}} u = R_z u + G_z^1(\Lambda_z^{\mathsf{B}} G_{\bar z}^* u)_1 + G_z^2(\Lambda_z^{\mathsf{B}} G_{\bar z}^* u)_2 = R_z u + G_z^2(\Lambda_z^{\mathsf{B}} G_{\bar z}^* u)_2. \quad (2.10)$$

Since the denseness of $\ker(\tau_2)$ implies

$$\operatorname{ran}(G_z^2) \cap \operatorname{dom}(A) = \{0\}$$

(see [16, Remark 2.9]), the relation (2.10) gives $G_z^2(\Lambda_z^{\mathsf{B}} G_{\bar z}^* u)_2 = 0$. Thus, $R_z^{\mathsf{B}} u = 0$ compels $R_z u = 0$ and hence $u = 0$.

Finally, the equality $Z_{\mathsf{B}} = \varrho(A_{\mathsf{B}}) \cap \varrho(A)$ is consequence of [7, Theorem 2.19 and Remark 2.20]. ∎

Remark 2.2. Looking at the previous proof, one notices that Theorem 2.1 holds without requiring the denseness of $\operatorname{ran}(\tau_2)$; that hypothesis comes into play in later results.

Remark 2.3. By (2.7), if $u \in \operatorname{dom}(A_{\mathsf{B}})$, then $u = u_0 + G_z(\phi_1 \oplus \phi_2)$ for some $u_0 \in \mathsf{H}_A$ and $\phi_1 \oplus \phi_2 \in \mathfrak{b}_1^* \oplus \mathfrak{b}_2^*$; hence, by (2.2),

$$\tau \colon \operatorname{dom}(A_{\mathsf{B}}) \to \mathfrak{b}_1 \oplus \mathfrak{b}_2.$$

2.2. An additive representation

At first, let us introduce the Hilbert space H_A^* defined as the completion of H endowed with the scalar product

$$\langle u, v\rangle_{\mathsf{H}_A^*} := \langle (A^2+1)^{-1/2} u, (A^2+1)^{-1/2} v\rangle_{\mathsf{H}}.$$

Notice that R_z extends to a bounded bijective map (which we denote by the same symbol) on H_A^* onto H. The linear operator A, being a densely defined bounded operator on H to H_A^*, extends to a bounded operator $\bar{A}: \mathsf{H} \to \mathsf{H}_A^*$ given by its closure. Moreover, denoting by $\langle\cdot,\cdot\rangle_{\mathsf{H}_A^*,\mathsf{H}_A}$ the pairing obtained by extending the scalar product in H, since A is self-adjoint and since $\mathrm{dom}(A)$ is dense in H,

$$\langle u, Av\rangle_{\mathsf{H}} = \langle \bar{A}u, v\rangle_{\mathsf{H}_A^*,\mathsf{H}_A}, \quad u \in \mathsf{H},\ v \in \mathsf{H}_A.$$

Further, we define $\tau^*: \mathfrak{h}_1^* \oplus \mathfrak{h}_2^* \to \mathsf{H}_A^*$ by

$$\langle \tau^*\phi, u\rangle_{\mathsf{H}_A^*,\mathsf{H}_A} = \langle \phi, \tau u\rangle_{\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*,\mathfrak{h}_1\oplus\mathfrak{h}_2}, \quad u \in \mathsf{H}_A,\ \phi \in h_1^* \oplus \mathfrak{h}_2^*.$$

Obviously, $\tau^*(\phi_1 \oplus \phi_2) = \tau_1^*\phi_1 + \tau_2^*\phi_2$, where $\tau_k^*: \mathfrak{h}_k \to \mathsf{H}_A^*$, $k = 1, 2$, are defined in the same way as τ^*.

Let us notice that $R_z: \mathsf{H}_A^* \to \mathsf{H}$ is the adjoint, with respect the pairing $\langle\cdot,\cdot\rangle_{\mathsf{H}_A^*,\mathsf{H}_A}$, of $R_{\bar{z}}: \mathsf{H}_A \to \mathsf{H}$ and it is the inverse of $(-\bar{A} + z): \mathsf{H} \to \mathsf{H}_A^*$; therefore

$$G_z = R_z\tau^*. \tag{2.11}$$

Lemma 2.4. *Let $A_{\mathsf{B}}: \mathrm{dom}(A_{\mathsf{B}}) \subseteq \mathsf{H} \to \mathsf{H}$ be the self-adjoint operator provided in Theorem 2.1 and define, for any $u \in \mathsf{H}$ and $z \in \varrho(A_{\mathsf{B}}) \cap \varrho(A)$,*

$$\rho_{\mathsf{B}}: \mathrm{dom}(A_{\mathsf{B}}) \to \mathfrak{h}_1^* \oplus \mathfrak{h}_2^*, \quad \rho_{\mathsf{B}}(R_z^{\mathsf{B}}u) := (\pi_1^* \oplus 1)\Lambda_z^{\mathsf{B}}G_{\bar{z}}^*u, \tag{2.12}$$

where π_1 denotes the orthogonal projection onto the subspace $\overline{\mathrm{ran}(\tau_1)}$. Then, the definition of ρ_{B} is well posed, i.e.,

$$R_{z_1}^{\mathsf{B}}u_1 = R_{z_2}^{\mathsf{B}}u_2 \implies (\pi_1^* \oplus 1)\Lambda_{z_1}^{\mathsf{B}}G_{\bar{z}_1}^*u_1 = (\pi_1^* \oplus 1)\Lambda_{z_2}^{\mathsf{B}}G_{\bar{z}_2}^*u_2$$

and

$$\langle u, A_{\mathsf{B}}v\rangle_{\mathsf{H}} = \langle Au, v\rangle_{\mathsf{H}} + \langle \tau u, \rho_{\mathsf{B}}v\rangle_{\mathfrak{h}_1\oplus\mathfrak{h}_2,\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*}, \quad u \in \mathrm{dom}(A),\ v \in \mathrm{dom}(A_{\mathsf{B}}). \tag{2.13}$$

Proof. Let $v = R_z^{\mathsf{B}}u = v_z + G_z\Lambda_z^{\mathsf{B}}\tau v_z$, where $v_z := R_z u$ (hence $\tau v_z = G_{\bar{z}}^*u$). Then

$$\begin{aligned}
&\langle u, A_{\mathsf{B}}v\rangle_{\mathsf{H}} - \langle Au, v\rangle_{\mathsf{H}}\\
&\quad= -\langle u, (-A_{\mathsf{B}} + z)v\rangle_{\mathsf{H}} + \langle(-A + \bar{z})u, v\rangle_{\mathsf{H}}\\
&\quad= -\langle u, (-A + z)v_z\rangle_{\mathsf{H}} + \langle(-A + \bar{z})u, v_z + G_z\Lambda_z^{\mathsf{B}}\tau v_z\rangle_{\mathsf{H}}\\
&\quad= \langle(-A + \bar{z})u, G_z\Lambda_z^{\mathsf{B}}\tau v_z\rangle_{\mathsf{H}} = \langle \tau u, \Lambda_z^{\mathsf{B}}\tau v_z\rangle_{\mathfrak{h}_1\oplus\mathfrak{h}_2,\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*}\\
&\quad= \langle(\pi_1 \oplus 1)\tau u, \Lambda_z^{\mathsf{B}}\tau v_z\rangle_{\mathfrak{h}_1\oplus\mathfrak{h}_2,\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*} = \langle \tau u, (\pi_1^* \oplus 1)\Lambda_z^{\mathsf{B}}\tau v_z\rangle_{\mathfrak{h}_1\oplus\mathfrak{h}_2,\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*}.
\end{aligned}$$

Suppose now that $R_{z_1}^{\mathsf{B}}u_1 = R_{z_2}^{\mathsf{B}}u_2$. Then, by the above identities, one gets, for any $u \in \mathrm{dom}(A)$,

$$\langle \tau^*(\pi_1^* \oplus 1)(\Lambda_{z_1}^{\mathsf{B}}G_{\bar{z}_1}^*u_1 - \Lambda_{z_2}^{\mathsf{B}}G_{\bar{z}_2}^*u_2), u\rangle_{\mathsf{H}_A^*,\mathsf{H}_A} = 0.$$

Hence,

$$\tau^*((\pi_1^*\oplus 1)\Lambda^{\mathsf{B}}_{z_1}G^*_{\bar z_1}u_1-(\pi_1^*\oplus 1)\Lambda^{\mathsf{B}}_{z_2}G^*_{\bar z_2}u_2)=0.$$

However, $\ker(\tau^*)\cap\operatorname{ran}((\pi_1^*\oplus 1))=\{0\}$ since $\pi_1^*\oplus 1$ is the projector onto the subspace orthogonal to $\ker(\tau^*)$. ■

The next lemma provides a sort of abstract boundary conditions holding for the elements in $\operatorname{dom}(A_{\mathsf{B}})$.

Lemma 2.5. *Let A_{B} be the self-adjoint operator in Theorem 2.1. Then, for any $z\in\varrho(A_{\mathsf{B}})\cap\varrho(A)$, one has the representation*

$$\operatorname{dom}(A_{\mathsf{B}})=\{u\in\mathsf{H}\colon u_z:=u-G_z\rho_{\mathsf{B}}u\in\operatorname{dom}(A)\},$$

$$(-A_{\mathsf{B}}+z)u=(-A+z)u_z.$$

Moreover,

$$u\in\operatorname{dom}(A_{\mathsf{B}})\implies(\pi_1^*B_1\oplus B_2)\tau u=(1\oplus B_0)\rho_{\mathsf{B}}u.$$

Proof. Since $G_z=R_z\tau^*$ (see (2.11) below) and $\pi_1^*\oplus 1$ is the projection onto the orthogonal to $\ker(\tau^*)$, one has $G_z=G_z(\pi_1^*\oplus 1)$. Hence, $u\in\operatorname{dom}(A_{\mathsf{B}})$ if and only if $u=R_zv+G_z(\pi_1^*\oplus 1)\Lambda^{\mathsf{B}}_zG^*_{\bar z}v=R_zv+G_z\rho_{\mathsf{B}}u$. Therefore,

$$\operatorname{dom}(A_{\mathsf{B}})=\{u\in\mathsf{H}\colon u=u_z+G_z\rho_{\mathsf{B}}u,\ \ u_z\in\operatorname{dom}(A)\}.$$

Moreover, given any $u\in\operatorname{dom}(A)$, $u=R^{\mathsf{B}}_zv$, one has

$$(-A+z)u_z=(-A+z)R_zv=(-A_{\mathsf{B}}+z)R^{\mathsf{B}}_zv=(-A_{\mathsf{B}}+z)u.$$

Finally, given $u=R^{\mathsf{B}}_zv\in\operatorname{dom}(A_{\mathsf{B}})$, one has

$$\begin{aligned}(\pi_1^*B_1\oplus B_2)\tau u&=(\pi_1^*\oplus 1)(B_1\oplus B_2)\tau R^{\mathsf{B}}_zv\\&=(\pi_1^*\oplus 1)\big((B_1\oplus B_2)G_{\bar z}v+(B_1\oplus B_2)\tau G_z((1\oplus B_0)\\&\qquad-(B_1\oplus B_2)\tau G_z)^{-1}(B_1\oplus B_2)G_{\bar z}v\big)\\&=(\pi_1^*\oplus 1)(1\oplus B_0)\Lambda^{\mathsf{B}}_zG_{\bar z}v\\&=(1\oplus B_0)(\pi_1^*\oplus 1)\Lambda^{\mathsf{B}}_zG_{\bar z}v\\&=(1\oplus B_0)\rho_{\mathsf{B}}u.\end{aligned}$$

■

Now, we provide an additive representation of the self-adjoint A_{B} in Theorem 2.1.

Theorem 2.6. *Let $A_{\mathsf{B}}\colon\operatorname{dom}(A_{\mathsf{B}})\subseteq\mathsf{H}\to\mathsf{H}$ be the self-adjoint operator appearing in Theorem 2.1. Then*

$$A_{\mathsf{B}}=\bar A+\tau^*\rho_{\mathsf{B}},$$

where ρ_{B} is defined in (2.12). In particular, if $B_0^{-1}\in\mathcal{B}(\mathfrak{b}^*_{2,2},\mathfrak{b}^*_2)$, then

$$A_{\mathsf{B}}=\bar A+\tau_1^*B_1\tau_1+\tau_2^*B_0^{-1}B_2\tau_2.$$

Proof. By (2.13), for any $u \in \mathrm{dom}(A_{\mathsf{B}})$ and $v \in \mathsf{H}_A$,

$$\begin{aligned}\langle A_{\mathsf{B}}u, v\rangle_{\mathsf{H}_A^*,\mathsf{H}_A} \equiv \langle A_{\mathsf{B}}u, v\rangle_{\mathsf{H}} &= \langle u, Av\rangle_{\mathsf{H}} + \langle \rho_{\mathsf{B}}u, \tau v\rangle_{\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*,\mathfrak{h}_1\oplus\mathfrak{h}_2}\\ &= \langle \bar{A}u + \tau^*\rho_{\mathsf{B}}u, v\rangle_{\mathsf{H}_A^*,\mathsf{H}_A}.\end{aligned}$$

By Lemma 2.5 and by $\tau_1^*\pi_1^* = (\pi_1\tau_1)^* = \tau_1^*$,

$$\tau^*\rho_{\mathsf{B}} = \tau^*(\pi_1^* B_1\tau_1 \oplus B_0^{-1}B_1\tau_2) = \tau_1^* B_1\tau_1 + \tau_2^* B_0^{-1}B_2\tau_2. \qquad \blacksquare$$

2.3. An alternative resolvent formula

At first, let us notice that hypothesis (2.2), can be re-written as

$$\tau_j G_z^k|\mathfrak{b}_k \in \mathcal{B}(\mathfrak{b}_k^*, \mathfrak{b}_j), \quad j,k = 1,2, \quad G_z^k := (\tau_k R_{\bar{z}})^*.$$

Moreover,

$$M_z^{\mathsf{B}} = (1\oplus B_0) + (B_1\oplus B_2)\tau G_z = \begin{bmatrix} M_z^{B_1} & B_1\tau_1 G_z^2\\ B_2\tau_2 G_z^1 & M_z^{B_0,B_2}\end{bmatrix}$$

where

$$M_z^{B_1} := 1 - B_1\tau_1 G_z^1, \quad M_z^{B_0,B_2} := B_0 - B_2\tau_2 G_z^2.$$

Then, supposing all the inverse operators appearing in the next formula exist, by the inversion formula for block operator matrices, one gets

$$(M_z^{\mathsf{B}})^{-1} = \begin{bmatrix} (M_z^{B_1})^{-1} + (M_z^{B_1})^{-1}B_1\tau_1 G_z^2(C_z^{\mathsf{B}})^{-1}B_2\tau_2 G_z^1(M_z^{B_1})^{-1} & (M_z^{B_1})^{-1}B_1\tau_1 G_z^2(C_z^{\mathsf{B}})^{-1}\\ (C_z^{\mathsf{B}})^{-1}B_2\tau_2 G_z^1(M_z^{B_1})^{-1} & (C_z^{\mathsf{B}})^{-1}\end{bmatrix}, \tag{2.14}$$

where C_z^{B} denotes the second Schur complement, i.e.,

$$\begin{aligned} C_z^{\mathsf{B}} :&= M_z^{B_0,B_2} - B_2\tau_2 G_z^1(M_z^{B_1})^{-1}B_1\tau_1 G_z^2\\ &= M_z^{B_0,B_2}(1 - (M_z^{B_0,B_2})^{-1}B_2\tau_2 G_z^1(M_z^{B_1})^{-1}B_1\tau_1 G_z^2)\\ &= M_z^{B_0,B_2}(1 - \Lambda_z^{B_0,B_2}\tau_2 G_z^1\Lambda_z^{B_1}\tau_1 G_z^2),\end{aligned}$$

$$\Lambda_z^{B_1} := (1 - B_1\tau_1 G_z^1)^{-1}B_1, \tag{2.15}$$

$$\Lambda_z^{B_0,B_2} := (B_0 - B_2\tau_2 G_z^2)^{-1}B_2. \tag{2.16}$$

Regarding the well-posedness of (2.14), taking into account the definition of C_z^{B}, one has

$$Z_{\mathsf{B}} = \{z \in \varrho(A) : (M_z^{\mathsf{B}})^{-1} \in \mathcal{B}(\mathfrak{b}_1^* \oplus \mathfrak{b}_{2,2}^*, \mathfrak{b}_1^* \oplus \mathfrak{b}_2^*),\ w = z, \bar{z}\} \supseteq \widehat{Z}_{\mathsf{B}},$$

where

$$\widehat{Z}_{\mathsf{B}} := \{z \in Z_{B_1} \cap Z_{B_0,B_2}\colon \\ (1 - \Lambda_w^{B_0,B_2}\tau_2 G_w^1 \Lambda_w^{B_1}\tau_1 G_w^2)^{-1} \in \mathcal{B}(\mathfrak{b}_2^*),\ w = z, \bar{z}\}, \tag{2.17}$$

$$Z_{B_1} := \{z \in \varrho(A)\colon (1 - B_1\tau_1 G_w^1)^{-1} \in \mathcal{B}(\mathfrak{b}_1^*),\ w = z, \bar{z}\}, \tag{2.18}$$

$$Z_{B_0,B_2} := \{z \in \varrho(A)\colon (B_0 - B_2\tau_2 G_w^2)^{-1} \in \mathcal{B}(\mathfrak{b}_{2,2}^*, \mathfrak{b}_2^*),\ w = z, \bar{z}\}. \tag{2.19}$$

Therefore, supposing that $\widehat{Z}_{\mathsf{B}}$ is not empty, for any $z \in \widehat{Z}_{\mathsf{B}}$, by (2.7) and by

$$(C_z^{\mathsf{B}})^{-1} B_2 = \Sigma_z^{\mathsf{B}} \Lambda_z^{B_0,B_2}, \quad \Sigma_z^{\mathsf{B}} := (1 - \Lambda_z^{B_0,B_2}\tau_2 G_z^1 \Lambda_z^{B_1}\tau_1 G_z^2)^{-1},$$

one has

$$\Lambda_z^{\mathsf{B}} = (M_z^{\mathsf{B}})^{-1}\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} \Lambda_z^{B_1} + \Lambda_z^{B_1}\tau_1 G_z^2 \Sigma_z^{\mathsf{B}} \Lambda_z^{B_0,B_2}\tau_2 G_z^1 \Lambda_z^{B_1} & \Lambda_z^{B_1}\tau_1 G_z^2 \Sigma_z^{\mathsf{B}} \Lambda_z^{B_0,B_2} \\ \Sigma_z^{\mathsf{B}} \Lambda_z^{B_0,B_2}\tau_2 G_z^1 \Lambda_z^{B_1} & \Sigma_z^{\mathsf{B}} \Lambda_z^{B_0,B_2} \end{bmatrix}.$$

Therefore,

$$R_z^{\mathsf{B}} = R_z + [\,G_z^1\ G_z^2\,]\begin{bmatrix} \Lambda_z^{B_1} + \Lambda_z^{B_1}\tau_1 G_z^2 \Sigma_z^{\mathsf{B}} \Lambda_z^{B_0,B_2}\tau_2 G_z^1 \Lambda_z^{B_1} & \Lambda_z^{B_1}\tau_1 G_z^2 \Sigma_z^{\mathsf{B}} \Lambda_z^{B_0,B_2} \\ \Sigma_z^{\mathsf{B}} \Lambda_z^{B_0,B_2}\tau_2 G_z^1 \Lambda_z^{B_1} & \Sigma_z^{\mathsf{B}} \Lambda_z^{B_0,B_2} \end{bmatrix}\begin{bmatrix} G_{\bar{z}}^{1*} \\ G_{\bar{z}}^{2*} \end{bmatrix}. \tag{2.20}$$

In particular, taking $\mathsf{B} = (1, B_1, 0)$, one gets, for any $z \in Z_{B_1}$,

$$R_z^{B_1} := R_z^{(1,B_1,0)} = R_z + [\,G_z^1\ G_z^2\,]\begin{bmatrix} \Lambda_z^{B_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_{\bar{z}}^{1*} \\ G_{\bar{z}}^{2*} \end{bmatrix} = R_z + G_z^1 \Lambda_z^{B_1} G_{\bar{z}}^{1*} \tag{2.21}$$

while, taking $\mathsf{B} = (B_0, 0, B_2)$, one gets, for any $z \in Z_{B_0,B_2}$,

$$R_z^{B_0,B_2} := R_z^{(B_0,0,B_2)} = R_z + [\,G_z^1\ G_z^2\,]\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_z^{B_0,B_2} \end{bmatrix}\begin{bmatrix} G_{\bar{z}}^{1*} \\ G_{\bar{z}}^{2*} \end{bmatrix} \\ = R_z + G_z^2 \Lambda_z^{B_0,B_2} G_{\bar{z}}^{2*}.$$

Therefore, by Theorem 2.1 with $\mathsf{B} = (1, B_1, 0)$, one gets the following result.

Corollary 2.7. *Let $\tau_1 \in \mathcal{B}(\mathsf{H}_A, \mathfrak{h}_1)$ be such that $\tau_1 G_z^1|\mathfrak{b}_1^* \in \mathcal{B}(\mathfrak{b}_1^*, \mathfrak{b}_1)$ and let $B_1 \in \mathcal{B}(\mathfrak{b}_1, \mathfrak{b}_1^*)$ be self-adjoint; suppose that Z_{B_1} defined in (2.18) is not empty. Then*

$$R_z^{B_1} = R_z + G_z^1 \Lambda_z^{B_1} G_{\bar{z}}^{1*}, \quad z \in Z_{B_1}, \tag{2.22}$$

where $\Lambda_z^{B_1}$ is defined in (2.15), is the resolvent of a self-adjoint operator A_{B_1} and $Z_{B_1} = \varrho(A_{B_1}) \cap \varrho(A)$.

By Theorem 2.1 with $\mathsf{B} = (B_0, 0, B_2)$, one gets the following result.

