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Sharp spectral stability for a class of singularly perturbed pseudo-differential operators

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Abstract. Let $a(x,\xi)$ be a real Hörmander symbol of the type $S^0_{0,0}(\mathbb{R}^d\times\mathbb{R}^d)$, let F be a smooth function with all its derivatives globally bounded, and let K_δ be the self-adjoint Weyl quantization of the perturbed symbols $a(x+F(\delta x),\xi)$, where $|\delta|\le 1$. First, we prove that the Hausdorff distance between the spectra of K_δ and K_0 is bounded by $\sqrt{|\delta|}$, and we give examples where spectral gaps of this magnitude can open when $\delta\neq 0$. Second, we show that the distance between the spectral edges of K_δ and K_0 (and also the edges of the inner spectral gaps, as long as they remain open at $\delta=0$) are of order $|\delta|$, and give a precise dependence on the width of the spectral gaps.

1. Introduction and main results

Let $a(x,\xi)$ be a real Hörmander symbol [14] of class $S^0_{0,0}(\mathbb{R}^d\times\mathbb{R}^d)$, i.e., a smooth function on $\mathbb{R}^{2d}$ satisfying the estimate

$$\sup_{x,\xi\in\mathbb{R}^d}|D^\alpha_x D^\beta_\xi a|<\infty,\quad \text{for all } \alpha,\beta\in\mathbb{N}^d. \tag{1.1}$$

For $|\delta|\le 1$ let $a_\delta(x,\xi)=a(\sqrt{1+\delta}x,\sqrt{1+\delta}\xi)\in\mathbb{R}$. It belongs to the same class. We denote by $H_\delta=\mathfrak{Op}^w(a_\delta)$ the self-adjoint operator generated by the Weyl quantization, which means that

$$\langle\psi,H_\delta\phi\rangle:=(2\pi)^{-d}\int_{\mathbb{R}^d}d\xi\int_{\mathbb{R}^{2d}}dx'\,dx e^{i\xi\cdot(x-x')}a_\delta((x+x')/2,\xi)\bar{\psi}(x)\phi(x'),$$

where $\psi,\phi\in\mathcal{S}(\mathbb{R}^d)$ and $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L^2(\mathbb{R}^d)$ (considered to be anti-linear in the first variable). The operator H_δ has a distribution kernel that can be written as the oscillatory integral

$$\mathfrak{H}_\delta(u,v):=(2\pi)^{-d}\int_{\mathbb{R}^d}a_\delta(u,\xi)e^{i\xi\cdot v}d\xi,\quad u=(x+x')/2,\ v=x-x'. \tag{1.2}$$

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In [11], Gröchenig et al. proved that the spectral edges of the spectrum $\sigma(H_\delta)$ are Lipschitz at $\delta = 0$. The problem is non-trivial because the map $\delta \mapsto H_\delta$ is not necessarily differentiable in the operator norm topology, which can already be seen at the level of the symbol: a should have some extra linear decay in both x and ξ in order to make sure that $H_\delta - H_0$ has a norm of order δ, which would imply that the Hausdorff distance between the spectra of H_δ and H_0 is Lipschitz continuous at zero.

Nevertheless, the authors of [11] show that such a strong decay is far from necessary if one is only interested in the spectral edges. Actually they even consider more general operators corresponding to symbols of Sjöstrand type, operators which belong to certain weighted modulation spaces; see [10] and references therein for an introduction to the subject.

A similar phenomenon appears in the case of long range magnetic perturbations [2, 4–6, 8, 9]. In fact, the two problems are very much related, see Section 3.2 of the current manuscript for more details.

1.1. A more general perturbation

In this manuscript we are interested in a more general perturbation of the symbol, where the dilation treated in [11] becomes just a particular case. In order to achieve that, we have to “rotate” the operators H_δ in the following way.

Lemma 1.1. *Denote by U_δ the unitary transformation in $L^2(\mathbb{R}^d)$ given by*

$$(U_\delta f)(x) = (1+\delta)^{-d/4} f((1+\delta)^{-1/2}x), \quad \textit{for all } f \in L^2(\mathbb{R}^d).$$

Then $U_\delta^* H_\delta U_\delta$ equals the Weyl quantization of the symbol $a(x+\delta x, \xi)$, and is isospectral with H_δ.

This result is straightforward and we will omit its proof. The advantage of working with symbols shifted only in x is that we can identify a larger class of perturbations, where the same spectral results as proved in [11] hold true. More precisely, instead of $x + \delta x$ we will consider $x + F(\delta x)$ where F satisfies the following assumptions.