Corollary 2.8. *Let* $\tau_2 \in \mathcal{B}(\mathsf{H}_A, \mathfrak{h}_2)$ *satisfy* (2.1) *be such that* $\tau_1 G^1_z | \mathfrak{b}^*_2 \in \mathcal{B}(\mathfrak{b}^*_2, \mathfrak{b}_2)$ *and let* $B_0 \in \mathcal{B}(\mathfrak{b}^*_2, \mathfrak{b}^*_{2,2})$*,* $B_2 \in \mathcal{B}(\mathfrak{b}_2, \mathfrak{b}^*_{2,2})$ *be such that* $B_0 B^*_2 = B_2 B^*_0$*; suppose that* Z_{B_0,B_2} *defined in* (2.19) *is not empty. Then*

$$R^{B_0,B_2}_z = R_z + G^2_z \Lambda^{B_0,B_2}_z G^{2*}_{\bar z}, \quad z \in Z_{B_0,B_2}, \tag{2.23}$$

where Λ^{B_0,B_2}_z *is defined in* (2.16)*, is the resolvent of a self-adjoint operator* A_{B_0,B_2} *and* $Z_{B_0,B_2} = \varrho(A_{B_0,B_2}) \cap \varrho(A)$*.*

Supposing $\widehat{Z}_{\mathsf{B}} \neq \varnothing$, by (2.20), by (2.21) and by the relations

$$\begin{aligned} G^{B_1}_z &:= (\tau_2 R^{B_1}_{\bar z})^* = (\tau_2 R_{\bar z} + \tau_2 G^1_{\bar z} \Lambda^{B_1}_{\bar z} G^{1*}_z)^* \\ &= G^2_z + G^1_z \Lambda^{B_1}_z \tau_1 G^2_z \end{aligned} \tag{2.24}$$

$$\begin{aligned} G^{B_1 *}_{\bar z} = \tau_2 R^{B_1}_z &= \tau_2 R_z + \tau_2 G^1_z \Lambda^{B_1}_z G^{1*}_{\bar z} \\ &= G^{2*}_{\bar z} + \tau_2 G^1_z \Lambda^{B_1}_z G^{1*}_{\bar z} \end{aligned} \tag{2.25}$$

$$\begin{aligned} \widehat{M}^{\mathsf{B}}_z = B_0 - B_2 \tau_2 G^{B_1}_z &= B_0 - B_2 \tau_2 G^2_z + \tau_2 G^1_z \Lambda^{B_1}_z \tau_1 G^2_z \\ &= M^{B_0,B_2}_z + B_2 \tau_2 G^1_z \Lambda^{B_1}_z \tau_1 G^2_z \\ &= M^{B_0,B_2}_z (1 + \Lambda^{B_0,B_1}_z \tau_2 G^1_z \Lambda^{B_1}_z \tau_1 G^2_z) \end{aligned}$$

$$\widehat{\Lambda}^{\mathsf{B}}_z := (\widehat{M}^{\mathsf{B}}_z)^{-1} B_2 = (B_0 - B_2 \tau_2 G^{B_1}_z)^{-1} B_2 = \Sigma^{\mathsf{B}}_z \Lambda^{B_0,B_2}_z$$

one gets

$$\Lambda^{\mathsf{B}}_z = \begin{bmatrix} \Lambda^{B_1}_z + \Lambda^{B_1}_z \tau_1 G^2_z \widehat{\Lambda}^{\mathsf{B}}_z \tau_2 G^1_z \Lambda^{B_1}_z & \Lambda^{B_1}_z \tau_1 G^2_z \widehat{\Lambda}^{\mathsf{B}}_z \\ \widehat{\Lambda}^{\mathsf{B}}_z \tau_2 G^1_z \Lambda^{B_1}_z & \widehat{\Lambda}^{\mathsf{B}}_z; \end{bmatrix} \tag{2.26}$$

$$= \left(1 + \begin{bmatrix} \Lambda^{B_1}_z & 0 \\ 0 & \widehat{\Lambda}^{\mathsf{B}}_z \end{bmatrix} \begin{bmatrix} \tau_1 G^2_z \widehat{\Lambda}^{\mathsf{B}}_z \tau_2 G^1_z & \tau_1 G^2_z \\ \tau_2 G^1_z & 0 \end{bmatrix} \right) \begin{bmatrix} \Lambda^{B_1}_z & 0 \\ 0 & \widehat{\Lambda}^{\mathsf{B}}_z \end{bmatrix}. \tag{2.27}$$

Therefore,

$$\begin{aligned} R^{\mathsf{B}}_z &= R_z + [\, G^1_z \;\; G^2_z \,] \begin{bmatrix} \Lambda^{B_1}_z + \Lambda^{B_1}_z \tau_1 G^2_z \widehat{\Lambda}^{\mathsf{B}}_z \tau_2 G^1_z \Lambda^{B_1}_z & \Lambda^{B_1}_z \tau_1 G^2_z \widehat{\Lambda}^{\mathsf{B}}_z \\ \widehat{\Lambda}^{\mathsf{B}}_z \tau_2 G^1_z \Lambda^{B_1}_z & \widehat{\Lambda}^{\mathsf{B}}_z \end{bmatrix} \begin{bmatrix} G^{1*}_{\bar z} \\ G^{2*}_{\bar z} \end{bmatrix} \\ &= R_z + [\, G^1_z \;\; G^2_z \,] \begin{bmatrix} \Lambda^{B_1}_z G^{1*}_{\bar z} + \Lambda^{B_1}_z \tau_1 G^2_z \widehat{\Lambda}^{\mathsf{B}}_z \tau_2 G^1_z \Lambda^{B_1}_z G^{1*}_{\bar z} + \Lambda^{B_1}_z \tau_1 G^2_z \widehat{\Lambda}^{\mathsf{B}}_z G^{2*}_{\bar z} \\ \widehat{\Lambda}^{\mathsf{B}}_z \tau_2 G^1_z \Lambda^{B_1}_z G^{1*}_{\bar z} + \widehat{\Lambda}^{\mathsf{B}}_z G^{2*}_{\bar z} \end{bmatrix} \\ &= R_z + G^1_z \Lambda^{B_1}_z G^{1*}_{\bar z} + G^1_z \Lambda^{B_1}_z \tau_1 G^2_z \widehat{\Lambda}^{\mathsf{B}}_z \tau_2 G^1_z \Lambda^{B_1}_z G^{1*}_{\bar z} + G^1_z \Lambda^{B_1}_z \tau_1 G^2_z \widehat{\Lambda}^{\mathsf{B}}_z G^{2*}_{\bar z} \\ &\quad + G^2_z \widehat{\Lambda}^{\mathsf{B}}_z \tau_2 G^1_z \Lambda^{B_1}_z G^{1*}_{\bar z} + G^2_z \widehat{\Lambda}^{\mathsf{B}}_z G^{2*}_{\bar z} \\ &= R^{B_1}_z + G^{B_1}_z \widehat{\Lambda}^{\mathsf{B}}_z G^{B_1 *}_{\bar z}. \end{aligned} \tag{2.28}$$

This also entails, by [7, Theorem 2.19 and Remark 2.20], that if $\widehat{Z}_{\mathsf{B}} \neq \varnothing$, then $\widehat{Z}_{\mathsf{B}} = Z_{\mathsf{B}} = \varrho(A_{\mathsf{B}}) \cap \varrho(A_{\mathsf{B}_1})$. Summing up, one has the following result.

Theorem 2.9. *Assume that hypotheses* (2.2), (2.3), *and* (2.4) *hold and that* $\widehat{Z}_{\mathsf{B}}$ *defined in* (2.17) *is not empty. Then, for any* $z \in \varrho(A_{\mathsf{B}}) \cap \varrho(A_{B_1})$, *the resolvent* R_z^{B} *in* (2.7) *has the representation* (2.28) *and*

$$R_z^{\mathsf{B}} = R_z^{B_1} + G_z^{B_1} \widehat{\Lambda}_z^{\mathsf{B}} G_{\bar{z}}^{B_1 *}, \quad z \in \varrho(A_{\mathsf{B}}) \cap \varrho(A_{B_1}), \tag{2.29}$$

where $R_z^{B_1}$, $G_z^{B_1}$ *and* $\widehat{\Lambda}_z^{\mathsf{B}}$ *are defined in* (2.22), (2.24), *and* (2.15).

Remark 2.10. Let us notice that the resolvent formula (2.29) is of the same kind of the one in (2.23), whenever one replaces A with A_{B_1}.

Let us now introduce the map

$$\hat{\rho}_{\mathsf{B}}\colon \operatorname{dom}(A_{\mathsf{B}}) \to \mathfrak{h}_2^*, \quad \hat{\rho}_{\mathsf{B}}(R_z^{\mathsf{B}} u) := \widehat{\Lambda}_z^{\mathsf{B}} G_{\bar{z}}^{B_1 *} u.$$

By the definition of ρ_{B} in (2.12) and by (2.25), (2.26), one obtains the relation

$$\rho_{\mathsf{B}} u = \pi_1^* B_1 \tau_1 u \oplus \hat{\rho}_{\mathsf{B}} u.$$

Then, by using the same kind of arguments as in the proofs of Lemma 2.5 and Theorem 2.6, one gets the following.

Theorem 2.11. *Let* A_{B} *be the self-adjoint operator in Theorem* 2.9. *Then, for any* $z \in \varrho(A_{\mathsf{B}}) \cap \varrho(A_{B_1})$, *one has the representation*

$$\operatorname{dom}(A_{\mathsf{B}}) = \{u \in \mathsf{H} \colon u_z := u - G_z^{B_1} \hat{\rho}_{\mathsf{B}} u \in \operatorname{dom}(A_{B_1})\},$$

$$(-A_{\mathsf{B}} + z)u = (-A_{B_1} + z)u_z.$$

Moreover,

$$A_{\mathsf{B}} = \bar{A} + \tau_1^* B_1 \tau_1 + \tau_2^* \hat{\rho}_{\mathsf{B}},$$

and

$$u \in \operatorname{dom}(A_{\mathsf{B}}) \implies B_2 \tau_2 u = B_0 \hat{\rho}_{\mathsf{B}} u.$$

3. The limiting absorption principle and the scattering matrix

Now, given the measure space $(M, \mathcal{B}, m)$, we suppose that $\mathsf{H} = L^2(M, \mathcal{B}, m) \equiv L^2(M)$. Given a measurable $\varphi\colon M \to [1, +\infty)$, we define the weighted L^2-space

$$L_\varphi^2(M, \mathcal{B}, m) \equiv L_\varphi^2(M) := \{u\colon M \to \mathbb{C} \text{ measurable}\colon \varphi u \in L^2(M)\}.$$

By $\varphi \geq 1$,

$$L_\varphi^2(M) \hookrightarrow L^2(M) \hookrightarrow L_{\varphi^{-1}}^2(M) \simeq L_\varphi^2(M)^*.$$

From now on, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the scalar product and the corresponding norm on $L^2(M)$; $\langle\cdot,\cdot\rangle_\varphi$ and $\|\cdot\|_\varphi$ denote the scalar product and the corresponding norm on $L^2_\varphi(M)$.

Then we introduce the following hypotheses.

(H.1) A_{B_1} is bounded from above and there exists a positive $\lambda_1 \geq \sup\sigma(A_{B_1})$, such that $R^{B_1}_z \in \mathcal{B}(L^2_\varphi(M))$ for any $z \in \varrho(A_{B_1})$ such that $\mathrm{Re}(z) > \lambda_1$.

(H.2) A_{B_1} satisfies a Limiting Absorption Principle (LAP for short), i.e., there exists a (eventually empty) closed set with zero Lebesgue measure $e(A_{B_1}) \subset \mathbb{R}$ such that, for all $\lambda \in \mathbb{R}\backslash e(A_{B_1})$, the limits

$$R^{B_1,\pm}_\lambda := \lim_{\epsilon\searrow 0} R^{B_1}_{\lambda\pm i\epsilon}$$

exist in $\mathcal{B}(L^2_\varphi(M), L^2_{\varphi^{-1}}(M))$ and the maps

$$z \mapsto R^{B_1,\pm}_z,$$

where $R^{B_1,\pm}_z \equiv R^{B_1}_z$ whenever $z \in \varrho(A_{B_1})$, are continuous on $(\mathbb{R}\backslash e(A_{B_1})) \cup \mathbb{C}_\pm$ to $\mathcal{B}(L^2_\varphi(M), L^2_{\varphi^{-1}}(M))$.

(H.3) For any compact set $K \subset \mathbb{R}\backslash e(A_{B_1})$ there exists $c_K > 0$ such that for any $\lambda \in K$ and for any $u \in L^2_{\varphi^2}(M) \cap \ker(R^{B_1,+}_\lambda - R^{B_1,-}_\lambda)$ one has

$$\|R^{B_1,\pm}_\lambda u\| \leq c_K \|u\|_{\varphi^2}.$$

We split next hypothesis (H4) in two separate points.

(H4.1) A_{B} is bounded from above.

(H4.2) The embedding $\mathfrak{h}_2 \hookrightarrow \mathfrak{b}_2$ is compact and there exists a positive number $\lambda_2 > \sup\sigma(A_{B_1})$, such that $G^{B_1}_z \in \mathcal{B}(\mathfrak{h}^*_2, L^2_{\varphi^{2+\eta}}(M))$ for some $\eta > 0$ and for any $z \in \varrho(A_{B_1})$ such that $\mathrm{Re}(z) > \lambda_2$.

Then, A_{B} satisfies a Limiting Absorption Principle as well.

Theorem 3.1. *Suppose hypotheses* (H1)–(H4) *hold. Then the limits*

$$R^{\mathsf{B},\pm}_\lambda := \lim_{\epsilon\searrow 0} R^{\mathsf{B}}_{\lambda\pm i\epsilon}$$

exist in $\mathcal{B}(L^2_\varphi(M), L^2_{\varphi^{-1}}(M))$ *for all* $\lambda \in \mathbb{R}\backslash e(A_{\mathsf{B}})$, *where* $e(A_{\mathsf{B}}) := e(A_{B_1}) \cup \sigma_{\mathrm{p}}(A_{\mathsf{B}})$, *and* $e(A_{\mathsf{B}})\backslash e(A_{B_1})$ *is a (possibly empty) discrete set in* $\mathbb{R}\backslash e(A_{B_1})$*; the maps* $z \mapsto R^{\mathsf{B},\pm}_z$, *where* $R^{\mathsf{B},\pm}_z \equiv R^{\mathsf{B}}_z$ *whenever* $z \in \varrho(A_{\mathsf{B}})$, *are continuous on* $(\mathbb{R}\backslash e(A_{\mathsf{B}})) \cup \mathbb{C}_\pm$ *to* $\mathcal{B}(L^2_\varphi(M), L^2_{\varphi^{-1}}(M))$. *Moreover,*

$$\sigma_{\mathrm{ess}}(A_{\mathsf{B}}) = \sigma_{\mathrm{ess}}(A_{B_1}). \tag{3.1}$$

Proof. We use [11, Theorem 3.1] (which builds on [18]). By (H1), (2.29), and (H4.2), $R_z^{B_1}$ and R_z^{B} are in $\mathcal{B}(L^2_\varphi(M))$ and $z \mapsto R_z^{B_1}$ and $z \mapsto R_z^{\mathsf{B}}$ are continuous since pseudo-resolvents in $\mathcal{B}(L^2_\varphi(M))$; A_{B} is bounded from above by (H4.1). Therefore, hypothesis [11, (H1)] holds true. Our hypotheses (H2) and (H3) coincides with the same ones in [11]. By (H4.2), the embedding $\mathfrak{b}_2^* \hookrightarrow \mathfrak{h}_2^*$ is compact. From $\widehat{\Lambda}_z^{\mathsf{B}} \in \mathcal{B}(\mathfrak{b}_2, \mathfrak{b}_2^*)$ and (2.29), it follows that $R_z^{\mathsf{B}} - R_z^{B_1} \in \mathfrak{S}_\infty(L^2(M), L^2_{\varphi^{2+\gamma}}(M))$. Therefore, hypothesis (H4) in [11] holds and the statement is a consequence of [11, Theorem 3.1]. Finally, (3.1) is an immediate consequence of Weyl's Theorem. ■

Let us now assume the following hypothesis.

(H.5) The limits

$$G_\lambda^{B_1,\pm} := \lim_{\epsilon \searrow 0} G_{\lambda \pm i\epsilon}^{B_1}$$

exist in $\mathcal{B}(\mathfrak{h}_2^*, L^2_{\varphi^{-1}}(M))$ for any $\lambda \in \mathbb{R}\backslash e(A_{B_1})$ and the maps $z \mapsto G_z^{B_1,\pm}$, where $G_z^{B_1,\pm} \equiv G_z^{B_1}$ if $z \in \varrho(A_{B_1})$, are continuous on $(\mathbb{R}\backslash e(A_{B_1})) \cup \mathbb{C}_\pm$ to $\mathcal{B}(\mathfrak{h}_2^*, L^2_{\varphi^{-1}}(M))$; moreover, the linear operators $G_z^{B_1,\pm}$ are injective.

Then, by [11, Lemma 3.6], one gets the following.

Lemma 3.2. *Assume that* (H1)–(H5) *hold. Then, for any open and bounded I such that $\overline{I} \subset \mathbb{R}\backslash e(A_{\mathsf{B}})$, one has*

$$\sup_{(\lambda,\epsilon)\in I\times(0,1)} \|\widehat{\Lambda}_{\lambda\pm i\epsilon}^{\mathsf{B}}\|_{\mathfrak{h}_2,\mathfrak{h}_2^*} < +\infty.$$

Moreover, for any $\lambda \in \mathbb{R}\backslash e(A_{\mathsf{B}})$, the limits

$$\widehat{\Lambda}_\lambda^{\mathsf{B},\pm} := \lim_{\epsilon \searrow 0} \widehat{\Lambda}_{\lambda\pm i\epsilon}^{\mathsf{B}} \tag{3.2}$$

exist in $\mathcal{B}(\mathfrak{h}_2, \mathfrak{h}_2^*)$ and

$$R_\lambda^{\mathsf{B},\pm} = R_\lambda^{B_1,\pm} + G_\lambda^{B_1,\pm} \widehat{\Lambda}_\lambda^{\mathsf{B},\pm} (G_\lambda^{B_1,\mp})^*.$$

By the same reasoning as at the end of [11, proof of Theorem 5.1], one can improve the result regarding (3.2).

Corollary 3.3. *Suppose hypotheses* (H1)–(H5) *hold. Then the limits* (3.2) *exist in* $\mathcal{B}(\mathfrak{b}_2, \mathfrak{b}_2^*)$.

Before stating the next results, we recall the following.

Definition 3.4. Given two self-adjoint operators A_1 and A_2 in the Hilbert space H, we say that completeness holds for the scattering couple (A_1, A_2) whenever the strong limits

$$W_\pm(A_1, A_2) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itA_1} e^{-itA_2} P_2^{\mathrm{ac}},$$
$$W_\pm(A_2, A_1) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itA_2} e^{-itA_1} P_1^{\mathrm{ac}},$$

exist everywhere in H and

$$\operatorname{ran}(W_\pm(A_1, A_2)) = \mathsf{H}_1^{\mathrm{ac}}, \quad \operatorname{ran}(W_\pm(A_2, A_1)) = \mathsf{H}_2^{\mathrm{ac}},$$

$$W_\pm(A_1, A_1)^* = W_\pm(A_2, A_1),$$

where P_k^{ac} denotes the orthogonal projector onto the absolutely continuous subspace $\mathsf{H}_k^{\mathrm{ac}}$ of A_k. Furthermore, we say the asymptotic completeness holds for the scattering couple (A_1, A_2) whenever, beside completeness, one has

$$\mathsf{H}_1^{\mathrm{ac}} = (\mathsf{H}_1^{\mathrm{pp}})^\perp, \quad \mathsf{H}_2^{\mathrm{ac}} = (\mathsf{H}_2^{\mathrm{pp}})^\perp,$$

where $\mathsf{H}_k^{\mathrm{pp}}$ denotes the pure point subspace of A_k; equivalently, whenever $\sigma_{\mathrm{sc}}(A_1) = \sigma_{\mathrm{sc}}(A_2) = \varnothing$.

Our next hypothesis is the following.

(H6) completeness hold for the scattering couple (A_{B_1}, A).

Theorem 3.5. *Suppose that* (H1)–(H6) *hold. Then completeness holds for the couple* (A_{B}, A)*. Furthermore, if* $\sigma_{\mathrm{sc}}(A) = \varnothing$ *and*

(i) *the set of accumulation points of* $e(A_{B_1}) \cap \mathring{\sigma}_{\mathrm{ess}}(A_{B_1})$ *is discrete in* $\mathring{\sigma}_{\mathrm{ess}}(A_{B_1})$,

(ii) *the boundary of* $\sigma_{\mathrm{ess}}(A_{B_1})$ *is countable,*

then asymptotic completeness holds for the couple (A_{B}, A).

Proof. By (2.29) and by the same proof as in Lemma 2.4, one gets, for any $u \in \operatorname{dom}(A_{B_1})$, $v \in \operatorname{dom}(A_{\mathsf{B}})$,

$$\langle u, A_{\mathsf{B}} v\rangle_{L^2(M)} - \langle A_{B_1} u, v\rangle_{L^2(M)} = \langle \tau_2 u, \hat\rho_{\mathsf{B}} v\rangle_{\mathfrak{h}_2, \mathfrak{h}_2^*}, \tag{3.3}$$

where

$$\hat\rho_{\mathsf{B}}\colon \operatorname{dom}(A_{\mathsf{B}}) \to \mathfrak{h}_2^*, \quad \hat\rho_{\mathsf{B}}(R_z^{\mathsf{B}} u) := \widehat{\Lambda}_z^{\mathsf{B}} G_{\bar z}^{B_1 *} u, \quad u \in \mathsf{H},\ z \in \varrho(A_{\mathsf{B}}) \cap \varrho(A_{B_1}).$$

Then, by hypotheses (H1)–(H5) and by [11, Theorems 2.8 and 3.8] (compare (3.3) and Lemma 3.2 here with (2.19) and Lemma 3.6 there and notice that hypothesis (H6) there is included in our hypothesis (H4)) one gets the completeness for the couple

$(A_{\mathsf{B}}, A_{B_1})$. By (H6) and the chain rule for the wave operators (see [10, Theorem 3.4, Chapter X]), one then gets completeness for the scattering couple (A_{B}, A).

To conclude the proof it remains to show that $\sigma_{sc}(A_{\mathsf{B}}) = \varnothing$. Let $\mathsf{H}_{\mathsf{B}}^{pp}$ denote the pure point subspace of A_{B} and, given $u \in (\mathsf{H}_{\mathsf{B}}^{pp})^{\perp}$, we denote by μ_u^{B} be the corresponding spectral measure. By our choice of u, one gets $\text{supp}(\mu_u^{\mathsf{B}}) \subseteq \sigma_{cont}(A_{\mathsf{B}}) \subseteq \sigma_{ess}(A_{\mathsf{B}}) = \sigma_{ess}(A_{B_1})$. Let us define

$$e_{ess}(A_{B_1}) := e(A_{B_1}) \cap \mathring{\sigma}_{ess}(A_{B_1}),$$
$$e_{ess}(A_{\mathsf{B}}) := (e(A_{B_1}) \cup \sigma_p(A_{\mathsf{B}})) \cap \mathring{\sigma}_{ess}(A_{B_1}),$$

and denote by $e'_{ess}(A_{B_1})$ the set of accumulation points of $e_{ess}(A_{B_1})$. Since an open set minus a discrete subset is still open, one has

$$\mathring{\sigma}_{ess}(A_{B_1}) \backslash e'_{ess}(A_{B_1}) = \bigcup_{n\geq 1} I_n,$$

where the I_n's are open intervals. Moreover, since $I_n \cap e'_{ess}(A_{B_1}) = \varnothing$, then $I_n \cap e_{ess}(A_{B_1})$ is discrete in I_n and so $I_n \backslash (I_n \cap e_{ess}(A_{B_1}))$ is open. This yields

$$I_n \backslash (I_n \cap e_{ess}(A_{B_1})) = \bigcup_{m\geq 1} I_{n,m},$$

where the $I_{n,m}$'s are open intervals. By Theorem 3.1, the set of accumulation points of $e(A_{\mathsf{B}}) \backslash e(A_{B_1})$ is contained in $e(A_{B_1})$; therefore $I_{n,m} \cap (e(A_{\mathsf{B}}) \backslash e(A_{B_1}))$ is discrete in $I_{n,m}$. As before,
$I_{n,m} \backslash (I_{n,m} \cap (e(A_{\mathsf{B}}) \backslash e(A_{B_1})))$ is open and we get

$$I_{n,m} \backslash (I_{n,m} \cap (e(A_{\mathsf{B}}) \backslash e(A_{B_1}))) = \bigcup_{\ell\geq 1} I_{n,m,\ell},$$

where the $I_{n,m,\ell}$'s are open intervals. Hence,

$$\begin{aligned}
\mathring{\sigma}_{ess}(A_{B_1}) \backslash e_{ess}(A_{\mathsf{B}}) &= \mathring{\sigma}_{ess}(A_{B_1}) \backslash (e_{ess}(A_{B_1}) \cup e_{ess}(A_{\mathsf{B}}) \backslash e_{ess}(A_{B_1})) \\
&= \Big(\bigcup_{n\geq 1} I_n \cup e'_{ess}(A_{B_1})\Big) \backslash (e_{ess}(A_{B_1}) \cup e_{ess}(A_{\mathsf{B}}) \backslash e_{ess}(A_{B_1})) \\
&= \Big(\Big(\bigcup_{n\geq 1} I_n \backslash e_{ess}(A_{B_1})\Big) \cup (e'_{ess}(A_{B_1}) \backslash e_{ess}(A_{B_1}))\Big) \backslash (e(A_{\mathsf{B}}) \backslash e(A_{B_1})) \\
&= \Big(\bigcup_{n,m\geq 1} I_{n,m} \cup (e'_{ess}(A_{B_1}) \backslash e_{ess}(A_{B_1}))\Big) \backslash (e(A_{\mathsf{B}}) \backslash e(A_{B_1})) \\
&= \Big(\bigcup_{n,m\geq 1} I_{n,m} \backslash (e(A_{\mathsf{B}}) \backslash (e(A_{B_1}))\Big) \cup (e'_{ess}(A_{B_1}) \backslash e_{ess}(A_{\mathsf{B}})) \\
&= \Big(\bigcup_{n,m,\ell\geq 1} I_{n,m,\ell}\Big) \cup (e'_{ess}(A_{B_1}) \backslash e_{ess}(A_{\mathsf{B}})).
\end{aligned}$$

This gives

$$\begin{aligned}\operatorname{supp}(\mu_u^{\mathsf{B}}) \subseteq \sigma_{\mathrm{ess}}(A_{B_1}) &= (\mathring{\sigma}_{\mathrm{ess}}(A_{B_1})\backslash e_{\mathrm{ess}}(A_{\mathsf{B}})) \cup \partial\sigma_{\mathrm{ess}}(A_{B_1}) \cup e_{\mathrm{ess}}(A_{\mathsf{B}}) \\ &= \Big(\bigcup_{n,m,\ell\geq 1} I_{n,m,\ell}\Big) \cup \partial\sigma_{\mathrm{ess}}(A_{B_1}) \cup e_{\mathrm{ess}}(A_{\mathsf{B}}) \cup e'_{\mathrm{ess}}(A_{B_1}).\end{aligned}$$

By standard arguments (see e.g. [1, proof of Theorem 6.1] or [17, top of p. 178]) applied to any of the open intervals $I_{n,m,\ell}$, one gets the absolute continuity of the spectral function $\lambda \mapsto \mu_u^{\mathsf{B}}(-\infty, \lambda]$ on any compact interval in $I_{n,m,\ell}$; hence

$$\begin{aligned}\operatorname{supp}((\mu_u^{\mathsf{B}})_{\mathrm{sing}}) &\subseteq \partial\sigma_{\mathrm{ess}}(A_{B_1}) \cup e_{\mathrm{ess}}(A_{\mathsf{B}}) \cup e'_{\mathrm{ess}}(A_{B_1}) \\ &= \partial\sigma_{\mathrm{ess}}(A_{B_1}) \cup e_{\mathrm{ess}}(A_{B_1}) \cup (e_{\mathrm{ess}}(A_{\mathsf{B}})\backslash e_{\mathrm{ess}}(A_{B_1})) \cup e'_{\mathrm{ess}}(A_{B_1}).\end{aligned}$$

By Theorem 3.1, $e(A_{\mathsf{B}})\backslash e(A_{B_1})$ is discrete (hence countable) in $\mathbb{R}\backslash e(A_{B_1})$; by (i) and (ii), the sets $e'_{\mathrm{ess}}(A_{B_1})$, $e_{\mathrm{ess}}(A_{B_1})$ and $\partial\sigma_{\mathrm{ess}}(A_{B_1})$ are countable. Henceforth, the support of the singular continuous component of μ_u^{B} is contained in a countable set. This implies $\operatorname{supp}((\mu_u^{\mathsf{B}})_{\mathrm{sing}}) = \varnothing$. Therefore, u has a null projection onto $\mathsf{H}_{\mathsf{B}}^{sc}$, the singular continuous subspace of A_{B}. This gives $(\mathsf{H}_{\mathsf{B}}^{\mathrm{pp}})^{\perp} = \mathsf{H}_{\mathsf{B}}^{\mathrm{ac}}$, where $\mathsf{H}_{\mathsf{B}}^{\mathrm{ac}}$ denote the absolutely continuous subspace of A_{B}. ■

3.1. A representation formula for the scattering matrix

According to Theorem 3.5, under the assumptions there stated, the scattering operator

$$\mathsf{S}_{\mathsf{B}} := W_+(A_{\mathsf{B}}, A)^* W_-(A_{\mathsf{B}}, A)$$

is a well-defined unitary map. Let

$$F\colon L^2(M)_{\mathrm{ac}} \to \int_{\sigma_{\mathrm{ac}}(A)}^{\oplus} (L^2(M)_{\mathrm{ac}})_\lambda \, d\eta(\lambda)$$

be a unitary map which diagonalizes the absolutely continuous component of A, i.e., a direct integral representation of $L^2(M)_{\mathrm{ac}}$, the absolutely continuous subspace relative to A, with respect to the spectral measure of the absolutely continuous component of A (see e.g. [2, Section 4.5.1]). We define the scattering matrix

$$\mathsf{S}_\lambda^{\mathsf{B}}\colon (L^2(M)_{\mathrm{ac}})_\lambda \to (L^2(M)_{\mathrm{ac}})_\lambda$$

by the relation (see e.g. [2, Section 9.6.2])

$$F\mathsf{S}_{\mathsf{B}}F^* u_\lambda = \mathsf{S}_\lambda^{\mathsf{B}} u_\lambda.$$

Now, following the same scheme as in [11], which uses the Birman–Kato invariance principle and the Birman–Yafaev general scheme in stationary scattering theory, we provide an explicit relation between $\mathsf{S}^{\mathsf{B}}_{\lambda}$ and $\Lambda^{\mathsf{B},+}_{\lambda} := \lim_{\epsilon\searrow 0}\Lambda^{\mathsf{B}}_{\lambda+i\epsilon}$.