Hypothesis 1.2. Let $F \in [C^\infty(\mathbb{R}^d)]^d$ be a smooth real vector-valued function with all its derivatives of all order uniformly bounded (thus F can grow linearly at infinity). Given any real symbol $a \in S^0_{0,0}(\mathbb{R}^d \times \mathbb{R}^d)$ as in (1.1) and $\delta \in \mathbb{R}$, let K_δ be the Weyl quantization of the symbol $a[F]_\delta(x,\xi) := a(x + F(\delta x), \xi)$, i.e., $K_\delta = \mathfrak{Op}^w(a[F]_\delta)$. Also, the distribution kernel of K_δ (defined as in (1.2)) is denoted by $\mathfrak{K}_\delta$. If $F(x) = x$, then K_δ and H_δ are unitarily equivalent and isospectral.

We will only work with symbols of class $S^0_{0,0}(\mathbb{R}^d \times \mathbb{R}^d)$ (included in the Sjöstrand class of symbols considered in [11]) since they are more suitable for the less symmetric perturbation which we consider.

1.2. The main results

We start by recalling the definition of the *Hausdorff distance* between any two compact sets $M, N \subset \mathbb{R}$:

$$d_h(M, N) = \max\{\sup_{\lambda \in M} \operatorname{dist}(\lambda, N),\ \sup_{\mu \in N} \operatorname{dist}(\mu, M)\}.$$

Our first main result gives a sharp upper bound on how much the spectra can "move" as sets.

Theorem 1.3. *Consider the notation introduced in Hypothesis 1.2. Then there exists $C > 0$ such that*

$$d_h(\sigma(K_\delta), \sigma(K_0)) \leq C\sqrt{|\delta|}$$

for all $|\delta| \leq 1$. This bound is sharp, in the sense that one can construct a K_0 such that $0 \in \sigma(K_0)$ while the spectrum of K_δ develops gaps of order $\sqrt{|\delta|}$ near zero.

The next straightforward corollary spells out in a detailed way how the interior non-trivial gaps in the spectrum of K_0 may vary with δ.

Corollary 1.4. *Assume that K_0 has an open spectral gap (λ_0, μ_0) with $\lambda_0, \mu_0 \in \sigma(K_0)$. Then there exists a constant $C > 0$ (the same as in Theorem 1.3), independent on the spectral gap, such that for all $|\delta| < (\mu_0 - \lambda_0)^2/(4C^2)$ the interval $[\lambda_0 + C\sqrt{|\delta|}, \mu_0 - C\sqrt{|\delta|}]$ is non-empty and belongs to the resolvent set of K_δ. Moreover, both intervals $[\lambda_0 - C\sqrt{|\delta|}, \lambda_0 + C\sqrt{|\delta|}]$ and $[\mu_0 - C\sqrt{|\delta|}, \mu_0 + C\sqrt{|\delta|}]$ have a non-empty intersection with $\sigma(K_\delta)$ for all $|\delta| < (\mu_0 - \lambda_0)^2/(4C^2)$.*

The next main result states that the spectral edges of K_δ have a Lipschitz variation at $\delta = 0$.

Theorem 1.5. *Let $E_+(\delta) := \sup \sigma(K_\delta)$ and $E_-(\delta) := \inf \sigma(K_\delta)$. There exists $C > 0$ such that $|E_\pm(\delta) - E_\pm(0)| \leq C|\delta|$, for all $|\delta| \leq 1$.*

The next corollary describes the variation of the edges of those interior gaps which remain open at $\delta = 0$, and gives a precise control with respect to the width of the spectral gap.

Corollary 1.6. *Consider the same setting and the same notation as in Corollary 1.4. Let $|\delta| < (\mu_0 - \lambda_0)^2/(4C^2)$. Since $(\mu_0 + \lambda_0)/2$ is in the resolvent set of K_δ, and both sets $\sigma(K_\delta) \cap (-\infty, (\mu_0 + \lambda_0)/2)$ and $\sigma(K_\delta) \cap ((\mu_0 + \lambda_0)/2, \infty)$ are non-empty, we may define*

$$\lambda_\delta := \sup(\sigma(K_\delta) \cap (-\infty, (\mu_0 + \lambda_0)/2)),$$
$$\mu_\delta := \inf(\sigma(K_\delta) \cap ((\mu_0 + \lambda_0)/2, \infty)).$$

Then there exists a constant $\widetilde{C} > 0$*, independent of* $\mu_0 - \lambda_0$*, and some* $0 < \delta_1 < (\mu_0 - \lambda_0)^2/(4C^2)$ *such that*

$$\max\{|\lambda_\delta - \lambda_0|, |\mu_\delta - \mu_0|\} \leq \frac{\widetilde{C}|\delta|}{\mu_0 - \lambda_0}, \quad \textit{for all } |\delta| \leq \delta_1.$$

Remark 1.7. Corollary 1.6 is stronger than Corollary 1.4 only when $\sqrt{|\delta|}$ is much smaller than the width of the gap $\mu_0 - \lambda_0$. An important point is that the constant C in Corollary 1.4 is independent of the gap, while the Lipschitz constant in Corollary 1.6 is inverse proportional with the width of the gap at $\delta = 0$. This is compatible with Theorem 1.3: when $|\delta|$ increases and becomes of order $(\mu_0 - \lambda_0)^2$, the gap might even close.