Given $\mu \in \varrho(A)\cap\varrho(A_{\mathsf{B}})$, we consider the scattering couple $(R^{\mathsf{B}}_{\mu}, R_{\mu})$ and the strong limits

$$W_{\pm}(R^{\mathsf{B}}_{\mu}, R_{\mu}) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itR^{\mathsf{B}}_{\mu}}e^{-itR_{\mu}}P^{\mu}_{\mathrm{ac}},$$

where P^{μ}_{ac} is the orthogonal projector onto the absolutely continuous subspace of R_{μ}; we prove below that such limits exist everywhere in $L^2(M)$. Let $\mathsf{S}^{\mu}_{\mathsf{B}}$ the corresponding scattering operator

$$\mathsf{S}^{\mu}_{\mathsf{B}} := W_{+}(R^{\mathsf{B}}_{\mu}, R_{\mu})^{*}W_{-}(R^{\mathsf{B}}_{\mu}, R_{\mu}).$$

Using the unitary operator F_{μ} which diagonalizes the absolutely continuous component of R_{μ}, i.e., $(F_{\mu}u)_{\lambda} := \frac{1}{\lambda}(Fu)_{\mu-\frac{1}{\lambda}}$, $\lambda \neq 0$ such that $\mu - \frac{1}{\lambda} \in \sigma_{\mathrm{ac}}(A)$, one defines the scattering matrix

$$\mathsf{S}^{\mathsf{B},\mu}_{\lambda} \colon (L^2(M)_{\mathrm{ac}})_{\mu-\frac{1}{\lambda}} \to (L^2(M)_{\mathrm{ac}})_{\mu-\frac{1}{\lambda}}$$

corresponding to the scattering operator $\mathsf{S}^{\mu}_{\mathsf{B}}$ by the relation

$$F_{\mu}\mathsf{S}^{\mu}_{\mathsf{B}}F^{*}_{\mu}u^{\mu}_{\lambda} = \mathsf{S}^{\mathsf{B},\mu}_{\lambda}u^{\mu}_{\lambda}.$$

We introduce a further hypothesis (H7), which we split in four separate points.

(H7.1) A is bounded from above and satisfies a Limiting Absorption Principle: there exists a (eventually empty) closed set $e(A) \subset \mathbb{R}$ of zero Lebesgue measure such that for all $\lambda \in \mathbb{R}\backslash e(A)$ the limits

$$R^{\pm}_{\lambda} := \lim_{\epsilon\searrow 0} R_{\lambda\pm i\epsilon}$$

exist in $\mathcal{B}(L^2_{\varphi}(M), L^2_{\varphi^{-1}}(M))$.

(H7.2) $G^1_z \in \mathcal{B}(\mathfrak{h}^*_1, L^2_{\varphi}(M))$ for any $z \in \varrho(A)$ and the limits

$$G^{1,\pm}_{\lambda} := \lim_{\epsilon\searrow 0} G^{1}_{\lambda\pm i\epsilon} \tag{3.4}$$

exist in $\mathcal{B}(\mathfrak{h}^*_1, L^2_{\varphi^{-1}}(M))$ for any $\lambda \in \mathbb{R}\backslash e(A)$.

(H7.3) The limits

$$\Lambda^{B_1,\pm}_{\lambda} := \lim_{\epsilon\searrow 0} \Lambda^{B_1,\pm}_{\lambda\pm i\epsilon}$$

exist in $\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}^*_1)$ for any $\lambda \in \mathbb{R}\backslash e(A_{B_1})$.

(H7.4) The limits

$$\tau_2 G_\lambda^{1,\pm} := \lim_{\epsilon\searrow 0} \tau_2 G^1_{\lambda\pm i\epsilon}$$

exist in $\mathcal{B}(\mathfrak{b}_1^*, \mathfrak{b}_2)$ for any $\lambda \in \mathbb{R}\backslash e(A_{B_1})$.

Remark 3.6. By

$$\tau_2 G_z^1 = \tau_2(\tau_1 R_{\bar z})^* = (\tau_1(\tau_2 R_z)^*)^* = (\tau_1 G^2_{\bar z})^*,$$

hypothesis (H7.4) entails the existence in $\mathcal{B}(\mathfrak{b}_2, \mathfrak{b}_1^*)$, for any $\lambda \in \mathbb{R}\backslash e(A_{B_1})$, of the limits

$$\tau_1 G_\lambda^{2,\pm} := \lim_{\epsilon\searrow 0} \tau_1 G^2_{\lambda\pm i\epsilon}.$$

Remark 3.7. Whenever one strengthens hypotheses (H7.2) as in (H5), then, by the same kind of proof that leads to the existence of the limit (3.2) (see [11, Lemma 3.6]), one gets the existence of the limits requested in hypotheses (H7.3).

Lemma 3.8. *Suppose that* (H1)–(H5) *and* (H7) *hold. Then*

$$R_\lambda^{B_1,\pm} = R_\lambda^\pm + G_\lambda^{1,\pm}\Lambda_\lambda^{B_1,\pm}(G_\lambda^{1,\mp})^*, \tag{3.5}$$

$$G_z^2 \in \mathcal{B}(\mathfrak{h}_2^*, L^2_\varphi(M)), \quad z \in \varrho(A_{B_1}) \cap \varrho(A); \tag{3.6}$$

the limits

$$G_\lambda^{2,\pm} := \lim_{\epsilon\searrow 0} G^2_{\lambda\pm i\epsilon} \tag{3.7}$$

exist in $\mathcal{B}(\mathfrak{h}_2^*, L^2_{\varphi^{-1}}(M))$ *for any* $\lambda \in \mathbb{R}\backslash e(A_{B_1})$ *and*

$$G_\lambda^{B_1,\pm} = G_\lambda^{2,\pm} + G_\lambda^{1,\pm}\Lambda_\lambda^{B_1,\pm}\tau_1 G_\lambda^{2,\pm}; \tag{3.8}$$

the limits

$$\Lambda_\lambda^{\mathsf{B},\pm} := \lim_{\epsilon\searrow 0} \Lambda^{\mathsf{B}}_{\lambda\pm i\epsilon}$$

exist in $\mathcal{B}(\mathfrak{h}_1 \oplus \mathfrak{b}_2, \mathfrak{h}_1^* \oplus \mathfrak{b}_2^*)$ *and*

$$\Lambda_\lambda^{\mathsf{B},\pm} = \begin{bmatrix} \Lambda_\lambda^{B_1,\pm} + \Lambda_\lambda^{B_1,\pm}\tau_1 G_\lambda^{2,\pm}\widehat\Lambda_\lambda^{\mathsf{B},\pm}\tau_2 G_\lambda^{1,\pm}\Lambda_\lambda^{B_1,\pm} & \Lambda_\lambda^{B_1,\pm}\tau_1 G_\lambda^{2,\pm}\widehat\Lambda_\lambda^{\mathsf{B},\pm} \\ \widehat\Lambda_\lambda^{\mathsf{B},\pm}\tau_2 G_\lambda^{1,\pm}\Lambda_\lambda^{B_1,\pm} & \widehat\Lambda_\lambda^{\mathsf{B},\pm} \end{bmatrix} \tag{3.9}$$

$$= \left(1 + \begin{bmatrix} \Lambda_\lambda^{B_1,\pm} & 0 \\ 0 & \widehat\Lambda_\lambda^{\mathsf{B},\pm} \end{bmatrix}\begin{bmatrix} \tau_1 G_\lambda^{2,\pm}\widehat\Lambda_\lambda^{\mathsf{B},\pm}\tau_2 G_\lambda^{1,\pm} & \tau_1 G_\lambda^{2,\pm} \\ \tau_2 G_\lambda^{1,\pm} & 0 \end{bmatrix}\right)\begin{bmatrix} \Lambda_\lambda^{B_1,\pm} & 0 \\ 0 & \widehat\Lambda_\lambda^{\mathsf{B},\pm} \end{bmatrix}. \tag{3.10}$$

Proof. The relation (3.5) is an immediate consequence of (2.22) and (H7.1)–(H7.3). By (2.24),

$$G_z^2 = G_z^{B_1} - G_z^1\Lambda_z^{B_1}\tau_1 G_z^2$$

and (3.6) follows from (H4.2) and (H7.2). Then, Remark 3.6, (H.5) and (H7.3) entail (3.7) and (3.8). Finally, (3.9) and (3.10) are consequence of (2.26), (2.27), Corollary 3.3, (H7.3), Remark 3.6 and (H7.4). ■

Before stating the next results, let us notice the relations

$$(-R_\mu + z)^{-1} = \frac{1}{z}\Big(1 + \frac{1}{z} R_{\mu-\frac{1}{z}}\Big), \quad (-R^{\mathsf{B}}_\mu + z)^{-1} = \frac{1}{z}\Big(1 + \frac{1}{z} R^{\mathsf{B}}_{\mu-\frac{1}{z}}\Big), \tag{3.11}$$

Therefore, by (H7.1) and Theorem 3.1, the limits

$$(-R_\mu + (\lambda \pm i0))^{-1} := \lim_{\epsilon \searrow 0} (-R_\mu + (\lambda \pm i\epsilon))^{-1}, \quad \lambda \neq 0,\ \mu - \frac{1}{\lambda} \in \mathbb{R}\backslash e(A), \tag{3.12}$$

$$(-R^{\mathsf{B}}_\mu + (\lambda \pm i0))^{-1} := \lim_{\epsilon \searrow 0} (-R^{\mathsf{B}}_\mu + (\lambda \pm i\epsilon))^{-1}, \quad \lambda \neq 0,\ \mu - \frac{1}{\lambda} \in \mathbb{R}\backslash e(A_{\mathsf{B}}), \tag{3.13}$$

exist in $\mathcal{B}(L^2_\varphi(M), L^2_{\varphi^{-1}}(M))$.

Theorem 3.9. *Suppose that hypotheses* (H1)–(H7) *hold. Then the strong limits*

$$W_\pm(R^{\mathsf{B}}_\mu, R_\mu) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itR^{\mathsf{B}}_\mu} e^{-itR_\mu} P^\mu_{\mathrm{ac}} \tag{3.14}$$

exist everywhere in $L^2(M)$. Moreover, for any $\lambda \neq 0$ such that $\mu - \frac{1}{\lambda} \in \sigma_{\mathrm{ac}}(A) \cap (\mathbb{R}\backslash e(A_{\mathsf{B}}))$, one has

$$\mathsf{S}^{\mathsf{B},\mu}_\lambda = 1 - 2\pi i\, \mathcal{L}^\mu_\lambda \Lambda^{\mathsf{B}}_\mu (1 + G^*_\mu(-R^{\mathsf{B}}_\mu + (\lambda + i0))^{-1} G_\mu \Lambda^{\mathsf{B}}_\mu)(\mathcal{L}^\mu_\lambda)^*, \tag{3.15}$$

where

$$\mathcal{L}^\mu_\lambda \colon \mathfrak{h}^*_1 \oplus \mathfrak{h}^*_2 \to (L^2(M)_{\mathrm{ac}})_{\mu-\frac{1}{\lambda}}, \quad \mathcal{L}^\mu_\lambda(\phi_1 \oplus \phi_2) := \frac{1}{\lambda}(FG_\mu(\phi_1 \oplus \phi_2))_{\mu-\frac{1}{\lambda}}.$$

Proof. By (2.7), one has $R^{\mathsf{B}}_\mu - R_\mu = G_\mu \Lambda^B_\mu G^*_\mu$ and we can use [22, Theorem 4′, p. 178] (notice that the maps there denoted by G and V corresponds to our G^*_μ and Λ^B_μ respectively). Let us check that the hypotheses there required are satisfied. Since $G^*_\mu \in \mathcal{B}(L^2(M), \mathfrak{h}_1 \oplus \mathfrak{h}_2)$, the operator G_μ is $|R_\mu|^{1/2}$-bounded. By (H7.2) and (3.6), one has $G_z \in \mathcal{B}(\mathfrak{h}^*_1 \oplus \mathfrak{h}^*_2, L^2_\varphi(M))$ for any $z \in \varrho(A_{B_1}) \cap \varrho(A) \supset [\lambda_1, +\infty) \ni \mu$. Therefore, by (3.12), (3.13), (H7.1), Theorem 3.1 and (H4), the limits

$$\lim_{\epsilon\searrow 0} G^*_\mu(-R_\mu + (\lambda \pm i\epsilon))^{-1},$$

$$\lim_{\epsilon\searrow 0} G^*_\mu(-R^{\mathsf{B}}_\mu + (\lambda \pm i\epsilon))^{-1},$$

$$\lim_{\epsilon\searrow 0} G^*_\mu(-R^{\mathsf{B}}_\mu + (\lambda \pm i\epsilon))^{-1} G_\mu$$

exist. Therefore, to get the thesis we need to check the validity of the remaining hypothesis in [22, Theorem 4′, p. 178]: G^*_μ is weakly-R_μ smooth, i.e., by [22, Lemma 2, p. 154],

$$\sup_{0<\epsilon<1} \epsilon\, \|G^*_\mu(-R_\mu + (\lambda \pm i\epsilon))^{-1}\|^2_{L^2(M),\mathfrak{h}_1\oplus\mathfrak{h}_2} \le c_\lambda < +\infty, \quad \text{a.e. } \lambda.$$

By (3.11), this is consequence of

$$\sup_{0<\epsilon<1} \epsilon\, \|G^*_\mu R_{\mu-\frac{1}{\lambda}\pm i\epsilon}\|^2_{L^2(M),\mathfrak{h}_1\oplus\mathfrak{h}_2} \le C_\lambda < +\infty, \quad \text{a.e. } \lambda. \tag{3.16}$$

By [11, (3.16)],

$$\begin{aligned} &\epsilon \|G_{\lambda\pm i\epsilon}\|^2_{\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*,L^2(M)} \\ &\quad\le \frac{1}{2}(|\mu-\lambda|+\epsilon)\, \|G_\mu\|_{\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*,L^2_\varphi(M)} \big(\|G_{\lambda-i\epsilon}\|_{\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*,L^2_{\varphi^{-1}}(M)} \\ &\qquad\qquad + \|G_{\lambda+i\epsilon}\|_{\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*,L^2_{\varphi^{-1}}(M)}\big). \end{aligned}$$

Then, (3.16) follows from (3.4), (3.7), and the equality

$$\begin{aligned} \|G^*_\mu R_z\|_{L^2(M),\mathfrak{h}_1\oplus\mathfrak{h}_2} &= \|\tau R_\mu R_z\|_{L^2(M),\mathfrak{h}_1\oplus\mathfrak{h}_2} = \|\tau R_z R_\mu\|_{L^2(M),\mathfrak{h}_1\oplus\mathfrak{h}_2} \\ &= \|R_\mu(\tau R_z)^*\|_{\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*,L^2(M)} \le \|R_\mu\|_{L^2(M),L^2(M)}\|G_{\bar z}\|_{\mathfrak{h}_1^*\oplus\mathfrak{h}_2^*,L^2(M)}. \end{aligned}$$

Thus, by [22, Theorem 4′, p. 178], the limits (3.14) exist everywhere in $L^2(M)$ and the corresponding scattering matrix is given by (3.15), where $\mathcal{L}^\mu_\lambda\phi := (F^\mu G_\mu\phi)_\lambda = \frac{1}{\lambda}(FG_\mu\phi)_{\mu-\frac{1}{\lambda}}$. ∎

Theorem 3.10. *Suppose that hypotheses* (H1)–(H7) *hold. Then the scattering matrix of the couple* (A_{B}, A) *has the representation*

$$\mathsf{S}^{\mathsf{B}}_\lambda = 1 - 2\pi i\, \mathcal{L}_\lambda \Lambda^{\mathsf{B},+}_\lambda \mathcal{L}^*_\lambda, \quad \lambda \in \sigma_{\mathrm{ac}}(A) \cap (\mathbb{R}\backslash e(A_{\mathsf{B}})),$$

where $\mathcal{L}_\lambda\colon \mathfrak{h}_1^* \oplus \mathfrak{h}_2^* \to (L^2(M)_{\mathrm{ac}})_\lambda$ *is the* μ*-independent linear operator defined by*

$$\mathcal{L}_\lambda(\phi_1 \oplus \phi_2) := (\mu-\lambda)(FG_\mu(\phi_1\oplus\phi_2))_\lambda \tag{3.17}$$

and $\Lambda^{\mathsf{B},+}_\lambda$ *is given in* (3.9).

Proof. By Theorem 3.5, Theorem 3.9 and by Birman–Kato invariance principle (see e.g. [2, Section II.3.3]), one has

$$W_\pm(A_{\mathsf{B}}, A) = W_\pm(R^{\mathsf{B}}_\mu, R_\mu)$$

and so

$$\mathsf{S}_{\mathsf{B}} = \mathsf{S}^\mu_{\mathsf{B}}.$$

Thus, since $(F^{\mu}u)_{\lambda} = \frac{1}{\lambda}(Fu)_{\mu-\frac{1}{\lambda}}$, one obtains (see also [22, equation (14), Section 6, Chapter 2])

$$\mathsf{S}^{\mathsf{B}}_{\lambda} = \mathsf{S}^{\mathsf{B},\mu}_{(-\lambda+\mu)^{-1}}. \tag{3.18}$$

By [11, Lemma 4.2], for any $z \neq 0$ such that $\mu - \frac{1}{z} \in \varrho(A_{\mathsf{B}}) \cap \varrho(A)$, there holds

$$\Lambda^{\mathsf{B}}_{\mu}(1 + G^{*}_{\mu}(-R^{\mathsf{B}}_{\mu} + z)^{-1} G_{\mu} \Lambda^{B}_{\mu}) = \Lambda^{\mathsf{B}}_{\mu-\frac{1}{z}}.$$

Hence, whenever $z = \lambda \pm i\epsilon$ and $\mu - \frac{1}{\lambda} \in \mathbb{R}\backslash e(A_{\mathsf{B}})$, one gets, as $\epsilon \downarrow 0$,

$$\Lambda^{\mathsf{B}}_{\mu}(1 + G^{*}_{\mu}(-R^{\mathsf{B}}_{\mu} + (\lambda \pm i0))^{-1} G_{\mu} \Lambda^{\mathsf{B}}_{\mu}) = \Lambda^{\mathsf{B},\pm}_{\mu-\frac{1}{\lambda}}.$$

The proof is then concluded by setting $\mathcal{L}_{\lambda} := \mathcal{L}^{\mu}_{(-\lambda+\mu)^{-1}}$, by Theorem 3.9 and by (3.18). The operator $\mathcal{L}_{\lambda}$ is μ-independent by invariance principle (see the proof in [11, Corollary 4.3] for an explicit check). ■

Remark 3.11. By (3.9),

$$\Lambda^{\mathsf{B},\pm}_{\lambda} = \begin{bmatrix} \Lambda^{B_1,\pm}_{z} & 0 \\ 0 & 0 \end{bmatrix} + \widetilde{\Lambda}^{\mathsf{B},\pm}_{\lambda},$$

where

$$\widetilde{\Lambda}^{\mathsf{B},\pm}_{\lambda} := \begin{bmatrix} \Lambda^{B_1,\pm}_{\lambda} \tau_1 G^{2,\pm}_{\lambda} \widehat{\Lambda}^{\mathsf{B},\pm}_{\lambda} \tau_2 G^{1,\pm}_{\lambda} \Lambda^{B_1,\pm}_{\lambda} & \Lambda^{B_1,\pm}_{\lambda} \tau_1 G^{2,\pm}_{\lambda} \widehat{\Lambda}^{\mathsf{B},\pm}_{\lambda} \\ \widehat{\Lambda}^{\mathsf{B},\pm}_{\lambda} \tau_2 G^{1,\pm}_{\lambda} \Lambda^{B_1,\pm}_{\lambda} & \widehat{\Lambda}^{\mathsf{B},\pm}_{\lambda} \end{bmatrix}.$$

Therefore, defining

$$\mathcal{L}^{1}_{\lambda}\phi_1 := \mathcal{L}_{\lambda}(\phi_1 \oplus 0),$$

one gets

$$\mathsf{S}^{\mathsf{B}}_{\lambda} = \mathsf{S}^{B_1}_{\lambda} - 2\pi i \, \mathcal{L}_{\lambda} \widetilde{\Lambda}^{\mathsf{B},+}_{\lambda} \mathcal{L}^{*}_{\lambda},$$

where

$$\mathsf{S}^{B_1}_{\lambda} = 1 - 2\pi i \, \mathcal{L}^{1}_{\lambda} \widetilde{\Lambda}^{B_1,+}_{\lambda} (\mathcal{L}^{1}_{\lambda})^{*} \tag{3.19}$$

is the scattering matrix relative to the couple (A_{B_1}, A). Moreover, in the case $B_1 = 0$, defining

$$\mathcal{L}^{2}_{\lambda}\phi_2 := \mathcal{L}_{\lambda}(0 \oplus \phi_2),$$

one gets the following representation formula for the scattering couple (A_{B_0,B_2}, A) (compare with [11, Corollary 4.3]):

$$\mathsf{S}^{B_0,B_2}_{\lambda} = 1 - 2\pi i \, \mathcal{L}^{2}_{\lambda} \Lambda^{B_0,B_2,+}_{\lambda} (\mathcal{L}^{2}_{\lambda})^{*}.$$

Let us further notice that, whenever A is the free Laplacian in $L^2(\mathbb{R}^3)$ and B_1 corresponds to a perturbation by a regular potential as in Section 5 below, then (3.19) gives the usual formula for the scattering matrix for a short-range potential (see, e.g., [21, Section 8]).

4. Kato–Rellich perturbations and their layers potentials

4.1. Potential perturbations

In this section we suppose that the real-valued potential v is of Kato–Rellich type, i.e., $\mathsf{v} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, equivalently,

$$\mathsf{v} = \mathsf{v}_2 + \mathsf{v}_\infty, \quad \mathsf{v}_2 \in L^2(\mathbb{R}^3),\ \mathsf{v}_\infty \in L^\infty(\mathbb{R}^3).$$

We use the same symbol v to denote both the potential function and the corresponding multiplication operator $u \mapsto \mathsf{v}u$.