Remark 1.8. When $F(x) = x$, the results of Theorem 1.5 and Corollary 1.6 are also obtained in [11]. On the other hand, the results of Theorem 1.3 and Corollary 1.4 are new. We note that if one is only interested in proving Lipschitz behavior of the inner gap edges λ_δ and μ_δ, one does not need the explicit estimate in our Theorem 1.3, but only some a priori knowledge of their continuity, as in [11].

2. Technical preliminaries

2.1. Known facts about the Hausdorff distance between spectra

The following lemma is well known but also very important, hence we prove it for completeness, see also [9].

Lemma 2.1. *Let* A *and* B *be self-adjoint and bounded. Let* $E_+(A) = \sup \sigma(A)$*,* $E_-(A) = \inf \sigma(A)$*, and* $E_\pm(B)$ *denotes the same for* B*. Then*

$$|E_\pm(A) - E_\pm(B)| \leq d_h(\sigma(A), \sigma(B)) \leq \|A - B\|.$$

Proof. Let us prove the first inequality but only for "E_+". Let us assume, without loss of generality, that $E_+(A) \leq E_+(B)$. Then

$$\begin{aligned} 0 &\leq E_+(B) - E_+(A) = \operatorname{dist}(E_+(B), \sigma(A)) \\ &\leq \sup_{\lambda \in \sigma(B)} \operatorname{dist}(\lambda, \sigma(A)) \leq d_h(\sigma(A), \sigma(B)). \end{aligned}$$

Now, let us prove the second inequality. Let $z \notin \sigma(A)$. We have

$$B - z\mathbb{1} = (\mathbb{1} + (B - A)(A - z\mathbb{1})^{-1})(A - z\mathbb{1}).$$

If $\operatorname{dist}(z, \sigma(A)) > \|A - B\|$, then

$$\|(B - A)(A - z\mathbb{1})^{-1}\| < 1$$

and $z \notin \sigma(B)$. This means that the spectrum of B is located within a neighborhood of width $\|A - B\|$ of the spectrum of A. The same conclusion holds for A replaced with B. ■

Another useful inequality is the following.

Lemma 2.2. *Let A, B, C, D be bounded self-adjoint operators. Then*

$$|E_\pm(A) - E_\pm(D)| \le \|A - B\| + |E_\pm(B) - E_\pm(C)| + \|C - D\|. \tag{2.1}$$

Proof. Direct application of the triangle inequality and of Lemma 2.1. ■

2.2. Reduction to compact support in the second variable of the distribution kernel

We refer to Hypothesis 1.2 for the notation involving K_δ and $\mathfrak{K}_\delta$.

Lemma 2.3. *Let $0 \le f \le 1$ be smooth and compactly supported, with $f(x) = 1$ in a neighborhood of 0. Let $\widetilde{K}_\delta$ be the operator with the integral kernel $\widetilde{\mathfrak{K}}_\delta(u, v) := f(\sqrt{|\delta|}v)\mathfrak{K}_\delta(u, v)$. Then the symbol $\tilde{a}[F]_\delta$ of $\widetilde{K}_\delta$ obeys (1.1) uniformly in $|\delta| \le 1$ and*

$$\|K_\delta - \widetilde{K}_\delta\| = \mathcal{O}(\delta^\infty).$$

Proof. We may assume $\delta \ge 0$. Denote by $\hat{f}$ the Fourier transform of f. The symbol of $\widetilde{K}_\delta$ is a convolution,

$$\tilde{a}[F]_\delta(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} a(x + F(\delta x), \xi - \xi') \frac{\hat{f}(\xi'/\sqrt{\delta})}{\delta^{d/2}} d\xi',$$

while the symbol of $K_\delta - \widetilde{K}_\delta$ is

$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} (a(x + F(\delta x), \xi) - a(x + F(\delta x), \xi - \xi')) \frac{\hat{f}(\xi'/\sqrt{\delta})}{\delta^{d/2}} d\xi',$$

where we used that $f(0) = 1$. Using the integral Taylor formula we have

$$\begin{aligned} &a(x + F(\delta x), \xi) - a(x + F(\delta x), \xi - \xi') \\ &\quad = \int_0^1 dr \frac{d}{dr} a(x + F(\delta x), \xi - \xi' + r\xi') \\ &\quad = (\xi' \cdot \nabla_\xi) a(x + F(\delta x), \xi) - \int_0^1 dr\, r (\xi' \cdot \nabla_\xi)^2 a(x + F(\delta x), \xi - \xi' + r\xi') \end{aligned}$$

$$= \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} (\xi' \cdot \nabla_\xi)^{j+1} a(x + F(\delta x), \xi)$$

$$+ \frac{(-r)^N}{N!} \int_0^1 dr (\xi' \cdot \nabla_\xi)^{N+1} a(x + F(\delta x), \xi - \xi' + r\xi')$$

for every $N \geq 1$. Using that all the partial derivatives of f at zero equal zero, we may write the symbol of $K_\delta - \widetilde{K}_\delta$ as

$$(2\pi)^{-d/2} \int_0^1 dr \frac{(-r)^N}{N!} \int_{\mathbb{R}^d} (\xi' \cdot \nabla_\xi)^{N+1} a(x + F(\delta x), \xi - \xi' + r\xi') \frac{\hat{f}(\xi'/\sqrt{\delta})}{\delta^{d/2}} d\xi',$$

for all $N \geq 1$. After a change of variables, this symbol reads as

$$\delta^{(N+1)/2} (2\pi)^{-d/2}$$
$$\times \int_0^1 dr \frac{(-r)^N}{N!} \int_{\mathbb{R}^d} (\xi' \cdot \nabla_\xi)^{N+1} a(x + F(\delta x), \xi - \sqrt{\delta}\xi' + r\sqrt{\delta}\xi') \hat{f}(\xi') d\xi',$$

for all $N \geq 1$.

This symbol obeys (1.1) where the supremum is bounded by $C_{N,\alpha,\beta}\delta^{(N+1)/2}$ for all N. An application of the Calderón–Vaillancourt Theorem [3] finishes the proof. ∎

2.3. A localization result

Let $0 \leq g \leq 1$ be smooth with compact support such that

$$\sum_{\gamma \in \mathbb{Z}^d} g^2(x - \gamma) = 1, \quad \text{for all } x \in \mathbb{R}^d. \tag{2.2}$$

Let $0 \leq \tilde{g} \leq 1$ be any other compactly supported function and define $g_{\delta,\gamma}(x) := g(\sqrt{|\delta|}x - \gamma)$ and $\tilde{g}_{\delta,\gamma}(x) := \tilde{g}(\sqrt{|\delta|}x - \gamma)$.

Lemma 2.4. *Let $\mathcal{T} := \{T_\gamma\}_{\gamma \in \mathbb{Z}^d}$ be any family of bounded operators on $L^2(\mathbb{R}^d)$ and let $|||T||| = \sup_{\gamma \in \mathbb{Z}^d} \|T_\gamma\|$. We define*

$$\Gamma_{\tilde{g}}(\mathcal{T}) := \sum_{\gamma \in \mathbb{Z}^d} \tilde{g}_{\delta,\gamma} T_\gamma g_{\delta,\gamma}.$$

Then there exists a constant C independent of $|\delta| \leq 1$ such that

$$\|\Gamma_{\tilde{g}}(\mathcal{T})\| \leq C |||T|||$$

Proof. Given $\gamma \in \mathbb{Z}^d$, we denote by V_γ the set of all $\gamma' \in \mathbb{Z}^d$ with the property that the support of $\tilde{g}_{\delta,\gamma'}$ has a non-empty overlap with the support of $\tilde{g}_{\delta,\gamma}$, including $\gamma' = \gamma$. Denote by $\nu \in \mathbb{N} \setminus \{0\}$ the cardinal of V_γ; it is clearly independent of γ and δ. For $\psi \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} &\|\Gamma_{\tilde{g}}(\mathcal{T})\psi\|^2 \\ &= \sum_{\gamma\in\mathbb{Z}^d} \sum_{\gamma'\in V_\gamma} \langle \tilde{g}_{\delta,\gamma} T_\gamma g_{\delta,\gamma}\psi, \tilde{g}_{\delta,\gamma'} T_\gamma g_{\delta,\gamma'}\psi\rangle \leq \frac{\nu+1}{2} \sum_{\gamma\in\mathbb{Z}^d} \|T_\gamma g_{\delta,\gamma}\psi\|^2 \\ &\leq \|\!|T|\!\|^2 \frac{\nu+1}{2} \sum_{\gamma\in\mathbb{Z}^d} \int_{\mathbb{R}^d} g^2(\sqrt{|\delta|}x-\gamma)|\psi(x)|^2 dx = \|\!|T|\!\|^2 \frac{\nu+1}{2} \|\psi\|^2, \end{aligned}$$

where in the last equality we used (2.2). ∎

3. Proof of Theorem 1.3

For simplicity, let $0 \leq \delta \leq 1$. From Lemma 2.3 and Lemma 2.1, we infer that the Hausdorff distance between the spectra of K_δ and $\widetilde{K}_\delta$ is of order δ^∞. Let us define the operator $\mathring{K}_\delta$ through its integral kernel given by $\mathring{\mathfrak{K}}_\delta(u,v) = f(\sqrt{\delta}v)\mathfrak{K}_0(u,v)$. Then with the same proof as in Lemma 2.3 one can show that $\|\mathring{K}_\delta - K_0\| = \mathcal{O}(\delta^\infty)$, and the same is true for the Hausdorff distance between their spectra. Therefore, according to the second inequality in (2.1), it is enough to prove that the Hausdorff distance between the spectra of $\widetilde{K}_\delta$ and $\mathring{K}_\delta$ is of order $\sqrt{\delta}$.