Given $\Omega \subset \mathbb{R}^3$, open and bounded with a Lipschitz boundary Γ, we define the Sobolev spaces $H^s(\mathbb{R}^3\backslash\Gamma) \hookleftarrow H^s(\mathbb{R}^3)$ by

$$H^s(\mathbb{R}^3\backslash\Gamma) := H^s(\Omega) \oplus H^s(\Omega_{\text{ex}}), \quad s \geq 0.$$

We refer to [15, Chapter 3] for the definition of the Sobolev spaces $H^s(\mathbb{R}^3)$, $H^s(\Omega)$ and $H^s(\Gamma)$. One has

$$H^s(\mathbb{R}^3\backslash\Gamma) = H^s(\mathbb{R}^3), \quad 0 \leq s < 1/2.$$

Since (see [15, Theorems 3.29 and 3.30]),

$$H^s(\mathcal{O})^* = H^{-s}_{\overline{\mathcal{O}}}(\mathbb{R}^3), \quad s \in \mathbb{R},$$

$H^{-s}_{\overline{\mathcal{O}}}(\mathbb{R}^3)$ denoting the set of distributions $H^{-s}(\mathbb{R}^3)$ with support in $\overline{\mathcal{O}}$, one has

$$H^s(\mathbb{R}^3\backslash\Gamma)^* = H^s(\Omega)^* \oplus H^s(\mathbb{R}^3\backslash\overline{\Omega})^* = H^{-s}_{\overline{\Omega}}(\mathbb{R}^3) \oplus H^{-s}_{\Omega^c}(\mathbb{R}^3) \hookrightarrow H^{-s}(\mathbb{R}^3).$$

Let us notice that

$$\mathcal{B}(H^s(\mathbb{R}^3\backslash\Gamma), H^t(\mathbb{R}^3\backslash\Gamma)^*) \hookrightarrow \mathcal{B}(H^s(\mathbb{R}^3), H^{-t}(\mathbb{R}^3)), \quad s,t \geq 0, \tag{4.1}$$

and

$$\mathcal{B}(H^{-s}(\mathbb{R}^3), H^t(\mathbb{R}^3)) \hookrightarrow \mathcal{B}(H^s(\mathbb{R}^3\backslash\Gamma)^*, H^t(\mathbb{R}^3\backslash\Gamma)), \quad s,t \geq 0.$$

Lemma 4.1. *We have*

$$\mathsf{v} \in \mathcal{B}(H^{1+s}(\mathbb{R}^3\backslash\Gamma), H^{1-s}(\mathbb{R}^3\backslash\Gamma)^*), \quad -1 \leq s \leq 1. \tag{4.2}$$

Proof. Given $u = u_{\text{in}} \oplus u_{\text{ex}} \in H^2(\mathbb{R}^3\backslash\Gamma)$ one has

$$\|\mathsf{v}_\infty u\|_{L^2(\mathbb{R}^3)} \leq \|\mathsf{v}\|_{L^\infty(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)} \leq \|\mathsf{v}\|_{L^\infty(\mathbb{R}^3)} \|u\|_{H^2(\mathbb{R}^3\backslash\Gamma)}.$$

and

$$\begin{aligned}\|\mathsf{v}_2 u\|_{L^2(\mathbb{R}^3)} &= \|\mathsf{v}_2\|_{L^2(\Omega)}\|u_{\mathrm{in}}\|_{L^\infty(\Omega)} + \|\mathsf{v}_2\|_{L^2(\mathbb{R}^3\setminus\bar{\Omega})}\|u_{\mathrm{ex}}\|_{L^\infty(\mathbb{R}^3\setminus\bar{\Omega})}\\ &\lesssim \|\mathsf{v}_2\|_{L^2(\Omega)}\|u_{\mathrm{in}}\|_{H^2(\Omega)} + \|\mathsf{v}_2\|_{L^2(\mathbb{R}^3\setminus\bar{\Omega})}\|u_{\mathrm{ex}}\|_{H^2(\mathbb{R}^3\setminus\bar{\Omega})}\\ &\lesssim \|\mathsf{v}_2\|_{L^2(\mathbb{R}^3)}\|u\|_{H^2(\mathbb{R}^3\setminus\Gamma)}.\end{aligned}$$

Hence, $\mathsf{v} \in \mathcal{B}(H^2(\mathbb{R}^3\setminus\Gamma), L^2(\mathbb{R}^3))$. Then, for any $u, v \in H^2(\mathbb{R}^3\setminus\Gamma)$, one has

$$\begin{aligned}|\langle \mathsf{v}u, v\rangle_{H^2(\mathbb{R}^3\setminus\Gamma)^*, H^2(\mathbb{R}^3\setminus\Gamma)}| &= |\langle \mathsf{v}u, v\rangle_{L^2(\mathbb{R}^3)}|\\ &= |\langle u, \mathsf{v}v\rangle_{L^2(\mathbb{R}^3)}|\\ &\le \|\mathsf{v}\|_{H^2(\mathbb{R}^3\setminus\Gamma), L^2(\mathbb{R}^3)}\|u\|_{L^2(\mathbb{R}^3)}\|v\|_{H^2(\mathbb{R}^3\setminus\Gamma)},\end{aligned}$$

and so $u \mapsto \mathsf{v}u$ extends to a map in $\mathcal{B}(L^2(\mathbb{R}^3), H^2(\mathbb{R}^3\setminus\Gamma)^*)$. The proof is then concluded by interpolation. ■

In the following, R_z denotes the resolvent of the free Laplacian, i.e.,

$$R_z := (-\Delta + z)^{-1} \in \mathcal{B}(H^s(\mathbb{R}^3), H^{s+2}(\mathbb{R}^3)), \quad s \in \mathbb{R}. \tag{4.3}$$

Since v is of Rellich–Kato type, one has (see, e.g., [10, Section 3, §5, Chapter V]) the following result.

Theorem 4.2. *The operator* $\Delta + \mathsf{v}\colon H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ *is self-adjoint and semi-bounded from above. Moreover, for* $z \in \mathbb{C}$ *sufficiently far away from* $(-\infty, 0]$,

$$\|\mathsf{v}R_z\|_{L^2(\mathbb{R}^3), L^2(\mathbb{R}^3)} < 1,$$

and

$$R_z^{\mathsf{v}} := (-(\Delta + \mathsf{v}) + z)^{-1} = R_z + R_z(1 - \mathsf{v}R_z)^{-1}\mathsf{v}R_z, \tag{4.4}$$

$$(1 - \mathsf{v}R_z)^{-1} = \sum_{k=0}^{+\infty} (\mathsf{v}R_z)^k \in \mathcal{B}(L^2(\mathbb{R}^3)). \tag{4.5}$$

Remark 4.3. Let us notice that Theorem 4.2 could be obtained by Corollary 2.7 by taking $\tau_1 u := u$ and $B_1 = \mathsf{v}$. Hence, (4.4) holds for any z in $\varrho(\Delta + \mathsf{v}) \cap \mathbb{C}\setminus(-\infty, 0]$ and $(1 + \mathsf{v}R_z)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^3))$ there.

Remark 4.4. By (4.3), (4.4), (4.5), (4.2) and (4.1), one has $R_{\bar{z}}^{\mathsf{v}} \in \mathcal{B}(L^2(\mathbb{R}^3), H^2(\mathbb{R}^3))$ and hence $(R_{\bar{z}}^{\mathsf{v}})^* \in \mathcal{B}(H^{-2}(\mathbb{R}^3), L^2(\mathbb{R}^3))$. Since $(\Delta + \mathsf{v})$ is self-adjoint in $L^2(\mathbb{R}^3)$, $(R_{\bar{z}}^{\mathsf{v}})^* | L^2(\mathbb{R}^3) = R_z^{\mathsf{v}}$. Therefore, $R_z^{\mathsf{v}}\colon L^2(\mathbb{R}^3) \subset H^{-2}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ extends to an operator in $\mathcal{B}(H^{-2}(\mathbb{R}^3), L^2(\mathbb{R}^3))$ which, by abuse of notation, we still denote by R_z^{v} and which coincides with $(R_{\bar{z}}^{\mathsf{v}})^*$. Then, by interpolation, one gets

$$R_z^{\mathsf{v}} \in \mathcal{B}(H^{s-1}(\mathbb{R}^3), H^{s+1}(\mathbb{R}^3)), \quad -1 \le s \le 1. \tag{4.6}$$

Remark 4.5. By (4.4),

$$(1-\mathsf{v}R_z)^{-1}\mathsf{v} = (-\Delta+z)R_z^{\mathsf{v}}(-\Delta+z) - (-\Delta+z).$$

Hence, by (4.6), $(1-\mathsf{v}R_z)^{-1}\mathsf{v} \in \mathcal{B}(H^2(\mathbb{R}^3), L^2(\mathbb{R}^3))$ extends to a map

$$\Lambda_z^{\mathsf{v}} \in \mathcal{B}(H^{s+1}(\mathbb{R}^3), H^{s-1}(\mathbb{R}^3)), \quad -1 \le s \le 1. \tag{4.7}$$

With such a notation, R_z^{v} in (4.6) has the representation

$$R_z^{\mathsf{v}} = R_z + R_z\Lambda_z^{\mathsf{v}}R_z, \quad \Lambda_z^{\mathsf{v}}|H^2(\mathbb{R}^3) = (1-\mathsf{v}R_z)^{-1}\mathsf{v}. \tag{4.8}$$

Remark 4.6. Since

$$\|R_z\mathsf{v}\|_{L^2(\mathbb{R}^3),L^2(\mathbb{R}^3)} = \|(R_z\mathsf{v})^*\|_{L^2(\mathbb{R}^3),L^2(\mathbb{R}^3)} = \|\mathsf{v}R_{\bar z}\|_{L^2(\mathbb{R}^3),L^2(\mathbb{R}^3)} < 1$$

whenever $z \in \mathbb{C}$ is sufficiently far away from $(-\infty, 0]$, one has

$$(1-R_z\mathsf{v})^{-1} = \sum_{k=0}^{+\infty}(R_z\mathsf{v})^k \in \mathcal{B}(L^2(\mathbb{R}^3)) \tag{4.9}$$

and

$$\mathsf{v}(1-R_z\mathsf{v})^{-1} \in \mathcal{B}(L^2(\mathbb{R}^3), H^{-2}(\mathbb{R}^3)).$$

Then,

$$((1-R_z\mathsf{v})^{-1}\mathsf{v})^* = (\mathsf{v}(1-R_z\mathsf{v})^{-1})^* = \mathsf{v}((1-R_z\mathsf{v})^*)^{-1} = \mathsf{v}(1-R_{\bar z}\mathsf{v})^{-1}$$

and so

$$\mathcal{B}(H^{-2}(\mathbb{R}^3), L^2(\mathbb{R}^3)) \ni (R_z^{\mathsf{v}})^* = R_{\bar z} + R_{\bar z}\mathsf{v}(1-R_{\bar z}\mathsf{v})^{-1}R_{\bar z} = R_{\bar z}^{\mathsf{v}} = R_{\bar z} + R_{\bar z}\Lambda_{\bar z}^{\mathsf{v}}R_{\bar z}.$$

Therefore

$$\Lambda_z^{\mathsf{v}}|L^2(\mathbb{R}^3) = \mathsf{v}(1-R_z\mathsf{v})^{-1}. \tag{4.10}$$

Lemma 4.7. *One has*

$$\Lambda_z^{\mathsf{v}} \in \mathcal{B}(H^{1+s}(\mathbb{R}^3\backslash\Gamma), H^{1-s}(\mathbb{R}^3\backslash\Gamma)^*), \quad -1 \le s \le 1. \tag{4.11}$$

Proof. By Lemma 4.1 and by (4.5), one has

$$\Lambda_z^{\mathsf{v}} = (1+\mathsf{v}R_z)^{-1}\mathsf{v} \in \mathcal{B}(H^2(\mathbb{R}^3\backslash\Gamma), L^2(\mathbb{R}^3)).$$

By Lemma 4.1, (4.9) and (4.10), $\Lambda_z^{\mathsf{v}} \in \mathcal{B}(L^2(\mathbb{R}^3), H^2(\mathbb{R}^3\backslash\Gamma)^*)$. The proof is then concluded by interpolation. ∎

By $H^{1-s}(\mathbb{R}^3\backslash\Gamma)^* \hookrightarrow H^{s-1}(\mathbb{R}^3)$ and (4.3) one has the following result.

Corollary 4.8. *For* $0 \le s \le 2$,

$$R_z \Lambda_z^{\mathsf{v}} \in \mathcal{B}(H^s(\mathbb{R}^3 \backslash \Gamma),\ H^s(\mathbb{R}^3)), \tag{4.12}$$

In later proofs, we will need the estimate provided in the following fact.

Lemma 4.9. *There exist* $c_1 > 0$, $c_2 > 0$ *such that, for any* $u \equiv u_{\text{in}} \oplus u_{\text{ex}} \in H^1(\mathbb{R}^3 \backslash \Gamma)$ *and for any* $\epsilon > 0$, *there holds*

$$\begin{aligned} &|\langle \mathsf{v}u, u\rangle_{H^1(\mathbb{R}^3\backslash\Gamma)^*, H^1(\mathbb{R}^3\backslash\Gamma)}| \\ &\quad \le c_1 \epsilon (\|\nabla u_{\text{in}}\|^2_{L^2(\Omega_{\text{in}})} + \|\nabla u_{\text{ex}}\|^2_{L^2(\Omega_{\text{ex}})}) + c_2(1+\epsilon^{-3})\|u\|^2_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{4.13}$$

Proof. By $H^1(\Omega_{\text{in/ex}}) \hookrightarrow H^{3/4}(\Omega_{\text{in/ex}}) \hookrightarrow L^4(\Omega_{\text{in/ex}})$, by the Gagliardo–Nirenberg inequalities (see [6] for the interior case and [8] for the exterior one)

$$\begin{aligned} &\|u_{\text{in}}\|_{L^4(\Omega_{\text{in}})} \lesssim \|u_{\text{in}}\|_{H^{3/4}(\Omega_{\text{in}})} \lesssim \|u_{\text{in}}\|^{3/4}_{H^1(\Omega_{\text{in}})} \|u_{\text{in}}\|^{1/4}_{L^2(\Omega_{\text{in}})}, \\ &\|u_{\text{ex}}\|_{L^4(\Omega_{\text{ex}})} \lesssim \|\nabla u_{\text{ex}}\|^{3/4}_{L^2(\Omega_{\text{ex}})} \|u_{\text{ex}}\|^{1/4}_{L^2(\Omega_{\text{ex}})} \end{aligned}$$

and, by Young's inequality,

$$xy \le \frac{1}{\alpha}(\epsilon x^\alpha + (\alpha-1)\epsilon^{-1/(\alpha-1)}\, y^{\frac{\alpha-1}{\alpha}}), \quad x, y, \epsilon > 0,\ \alpha > 1,$$

one gets

$$\|u\|^2_{L^4(\mathbb{R}^3)} \lesssim \epsilon(\|\nabla u_{\text{in}}\|^2_{L^2(\Omega_{\text{in}})} + \|u\|^2_{L^2(\Omega_{\text{in}})} + \|\nabla u_{\text{ex}}\|^2_{L^2(\Omega_{\text{ex}})}) + \frac{1}{3}\,\epsilon^{-3}\|u\|^2_{L^2(\mathbb{R}^3)}.$$

The proof is then concluded by

$$\begin{aligned} &|\langle \mathsf{v}u, u\rangle_{H^1(\mathbb{R}^3\backslash\Gamma)^*, H^1(\mathbb{R}^3\backslash\Gamma)}| \\ &\quad \le \|\mathsf{v}_2\|_{L^2(\mathbb{R}^3)} \|u\|^2_{L^4(\mathbb{R}^3)} + \|\mathsf{v}_\infty\|_{L^\infty(\mathbb{R}^3)} \|u\|^2_{L^2(\mathbb{R}^3)}. \end{aligned}$$ ∎

Lemma 4.10. *For any* $z \in \mathbb{C}$ *sufficiently far away from* $(-\infty, 0]$, *one has*

$$\|\mathsf{v}R_z\|_{H^{-1}(\mathbb{R}^3), H^{-1}(\mathbb{R}^3)} < 1$$

and

$$(1 - \mathsf{v}R_z)^{-1} = \sum_{k=0}^{+\infty} \frac{1}{k!}\,(\mathsf{v}R_z)^k \in \mathcal{B}(H^{-1}(\mathbb{R}^3)).$$

Furthermore,

$$(1 - \mathsf{v}R_z)^{-1} \in \mathcal{B}(H^1(\mathbb{R}^3 \backslash \Gamma)^*).$$

Proof. By (4.13) and by the polarization identity, for any u and v in $H^1(\mathbb{R}^3)$ one has

$$|\langle \mathsf{v}u, v\rangle_{H^{-1}(\mathbb{R}^3),H^1(\mathbb{R}^3)}| \\ \le \frac{1}{4}(c_1\epsilon\,\langle -\Delta u, v\rangle_{H^{-1}(\mathbb{R}^3),H^1(\mathbb{R}^3)} + c_2(1+\epsilon^{-3})\langle u, v\rangle_{H^{-1}(\mathbb{R}^3),H^1(\mathbb{R}^3)})$$

which gives

$$\|\mathsf{v}u\|_{H^{-1}(\mathbb{R}^3)} \\ \le \frac{1}{4}(c_1\epsilon\,\|-\Delta u\|_{H^{-1}(\mathbb{R}^3)} + c_2(1+\epsilon^{-3})\|u\|_{H^{-1}(\mathbb{R}^3)}) \\ \le \frac{1}{4}(c_1\epsilon\,\|(-\Delta+z)u\|_{H^{-1}(\mathbb{R}^3)} + (c_1\epsilon\,|z| + c_2(1+\epsilon^{-3}))\|u\|_{H^{-1}(\mathbb{R}^3)}).$$

The proof is then concluded by taking $u = R_z u_\circ$, $u_\circ \in H^{-1}(\mathbb{R}^3)$, and by

$$\begin{aligned}\|R_z u_\circ\|_{H^{-1}(\mathbb{R}^3)} &= \|R_1^{1/2} R_z u_\circ\|_{L^2(\mathbb{R}^3)} = \|R_z R_1^{1/2} u_\circ\|_{L^2(\mathbb{R}^3)} \\ &\le \|R_z\|_{L^2(\mathbb{R}^3),L^2(\mathbb{R}^3)} \|u_\circ\|_{H^{-1}(\mathbb{R}^3)} \\ &\le d_z^{-1}\,\|u_\circ\|_{H^{-1}(\mathbb{R}^3)},\end{aligned}$$

where d_z is the distance of z from $[0,+\infty)$.

Let us now recall the well-known resolvent identity in $\mathcal{B}(L^2(\mathbb{R}^3))$:

$$(1 - \mathsf{v}R_z)^{-1} = 1 - \mathsf{v}R_z^{\mathsf{v}}. \tag{4.14}$$

Since the operators in both sides of the above identity are in $\mathcal{B}(H^{-1}(\mathbb{R}^3))$, it extends to $\mathcal{B}(H^{-1}(\mathbb{R}^3))$. By (4.6),

$$R_z^{\mathsf{v}} \in \mathcal{B}(H^{-1}(\mathbb{R}^3), H^1(\mathbb{R}^3)) \hookrightarrow \mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*, H^1(\mathbb{R}^3\backslash\Gamma));$$

by (4.2),

$$\mathsf{v} \in \mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma), H^1(\mathbb{R}^3\backslash\Gamma)^*);$$

then

$$(1 - \mathsf{v}R_z^{\mathsf{v}}) \in \mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*).$$

By (4.14), this implies that $1 - \mathsf{v}R_z$ is a bounded bijection from $H^1(\mathbb{R}^3\backslash\Gamma)^*$ onto itself. Therefore, by the Inverse Mapping Theorem, $(1-\mathsf{v}R_z)^{-1} \in \mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*)$ and (4.14) holds in $\mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*)$. ■

Remark 4.11. By Lemma 4.10,

$$\Lambda_z^{\mathsf{v}} | H^1(\mathbb{R}^3\backslash\Gamma) = (1 - \mathsf{v}R_z)^{-1}\mathsf{v}.$$

By $(1 - \mathsf{v}R_z)^{-1} \in \mathcal{B}(H^{-1}(\mathbb{R}^3))$ and by $\mathsf{v} \in \mathcal{B}(H^1(\mathbb{R}^3), H^{-1}(\mathbb{R}^3))$ one gets

$$(1 - \mathsf{v}R_z)^{-1}\mathsf{v} \in \mathcal{B}(H^1(\mathbb{R}^3), H^{-1}(\mathbb{R}^3)).$$

Thus, by (4.7) and (4.8),

$$\Lambda_z^{\mathsf{v}}|H^s(\mathbb{R}^3) = (1 - \mathsf{v}R_z)^{-1}\mathsf{v}, \quad 1 \le s \le 2.$$

By duality, similarly to Remark 4.6, $(1 - R_z\mathsf{v})^{-1} \in \mathcal{B}(H^1(\mathbb{R}^3))$ and (4.10) improves to

$$\Lambda_z^{\mathsf{v}}|H^s(\mathbb{R}^3) = \mathsf{v}(1 - \mathsf{v}R_z)^{-1}, \quad 0 \le s \le 1.$$

4.2. Boundary layer operators

We introduce the interior/exterior Dirichlet and Neumann trace operators

$$\gamma_0^{\text{in/ex}}\colon H^{s+1/2}(\Omega_{\text{in/ex}}) \to B^s_{2,2}(\Gamma), \quad s > 0,$$

$$\gamma_1^{\text{in/ex}}\colon H^{s+3/2}(\Omega_{\text{in/ex}}) \to B^s_{2,2}(\Gamma), \quad s > 0,$$

where $\Omega_{\text{in}} \equiv \Omega$ and $\Omega_{\text{ex}} := \Omega_{\text{ex}}$. The Besov-like trace spaces $B^s_{2,2}(\Gamma)$ identify with $H^s(\Gamma)$ when $|s| \le k + 1$ and Γ is of class $\mathcal{C}^{k,1}$ (see [9]). Then, we define, for any $s > 0$, the bounded linear operators

$$\gamma_0\colon H^{s+1/2}(\mathbb{R}^3\backslash\Gamma) \to B^s_{2,2}(\Gamma), \quad \gamma_0 u := \frac{1}{2}(\gamma_0^{\text{in}}(u|\Omega_{\text{in}}) + \gamma_0^{\text{ex}}(u|\Omega_{\text{ex}})), \tag{4.15}$$

$$\gamma_1\colon H^{s+3/2}(\mathbb{R}^3\backslash\Gamma) \to B^s_{2,2}(\Gamma), \quad \gamma_1 u := \frac{1}{2}(\gamma_1^{\text{in}}(u|\Omega_{\text{in}}) + \gamma_1^{\text{ex}}(u|\Omega_{\text{ex}})).$$

The corresponding trace jump bounded operators are defined by

$$[\gamma_0]\colon H^{s+1/2}(\mathbb{R}^3\backslash\Gamma) \to B^s_{2,2}(\Gamma), \quad [\gamma_0]u := \gamma_0^{\text{in}}(u|\Omega_{\text{in}}) - \gamma_0^{\text{ex}}(u|\Omega_{\text{ex}}),$$
$$[\gamma_1]\colon H^{s+3/2}(\mathbb{R}^3\backslash\Gamma) \to B^s_{2,2}(\Gamma), \quad [\gamma_1]u := \gamma_1^{\text{in}}(u|\Omega_{\text{in}}) - \gamma_1^{\text{ex}}(u|\Omega_{\text{ex}}).$$

By [15, Lemma 4.3], the trace maps $\gamma_1^{\text{in/ex}}$ can be extended to the spaces

$$H^1_\Delta(\Omega_{\text{in/ex}}) := \{u_{\text{in/ex}} \in H^1(\Omega_{\text{in/ex}})\colon \Delta_{\Omega_{\text{in/ex}}} u_{\text{in/ex}} \in L^2(\Omega_{\text{in/ex}})\}$$

as $H^{-1/2}(\Gamma)$-valued bounded operators:

$$\gamma_1^{\text{in/ex}}\colon H^1_\Delta(\Omega_{\text{in/ex}}) \to H^{-1/2}(\Gamma).$$

This gives the extensions of the maps γ_1 and $[\gamma_1]$ defined on

$$H^1_\Delta(\mathbb{R}^3\backslash\Gamma) := H^1_\Delta(\Omega_{\text{in}}) \oplus H^1_\Delta(\Omega_{\text{ex}})$$

with values in $H^{-1/2}(\Gamma)$.

Then, for any $z \in \mathbb{C}\backslash(-\infty, 0]$, one defines the single and double-layer operators

$$\mathrm{SL}_z := (\gamma_0 R_{\bar z})^* = R_z \gamma_0^* \in \mathcal{B}(B^{-s}_{2,2}(\Gamma), H^{3/2-s}(\mathbb{R}^3)), \quad s > 0, \tag{4.16}$$

$$\mathrm{DL}_z := (\gamma_1 R_{\bar z})^* = R_z \gamma_1^* \in \mathcal{B}(B^{-s}_{2,2}(\Gamma), H^{1/2-s}(\mathbb{R}^3)), \quad s > 0. \tag{4.17}$$

By (4.15), one has

$$S_z := \gamma_0 \,\mathrm{SL}_z \in \mathcal{B}((H^{s-1/2}(\Gamma), H^{s+1/2}(\Gamma))), \quad -1/2 < s < 1/2.$$

By the mapping properties of the double-layer operator, one gets[1] (see [15, Theorem 6.11])

$$\mathrm{DL}_z \in \mathcal{B}(H^{1/2}(\Gamma), H^1(\mathbb{R}^3\backslash\Gamma)).$$

Hence, by

$$(-(\Delta_{\Omega_{\rm in}} \oplus \Delta_{\Omega_{\rm ex}}) + z)\,\mathrm{DL}_z = 0,$$

one gets

$$\mathrm{DL}_z \in \mathcal{B}(H^{1/2}(\Gamma), H^1_\Delta(\mathbb{R}^3\backslash\Gamma)).$$

Thus

$$D_z := \gamma_1 \,\mathrm{DL}_z \in \mathcal{B}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma)).$$

These mapping properties can be extended to a larger range of Sobolev spaces (see [15, Theorem 6.12 and successive remarks]):

$$\mathrm{SL}_z \in \mathcal{B}(H^{s-1/2}(\Gamma), H^{s+1}(\mathbb{R}^3)), \qquad -1/2 \le s \le 1/2, \tag{4.18}$$

$$S_z \in \mathcal{B}(H^{s-1/2}(\Gamma), H^{s+1/2}(\Gamma)), \qquad -1/2 \le s \le 1/2, \tag{4.19}$$

$$\mathrm{DL}_z \in \mathcal{B}(H^{s+1/2}(\Gamma), H^{s+1}(\mathbb{R}^3\backslash\Gamma)), \qquad -1/2 \le s \le 1/2, \tag{4.20}$$

$$D_z \in \mathcal{B}(H^{s+1/2}(\Gamma), H^{s-1/2}(\Gamma)), \qquad -1/2 \le s \le 1/2, \tag{4.21}$$

and, whenever $s \ge 0$ in (4.18), (4.20) above, the following jump relations hold (see [15, Theorem 6.11]):

$$[\gamma_0]\,\mathrm{SL}_z = 0, \quad [\gamma_1]\,\mathrm{SL}_z = -1,$$
$$[\gamma_0]\,\mathrm{DL}_z = 1, \quad [\gamma_1]\,\mathrm{DL}_z = 0.$$

[1] here and below we can avoid the introduction of the cutoff function χ appearing in [15, Theorems 6.11–6.13] since we are dealing with the constant coefficients strongly elliptic operator $-\Delta + z$ (compare [15, Theorem 6.1] with [15, equation (6.10)])

Whenever the boundary Γ is of class $\mathcal{C}^{1,1}$ one gets an improvement as regards the regularity properties of the single- and double-layer operators (see [15, Theorem 6.13 and Corollary 6.14]):

$$\mathrm{SL}_z \in \mathcal{B}(H^{s-1/2}(\Gamma), H^{s+1}(\mathbb{R}^3\backslash\Gamma)), \quad 1/2 < s \leq 1, \tag{4.22}$$
$$\mathrm{DL}_z \in \mathcal{B}(H^{s+1/2}(\Gamma), H^{s+1}(\mathbb{R}^3\backslash\Gamma)), \quad 1/2 < s \leq 1. \tag{4.23}$$

By (4.16), (4.17), and (4.12) one has the following result.

Lemma 4.12. *For any* $z \in \varrho(\Delta + \mathrm{v}) \cap (\mathbb{C}\backslash(-\infty, 0])$,

$$\mathrm{SL}_z^{\mathrm{v}} := R_z^{\mathrm{v}} \gamma_0^* = \mathrm{SL}_z + R_z \Lambda_z^{\mathrm{v}} \mathrm{SL}_z \in \mathcal{B}(B_{2,2}^{-s}(\Gamma), H^{3/2-s}(\mathbb{R}^3)), \quad 0 < s \leq 3/2, \tag{4.24}$$
$$\mathrm{DL}_z^{\mathrm{v}} := R_z^{\mathrm{v}} \gamma_1^* = \mathrm{DL}_z + R_z \Lambda_z^{\mathrm{v}} \mathrm{DL}_z \in \mathcal{B}(B_{2,2}^{-s}(\Gamma), H^{1/2-s}(\mathbb{R}^3)), \quad 0 < s \leq 1/2.$$

By (4.18), (4.20), and (4.12), one has the following result.

Lemma 4.13. *We have*

$$\mathrm{SL}_z^{\mathrm{v}} \in \mathcal{B}(H^{s-1/2}(\Gamma), H^{s+1}(\mathbb{R}^3)), \quad -1/2 \leq s \leq 1/2, \tag{4.25}$$
$$\mathrm{DL}_z^{\mathrm{v}} \in \mathcal{B}(H^{s+1/2}(\Gamma), H^{s+1}(\mathbb{R}^3\backslash\Gamma)), \quad -1/2 \leq s \leq 1/2. \tag{4.26}$$

By (4.22), (4.23), and (4.12), one has

Lemma 4.14. *Let* $\Gamma \in \mathcal{C}^{1,1}$. *Then*

$$\mathrm{SL}_z^{\mathrm{v}} \in \mathcal{B}(H^{s-1/2}(\Gamma), H^{s+1}(\mathbb{R}^3\backslash\Gamma)), \quad 1/2 < s \leq 1,$$
$$\mathrm{DL}_z^{\mathrm{v}} \in \mathcal{B}(H^{s+1/2}(\Gamma), H^{s+1}(\mathbb{R}^3\backslash\Gamma)), \quad 1/2 < s \leq 1. \tag{4.27}$$

By either (4.24) or (4.25) one has

$$\gamma_0 \mathrm{SL}_z^{\mathrm{v}} = S_z + \gamma_0 R_z \Lambda_z^{\mathrm{v}} \mathrm{SL}_z \in \mathcal{B}(H^{s-1/2}(\Gamma), H^{s+1/2}(\Gamma)), \quad -1/2 < s < 1/2. \tag{4.28}$$

Since $\gamma_0 R_z = (R_{\bar z} \gamma_0^*)^* = \mathrm{SL}_{\bar z}^*$, one gets the following improvement of (4.28).

Lemma 4.15. *We have*

$$S_z^{\mathrm{v}} := S_z + \mathrm{SL}_{\bar z}^* \Lambda_z^{\mathrm{v}} \mathrm{SL}_z \in \mathcal{B}(H^{s-1/2}(\Gamma), H^{s+1/2}(\Gamma)), \quad -1/2 \leq s \leq 1/2.$$

Proof. By (4.18) and duality, $\mathrm{SL}_{\bar z}^* \in \mathcal{B}(H^{-1-s}(\mathbb{R}^3), H^{1/2-s}(\Gamma))$. The proof is then concluded by (4.19), (4.11), and (4.18). ■

If $\Gamma \in \mathcal{C}^{1,1}$, then, by (4.27),

$$\gamma_1 \mathrm{DL}_z^{\mathrm{v}} = D_z + \gamma_1 R_z \Lambda_z^{\mathrm{v}} \mathrm{DL}_z \in \mathcal{B}(H^{s+1/2}(\Gamma), H^{s-1/2}(\Gamma)), \quad 1/2 < s \leq 1. \tag{4.29}$$

Since $\gamma_1 R_z = (R_{\bar{z}}\gamma_1^*)^* = \mathrm{DL}_{\bar{z}}^*$, one can improve (4.29) even without requiring $\Gamma \in \mathcal{C}^{1,1}$.

Lemma 4.16. *We have*

$$D_z^{\mathsf{v}} := D_z + \mathrm{DL}_{\bar{z}}^* \,\Lambda_z^{\mathsf{v}}\, \mathrm{DL}_z \in \mathcal{B}(H^{s+1/2}(\Gamma), H^{s-1/2}(\Gamma)), \quad -1/2 \le s \le 1/2.$$

Proof. By (4.20) and duality, $\mathrm{DL}_{\bar{z}}^* \in \mathcal{B}(H^{s+1}(\mathbb{R}^3\backslash\Gamma)^*, H^{-s-1/2}(\Gamma))$. The proof is then concluded by (4.21), (4.11), and (4.20). ∎

In order to prove the jump relations of the double-layer operator relative to $\Delta + \mathsf{v}$ we need a technical result.

Lemma 4.17. *If $v \in H^1(\mathbb{R}^3\backslash\Gamma)^*$, then we have $[\gamma_1]R_z v = 0$ in $H^{-1/2}(\Gamma)$ for any $z \in \mathbb{C}\backslash(-\infty, 0]$.*

Proof. At first let us notice that it suffices to show that the result holds for a single $z \in \mathbb{C}\backslash(-\infty, 0]$. Indeed, by the resolvent identity $R_w v = R_z v + (z-w)R_w R_z v$, one gets $R_w R_z v \in H^3(\mathbb{R}^3) \subset \ker([\gamma_1])$. In particular, we choose z such that $\ker(S_z) = \{0\}$ (see, e.g., Lemma (4.19) below).

Given $v \in H^1(\mathbb{R}^3\backslash\Gamma)^* = H^{-1}_{\overline{\Omega}}(\mathbb{R}^3) \oplus H^{-1}_{\Omega^c}(\mathbb{R}^3) \subseteq H^{-1}(\mathbb{R}^3)$ and $\chi \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^3)$ such that $\chi = 1$ on a compact set containing an open neighborhood of $\overline{\Omega}$, let us set $u := \chi R_z v$. Since $\gamma_1^{\mathrm{in/ex}} u = \gamma_1^{\mathrm{in/ex}} R_z v$, it suffices to show that $[\gamma_1]u = 0$. Let us define $u_{\mathrm{in/ex}} := \chi R_z v|\Omega_{\mathrm{in/ex}} \in H^1(\Omega_{\mathrm{in/ex}})$, $f_{\mathrm{in/ex}} := ((-\Delta + z)\chi R_z v)|\Omega_{\mathrm{in/ex}} \in H^1(\Omega_{\mathrm{in/ex}})$ and $g_{\mathrm{in/ex}} := \gamma_0^{\mathrm{in/ex}} u_{\mathrm{in/ex}} \in H^{1/2}(\Gamma)$. Then $u_{\mathrm{in/ex}}$ solves the Dirichlet boundary value problems

$$\begin{cases} (-\Delta_{\Omega_{\mathrm{in/ex}}} + z)u_{\mathrm{in/ex}} = f_{\mathrm{in/ex}}, \\ \gamma_0^{\mathrm{in/ex}} u_{\mathrm{in/ex}} = g_{\mathrm{in/ex}}, \end{cases}$$

and so, by [15, Theorems 7.5 and 7.15] (notice that both u_{ex} and f_{ex} have a compact support; in particular, the radiation condition $\mathcal{M}u_{\mathrm{ex}} = 0$ there required is here satisfied), $\psi_{\mathrm{in/ex}} := \gamma_1^{\mathrm{in/ex}} u_{\mathrm{in/ex}} \in H^{-1/2}(\Gamma)$ satisfy the equations

$$S_z \psi_{\mathrm{in/ex}} = \frac{1}{2}(1 + D_z) g_{\mathrm{in/ex}} - \gamma_0 R_z v.$$

Since $u_{\mathrm{in}} \oplus u_{\mathrm{ex}} = \chi R_z v \in H^1(\mathbb{R}^3)$, one has $g_{\mathrm{in}} = g_{\mathrm{ex}}$ and so $[\gamma_1]R_z v = \psi_{\mathrm{in}} - \psi_{ex} = 0$ is consequence of $\ker(S_z) = \{0\}$. ∎

Lemma 4.18. *If $s \ge 0$ in (4.25) and (4.26), then*

$$[\gamma_0]\,\mathrm{SL}_z^{\mathsf{v}} = 0, \quad [\gamma_1]\,\mathrm{SL}_z^{\mathsf{v}} = -1, \tag{4.30}$$

$$[\gamma_0]\,\mathrm{DL}_z^{\mathsf{v}} = 1, \quad [\gamma_1]\,\mathrm{DL}_z^{\mathsf{v}} = 0. \tag{4.31}$$

Proof. $[\gamma_0]\,\mathrm{SL}_z^{\mathsf{v}}=0$ is consequence of $\mathrm{ran}(\mathrm{SL}_z^{\mathsf{v}})\subseteq H^1(\mathbb{R}^3)$ and, by (4.12), one gets $\mathrm{ran}(R_z\Lambda_z^{\mathsf{v}}\,\mathrm{DL}_z)\subseteq H^1(\mathbb{R}^3)$; so $[\gamma_0]\,\mathrm{DL}_z^{\mathsf{v}}=[\gamma_0]\,\mathrm{DL}_z+[\gamma_0]R_z\Lambda_z^{\mathsf{v}}\,\mathrm{DL}_z=[\gamma_0]\,\mathrm{DL}_z=1$.

Since

$$\Lambda_z^{\mathsf{v}}\,\mathrm{SL}_z\in\mathcal{B}(H^{s-1/2}(\Gamma),H^{1-s}(\mathbb{R}^3\backslash\Gamma)^*)$$

and

$$\Lambda_z^{\mathsf{v}}\,\mathrm{DL}_z\in\mathcal{B}(H^{s+1/2}(\Gamma),H^{1-s}(\mathbb{R}^3\backslash\Gamma)^*),$$

by Lemma 4.17 one gets

$$[\gamma_1]\,\mathrm{SL}_z^{\mathsf{v}}=[\gamma_1]\,\mathrm{SL}_z+[\gamma_1]R_z\Lambda_z^{\mathsf{v}}\,\mathrm{SL}_z=[\gamma_1]\,\mathrm{SL}_z=-1$$

and

$$[\gamma_1]\,\mathrm{DL}_z^{\mathsf{v}}=[\gamma_1]\,\mathrm{DL}_z+[\gamma_1]R_z\Lambda_z^{\mathsf{v}}\,\mathrm{DL}_z=[\gamma_1]\,\mathrm{DL}_z=0. \qquad ■$$

When $\mathsf{v}=0$, it is well known that the boundary layer operators have bounded inverses. This property is next extended to the operators relative to $\Delta+\mathsf{v}$.

Lemma 4.19. *There exist $Z^\circ_{\mathsf{v},d}$ and $Z^\circ_{\mathsf{v},n}$, not empty open subsets of $\varrho(\Delta+\mathsf{v})$, such that*

$$(S_z^{\mathsf{v}})^{-1}\in\mathcal{B}(H^{1/2}(\Gamma),\ H^{-1/2}(\Gamma))\quad\textit{for all } z\in Z^\circ_{\mathsf{v},d},$$
$$(D_z^{\mathsf{v}})^{-1}\in\mathcal{B}(H^{-1/2}(\Gamma),\ H^{1/2}(\Gamma))\quad\textit{for all } z\in Z^\circ_{\mathsf{v},n}.$$

In particular, there exists $\lambda_{\mathsf{v}}>\sup\sigma(\Delta+\mathsf{v})$ such that $[\lambda_{\mathsf{v}},+\infty)\subset Z^\circ_{\mathsf{v},d}\cap Z^\circ_{\mathsf{v},n}$; furthermore, $Z^\circ_{\mathsf{v},d}\cap Z^\circ_{0,d}\neq\varnothing$, $Z^\circ_{\mathsf{v},n}\cap Z^\circ_{0,n}\neq\varnothing$, and both $Z^\circ_{\mathsf{v},d}$ and $Z^\circ_{\mathsf{v},n}$ can be chosen to be symmetric with respect to the real axis.

Proof. At first, let us notice that it suffices to show that the bounded inverses exist for any real $\lambda\geq\lambda_{\mathsf{v}}$ for some $\lambda_{\mathsf{v}}>\sup\sigma(\Delta+\mathsf{v})$. Then, by the continuity of the maps $z\mapsto S_z^{\mathsf{v}}$ and $z\mapsto D_z^{\mathsf{v}}$, the bounded inverses exist in a complex open neighborhood of $[\lambda_{\mathsf{v}},+\infty)$.

We proceed as in the proof of [12, Lemma 3.2]. By

$$(-(\Delta+\mathsf{v})+\lambda)\,\mathrm{SL}_\lambda^{\mathsf{v}}\,|\Omega_{\mathrm{in/ex}}=0,$$

by Green's formula and by (4.30), one gets, for any $\phi\in H^{-1/2}(\Gamma)$,

$$\begin{aligned}0&=\|\nabla\,\mathrm{SL}_\lambda^{\mathsf{v}}\,\phi\|^2_{L^2(\mathbb{R}^3)}-\langle\mathsf{v}\,\mathrm{SL}_\lambda^{\mathsf{v}}\,\phi,\mathrm{SL}_\lambda^{\mathsf{v}}\,\phi\rangle_{H^{-1}(\mathbb{R}^3),H^1(\mathbb{R}^3)}+\lambda\,\|\,\mathrm{SL}_\lambda^{\mathsf{v}}\,\phi\|^2_{L^2(\mathbb{R}^3)}\\&\quad+\langle[\gamma_1]\,\mathrm{SL}_\lambda^{\mathsf{v}}\,\phi,\gamma_0\,\mathrm{SL}_\lambda\,\phi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\\&=\|\nabla\,\mathrm{SL}_\lambda^{\mathsf{v}}\,\phi\|^2_{L^2(\mathbb{R}^3)}-\langle\mathsf{v}\,\mathrm{SL}_\lambda^{\mathsf{v}}\,\phi,\mathrm{SL}_\lambda^{\mathsf{v}}\,\phi\rangle_{H^{-1}(\mathbb{R}^3),H^1(\mathbb{R}^3)}+\lambda\,\|\,\mathrm{SL}_\lambda^{\mathsf{v}}\,\phi\|^2_{L^2(\mathbb{R}^3)}\\&\quad-\langle\phi,S_\lambda^{\mathsf{v}}\phi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}.\end{aligned}$$

Then, by (4.13),

$$\begin{aligned}&\langle\phi,\gamma_0 S^{\mathsf{v}}_\lambda\phi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\\&\quad\ge(1-c_1\epsilon)\|\nabla\,\mathrm{SL}^{\mathsf{v}}_\lambda\,\phi\|^2_{L^2(\mathbb{R}^3)}+(\lambda-c_2(1+\epsilon^{-3}))\|\,\mathrm{SL}^{\mathsf{v}}_\lambda\,\phi\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Choosing $\epsilon>0$ such that $c_1\epsilon<1$ and then $\lambda\in\varrho(\Delta+\mathsf{v})$ such that $\lambda>c_2(1+\epsilon^{-3})$ (this is always possible since $\Delta+\mathsf{v}$ in bounded from above), one gets

$$\langle\phi,S^{\mathsf{v}}_\lambda\phi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\gtrsim\|\,\mathrm{SL}^{\mathsf{v}}_\lambda\,\phi\|^2_{H^1(\mathbb{R}^3)}.$$

By $\mathsf{v}\in\mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma),H^1(\mathbb{R}^3\backslash\Gamma)^*)$, Green's formula applies to a couple $u_{\text{in/ex}},v_{\text{in/ex}}\in H^1(\Omega_{\text{in/ex}})$ with $\Delta u_{\text{in/ex}}\in L^2(\Omega_{\text{in/ex}})$,

$$\begin{aligned}&\langle(-(\Delta+\mathsf{v})+\lambda)u_{\text{in/ex}},v_{\text{in/ex}}\rangle_{H^1(\Omega_{\text{in/ex}})^*,H^1(\Omega_{\text{in/ex}})}\\&\quad=\langle\nabla u_{\text{in/ex}},\nabla v_{\text{in/ex}}\rangle_{L^2(\Omega_{\text{in/ex}})}\\&\qquad-\langle\mathsf{v}u_{\text{in/ex}},v_{\text{in/ex}}\rangle_{H^1(\Omega_{\text{in/ex}})^*,H^1(\Omega_{\text{in/ex}})}+\lambda\,\langle u_{\text{in/ex}},v_{\text{in/ex}}\rangle_{L^2(\Omega_{\text{in/ex}})}\\&\qquad\pm\langle\gamma_1^{\text{in/ex}}u_{\text{in/ex}},\gamma_0^{\text{in/ex}}v_{\text{in/ex}}\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}.\end{aligned}\tag{4.32}$$

By

$$\left|\langle\mathsf{v}u_{\text{in/ex}},v_{\text{in/ex}}\rangle_{H^1(\Omega_{\text{in/ex}})^*,H^1(\Omega_{\text{in/ex}})}\right|\lesssim\|u_{\text{in/ex}}\|_{H^1(\Omega_{\text{in/ex}})}\|v_{\text{in/ex}}\|_{H^1(\Omega_{\text{in/ex}})},$$

equation (4.32) gives

$$\begin{aligned}&|\langle\gamma_1^{\text{in/ex}}u_{\text{in/ex}},\gamma_0^{\text{in/ex}}v_{\text{in/ex}}\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}|\\&\quad\lesssim(\|u_{\text{in/ex}}\|_{H^1(\Omega_{\text{in/ex}})}+\|(-(\Delta+\mathsf{v})+\lambda)u_{\text{in/ex}}\|_{H^1(\Omega_{\text{in/ex}})^*})\|v_{\text{in/ex}}\|_{H^1(\Omega_{\text{in/ex}})}.\end{aligned}$$

Since $\gamma_0^{\text{in/ex}}\colon H^1(\Omega_{\text{in/ex}})\to H^{1/2}(\Gamma)$ is surjective, finally one gets

$$\|\gamma_1^{\text{in/ex}}u_{\text{in/ex}}\|_{H^{-1/2}(\Gamma)}\lesssim\|u_{\text{in/ex}}\|_{H^1(\Omega_{\text{in/ex}})}+\|(-(\Delta+\mathsf{v})+\lambda)u_{\text{in/ex}}\|_{H^1(\Omega_{\text{in/ex}})^*}.\tag{4.33}$$

Then, proceeding as in [12, Lemma 3.2] (compare (3.31) there with (4.33) here), this yields

$$\langle\phi,S^{\mathsf{v}}_\lambda\phi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\gtrsim\|\phi\|^2_{H^{-1/2}(\Gamma)}$$

and so $(S^{\mathsf{v}}_\lambda)^{-1}\in\mathcal{B}(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))$ by the Lax–Milgram theorem.

As regards D^{v}_λ, the proof is almost the same. By $(-(\Delta+\mathsf{v})+\lambda)\,\mathrm{DL}^{\mathsf{v}}_\lambda\,|\Omega_{\text{in/ex}}=0$, by Green's formula and by (4.31), one gets, for any $\phi\in H^{1/2}(\Gamma)$,

$$\begin{aligned}0&=\|\nabla\,\mathrm{DL}^{\mathsf{v}}_\lambda\,\phi\|^2_{L^2(\Omega_{\text{in}})}+\|\nabla\,\mathrm{DL}^{\mathsf{v}}_\lambda\,\phi\|^2_{L^2(\Omega_{\text{ex}})}\\&\quad-\langle\mathsf{v}\,\mathrm{DL}^{\mathsf{v}}_\lambda\,\phi,\mathrm{DL}^{\mathsf{v}}_\lambda\,\phi\rangle_{H^1(\mathbb{R}^3\backslash\Gamma)^*,H^1(\mathbb{R}^3\backslash\Gamma)}+\lambda\,\|\,\mathrm{DL}^{\mathsf{v}}_\lambda\,\phi\|^2_{L^2(\mathbb{R}^3)}\\&\quad+\langle D^{\mathsf{v}}_\lambda\phi,\phi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)},\end{aligned}$$

which leads to

$$-\langle D^{\mathsf{v}}_{\lambda}\phi,\phi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \gtrsim \|\,\mathrm{DL}^{\mathsf{v}}_{\lambda}\,\phi\|^2_{H^1(\mathbb{R}^3\backslash\Gamma)}.$$

Then, proceeding as in [12, Lemma 3.2], by (4.33), this yields

$$-\langle D^{\mathsf{v}}_{\lambda}\phi,\phi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \gtrsim \|\phi\|^2_{H^{1/2}(\Gamma)}$$

and so $(D^{\mathsf{v}}_{\lambda})^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))$ by the Lax–Milgram theorem. ■

5. Laplacians with regular and singular perturbations

Here we apply the abstract results in Section 2, presenting various examples were the self-adjoint operator A is the free Laplacian $\Delta\colon H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ and $A_{B_1} = \Delta + \mathsf{v}$. All over this section we consider a Kato–Rellich potential $\mathsf{v} = \mathsf{v}_2 + \mathsf{v}_\infty$ of short-range type, i.e.,

$$\mathsf{v}_2 \in L^2(\mathbb{R}^3),\quad \operatorname{supp}(\mathsf{v}_2)\text{ bounded},\quad |\mathsf{v}_\infty(x)| \lesssim (1+|x|)^{-\kappa(1+\epsilon)},\quad \kappa \ge 1,\quad \epsilon > 0. \tag{5.1}$$

We take

$$\mathfrak{h}_1 = H^2(\mathbb{R}^3) \hookrightarrow \mathfrak{b}_1 = H^1(\mathbb{R}^3\backslash\Gamma) \hookrightarrow \mathfrak{h}_1^\circ = L^2(\mathbb{R}^3),$$

and, introducing the multiplication operator $\langle x\rangle$ by $\langle x\rangle u\colon x \mapsto (1+|x|^2)^{1/2}u(x)$, we define

$$\tau_1\colon H^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3),\quad \tau_1 u := \langle x\rangle^{-s}u,\quad s \ge 0, \tag{5.2}$$

and

$$B_1 u := \langle x\rangle^{2s}\mathsf{v}u,\quad 2s < 1+\epsilon. \tag{5.3}$$

Further, we take either

$$\tau_2 = \gamma_0\colon H^2(\mathbb{R}^3) \to \mathfrak{h}_2 = B^{3/2}_{2,2}(\Gamma) \hookrightarrow \mathfrak{b}_2 = H^{s_\circ}(\Gamma),\quad 0 < s_\circ \le 1/2, \tag{5.4}$$

or

$$\tau_2 = \gamma_1\colon H^2(\mathbb{R}^3) \to \mathfrak{h}_2 = H^{1/2}(\Gamma) \hookrightarrow \mathfrak{b}_2 = H^{-1/2}(\Gamma). \tag{5.5}$$

Hence, by what is recalled in Section 4.2, either $G^2_z = \mathrm{SL}_z$ or $G^2_z = \mathrm{DL}_z$ and either

$$\tau G_z(u \oplus \phi) = \langle x\rangle^{-s}R_z\langle x\rangle^{-s}u + S_z\phi$$

or

$$\tau G_z(u \oplus \phi) = \langle x\rangle^{-s}R_z\langle x\rangle^{-s}u + D_z\phi.$$

Thus, (2.2) holds. Notice that $\gamma_0^*\phi$ and $\gamma_1^*\phi$, whenever $\phi \in L^2(\Gamma)$, identify with the tempered distributions which act on a test function f respectively as

$$(\phi\delta_\Gamma)f := \int_\Gamma \phi(x) f(x)\, d\sigma_\Gamma(x), \quad (\phi\delta'_\Gamma)f := \int_\Gamma \phi(x)\nu(x)\cdot\nabla f(x)\, d\sigma_\Gamma(x),$$

where ν is the exterior normal to Γ. By a slight abuse of notation, in the following we set $\gamma_0^*\phi \equiv \phi\delta_\Gamma$ and $\phi\gamma_0^* \equiv \delta'_\Gamma\phi$ and so, either

$$\tau^*(u \oplus \phi) = \langle x\rangle^{-s}u + \phi\delta_\Gamma$$

or

$$\tau^*(u \oplus \phi) = \langle x\rangle^{-s}u + \phi\delta'_\Gamma.$$

In this framework, given a couple of linear operators B_0 and B_2 as in (2.3) and such that the triple $\mathsf{B} = (B_0, B_1, B_2)$ satisfies the hypotheses in Theorem 2.1, equation (2.7) defines a self-adjoint operator Δ_B representing a Laplacian with a Kato–Rellich potential and a distributional one supported on Γ. Let us remark that, although τ_1 and B_1 depend on the index s, the operator Δ_B is s-independent whenever B_0 and B_2 are (see the next subsections). The choice $s \neq 0$ is a technical trick which we use to obtain LAP and a representation formula for the scattering couple $(\Delta_\mathsf{B}, \Delta)$; whenever one is only interested in providing a resolvent formula for Δ_B, then the choice $s = 0$ is preferable. In particular, the resolvent formula for Δ_B holds in the setting $s = 0$ for any Kato–Rellich potential.