Let τ_α be the unitary operator induced by the translation with $-\alpha$, i.e., $(\tau_\alpha\psi)(x) = \psi(x-\alpha)$; we use the notations introduced in Lemma 2.4 and work with Γ_g, i.e., with $\tilde{g} = g$. We shall prove the following statement.

Proposition 3.1. *Let $\mathfrak{z} \in \mathbb{C}$ be in the resolvent set of $\mathring{K}_\delta$ defined above. Let us define*

$$T_\gamma(\mathfrak{z}) = \tau_{-F(\sqrt{\delta}\gamma)}(\mathring{K}_\delta - \mathfrak{z}\mathbb{1})^{-1}\tau_{F(\sqrt{\delta}\gamma)}$$

the associated family $\mathcal{T}(\mathfrak{z}) := \{T_\gamma(\mathfrak{z})\}_{\gamma\in\mathbb{Z}^d}$ as in Lemma 2.4 and the "remainder operator"

$$R_\delta(\mathfrak{z}) := (\widetilde{K}_\delta - \mathfrak{z}\mathbb{1})\Gamma_g(\mathcal{T}(\mathfrak{z})) - \mathbb{1}.$$

Then there exists a constant C such that for all $0 \leq \delta \leq 1$ we have

$$\|R_\delta(\mathfrak{z})\| \leq C \frac{\sqrt{\delta}}{dist(\mathfrak{z}, \sigma(\mathring{K}_\delta))}. \tag{3.1}$$

In particular, this implies that the spectrum of $\widetilde{K}_\delta$ belongs to a neighborhood of width $C\sqrt{\delta}$ of the spectrum of $\mathring{K}_\delta$.

3.1. Proof of Proposition 3.1

Here we use some of the ideas employed in [1]. We need to investigate the operator $\widetilde{K}_\delta g_{\delta,\gamma}$ and compare it with $g_{\delta,\gamma}\tau_{-F(\sqrt{\delta}\gamma)}\mathring{K}_\delta\tau_{F(\sqrt{\delta}\gamma)}$. If they were equal to each other, then $R_\delta(\mathfrak{z})$ would equal zero. In fact, there are two contributions to $R_\delta(\mathfrak{z})$: one coming from replacing $\widetilde{K}_\delta g_{\delta,\gamma}$ with $\tau_{-F(\sqrt{\delta}\gamma)}\mathring{K}_\delta\tau_{F(\sqrt{\delta}\gamma)}g_{\delta,\gamma}$, and the other one coming from the commutator $[\tau_{-F(\sqrt{\delta}\gamma)}\mathring{K}_\delta\tau_{F(\sqrt{\delta}\gamma)}, g_{\delta,\gamma}]$.

Lemma 3.2. *There exists a constant $C > 0$ such that*

$$\|\widetilde{K}_\delta g_{\delta,\gamma} - \tau_{-F(\sqrt{\delta}\gamma)}\mathring{K}_\delta\tau_{F(\sqrt{\delta}\gamma)}g_{\delta,\gamma}\| \leq C\sqrt{\delta}, \quad \textit{for all } \gamma \in \mathbb{Z}^d.$$

Proof. The distribution kernel of the operator

$$L_{\delta,\gamma} := \widetilde{K}_\delta g_{\delta,\gamma} - \tau_{-F(\sqrt{\delta}\gamma)}\mathring{K}_\delta\tau_{F(\sqrt{\delta}\gamma)}g_{\delta,\gamma}$$

from the statement of the Lemma is given by

$$f(\sqrt{\delta}v)(\mathfrak{K}_0(u + F(\delta u), v) - \mathfrak{K}_0(u + F(\sqrt{\delta}\gamma), v))g(\sqrt{\delta}(u - v/2) - \gamma).$$

Let us denote

$$h_y(x) := f(x)g(-x/2 + y).$$

We see that h_y is identically zero if $|y|$ is large enough. We may find some smooth and compactly supported function $0 \leq \tilde{h} \leq 1$ such that

$$h_y(x)\tilde{h}(y) = h_y(x), \quad \text{for all } x, y \in \mathbb{R}^d.$$

The role of y is played by $\sqrt{\delta}u - \gamma$, which means that the quantity $|\sqrt{\delta}u - \gamma|$ remains uniformly bounded in both δ and γ due to the presence of $\tilde{h}$.