5.1. The Schrödinger operator

By our hypotheses on v, one has $\langle x\rangle^{2s}\mathsf{v} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ and so, by Lemma 4.1,

$$B_1 \in \mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma), H^1(\mathbb{R}^3\backslash\Gamma)^*).$$

Considering the weight $\varphi(x) = (1 + |x|^2)^{w/2}$, $w \in \mathbb{R}$, we use the notation $L^2_\varphi(\mathbb{R}^3) \equiv L^2_w(\mathbb{R}^3)$; $H^k_w(\mathbb{R}^3)$, $H^k_w(\mathbb{R}^3\backslash\Gamma)$ denotes the corresponding scales of weighted Sobolev spaces.

Since

$$\langle x\rangle^w \in \mathcal{B}(H^1_{w'}(\mathbb{R}^3\backslash\Gamma), H^1_{w'-w}(\mathbb{R}^3\backslash\Gamma))$$

and, by duality,

$$\langle x\rangle^w \in \mathcal{B}(H^1_{w'}(\mathbb{R}^3\backslash\Gamma)^*, H^1_{w'+w}(\mathbb{R}^3\backslash\Gamma)^*),$$

one gets

$$\langle x\rangle^{-w-2s}B_1\langle x\rangle^w = \mathsf{v} \in \mathcal{B}(H^1_w(\mathbb{R}^3\backslash\Gamma), H^1_{-w-2s}(\mathbb{R}^3\backslash\Gamma)^*). \tag{5.6}$$

Since

$$R_z \in \mathcal{B}(H_w^{-1}(\mathbb{R}^3), H_w^1(\mathbb{R}^3)) \hookrightarrow \mathcal{B}(H_{-w}^1(\mathbb{R}^3\backslash\Gamma)^*, H_w^1(\mathbb{R}^3\backslash\Gamma)), \tag{5.7}$$

one has

$$\tau_1 G_z^1 = \langle x\rangle^{-s} R_z \langle x\rangle^{-s} \in \mathcal{B}(H_{-w}^1(\mathbb{R}^3\backslash\Gamma)^*, H_{w+2s}^1(\mathbb{R}^3\backslash\Gamma)).$$

In particular, this gives

$$\tau_1 G_z^1 \in \mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*), H^1(\mathbb{R}^3\backslash\Gamma)).$$

For $0 \le 2s < 1+\epsilon$ we define

$$\begin{aligned} M_z^{B_1} &= 1 - B_1\tau_1 G_z^1 = 1 - \langle x\rangle^s \mathsf{v} R_z \langle x\rangle^{-s} \\ &= \langle x\rangle^s (1-\mathsf{v}R_z)\langle x\rangle^{-s} \in \mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*). \end{aligned}$$

Lemma 5.1. *Let* v *be as in* (5.1)*, with* $\kappa = 1$*. Then, for* s *such that* $0 \le 2s < 1+\epsilon$ *and for* $z \in \mathbb{C}$ *sufficiently far away from* $(-\infty, 0]$,

$$(1-\mathsf{v}R_z)^{-1} \in \mathcal{B}(H_{-s}^1(\mathbb{R}^3\backslash\Gamma)^*).$$

Equivalently,

$$(M_z^{B_1})^{-1} \in \mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*).$$

Proof. Here we use the same kind of arguments as in the second part of the proof of Lemma 4.10. Thus, we start from the resolvent identity

$$(1-\mathsf{v}R_z)^{-1} = 1 - \mathsf{v}R_z^{\mathsf{v}}. \tag{5.8}$$

By Lemma 4.10, such an equality holds in $\mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*)$. By (4.6),

$$\begin{aligned} R_z^{\mathsf{v}} \in \mathcal{B}(H^{-1}(\mathbb{R}^3), H^1(\mathbb{R}^3)) &\hookrightarrow \mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*, H^1(\mathbb{R}^3\backslash\Gamma)) \\ &\hookrightarrow \mathcal{B}(H_{-s}^1(\mathbb{R}^3\backslash\Gamma)^*, H_{-s}^1(\mathbb{R}^3\backslash\Gamma)); \end{aligned}$$

by (5.6),

$$\mathsf{v} \in \mathcal{B}(H_{-s}^1(\mathbb{R}^3\backslash\Gamma), H_{-s}^1(\mathbb{R}^3\backslash\Gamma)^*);$$

then

$$(1-\mathsf{v}R_z^{\mathsf{v}}) \in \mathcal{B}(H_{-s}^1(\mathbb{R}^3\backslash\Gamma)^*).$$

Analogously,

$$(1-\mathsf{v}R_z) \in \mathcal{B}(H_{-s}^1(\mathbb{R}^3\backslash\Gamma)^*).$$

By (5.8), this implies that $1-\mathsf{v}R_z$ is a bounded bijection from $H_{-s}^1(\mathbb{R}^3\backslash\Gamma)^*$ onto itself. Therefore, by the Inverse Mapping Theorem, $(1-\mathsf{v}R_z)^{-1} \in \mathcal{B}(H_{-s}^1(\mathbb{R}^3\backslash\Gamma)^*)$ and (4.14) holds in $\mathcal{B}(H_{-s}^1(\mathbb{R}^3\backslash\Gamma)^*)$. ∎

Choosing $\mathsf{B} = (1, B_1, 0)$, and whenever $Z_{B_1} \neq \varnothing$, by Corollary 2.7 the operator $\Delta_{B_1} := \Delta_{(1,B_1,0)}$ is defined according to the relation

$$R_z^{B_1} := (-\Delta_{B_1} + z)^{-1} = R_z + R_z \langle x\rangle^{-s} (M_z^{B_1})^{-1} B_1 \langle x\rangle^{-s} R_z,$$

where $z \in Z_{B_1} = \varrho(\Delta_{B_1}) \cap (\mathbb{C}\backslash(-\infty, 0])$. By Lemma 5.1, $Z_{B_1} \neq \varnothing$ and by the relation

$$\Lambda_z^{B_1} = (M_z^{B_1})^{-1} B_1 = \langle x\rangle^s (1 - \mathsf{v} R_z)^{-1} \langle x\rangle^s \mathsf{v} = \langle x\rangle^s \Lambda_z^{\mathsf{v}} \langle x\rangle^s, \tag{5.9}$$

one has

$$\Lambda_z^{B_1} \in \mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma), H^1(\mathbb{R}^3\backslash\Gamma)^*). \tag{5.10}$$

Therefore, Theorem 4.2 (see also Remark 4.3) yields

$$R_z^{\mathsf{v}} = (-(\Delta + \mathsf{v}) + z)^{-1} = R_z + R_z \Lambda_z^{\mathsf{v}} R_z = (-\Delta_{B_1} + z)^{-1}$$

for $z \in \varrho(\Delta + \mathsf{v}) \cap \mathbb{C}\backslash(-\infty, 0]$. The above relation shows that Δ_{B_1} coincides with the Schrödinger operator $\Delta + \mathsf{v}$ provided by the Kato–Rellich theorem. This also shows that Δ_{B_1} is s-independent. Nevertheless, the operator $\Lambda_z^{B_1}$ depends on the choice of s and the relations (5.9) and (5.10) with $s \neq 0$ are key objects in our analysis of LAP and scattering theory in the general case.

5.2. Asymptotic completeness and scattering matrix

Before discussing the validity of our assumptions, we provide the following general results on the scattering couple $(\Delta_{\mathsf{B}}, \Delta)$.

Theorem 5.2. *Assume* (5.1) *with* $\kappa = 1$ *and let* τ_1, τ_2 *and* B_1 *be defined as in* (5.2)–(5.5). *If* B *is such that* (H1)–(H6) *hold, then the scattering couple* $(\Delta_{\mathsf{B}}, \Delta)$ *is asymptotically complete.*

Proof. By hypothesis (5.1) with $\kappa = 1$, it is well known that for $\Delta_{B_1} = \Delta + \mathsf{v}$ one has $\sigma_{\mathrm{ess}}(\Delta_{B_1}) = (-\infty, 0]$; moreover, by [1, Theorem 3.1], $\sigma_{\mathrm{p}}(\Delta_{B_1}) \cap (-\infty, 0)$ is discrete in $(-\infty, 0)$. Hence, by [1, Theorem 4.2], $e(\Delta_{B_1}) \cap (-\infty, 0)$ is countable with $\{0\}$ as the eventual set of accumulations points. Therefore, by Theorem 3.5, $\sigma_{\mathrm{sc}}(\Delta_{\mathsf{B}}) = \varnothing$ and $(\Delta_{\mathsf{B}}, \Delta)$ is asymptotically complete. ■

In the framework of this section, Theorem 3.10 rephrases as follows.

Theorem 5.3. *Assume* (5.1) *with* $\kappa = 1$ *and let* τ_1, τ_2 *and* B_1 *be defined as in* (5.2)–(5.5). *If* B *is such that* (H1)–(H7) *hold, then the scattering matrix of the couple* $(\Delta_{\mathsf{B}}, \Delta)$ *has the representation*

$$\mathsf{S}_\lambda^{\mathsf{B}} = 1 - 2\pi i\, \mathsf{L}_\lambda \mathbf{\Lambda}_\lambda^{\mathsf{B},+} \mathsf{L}_\lambda^*, \quad \lambda \in (-\infty, 0] \cap (\mathbb{R}\backslash e(\Delta_{\mathsf{B}})),$$

where

$$\boldsymbol{\Lambda}^{\mathsf{B},\pm}_{\lambda} = \lim_{\epsilon\searrow 0} \boldsymbol{\Lambda}^{\mathsf{B}}_{\lambda\pm i\epsilon},$$

the limit existing in $\mathcal{B}(H^1_s(\mathbb{R}^3\backslash\Gamma) \oplus H^t(\Gamma), H^1_s(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{-t}(\Gamma))$,

$$\boldsymbol{\Lambda}^{\mathsf{B}}_z := \begin{bmatrix} \Lambda^{\mathsf{V}}_z + \Lambda^{\mathsf{V}}_z G^2_z \widehat{\Lambda}^{\mathsf{B}}_z (G^2_{\bar z})^* \Lambda^{\mathsf{V}}_z & \Lambda^{\mathsf{V}}_z G^2_z \widehat{\Lambda}^{\mathsf{B}}_z \\ \widehat{\Lambda}^{\mathsf{B}}_z (G^2_{\bar z})^* \Lambda^{\mathsf{V}}_z & \widehat{\Lambda}^{\mathsf{B}}_z; \end{bmatrix}$$
$$= \left(1 + \begin{bmatrix} \Lambda^{\mathsf{V}}_z & 0 \\ 0 & \widehat{\Lambda}^{\mathsf{B}}_z \end{bmatrix} \begin{bmatrix} G^2_z \widehat{\Lambda}^{\mathsf{B}}_z (G^2_{\bar z})^* & G^2_z \\ (G^2_{\bar z})^* & 0 \end{bmatrix}\right) \begin{bmatrix} \Lambda^{\mathsf{V}}_z & 0 \\ 0 & \widehat{\Lambda}^{\mathsf{B}}_z \end{bmatrix}$$

and

$$\mathsf{L}_\lambda \colon H^1_s(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{-t}(\Gamma) \to (L^2(M)_{\mathrm{ac}})_\lambda, \quad \mathsf{L}_\lambda(u \oplus \phi) := \frac{|\lambda|^{\frac14}}{2^{\frac12}}(\mathsf{L}^1_\lambda u + \mathsf{L}^2_\lambda \phi),$$

with

$$G^2_z = \mathrm{SL}_z, \quad t = s_\circ, \quad \textit{whenever } \tau_2 = \gamma_0,$$
$$G^2_z = \mathrm{DL}_z, \quad t = \frac{1}{2}, \quad \textit{whenever } \tau_2 = \gamma_1,$$
$$\mathsf{L}^1_\lambda u(\xi) := \hat u(|\lambda|^{1/2}\xi), \quad \mathsf{L}^2_\lambda \phi(\xi) := \frac{1}{(2\pi)^{\frac32}} \langle \tau_2(\chi u^\xi_\lambda), \phi\rangle_{H^t(\Gamma), H^{-t}(\Gamma)}.$$

Here $\hat u$ *denotes the Fourier transform of* u *and* $u^\xi_\lambda(x) := e^{i\,|\lambda|^{\frac12}\xi\cdot x}$ *is the plane wave with direction* ξ *in the 2-dimensional unitary sphere* $\mathbb{S}^2 \subset \mathbb{R}^3$ *and wavenumber* $|\lambda|^{\frac12}$*;* $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^3)$ *is such that* $\chi|\Gamma = 1$.

Proof. Taking into account the definition in (3.17), let us set

$$\mathsf{L}_\lambda(u\oplus\phi) := -\mathscr{L}_\lambda(\langle x\rangle^s u \oplus \phi) = -(\mu-\lambda)(FG_\mu(\langle x\rangle^s u \oplus \phi))_\lambda$$
$$= -(\mu-\lambda)(FR_\mu \tau_1^* \langle x\rangle^s u + FR_\mu \tau_2^* \phi))_\lambda.$$

The unitary map $F\colon L^2(\mathbb{R}^3) \to \int^{\oplus}_{(-\infty,0)} L^2(\mathbb{S}^2)\, d\lambda \equiv L^2((-\infty,0); L^2(\mathbb{S}^2))$ diagonalizing $A = \Delta$ is given by

$$(Fu)_\lambda(\xi) := -\frac{|\lambda|^{\frac14}}{2^{\frac12}}\, \hat u(|\lambda|^{1/2}\xi).$$

Therefore, by $(\mu-\lambda)\widehat{R_\mu f}(|\lambda|^{1/2}\xi) = -\hat f(|\lambda|^{1/2}\xi)$, one gets

$$(\mu-\lambda)(FR_\mu\tau_1^*\langle x\rangle^s u)_\lambda(\xi) = -\frac{|\lambda|^{\frac14}}{2^{\frac12}} \hat u(|\lambda|^{1/2}\xi).$$

This gives L^1_λ. As regards L^2_λ, the computation was given in [11, Theorem 5.1].

The results about $\boldsymbol{\Lambda}^{\mathsf{B}}_z$ are direct consequences of the definition of L_λ, Theorem 3.10 and relations (2.26), (2.27), (5.9). ∎

Remark 5.4. Let us notice that, whenever $u \in L^2_w(\mathbb{R}^3)$, $w > 3/2$,

$$\mathsf{L}^1_\lambda u(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \langle u^\xi_\lambda, u\rangle_{L^2_{-w}(\mathbb{R}^3), L^2_w(\mathbb{R}^3)}$$

and so L^1_λ and L^2_λ have a similar structure.

5.3. Checking the conditions (H1)–(H7)

Next we discuss the validity of (H1)–(H7) in our framework. In particular, we show that (H1), (H2), (H4.2)–(H7) hold with the choice $\kappa = 1$ in (5.1), without the need to specify the operators B_0 and B_2. We prove (H3) with $\kappa = 2$, while the validity of (H4.1), i.e., the semi-boundedness of A_{B}, will be checked case by case in the analysis of each model.

As in the previous subsections we use the weight $\varphi(x) = (1 + |x|^2)^{w/2}$, $w \in \mathbb{R}$; the notation for the corresponding weighted spaces are $L^2_w(\mathbb{R}^3)$, $H^k_w(\mathbb{R}^3)$, and $H^k_w(\mathbb{R}^3 \backslash \Gamma)$. From now on, the parameter s in the definitions (5.2) and (5.3) is restricted to the range

$$1 < 2s < 1 + \epsilon. \tag{5.11}$$

Be aware that in the following proofs the index s labeling the weighted spaces fulfills the bounds (5.11).

Lemma 5.5. *Let* v *be short-range as in* (5.1)*, with* $\kappa = 1$*. Then hypotheses* (H1), (H2), (H6), (H7.1), (H7.2), (H7.3) *hold true.*

Proof. By [17, Lemma 1, p. 170], $R_z = (-\Delta + z)^{-1} \in \mathcal{B}(L^2_s(\mathbb{R}^3))$ for any $z \in \mathbb{C}\backslash(-\infty, 0]$. Therefore, by the resolvent identity $R^{\mathsf{v}}_z = R_z(1 - \mathsf{v}R^{\mathsf{v}}_z)$, $z \in \varrho(\Delta + \mathsf{v})$, and by $R^{\mathsf{v}}_z \in \mathcal{B}(L^2_s(\mathbb{R}^3), H^2(\mathbb{R}^3))$, hypothesis (H1) is consequence of $\mathsf{v} = \mathsf{v}_2 + \mathsf{v}_\infty \in \mathcal{B}(H^2(\mathbb{R}^3), L^2_s(\mathbb{R}^3))$. Since v_2 has a compact support, $\mathsf{v}_2 \in \mathcal{B}(H^2(\mathbb{R}^3), L^2_s(\mathbb{R}^3))$ by Lemma 4.1. As regards v_∞, one has

$$\begin{aligned}\|\mathsf{v}_\infty u\|^2_{L^2_s(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\mathsf{v}_\infty u|^2 (1 + |x|^2)^s \, dx \\ &\le c \int_{\mathbb{R}^3} (1 + |x|)^{-2(1+\epsilon)} (1 + |x^2|)^s |u|^2 \, dx \\ &\le c \|u\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

By [1, Theorem 4.1], LAP holds for $A = \Delta$; hence (H7.1) is satisfied. By the short-range hypothesis on v and by [1, Theorem 4.2], LAP holds for $A_{B_1} \equiv \Delta + \mathsf{v}$ as well and, by [1, Theorems 6.1 and 7.1] asymptotic completeness holds for the scattering couple $(\Delta_{B_1}, A) \equiv (\Delta + \mathsf{v}, \Delta)$. Hence, hypotheses (H1), (H2) and (H6) are verified.

By $R_z \in \mathcal{B}(L^2_{-s}(\mathbb{R}^3), H^2_{-s}(\mathbb{R}^3))$, one gets

$$G_z^{1*} = \langle x\rangle^{-s} R_z \in \mathcal{B}(L^2_{-s}(\mathbb{R}^3), H^2(\mathbb{R}^3))$$

and so, by duality, $G_z^1 \in \mathcal{B}(H^{-2}(\mathbb{R}^3), L^2_s(\mathbb{R}^3))$; moreover, by a similar duality argument and by $R_\lambda^\pm \in \mathcal{B}(L^2_s(\mathbb{R}^3), H^2_{-s}(\mathbb{R}^3))$, one gets $G_\lambda^{1,\pm} \in \mathcal{B}(H^{-2}(\mathbb{R}^3), L^2_{-s}(\mathbb{R}^3))$. Thus, hypothesis (H7.2) holds.

By (5.9) and (5.10), we have that hypothesis (H7.3) is equivalent to the existence in $\mathcal{B}(H^2_{-s}(\mathbb{R}^3), H_s^{-2}(\mathbb{R}^3))$ of $\lim_{\epsilon\searrow 0} \Lambda^{\mathsf{v}}_{\lambda\pm i\epsilon} = \lim_{\epsilon\searrow 0}(1-\mathsf{v}R_{\lambda\pm i\epsilon})^{-1}\mathsf{v}$. By (5.1), $\mathsf{v} \in \mathcal{B}(H^2_{-s}(\mathbb{R}^3), L^2_s(\mathbb{R}^3))$. Then, $\lim_{\epsilon\searrow 0}(1-\mathsf{v}R_{\lambda\pm i\epsilon})^{-1}$ exists in $\mathcal{B}(L^2_s(\mathbb{R}^3))$ (see [17, proof of Theorem XIII.33, p. 177]) and so (H7.3) holds. ■

Lemma 5.6. *Let* v *be short-range as in* (5.1)*, with* $\kappa = 2$*. Then hypothesis* (H3) *holds true.*

Proof. The proof is the same as the one for [14, Lemma 4.5], once one proves that

$$\mathsf{v}R_\lambda^{\mathsf{v},\pm} \in \mathcal{B}(L^2_{2s}(\mathbb{R}^3)). \tag{5.12}$$

Since $R_\lambda^{\mathsf{v},\pm} \in \mathcal{B}(L^2_{2s}(\mathbb{R}^3), H^2_{-2s}(\mathbb{R}^3))$, (5.12) is consequence of

$$\mathsf{v} = \mathsf{v}_2 + \mathsf{v}_\infty \in \mathcal{B}(H^2_{-2s}(\mathbb{R}^3), L^2_{2s}(\mathbb{R}^3)). \tag{5.13}$$

Lemma 4.1 entails $\mathsf{v}_2 \in \mathcal{B}(H^2(\mathbb{R}^3), L^2(\mathbb{R}^3))$ and so, since v_2 has a compact support, one gets that v_2 satisfies (5.13). As regards v_∞, one has, by $1 < 2s < 1+\epsilon$,

$$\begin{aligned}
\|\mathsf{v}_\infty u\|^2_{L^2_{2s}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\mathsf{v}_\infty u|^2 (1+|x|^2)^{2s}\, dx \\
&\le c \int_{\mathbb{R}^3} (1+|x|)^{-4(1+\epsilon)}(1+|x|^2)^{4s}|u|^2(1+|x|^2)^{-2s}\, dx \\
&\le c\, \|u\|^2_{L^2_{-2s}(\mathbb{R}^3)} \le c\, \|u\|^2_{H^2_{-2s}(\mathbb{R}^3)}
\end{aligned}$$

and so v_∞ satisfies (5.13) as well. ■

Lemma 5.7. *Let* v *be short-range as in* (5.1)*, with* $\kappa = 1$ *and let* τ_2 *be either as in* (5.4) *or as in* (5.5)*. Then hypotheses* (H4.2), (H5)*, and* (H7.4) *hold true.*

Proof. By the continuity of $z \mapsto R_z^\pm$ as a $\mathcal{B}(H_s^{-1}(\mathbb{R}^3), H^1_{-s}(\mathbb{R}^3))$-valued map, one gets the continuity of $z \mapsto G_z^{1,\pm} = R_z^\pm\langle x\rangle^{-s}$ as a $\mathcal{B}(H^{-1}(\mathbb{R}^3), H^1_{-s}(\mathbb{R}^3))$-valued map. Hence, given $\chi \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^3)$ such that $\chi = 1$ on a compact set containing an open neighborhood of $\overline{\Omega}$, one gets the continuity of $z \mapsto \chi R_z^\pm\langle x\rangle^{-s}$ as a $\mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*$, $H^1(\mathbb{R}^3))$-valued map. Therefore, $z \mapsto \gamma_0 G_z^{1,\pm} = \gamma_0 R_z^\pm\langle x\rangle^{-s} = \gamma_0\chi R_z^\pm\langle x\rangle^{-s}$ is continuous as a $\mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*, H^{1/2}(\Gamma))$-valued map. The continuity of $z \mapsto \gamma_1 G_z^{1,\pm} =$

$\gamma_1 R_z^{\pm}\langle x\rangle^{-s} = \gamma_1 \chi R_z^{\pm}\langle x\rangle^{-s}$ as a $\mathcal{B}(H^1(\mathbb{R}^3\backslash\Gamma)^*, H^{-1/2}(\Gamma))$-valued map follows in an analogous way using the same reasoning as in the proof of Lemma 4.17. In conclusion, hypothesis (H7.4) holds true.

Since Γ is compact, the embeddings $\mathfrak{h}_2 \hookrightarrow \mathfrak{b}_2$, where $\mathfrak{h}_2$ and $\mathfrak{b}_2$ are as in (5.4) and (5.5), are compact by standard results on Sobolev embeddings.

Since $\mathsf{v} \in \mathcal{B}(L^2_{-(2s+\eta)}(\mathbb{R}^3), L^2_{1+\epsilon-(2s+\eta)}(\mathbb{R}^3))$ and $(1+\mathsf{v}R_z)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^3))$, by taking $\eta = 1+\epsilon-2s > 0$, one gets $\Lambda_z^{\mathsf{v}} \in \mathcal{B}(L^2_{-(2s+\eta)}(\mathbb{R}^3), L^2(\mathbb{R}^3))$. Hence, by the resolvent formula (4.8) and by $R_z \in \mathcal{B}(L^2_{-(2s+\eta)}(\mathbb{R}^3), H^2_{-(2s+\eta)}(\mathbb{R}^3))$, one gets $R_z^{\mathsf{v}} \in \mathcal{B}(L^2_{-(2s+\eta)}(\mathbb{R}^3), H^2_{-(2s+\eta)}(\mathbb{R}^3))$. This entails

$$\gamma_0 R_z^{B_1} = \gamma_0 R_z^{\mathsf{v}} = \gamma_0 \chi R_z^{\mathsf{v}} \in \mathcal{B}(L^2_{-(2s+\gamma)}(\mathbb{R}^3), B^2_{2,2}(\Gamma))$$

and

$$\gamma_1 R_z^{B_1} = \gamma_1 R_z^{\mathsf{v}} = \gamma_1 \chi R_z^{\mathsf{v}} \in \mathcal{B}(L^2_{-(2s+\eta)}(\mathbb{R}^3), H^{1/2}(\Gamma)).$$

Then, by duality, one gets $G_z^{B_1} \in \mathcal{B}(\mathfrak{h}_2^*, L^2_{2s+\eta}(\mathbb{R}^3))$. This shows that (H4.2) holds.

By [1, Theorem 4.2], the map

$$(\mathbb{R}\backslash e(A_{B_1})) \cup \mathbb{C}_\pm \ni z \mapsto R_z^{B_1,\pm} = R_z^{\mathsf{v},\pm} \in \mathcal{B}(L^2_s(\mathbb{R}^3), H^2_{-s}(\mathbb{R}^3))$$

is continuous. Hence,

$$z \mapsto \gamma_0 R_z^{B_1,\pm} = \gamma_0 R_z^{\mathsf{v},\pm} = \gamma_0 \chi R_z^{\mathsf{v},\pm}$$

and

$$z \mapsto \gamma_1 R_z^{B_1,\pm} = \gamma_1 R_z^{\mathsf{v},\pm} = \gamma_1 \chi R_z^{\mathsf{v},\pm}$$

are continuous as $\mathcal{B}(L^2_s(\mathbb{R}^3), B^{3/2}_{2,2}(\Gamma))$-valued and $\mathcal{B}(L^2_s(\mathbb{R}^3), H^{1/2}(\Gamma))$-valued maps respectively. Then, by duality, we have that $z \mapsto G^{B_1,\pm}$ is continuous on $(\mathbb{R}\backslash e(A_{B_1})) \cup \mathbb{C}_\pm$ as a $\mathcal{B}(\mathfrak{h}_2^*, L^2_{-s}(\mathbb{R}^3))$-valued map. Since $\gamma_0\colon H^2(\mathbb{R}^2) \to B^{3/2}_{2,2}(\Gamma)$ and $\gamma_1\colon H^2(\mathbb{R}^2) \to H^{1/2}(\Gamma)$ are surjective, $G_z^{B_1,\pm} \in \mathcal{B}(\mathfrak{h}_2^*, L^2_{-s}(\mathbb{R}^3))$ is the adjoint of a surjective map and hence is injective. Thus, we proved that (H5) holds. ∎

6. Applications

6.1. Short-range potentials and semi-transparent boundary conditions of δ_Γ-type

Here we take

$$\mathfrak{h}_2 = B^{3/2}_{2,2}(\Gamma) \hookrightarrow \mathfrak{b}_2 = \mathfrak{b}_{2,2} = H^{s_\circ}(\Gamma) \hookrightarrow \mathfrak{h}_2^\circ = L^2(\Gamma), \quad 0 < s_\circ < 1/2,$$

$$\tau_2 = \gamma_0\colon H^2(\mathbb{R}^3) \to B^{3/2}_{2,2}(\Gamma), \quad B_0 = 1, \quad B_2 = \alpha,$$

where

$$\alpha \in \mathcal{B}(H^{s_\circ}(\Gamma), H^{-s_\circ}(\Gamma)), \quad \alpha^* = \alpha.$$

Let us notice (see [14, Remark 2.6]) that in the case α is the multiplication operator associated to a real-valued function α, then $\alpha \in L^p(\Gamma)$, $p > 2$, fulfills our hypothesis.

For any $z \in \mathbb{C}\backslash(-\infty, 0]$, one has

$$\begin{aligned} M_z^{\mathsf{B}} &= 1 - \begin{bmatrix} \langle x\rangle^{2s}\mathsf{v} & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} \langle x\rangle^{-s} R_z \langle x\rangle^{-s} & \langle x\rangle^{-s} R_z \gamma_0^* \\ \gamma_0 R_z \langle x\rangle^{-s} & \gamma_0 R_z \gamma_0^* \end{bmatrix} \\ &= \begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix} M_z^{\mathsf{v},\alpha} \begin{bmatrix} \langle x\rangle^{-s} & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

$$M_z^{\mathsf{v},\alpha} := \begin{bmatrix} 1-\mathsf{v}R_z & -\mathsf{v}\,\mathrm{SL}_z \\ -\alpha\,\mathrm{SL}_{\bar z}^* & 1-\alpha S_z \end{bmatrix}.$$

By the mapping properties provided in Sections 4.1 and 4.2, by (5.6) and (5.7) with $w = -s$, one gets

$$M_z^{\mathsf{v},\alpha} \in \mathcal{B}(H^1_{-s}(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{-s_\circ}(\Gamma)).$$

According to [11, Lemma 5.8], for any $z \in \mathbb{C}\backslash((-\infty, 0] \cup \sigma_\alpha)$, where $\sigma_\alpha \subset (0, +\infty)$ is discrete in $(0, +\infty)$, one has

$$(M_z^{B_0,B_2})^{-1} = (M_z^{\alpha})^{-1} := (1 - \alpha S_z)^{-1} \in \mathcal{B}(H^{-s_\circ}(\Gamma)).$$

Thus

$$\begin{aligned} Z_{B_0,B_2} = Z_\alpha &:= \{z \in \mathbb{C}\backslash(-\infty, 0] \colon (M_w^{\alpha})^{-1} \in \mathcal{B}(H^{-s_\circ}(\Gamma)),\ w = z, \bar z\} \\ &\supseteq \mathbb{C}\backslash((-\infty, 0] \cup \sigma_\alpha) \end{aligned}$$

and

$$\Lambda_z^{B_0,B_2} = (M_z^{B_0,B_2})^{-1} B_2 = \Lambda_z^{\alpha} := (1 - \alpha S_z)^{-1}\alpha \in \mathcal{B}(H^{s_\circ}(\Gamma), H^{-s_\circ}(\Gamma)).$$

By [14, Corollary 2.4], for any $z \in \varrho(\Delta + \mathsf{v})\backslash\sigma_{\mathsf{v},\alpha}$, where $\sigma_{\mathsf{v},\alpha} \subset \varrho(\Delta + \mathsf{v}) \cap \mathbb{R}$ is discrete in $\varrho(\Delta + \mathsf{v}) \cap \mathbb{R}$,

$$(\widehat M_z^{\mathsf{B}})^{-1} = (\widehat M_z^{\mathsf{v},\alpha})^{-1} := (1 - \alpha S_z^{\mathsf{v}})^{-1} \in \mathcal{B}(H^{-s_\circ}(\Gamma)).$$

Thus

$$\begin{aligned} \widehat Z_{\mathsf{B}} = \widehat Z_{\mathsf{v},\alpha} &:= \{z \in \varrho(\Delta + \mathsf{v}) \colon (\widehat M_w^{\mathsf{v},\alpha})^{-1} \in \mathcal{B}(H^{-s_\circ}(\Gamma)),\ w = z, \bar z\} \\ &\supseteq \varrho(\Delta + \mathsf{v})\backslash\sigma_{\mathsf{v},\alpha} \end{aligned}$$

and

$$\widehat\Lambda_z^{\mathsf{B}} = (\widehat M_z^{\mathsf{B}})^{-1} B_2 = \widehat\Lambda_z^{\mathsf{v},\alpha} := (1 - \alpha S_z^{\mathsf{v}})^{-1}\alpha \in \mathcal{B}(H^{s_\circ}(\Gamma), H^{-s_\circ}(\Gamma)).$$

Hence,

$$\Lambda_z^{\mathsf{B}} = \begin{bmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{bmatrix} (M_z^{\mathsf{v},\alpha})^{-1} \begin{bmatrix} \langle x\rangle^{-s} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \langle x\rangle^{2s}\mathsf{v} & 0 \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{bmatrix} \mathbf{\Lambda}_z^{\mathsf{B}} \begin{bmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{bmatrix},$$

where, by Theorem 5.3,

$$\begin{aligned} \mathbf{\Lambda}_z^{\mathsf{B}} = \mathbf{\Lambda}_z^{\mathsf{v},\alpha} &:= \begin{bmatrix} \Lambda_z^{\mathsf{v}} + \Lambda_z^{\mathsf{v}}\,\mathrm{SL}_z\,\widehat{\Lambda}_z^{\mathsf{v},\alpha}\,\mathrm{SL}_{\bar z}^*\,\Lambda_z^{\mathsf{v}} & \Lambda_z^{\mathsf{v}}\,\mathrm{SL}_z\,\widehat{\Lambda}_z^{\mathsf{v},\alpha} \\ \widehat{\Lambda}_z^{\mathsf{v},\alpha}\,\mathrm{SL}_{\bar z}^*\,\Lambda_z^{\mathsf{v}} & \widehat{\Lambda}_z^{\mathsf{v},\alpha} \end{bmatrix} \\ &= \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & \widehat{\Lambda}_z^{\mathsf{v},\alpha} \end{bmatrix} \left(1 + \begin{bmatrix} \mathrm{SL}_z\,\widehat{\Lambda}_z^{\mathsf{v},\alpha}\,\mathrm{SL}_{\bar z}^* & \mathrm{SL}_z \\ \mathrm{SL}_{\bar z}^* & 0 \end{bmatrix} \right) \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & \widehat{\Lambda}_z^{\mathsf{v},\alpha} \end{bmatrix}. \end{aligned}$$

One has

$$\mathbf{\Lambda}_z^{\mathsf{v},\alpha} \in \mathscr{B}(H_{-s}^1(\mathbb{R}^3\backslash\Gamma) \oplus H^{s_\circ}(\Gamma), H_{-s}^1(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{-s_\circ}(\Gamma)).$$

By Theorems 2.1 and 2.9, there follows

$$\begin{aligned} R_z^{\mathsf{v},\alpha} &= R_z + \begin{bmatrix} R_z\langle x\rangle^{-s} & \mathrm{SL}_z \end{bmatrix} \begin{bmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{bmatrix} \mathbf{\Lambda}_z^{\mathsf{v},\alpha} \begin{bmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \langle x\rangle^{2s}\mathsf{v}\langle x\rangle^{-s} R_z \\ \alpha\,\mathrm{SL}_{\bar z}^* \end{bmatrix} &(6.1) \\ &= R_z + \begin{bmatrix} R_z & \mathrm{SL}_z \end{bmatrix} \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & \widehat{\Lambda}_z^{\mathsf{v},\alpha} \end{bmatrix} \left(1 + \begin{bmatrix} \mathrm{SL}_z\,\widehat{\Lambda}_z^{\mathsf{v},\alpha}\,\mathrm{SL}_{\bar z}^* & \mathrm{SL}_z \\ \mathrm{SL}_{\bar z}^* & 0 \end{bmatrix} \right) \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & \widehat{\Lambda}_z^{\mathsf{v},\alpha} \end{bmatrix} \begin{bmatrix} R_z \\ \mathrm{SL}_{\bar z}^* \end{bmatrix} &(6.2) \\ &= R_z^{\mathsf{v}} + \mathrm{SL}_z^{\mathsf{v}}\,\widehat{\Lambda}_z^{\mathsf{v},\alpha}\mathrm{SL}_{\bar z}^{\mathsf{v}\,*}. &(6.3) \end{aligned}$$

is the resolvent of a self-adjoint operator $\Delta^{\mathsf{v},\delta,\alpha}$; (6.1) holds for any $z \in \varrho(\Delta^{\mathsf{v},\delta,\alpha}) \cap \mathbb{C}\backslash(-\infty, 0]$, both (6.2) and (6.3) hold for any $z \in \varrho(\Delta^{\mathsf{v},\delta,\alpha}) \cap \varrho(\Delta + \mathsf{v})$.

By Theorem 2.6,

$$\Delta^{\mathsf{v},\delta,\alpha}u = \Delta u + \mathsf{v}u + (\alpha\gamma_0 u)\delta_\Gamma.$$

By (6.3) and by the mapping properties of $\mathrm{SL}_z^{\mathsf{v}}$, one has

$$\mathrm{dom}(\Delta^{\mathsf{v},\delta,\alpha}) \subseteq H^{3/2-s_\circ}(\mathbb{R}^3).$$

Moreover, by $R_z^{\mathsf{v}}u \in H^2(\mathbb{R}^3)$, so that $[\gamma_1]R_z^{\mathsf{v}}u = 0$, and by (4.30), one gets

$$[\gamma_1]R_z^{\mathsf{v},\alpha}u = -\widehat{\Lambda}_z^{\mathsf{v},\alpha}\mathrm{SL}_{\bar z}^{\mathsf{v}\,*}u = -\hat{\rho}_{\mathsf{B}}(R_z^{\mathsf{v},\alpha}u).$$

Hence, by Theorem 2.11,

$$u \in \mathrm{dom}(\Delta^{\mathsf{v},\delta,\alpha}) \implies \alpha\gamma_0 u + [\gamma_1]u = 0.$$

Since $\widehat{Z}_{\mathsf{v},\alpha}$ contains a positive half-line, $\Delta^{\mathsf{v},\delta,\alpha}$ is bounded from above and hypothesis (H4.1) holds. The scattering couple $(\Delta^{\mathsf{v},\delta,\alpha}, \Delta)$ is asymptotically complete and the corresponding scattering matrix is given by

$$\mathsf{S}_\lambda^{\mathsf{v},\alpha} = 1 - 2\pi i\,\mathsf{L}_\lambda \mathbf{\Lambda}_\lambda^{\mathsf{v},\alpha,+}\mathsf{L}_\lambda^*, \quad \lambda \in (-\infty, 0]\backslash(\sigma_{\mathrm{p}}^-(\Delta + \mathsf{v}) \cup \sigma_{\mathrm{p}}^-(\Delta^{\mathsf{v},\delta,\alpha})),$$

where L_λ is given in Theorem 5.3 and $\boldsymbol{\Lambda}_\lambda^{\mathsf{v},\alpha,+} := \lim_{\epsilon\searrow 0}\boldsymbol{\Lambda}_{\lambda+i\epsilon}^{\mathsf{v},\alpha}$. This latter limit exists by Lemma 3.8; in particular, by (3.10),

$$\boldsymbol{\Lambda}^{\mathsf{v},\alpha,+} = \left(1 + \begin{bmatrix} (1-\mathsf{v}R_\lambda^+)^{-1}\mathsf{v} & 0 \\ 0 & (1-\alpha S_\lambda^{\mathsf{v},+})^{-1}\alpha \end{bmatrix}\begin{bmatrix} \mathrm{SL}_\lambda^+(1-\alpha S_\lambda^{\mathsf{v},+})^{-1}\alpha(\mathrm{SL}_\lambda^-)^* & \mathrm{SL}_\lambda^+ \\ (\mathrm{SL}_\lambda^-)^* & 0 \end{bmatrix}\right)$$
$$\times \begin{bmatrix} (1-\mathsf{v}R_\lambda^+)^{-1}\mathsf{v} & 0 \\ 0 & (1-\alpha S_\lambda^{\mathsf{v},+})^{-1}\alpha \end{bmatrix},$$

where

$$R_\lambda^\pm := \lim_{\epsilon\searrow 0} R_{\lambda\pm i\epsilon}, \quad \mathrm{SL}_\lambda^\pm := \lim_{\epsilon\searrow 0}\mathrm{SL}_{\lambda\pm i\epsilon}, \quad S_\lambda^{\mathsf{v},\pm} := \lim_{\epsilon\searrow 0}\gamma_0\,\mathrm{SL}_{\lambda\pm i\epsilon}^{\mathsf{v}}.$$

6.2. Short-range potentials and Dirichlet boundary conditions

Here we take

$$\mathfrak{h}_2 = B^{3/2}_{2,2}(\Gamma) \hookrightarrow \mathfrak{b}_2 = H^{1/2}(\Gamma) \hookrightarrow \mathfrak{h}_2^\circ = L^2(\Gamma) \hookrightarrow \mathfrak{b}_{2,2} = \mathfrak{b}_2^* = H^{-1/2}(\Gamma),$$

$$\tau_2 = \gamma_0\colon H^2(\mathbb{R}^3) \to B^{3/2}_{2,2}(\Gamma), \quad B_0 = 0, \quad B_2 = 1.$$

For any $z \in \mathbb{C}\backslash(-\infty,0]$, one has

$$\begin{aligned} M_z^{\mathsf{B}} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \langle x\rangle^{2s}\mathsf{v} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \langle x\rangle^{-s}R_z\langle x\rangle^{-s} & \langle x\rangle^{-s}R_z\gamma_0^* \\ \gamma_0 R_z\langle x\rangle^{-s} & \gamma_0 R_z\gamma_0^* \end{bmatrix} \\ &= \begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix} M_z^{\mathsf{v},d}\begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

$$M_z^{\mathsf{v},d} := \begin{bmatrix} 1-\mathsf{v}R_z & -\mathsf{v}\,\mathrm{SL}_z \\ -\mathrm{SL}_{\bar z}^* & -S_z \end{bmatrix}.$$

By the mapping properties provided in Sections 4.1 and 4.2, by (5.6) and (5.7) with $w = -s$, one gets

$$M_z^{\mathsf{v},d} \in \mathscr{B}(H^1_{-s}(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{-1/2}(\Gamma), H^1_{-s}(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{1/2}(\Gamma)).$$

By Lemma 4.19 with $\mathsf{v} = 0$, for any $z \in Z^\circ_{0,d} \neq \varnothing$,

$$(M_z^{B_0,B_2})^{-1} = \Lambda_z^{B_0,B_2} = (M_z^d)^{-1} = \Lambda_z^d := -S_z^{-1} \in \mathscr{B}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma)).$$

Thus,

$$Z_{B_0,B_2} = Z_d := \{z \in \mathbb{C}\backslash(-\infty,0] \colon (M_z^d)^{-1} \in \mathscr{B}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))\} \supseteq Z^\circ_{0,d}.$$

By Lemma 4.19 again, for any $z \in Z^\circ_{\mathsf{v},d} \neq \varnothing$,

$$\begin{aligned} (\widehat{M}_z^{B_0,B_2})^{-1} &= (\widehat{\Lambda}_z^{B_0,B_2})^{-1} = (\widehat{M}_z^{\mathsf{v},d})^{-1} \\ &= \widehat{\Lambda}_z^{\mathsf{v},d} := -(S_z^{\mathsf{v}})^{-1} \in \mathscr{B}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma)). \end{aligned}$$

Thus,

$$\widehat{Z}_{\mathsf{B}} = \widehat{Z}_{\mathsf{v},d} := \{z \in \varrho(\Delta + \mathsf{v}) \colon (\widehat{M}_z^{\mathsf{v},d})^{-1} \in \mathcal{B}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))\} \supseteq Z^{\circ}_{\mathsf{v},d}.$$

Hence,

$$\Lambda_z^{\mathsf{B}} = \left[\begin{smallmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{smallmatrix}\right](M_z^{\mathsf{v},d})^{-1}\left[\begin{smallmatrix} \langle x\rangle^{-s} & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} \langle x\rangle^{2s}\mathsf{v} & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{smallmatrix}\right]\mathbf{\Lambda}_z^{\mathsf{B}}\left[\begin{smallmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{smallmatrix}\right],$$

where, by Theorem 5.3,

$$\begin{aligned}\mathbf{\Lambda}_z^{\mathsf{B}} = \mathbf{\Lambda}_z^{\mathsf{v},d} &:= \begin{bmatrix} \Lambda_z^{\mathsf{v}} - \Lambda_z^{\mathsf{v}}\, \mathrm{SL}_z (S_z^{\mathsf{v}})^{-1}\, \mathrm{SL}_{\bar z}^*\, \Lambda_z^{\mathsf{v}} & -\Lambda_z^{\mathsf{v}}\, \mathrm{SL}_z (S_z^{\mathsf{v}})^{-1} \\ -(S_z^{\mathsf{v}})^{-1}\, \mathrm{SL}_{\bar z}^*\, \Lambda_z^{\mathsf{v}} & -(S_z^{\mathsf{v}})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & -(S_z^{\mathsf{v}})^{-1} \end{bmatrix}\left(1 + \begin{bmatrix} -\mathrm{SL}_z (S_z^{\mathsf{v}})^{-1}\, \mathrm{SL}_{\bar z}^* & \mathrm{SL}_z \\ \mathrm{SL}_{\bar z}^* & 0 \end{bmatrix}\right)\begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & -(S_z^{\mathsf{v}})^{-1} \end{bmatrix}.\end{aligned}$$

One has

$$\mathbf{\Lambda}_z^{\mathsf{v},d} \in \mathcal{B}(H^1_{-s}(\mathbb{R}^3\backslash\Gamma) \oplus H^{1/2}(\Gamma), H^1_{-s}(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{-1/2}(\Gamma)).$$

By Theorems 2.1 and 2.9, there follows that

$$R_z^{\mathsf{v},d} = R_z + \left[\, R_z \langle x\rangle^{-s}\ \ \mathrm{SL}_z \,\right]\left[\begin{smallmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{smallmatrix}\right]\mathbf{\Lambda}_z^{\mathsf{v},d}\left[\begin{smallmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{smallmatrix}\right]\begin{bmatrix} \langle x\rangle^{2s}\mathsf{v}\langle x\rangle^{-s} R_z \\ \mathrm{SL}_{\bar z}^* \end{bmatrix} \tag{6.4}$$

$$\begin{aligned} &= R_z + \left[\, R_z\ \ \mathrm{SL}_z \,\right]\begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & -(S_z^{\mathsf{v}})^{-1} \end{bmatrix}\left(1 + \begin{bmatrix} -\mathrm{SL}_z (S_z^{\mathsf{v}})^{-1}\, \mathrm{SL}_{\bar z}^* & \mathrm{SL}_z \\ \mathrm{SL}_{\bar z}^* & 0 \end{bmatrix}\right) \\ &\quad \times \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & -(S_z^{\mathsf{v}})^{-1} \end{bmatrix}\begin{bmatrix} R_z \\ \mathrm{SL}_{\bar z}^* \end{bmatrix} \end{aligned} \tag{6.5}$$

$$= R_z^{\mathsf{v}} - \mathrm{SL}^{\mathsf{v}}(S_z^{\mathsf{v}})^{-1}\mathrm{SL}_{\bar z}^{\mathsf{v}\,*} \tag{6.6}$$

is the resolvent of a self-adjoint operator $\Delta^{\mathsf{v},d}$; (6.4) holds for any $z \in \varrho(\Delta^{\mathsf{v},d}) \cap \mathbb{C}\backslash(-\infty, 0]$, both (6.5) and (6.6) hold for any $z \in \varrho(\Delta^{\mathsf{v},d}) \cap \varrho(\Delta + \mathsf{v})$. By (6.3) and by the mapping properties of $\mathrm{SL}_z^{\mathsf{v}}$, one has

$$\mathrm{dom}(\Delta^{\mathsf{v},d}) \subseteq H^1(\mathbb{R}^3).$$

By Theorem 2.11 and by $[\gamma_1]u = -\hat\rho_{\mathsf{B}}u$, for any $u \in \mathrm{dom}(\Delta^{\mathsf{v},d})$, one gets

$$\Delta^{\mathsf{v},d}\, u = \Delta u + \mathsf{v}u - ([\gamma_1]u)\delta_\Gamma$$

and

$$u \in \mathrm{dom}(\Delta^{\mathsf{v},d}) \implies \gamma_0 u = 0.$$

Therefore, $\mathrm{dom}(\Delta^{\mathsf{v},d}) \subseteq H^1_0(\Omega_{\mathrm{in}}) \oplus H^1_0(\Omega_{\mathrm{ex}})$. Since $\widehat{Z}_{\mathsf{v},\alpha}$ contains a positive half-line, $\Delta^{\mathsf{v},d}$ is bounded from above and hypothesis (H4.1) holds. The scattering couple $(\Delta^{\mathsf{v},d}, \Delta)$ is asymptotically complete and the corresponding scattering matrix is given by

$$\mathsf{S}_\lambda^{\mathsf{v},d} = 1 - 2\pi i\, \mathsf{L}_\lambda \mathbf{\Lambda}_\lambda^{\mathsf{v},d,+}\mathsf{L}_\lambda^*, \quad \lambda \in (-\infty, 0]\backslash(\sigma_{\mathrm{p}}^-(\Delta + \mathsf{v}) \cup \sigma_{\mathrm{p}}^-(\Delta^{\mathsf{v},d})),$$

where L_λ is given in Theorem 5.3 and $\boldsymbol{\Lambda}_\lambda^{\mathsf{v},d,+} := \lim_{\epsilon\searrow 0}\boldsymbol{\Lambda}_{\lambda+i\epsilon}^{\mathsf{v},d}$. This latter limit exists by Lemma 3.8; in particular, by (3.10),

$$\boldsymbol{\Lambda}^{\mathsf{v},d,+} = \left(1 + \begin{bmatrix} (1-\mathsf{v}R_\lambda^+)^{-1}\mathsf{v} & 0 \\ 0 & -(S_\lambda^{\mathsf{v},+})^{-1} \end{bmatrix}\begin{bmatrix} -\mathrm{SL}_\lambda^+(S_\lambda^{\mathsf{v},+})^{-1}(\mathrm{SL}_\lambda^-)^* & \mathrm{SL}_\lambda^+ \\ (\mathrm{SL}_\lambda^-)^* & 0 \end{bmatrix}\right)$$
$$\times\begin{bmatrix} (1-\mathsf{v}R_\lambda^+)^{-1}\mathsf{v} & 0 \\ 0 & -(S_\lambda^{\mathsf{v},+})^{-1} \end{bmatrix},$$

where

$$R_\lambda^\pm := \lim_{\epsilon\searrow 0} R_{\lambda\pm i\epsilon}, \quad \mathrm{SL}_\lambda^\pm := \lim_{\epsilon\searrow 0}\mathrm{SL}_{\lambda\pm i\epsilon}, \quad S_\lambda^{\mathsf{v},\pm} := \lim_{\epsilon\searrow 0}\gamma_0\,\mathrm{SL}_{\lambda\pm i\epsilon}^{\mathsf{v}}.$$