Denote by $\nabla_1 a(x, \xi)$ the partial gradient of $a(x, \xi)$ with respect to the spatial variables x. Then by denoting

$$\begin{aligned} &\alpha_{\delta,\gamma}(u, \xi) \\ &\quad := a(u + F(\delta u), \xi) - a(u + F(\sqrt{\delta}\gamma), \xi) \\ &\quad = (F(\delta u) - F(\sqrt{\delta}\gamma)) \cdot \int_0^1 dr \nabla_1 a(u + F(\sqrt{\delta}\gamma) + r[F(\delta u) - F(\sqrt{\delta}\gamma)], \xi), \end{aligned}$$

the symbol of our operator $L_{\delta,\gamma}$ is

$$b_{\delta,\gamma}(u, \xi) := (2\pi)^{-d/2}\tilde{h}(\sqrt{\delta}u - \gamma) \int_{\mathbb{R}^d} \alpha_{\delta,\gamma}(u, \xi - \xi') \frac{\hat{h}_{\sqrt{\delta}t-\gamma}(\xi'/\sqrt{\delta})}{\delta^{d/2}} d\xi'.$$

An important observation is that the function

$$\tilde{h}(\sqrt{\delta}u - \gamma)(F(\delta u) - F(\sqrt{\delta}\gamma))$$

is uniformly bounded by $\sqrt{\delta}$ together with all its derivatives. Thus, all the seminorms of the above symbol will have (at least) a factor $\sqrt{\delta}$, uniformly in γ and we are done after an application of Calderón–Vaillancourt. ∎

Lemma 3.3. *There exists a constant* $C > 0$ *such that*

$$\|[g_{\delta,\gamma},\, \tau_{-F(\sqrt{\delta}\gamma)} \mathring{K}_\delta \tau_{F(\sqrt{\delta}\gamma)}]\| \leq C\sqrt{\delta}, \quad \textit{for all } \gamma \in \mathbb{Z}^d.$$

Proof. The distribution kernel of the above commutator is

$$f(\sqrt{\delta}v)\mathfrak{K}_0(u + F(\sqrt{\delta}\gamma), v)(g(\sqrt{\delta}(u + v/2) - \gamma) - g(\sqrt{\delta}(u - v/2) - \gamma))$$

which equals

$$\sqrt{\delta}2^{-1} \int_{-1/2}^{1/2} dr\, f(\sqrt{\delta}v)\mathfrak{K}_0(u + F(\sqrt{\delta}\gamma), v)v \cdot \nabla g(\sqrt{\delta}u - \gamma + r\sqrt{\delta}v/2).$$

The v appearing in the factor $v \cdot \nabla g$ has to be coupled with $\mathfrak{K}_0$, in the sense that when we write the symbol of the commutator as a convolution, by using integration by parts, the factor $\mathfrak{K}_0(u + F(\sqrt{\delta}\gamma), v)v$ becomes $\nabla_\xi a(u + F(\sqrt{\delta}\gamma), \xi - \xi')$ in the convolution. It turns out that again, all the seminorms of the commutator symbol will be of order $\sqrt{\delta}$ uniformly in γ. ∎

We are now ready to complete the proof of Proposition 3.1. Let $\Omega \subset \mathbb{R}^d$ be a ball which contains the set

$$\{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \operatorname{supp}(g)) \leq 1\}$$

and let us denote by $\tilde{g}$ the indicator function of Ω.

Remark 3.4. Due to our choice of f with support in the unit ball, the presence of $f(\sqrt{\delta}v)$ in the distribution kernels of both $\widetilde{K}_\delta$ and $\mathring{K}_\delta$ implies that

$$\widetilde{K}_\delta g_{\delta,\gamma} = \tilde{g}_{\delta,\gamma}\widetilde{K}_\delta g_{\delta,\gamma}, \tag{3.2a}$$

$$\tau_{-F(\sqrt{\delta}\gamma)}\mathring{K}_\delta\tau_{F(\sqrt{\delta}\gamma)} g_{\delta,\gamma} = \tilde{g}_{\delta,\gamma}\tau_{-F(\sqrt{\delta}\gamma)}\mathring{K}_\delta\tau_{F(\sqrt{\delta}\gamma)} g_{\delta,\gamma}. \tag{3.2b}$$

Now, let $\{M_\gamma(\mathfrak{z})\}_{\gamma\in\mathbb{Z}^d}$ be a family $\mathcal{M}(\mathfrak{z})$ of operators given by

$$M_\gamma(\mathfrak{z}) = (\widetilde{K}_\delta - \tau_{-F(\sqrt{\delta}\gamma)}\mathring{K}_\delta\tau_{F(\sqrt{\delta}\gamma)})g_{\delta,\gamma}T_\gamma(\mathfrak{z}) - [g_{\delta,\gamma},\, \tau_{-F(\sqrt{\delta}\gamma)}\mathring{K}_\delta\tau_{F(\sqrt{\delta}\gamma)}]T_\gamma(\mathfrak{z}).$$

A short computation using also (3.2) shows that

$$R_\delta(\mathfrak{z}) = \Gamma_{\tilde{g}}(\mathcal{M}(\mathfrak{z})),$$

hence an application of Lemma 2.4 with $\mathcal{T}$ replaced by $\mathcal{M}$ finishes the proof of (3.1).

Now, let us investigate the spectral consequences. If $\mathfrak{z} \in \mathbb{C}$ is in the resolvent set of $\mathring{K}_\delta$, we have the identity

$$(\widetilde{K}_\delta - \mathfrak{z}\mathbb{1})\Gamma_g(\mathcal{T}(\mathfrak{z})) = \mathbb{1} + R_\delta(\mathfrak{z}).$$

Now, if we also impose that $\operatorname{dist}(\mathfrak{z}, \sigma(\mathring{K}_\delta)) > C\sqrt{\delta}$, then $\|R_\delta(z)\| < 1$ and $(\widetilde{K}_\delta - \mathfrak{z}\mathbb{1})$ is invertible, thus $\mathfrak{z}$ cannot belong to the spectrum of $\sigma(\widetilde{K}_\delta)$. Thus, if $\lambda \in \sigma(\widetilde{K}_\delta)$, then $\operatorname{dist}(\mathfrak{z}, \sigma(\mathring{K}_\delta)) \leq C\sqrt{\delta}$. This ends the proof of Proposition 3.1. ∎

3.2. Concluding the proof of Theorem 1.3

We have seen in Proposition 3.1 that if $\mathfrak{z} \in \mathbb{C}$ is at a distance larger than a constant times $\sqrt{\delta}$ from the spectrum of $\mathring{K}_\delta$, then $\mathfrak{z}$ is also in the resolvent set of $\widetilde{K}_\delta$. Exactly the same type of proof can be used when we swap the roles of $a(x + F(\delta x), \xi)$ and $a(x, \xi)$, namely by putting

$$\tilde{a}(x, \xi) := a(x + F(\delta x), \xi)$$

and

$$\tilde{a}_\delta(x, \xi) := \tilde{a}(x - F(\delta x), \xi).$$

Thus, this proves that the Hausdorff distance between $\sigma(\mathring{K}_\delta)$ and $\sigma(\widetilde{K}_\delta)$ goes like $\sqrt{\delta}$.

This bound cannot be made better in general. Let $d = 2$ and let

$$a_\delta(x, \xi) = \cos(\xi_1) + \cos(\xi_2 + (1 + \delta)x_1).$$

Through Weyl quantization, this symbol generates an operator which is isospectral with the Hofstadter model, in the Landau gauge, with a constant magnetic field $b = 1 + \delta$. It is known [12, 13] that $\mathfrak{Op}(a_0)$ corresponds to the "half-flux case," and the operator has an absolutely continuous gap-less spectrum which contains the origin. If $\delta \neq 0$ is small, then the spectrum of $\mathfrak{Op}(a_\delta)$ develops gaps near zero of width $\sqrt{\delta}$. A recent detailed analysis regarding the magnetic perturbations of "Dirac cones" which produce gaps of order $\sqrt{\delta}$ may be found in [7].

4. Proof of Theorem 1.5

We only prove the theorem for E_+ and $0 \leq \delta \leq 1$. Due to (2.1), it is enough to prove the statement for the pair of operators $\widetilde{K}_\delta$ and $\mathring{K}_\delta$, where as before, $\widetilde{K}_\delta$ corresponds to the integral kernel $f(\sqrt{\delta}v)\mathfrak{K}_0(u + F(\delta u), v)$, while $\mathring{K}_\delta$ corresponds to the integral kernel $f(\sqrt{\delta}v)\mathfrak{K}_0(u, v)$.

Lemma 4.1. *Let $\delta > 0$. For every $\psi \in L^2(\mathbb{R}^d)$ we define*

$$\Psi_\delta(x, y) := \left(\frac{\delta}{4\pi}\right)^{d/4} e^{-\delta\frac{|x-y|^2}{8}} \psi(x).$$

Then $\Psi_\delta \in L^2(\mathbb{R}^{2d})$ and $\|\Psi_\delta\|_{L^2(\mathbb{R}^{2d})} = \|\psi\|_{L^2(\mathbb{R}^d)}$.