6.3. Short-range potentials and Neumann boundary conditions

Here we take

$$\mathfrak{h}_2 = \mathfrak{b}_2^* = \mathfrak{b}_{2,2} = H^{1/2}(\Gamma) \hookrightarrow \mathfrak{h}_2^\circ = L^2(\Gamma) \hookrightarrow \mathfrak{b}_2 = \mathfrak{h}_2^* = \mathfrak{b}_{2,2}^* = H^{-1/2}(\Gamma),$$

$$\tau_2 = \gamma_1 \colon H^2(\mathbb{R}^3) \to H^{1/2}(\Gamma), \quad B_0 = 0, \quad B_2 = 1.$$

For any $z \in \mathbb{C}\backslash(-\infty, 0]$, one has

$$M_z^{\mathsf{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \langle x\rangle^{2s}\mathsf{v} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \langle x\rangle^{-s}R_z\langle x\rangle^{-s} & \langle x\rangle^{-s}R_z\gamma_1^* \\ \gamma_1 R_z\langle x\rangle^{-s} & \gamma_1 R_z\gamma_0^* \end{bmatrix}$$
$$= \begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix} M_z^{\mathsf{v},n}\begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix},$$

$$M_z^{\mathsf{v},n} := \begin{bmatrix} 1-\mathsf{v}R_z & -\mathsf{v}\,\mathrm{DL}_z \\ -\mathrm{DL}_{\bar z}^* & -D_z \end{bmatrix}.$$

By the mapping properties provided in Sections 4.1 and 4.2, by (5.6) and (5.7) with $w = -s$, one gets

$$M_z^{\mathsf{v},n} \in \mathcal{B}(H^1_{-s}(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{1/2}(\Gamma), H^1_{-s}(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{-1/2}(\Gamma)).$$

By Lemma 4.19 with $\mathsf{v} = 0$, for any $z \in Z_{0,n}^\circ \neq \varnothing$,

$$(M_z^{B_0,B_2})^{-1} = \Lambda_z^{B_0,B_2} = (M_z^n)^{-1} = \Lambda_z^n := -D_z^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma)).$$

Thus,

$$Z_{B_0,B_2} = Z_n := \{z \in \mathbb{C}\backslash(-\infty, 0] \colon (M_z^n)^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))\} \supseteq Z_{0,n}^\circ.$$

By Lemma 4.19 again, for any $z \in Z_{\mathsf{v},n}^\circ \neq \varnothing$,

$$(\widehat{M}_z^{B_0,B_2})^{-1} = (\widehat{\Lambda}_z^{B_0,B_2})^{-1} = (\widehat{M}_z^{\mathsf{v},n})^{-1}$$
$$= \widehat{\Lambda}_z^{\mathsf{v},n} := -(D_z^{\mathsf{v}})^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma)).$$

Thus,

$$\widehat{Z}_{\mathsf{B}} = \widehat{Z}_n := \{z \in \varrho(\Delta + \mathsf{v})\colon (\widehat{M}_z^{\mathsf{v},n})^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))\} \supseteq Z^{\circ}_{\mathsf{v},n}.$$

Hence,

$$\Lambda_z^{\mathsf{B}} = \begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix} (M_z^{\mathsf{v},n})^{-1} \begin{bmatrix} \langle x\rangle^{-s} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \langle x\rangle^{2s}\mathsf{v} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{\Lambda}_z^{\mathsf{B}} \begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix},$$

where, by Theorem 5.3,

$$\begin{aligned} \mathbf{\Lambda}_z^{\mathsf{B}} = \Lambda_z^{\mathsf{v},n} &:= \begin{bmatrix} \Lambda_z^{\mathsf{v}} - \Lambda_z^{\mathsf{v}}\,\mathrm{DL}_z (D_z^{\mathsf{v}})^{-1}\,\mathrm{DL}_{\bar z}^{*}\,\Lambda_z^{\mathsf{v}} & -\Lambda_z^{\mathsf{v}}\,\mathrm{DL}_z (D_z^{\mathsf{v}})^{-1} \\ -(D_z^{\mathsf{v}})^{-1}\,\mathrm{DL}_{\bar z}^{*}\,\Lambda_z^{\mathsf{v}} & -(D_z^{\mathsf{v}})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & -(D_z^{\mathsf{v}})^{-1} \end{bmatrix} \left(1 + \begin{bmatrix} -\mathrm{DL}_z (D_z^{\mathsf{v}})^{-1}\,\mathrm{DL}_{\bar z}^{*} & \mathrm{DL}_z \\ \mathrm{DL}_{\bar z}^{*} & 0 \end{bmatrix}\right) \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & -(D_z^{\mathsf{v}})^{-1} \end{bmatrix}. \end{aligned}$$

One has

$$\mathbf{\Lambda}_z^{\mathsf{v},n} \in \mathcal{B}(H^1_{-s}(\mathbb{R}^3\backslash\Gamma) \oplus H^{-1/2}(\Gamma), H^1_{-s}(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{1/2}(\Gamma)).$$

By Theorems 2.1 and 2.9, there follows that

$$\begin{aligned} R_z^{\mathsf{v},n} &= R_z + \begin{bmatrix} R_z \langle x\rangle^{-s} & \mathrm{DL}_z \end{bmatrix} \begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{\Lambda}_z^{\mathsf{v},n} \begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \langle x\rangle^{2s}\mathsf{v}\langle x\rangle^{-s} R_z \\ \mathrm{DL}_{\bar z}^{*} \end{bmatrix} &\quad (6.7) \\ &= R_z + \begin{bmatrix} R_z & \mathrm{DL}_z \end{bmatrix} \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & -(D_z^{\mathsf{v}})^{-1} \end{bmatrix} \left(1 + \begin{bmatrix} -\mathrm{DL}_z (D_z^{\mathsf{v}})^{-1}\,\mathrm{DL}_{\bar z}^{*} & \mathrm{DL}_z \\ \mathrm{DL}_{\bar z}^{*} & 0 \end{bmatrix}\right) \\ &\quad \times \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & -(D_z^{\mathsf{v}})^{-1} \end{bmatrix} \begin{bmatrix} R_z \\ \mathrm{DL}_{\bar z}^{*} \end{bmatrix} &\quad (6.8) \\ &= R_z^{\mathsf{v}} - \mathrm{DL}_z^{\mathsf{v}} (D_z^{\mathsf{v}})^{-1} \mathrm{DL}_{\bar z}^{\mathsf{v}\,*} &\quad (6.9) \end{aligned}$$

is the resolvent of a self-adjoint operator $\Delta^{\mathsf{v},n}$; (6.7) holds for any $z \in \varrho(\Delta^{\mathsf{v},n}) \cap \mathbb{C}\backslash(-\infty, 0]$, both (6.8) and (6.9) hold for any $z \in \varrho(\Delta^{\mathsf{v},n}) \cap \varrho(\Delta + \mathsf{v})$. By (6.3) and by the mapping properties of $\mathrm{DL}_z^{\mathsf{v}}$, one has

$$\mathrm{dom}(\Delta^{\mathsf{v},n}) \subseteq H^1(\mathbb{R}^3\backslash\Gamma).$$

By Theorem 2.11 and by $[\gamma_0]u = \hat\rho_{\mathsf{B}} u$ for any $u \in \mathrm{dom}(\Delta^{\mathsf{v},n})$, one gets

$$\Delta^{\mathsf{v},n} u = \Delta u + \mathsf{v}u + ([\gamma_0]u)\delta'_\Gamma$$

and

$$u \in \mathrm{dom}(\Delta^{\mathsf{v},n}) \implies \gamma_1 u = 0.$$

Since $\widehat{Z}_{\mathsf{v},n}$ contains a positive half-line, $\Delta^{\mathsf{v},n}$ is bounded from above and hypothesis (H4.1) holds. The scattering couple $(\Delta^{\mathsf{v},n}, \Delta)$ is asymptotically complete and the corresponding scattering matrix is given by

$$\mathsf{S}_\lambda^{\mathsf{v},n} = 1 - 2\pi i\, \mathsf{L}_\lambda \mathbf{\Lambda}_\lambda^{\mathsf{v},n,+} \mathsf{L}_\lambda^{*}, \quad \lambda \in (-\infty, 0]\backslash(\sigma_{\mathrm{p}}^{-}(\Delta + \mathsf{v}) \cup \sigma_{\mathrm{p}}^{-}(\Delta^{\mathsf{v},n})),$$

where L_λ is given in Theorem 5.3 and $\mathbf{\Lambda}_\lambda^{\mathsf{v},n,+} := \lim_{\epsilon\searrow 0} \mathbf{\Lambda}_{\lambda+i\epsilon}^{\mathsf{v},n}$. This latter limit exists by Lemma 3.8; in particular, by (3.10),

$$\mathbf{\Lambda}^{\mathsf{v},n,+} = \left(1 + \begin{bmatrix} (1-\mathsf{v}R_\lambda^+)^{-1}\mathsf{v} & 0 \\ 0 & -(D_\lambda^{\mathsf{v},+})^{-1} \end{bmatrix}\begin{bmatrix} -\mathrm{DL}_\lambda^+(D_\lambda^{\mathsf{v},+})^{-1}(\mathrm{DL}_\lambda^-)^* & \mathrm{DL}_\lambda^+ \\ (\mathrm{DL}_\lambda^-)^* & 0 \end{bmatrix}\right) \times \begin{bmatrix} (1-\mathsf{v}R_\lambda^+)^{-1}\mathsf{v} & 0 \\ 0 & -(D_\lambda^{\mathsf{v},+})^{-1} \end{bmatrix},$$

where

$$R_\lambda^\pm := \lim_{\epsilon\searrow 0} R_{\lambda\pm i\epsilon}, \quad \mathrm{DL}_\lambda^\pm := \lim_{\epsilon\searrow 0} \mathrm{DL}_{\lambda\pm i\epsilon}, \quad D_\lambda^{\mathsf{v},\pm} := \lim_{\epsilon\searrow 0} \gamma_1 \mathrm{DL}_{\lambda\pm i\epsilon}^{\mathsf{v}}.$$

6.4. Short-range potentials and semi-transparent boundary conditions of δ'_Γ-type

Here we take

$$\mathfrak{h}_2 = \mathfrak{b}_2^* = \mathfrak{b}_{2,2} = H^{1/2}(\Gamma) \hookrightarrow \mathfrak{h}_2^\circ = L^2(\Gamma) \hookrightarrow \mathfrak{b}_2 = \mathfrak{h}_2^* = \mathfrak{b}_{2,2}^* = H^{-1/2}(\Gamma),$$

$$\tau_2 = \gamma_1 \colon H^2(\mathbb{R}^3) \to H^{1/2}(\Gamma), \quad B_0 = \theta, \quad B_2 = 1,$$

where

$$\theta \in \mathcal{B}(H^{s_\circ}(\Gamma), H^{-s_\circ}(\Gamma)), \quad 0 < s_\circ < 1/2, \quad \theta^* = \theta.$$

Let us notice (see [14, Remark 2.6]) that in the case θ is the multiplication operator associated to a real-valued function θ, then $\theta \in L^p(\Gamma)$, $p > 2$, fulfills our hypothesis. Let us also remark that $\mathcal{B}(H^{s_\circ}(\Gamma), H^{-s_\circ}(\Gamma)) \subseteq \mathcal{B}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma)) = \mathcal{B}(\mathfrak{b}_2^*, \mathfrak{b}_{2,2}^*)$.

For any $z \in \mathbb{C}\backslash(-\infty, 0]$, one has

$$\begin{aligned} M_z^{\mathsf{B}} &= \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix} - \begin{bmatrix} \langle x\rangle^{2s}\mathsf{v} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \langle x\rangle^{-s}R_z\langle x\rangle^{-s} & \langle x\rangle^{-s}R_z\gamma_1^* \\ \gamma_1 R_z\langle x\rangle^{-s} & \gamma_1 R_z\gamma_1^* \end{bmatrix} \\ &= \begin{bmatrix} \langle x\rangle^{s} & 0 \\ 0 & 1 \end{bmatrix} M_z^{\mathsf{v},\theta} \begin{bmatrix} \langle x\rangle^{-s} & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

$$M_z^{\mathsf{v},\theta} := \begin{bmatrix} 1-\mathsf{v}R_z & -\mathsf{v}\,\mathrm{DL}_z \\ -\mathrm{DL}_{\bar z}^* & \theta - D_z \end{bmatrix}.$$

By the mapping properties provided in Sections 4.1 and 4.2, by (5.6) and (5.7) with $w = -s$, one gets

$$M_z^{\mathsf{v},\theta} \in \mathcal{B}(H^1_{-s}(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{1/2}(\Gamma), H^1_{-s}(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{-1/2}(\Gamma)).$$

Lemma 6.1. *Let $Z^\circ_{\mathsf{v},n} \neq \varnothing$ be given as in Lemma 4.19. Then,*

$$(1 - \theta(D_z^{\mathsf{v}})^{-1})^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma)) \quad \textit{for all } z \in \widehat{Z}^\circ_{\mathsf{v},n} := Z^\circ_{\mathsf{v},n} \cap \mathbb{C}\backslash\mathbb{R}.$$

Proof. We follow the same the arguments as in the proof of [11, Lemma 5.4]. Since, by the compact embedding $H^{-s_\circ}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)$, $\theta(D_z^{\mathsf{v}})^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma))$ is compact, by the Fredholm alternative, $1 - \theta(D_z^{\mathsf{v}})^{-1}$ has a bounded inverse if and only if it has trivial kernel. Let $\varphi \in H^{-1/2}(\Gamma)$ be such that $D_z^{\mathsf{v}}\varphi = \theta\varphi$; using the self-adjointness of θ, we get

$$(D_z^{\mathsf{v}} - D_{\bar z}^{\mathsf{v}})\varphi = 0.$$

By the resolvent identity,

$$\mathrm{Im}(z)\gamma_1 R_{\bar z}^{\mathsf{v}} R_z^{\mathsf{v}} \gamma_1^* \varphi = 0.$$

This gives

$$\|R_z^{\mathsf{v}}\gamma_1^*\varphi\|_{L^2(\mathbb{R}^3)} = 0.$$

Since $(R_z^{\mathsf{v}}\gamma_1^*)^* = \gamma_1 R_{\bar z}^{\mathsf{v}} \in \mathcal{B}(L^2(\mathbb{R}^3), H^{1/2}(\Gamma))$ is surjective, the range of $R_z^{\mathsf{v}}\gamma_1^*$ is closed by the closed range theorem and, by [10, Theorem 5.2, p. 231],

$$\|R_z^{\mathsf{v}}\gamma_1^*\varphi\|_{L^2(\mathbb{R}^3)} \gtrsim \|\varphi\|_{H^{-1/2}(\Gamma)}.$$

Thus, $\ker(1 - \theta(D_z^{\mathsf{v}})^{-1}) = \{0\}$ and the proof is done. ■

According to Lemma 6.1 with $\mathsf{v} = 0$, for any $z \in \widehat{Z}^\circ_{0,n} \neq \varnothing$,

$$\begin{aligned}(M_z^{B_0,B_2})^{-1} &= (M_z^\theta)^{-1} = \Lambda_z^\theta := (\theta - D_z)^{-1}\\ &= -D_z^{-1}(1 - \theta D_z^{-1})^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma)).\end{aligned}$$

Thus,

$$Z_{B_0,B_2} = Z_\theta := \{z \in \mathbb{C}\backslash(-\infty, 0], \quad (M_z^\theta)^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma))\} \supseteq \widehat{Z}^\circ_{0,n}.$$

According to Lemma 6.1 again, for any $z \in \widehat{Z}^\circ_{\mathsf{v},n} \neq \varnothing$,

$$\begin{aligned}(\widehat{M}_z^{B_0,B_2})^{-1} &= (\widehat{M}_z^{\mathsf{v},\theta})^{-1} = \widehat{\Lambda}_z^{\mathsf{v},\theta} := (\theta - D_z^{\mathsf{v}})^{-1}\\ &= -(D_z^{\mathsf{v}})^{-1}(1 - \theta(D_z^{\mathsf{v}})^{-1})^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma)).\end{aligned}$$

Thus

$$\widehat{Z}_{\mathsf{B}} = \widehat{Z}_{\mathsf{v},\theta} := \{z \in \varrho(\Delta + \mathsf{v}) \colon (\widehat{M}_z^{\mathsf{v},\theta})^{-1} \in \mathcal{B}(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma))\} \supseteq \widehat{Z}^\circ_{\mathsf{v},n}.$$

Hence,

$$\Lambda_z^{\mathsf{B}} = \begin{bmatrix} \langle x\rangle^s & 0\\ 0 & 1\end{bmatrix}(M_z^{\mathsf{v},\theta})^{-1}\begin{bmatrix} \langle x\rangle^{-s} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} \langle x\rangle^{2s}\mathsf{v} & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix} \langle x\rangle^s & 0\\ 0 & 1\end{bmatrix}\mathbf{\Lambda}_z^{\mathsf{B}}\begin{bmatrix} \langle x\rangle^s & 0\\ 0 & 1\end{bmatrix},$$

where, by Theorem 5.3,

$$\mathbf{\Lambda}_z^{\mathsf{B}} = \mathbf{\Lambda}_z^{\mathsf{v},\theta} := \begin{bmatrix} \Lambda_z^{\mathsf{v},\theta}\,\Lambda_z^{\mathsf{v}} + \Lambda_z^{\mathsf{v}}\,\mathrm{DL}_z\,\widehat{\Lambda}_z^{\mathsf{v},\theta}\,\mathrm{DL}_{\bar z}^*\,\Lambda_z^{\mathsf{v}} & \Lambda_z^{\mathsf{v}}\,\mathrm{DL}_z\,\widehat{\Lambda}_z^{\mathsf{v},\theta} \\ \widehat{\Lambda}_z^{\mathsf{v},\theta}\,\mathrm{DL}_{\bar z}^*\,\Lambda_z^{\mathsf{v}} & \widehat{\Lambda}_z^{\mathsf{v},\theta} \end{bmatrix}$$
$$= \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & \widehat{\Lambda}_z^{\mathsf{v},\theta} \end{bmatrix} \left(1 + \begin{bmatrix} \mathrm{DL}_z\,\widehat{\Lambda}_z^{\mathsf{v},\theta}\,\mathrm{DL}_{\bar z}^* & \mathrm{DL}_z \\ \mathrm{DL}_{\bar z}^* & 0 \end{bmatrix}\right) \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & \widehat{\Lambda}_z^{\mathsf{v},\theta} \end{bmatrix}.$$

One has

$$\mathbf{\Lambda}_z^{\mathsf{v},\theta} \in \mathcal{B}(H^1_{-s}(\mathbb{R}^3\backslash\Gamma) \oplus H^{-1/2}(\Gamma), H^1_{-s}(\mathbb{R}^3\backslash\Gamma)^* \oplus H^{1/2}(\Gamma)).$$

By Theorems 2.1 and 2.9, there follows that

$$R_z^{\mathsf{v},\theta} = R_z + \begin{bmatrix} R_z\langle x\rangle^{-s} & \mathrm{DL}_z \end{bmatrix} \begin{bmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{bmatrix} \mathbf{\Lambda}_z^{\mathsf{v},\theta} \begin{bmatrix} \langle x\rangle^s & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \langle x\rangle^{2s}\mathsf{v}\langle x\rangle^{-s}R_z \\ \mathrm{DL}_{\bar z}^* \end{bmatrix} \tag{6.10}$$
$$= R_z + \begin{bmatrix} R_z & \mathrm{DL}_z \end{bmatrix} \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & \widehat{\Lambda}_z^{\mathsf{v},\theta} \end{bmatrix} \left(1 + \begin{bmatrix} \mathrm{DL}_z\,\widehat{\Lambda}_z^{\mathsf{v},\theta}\,\mathrm{DL}_{\bar z}^* & \mathrm{DL}_z \\ \mathrm{DL}_{\bar z}^* & 0 \end{bmatrix}\right) \begin{bmatrix} \Lambda_z^{\mathsf{v}} & 0 \\ 0 & \widehat{\Lambda}_z^{\mathsf{v},\theta} \end{bmatrix} \begin{bmatrix} R_z \\ \mathrm{DL}_{\bar z}^* \end{bmatrix} \tag{6.11}$$
$$= R_z^{\mathsf{v}} + \mathrm{DL}_z^{\mathsf{v}}\,\widehat{\Lambda}_z^{\mathsf{v},\theta}\mathrm{DL}_{\bar z}^{\mathsf{v}\,*}. \tag{6.12}$$

is the resolvent of a self-adjoint operator $\Delta^{\mathsf{v},\delta',\theta}$; (6.10) holds for any $z \in \varrho(\Delta^{\mathsf{v},\delta',\theta}) \cap \mathbb{C}\backslash(-\infty, 0]$, both (6.11) and (6.12) hold for any $z \in \varrho(\Delta^{\mathsf{v},\delta',\theta}) \cap \varrho(\Delta + \mathsf{v})$. By (6.3) and by the mapping properties of $\mathrm{DL}_z^{\mathsf{v}}$, one has

$$\mathrm{dom}(\Delta^{\mathsf{v},\delta',\theta}) \subseteq H^1(\mathbb{R}^3\backslash\Gamma).$$

By $R_z^{\mathsf{v}}u \in H^2(\mathbb{R}^3)$, so that $[\gamma_1]R_z^{\mathsf{v}}u = 0$, and by (4.31), one gets

$$[\gamma_0]R_z^{\mathsf{v},\theta}u = \widehat{\Lambda}_z^{\mathsf{v},\theta}\mathrm{DL}_{\bar z}^{\mathsf{v}\,*}u = \hat{\rho}_{\mathsf{B}}(R_z^{\mathsf{v},\theta}u).$$

Hence, by Theorem 2.11,

$$\Delta^{\mathsf{v},\delta',\theta}u = \Delta u + \mathsf{v}u + ([\gamma_0]u)\delta'_\Gamma$$

and

$$u \in \mathrm{dom}(\Delta^{\mathsf{v},\delta',\theta}) \implies \gamma_1 u = \theta[\gamma_0]u.$$

Proceeding as in [11, Section 5.5] (see the proof of Theorem 5.15 there), $\Delta^{\mathsf{v},\delta',\theta}$ is bounded from above and so hypothesis (H4.1) holds. The scattering couple $(\Delta^{\mathsf{v},\delta',\theta}, \Delta)$ is asymptotically complete and the corresponding scattering matrix is given by

$$\mathsf{S}_\lambda^{\mathsf{v},\theta} = 1 - 2\pi i\, \mathsf{L}_\lambda \mathbf{\Lambda}_\lambda^{\mathsf{v},\theta,+} \mathsf{L}_\lambda^*, \quad \lambda \in (-\infty, 0]\backslash(\sigma_{\mathrm{p}}^-(\Delta + \mathsf{v}) \cup \sigma_{\mathrm{p}}^-(\Delta^{\mathsf{v},\delta',\theta})),$$

where L_λ is given in Theorem 5.3 and $\mathbf{\Lambda}^{\mathsf{v},\theta,+}_\lambda := \lim_{\epsilon\searrow 0}\mathbf{\Lambda}^{\mathsf{v},\theta}_{\lambda+i\epsilon}$. This latter limit exists by Lemma 3.8; in particular, by (3.10),

$$\mathbf{\Lambda}^{\mathsf{v},\theta,+} = \left(1+\begin{bmatrix}(1-\mathsf{v}R^+_\lambda)^{-1}\mathsf{v} & 0\\ 0 & (\theta-D^{\mathsf{v},+}_\lambda)^{-1}\end{bmatrix}\begin{bmatrix}\mathrm{DL}^+_\lambda(\theta-D^{\mathsf{v},+}_\lambda)^{-1}\alpha(\mathrm{DL}^-_\lambda)^* & \mathrm{DL}^+_\lambda\\ (\mathrm{DL}^-_\lambda)^* & 0\end{bmatrix}\right)$$
$$\times\begin{bmatrix}(1-\mathsf{v}R^+_\lambda)^{-1}\mathsf{v} & 0\\ 0 & (\theta-D^{\mathsf{v},+}_\lambda)^{-1}\alpha\end{bmatrix},$$

where

$$R^\pm_\lambda := \lim_{\epsilon\searrow 0} R_{\lambda\pm i\epsilon},\quad \mathrm{DL}^\pm_\lambda := \lim_{\epsilon\searrow 0}\mathrm{SL}_{\lambda\pm i\epsilon},\quad D^{\mathsf{v},\pm}_\lambda := \lim_{\epsilon\searrow 0}\gamma_0\,\mathrm{DL}^{\mathsf{v}}_{\lambda\pm i\epsilon}.$$

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