Proof. Direct computation. ∎

Lemma 4.2. *Let $x, y, z \in \mathbb{R}^d$. Then*

$$2^{-1}|x + y/2 - z|^2 + 2^{-1}|x - y/2 - z|^2 - y^2/4 = |x - z|^2.$$

Proof. Direct computation (the parallelogram identity). ∎

For $\psi \in \mathcal{S}(\mathbb{R}^d)$ we notice that we have the following identity:

$$\begin{aligned}\langle \psi, \widetilde{K}_\delta \psi\rangle &= \int_{\mathbb{R}^{2d}} du\, dv \bar{\psi}(u + v/2) f(\sqrt{\delta}v)\mathfrak{K}_0(u + F(\delta u), v)\psi(u - v/2) \\ &= \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^{2d}} du\, dv \bar{\psi}(u + v/2) f(\sqrt{\delta}v)\mathfrak{K}_0(u + F(\delta u), v) \\ &\quad \times \left(\frac{\delta}{4\pi}\right)^{d/2} e^{-\delta\frac{|u-y|^2}{4}} \psi(u - v/2),\end{aligned}$$

where we used that the y-integral of the heat kernel equals 1.

The next lemma is very important, and says that we may replace $F(\delta u)$ with $F(\delta y)$, making only an error of order δ.

Lemma 4.3. *There exists $C > 0$ such that*

$$\begin{aligned}\langle \psi, \widetilde{K}_\delta \psi\rangle \leq &\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^{2d}} du dv \bar{\psi}(u + v/2) f(\sqrt{\delta}v)\mathfrak{K}_0(u + F(\delta y), v) \\ &\times \left(\frac{\delta}{4\pi}\right)^{d/2} e^{-\delta\frac{|u-y|^2}{4}} \psi(u - v/2) + C\delta\|\psi\|^2, \quad 0 < \delta \leq 1.\end{aligned}$$

Proof. Let us consider the following distribution kernel:

$$\int_{\mathbb{R}^d} dy\, f(\sqrt{\delta} v)(\mathfrak{K}_0(u + F(\delta y), v) - \mathfrak{K}_0(u + F(\delta u), v))\Big(\frac{\delta}{4\pi}\Big)^{d/2} e^{-\delta \frac{|u-y|^2}{4}}.$$

Denoting by ∇_1 the partial gradient with respect to the "$u \in \mathbb{R}^d$" variables, we can write the above distribution kernel as

$$\begin{aligned}
&\int_{\mathbb{R}^d} dy\, f(\sqrt{\delta} v)(F(\delta y) - F(\delta u)) \cdot \nabla_1 \mathfrak{K}_0(u + F(\delta u), v)\Big(\frac{\delta}{4\pi}\Big)^{d/2} e^{-\delta \frac{|u-y|^2}{4}} \\
&\quad + \int_{\mathbb{R}^d} dy \int_0^1 dr (1 - r)((F(\delta y) - F(\delta u)) \cdot \nabla_1)^2 \\
&\qquad \times f(\sqrt{\delta} v) \mathfrak{K}_0(u + F(\delta u) + r(F(\delta y) - F(\delta u)), v)\Big(\frac{\delta}{4\pi}\Big)^{d/2} e^{-\delta \frac{|u-y|^2}{4}}.
\end{aligned}$$

From our Hypothesis 1.2, we have $|F(\delta y) - F(\delta u)| \le C\delta |u - y|$, hence both above kernels correspond to $S^0_{0,0}$ symbols due to the fact that the growth in $|u - y|$ is controlled by the Gaussian factor.

Moreover, in the second kernel we can couple one power of δ with the quadratic term $|y - u|^2$ and thus we can bound this second kernel by a constant times δ and conclude that all the seminorms of its associated symbol will be of order δ.

For the first kernel, we use the Taylor expansion

$$F(\delta y) - F(\delta u) = \delta \nabla F(\delta u) \cdot (y - u) + \mathcal{O}(\delta^2 |y - u|^2).$$

The remarkable fact is that the linear term vanishes identically after integration in y. The quadratic term can be dealt with as we did with the second kernel concluding that it will generate an operator with norm of order δ. ■

Using the notation from Lemmas 4.1 and 4.2, the inequality from Lemma 4.3 reads as

$$\begin{aligned}
\langle \psi, \widetilde{K}_\delta \psi \rangle \le \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^{2d}} du\, dv\, \overline{\Psi_\delta}(u + v/2, y) f(\sqrt{\delta} v) e^{\frac{\delta v^2}{16}} \mathfrak{K}_0(u + F(\delta y), v) \\
\times \Psi_\delta(u - v/2, y) + C\delta \|\psi\|^2. \qquad (4.1)
\end{aligned}$$

Another crucial observation is that the operator with the integral kernel given by

$$f(\sqrt{\delta} v) e^{\frac{\delta v^2}{16}} \mathfrak{K}_0(u + F(\delta y), v), \quad \text{for all } y \in \mathbb{R}^d,$$

appearing in (4.1), is unitarily equivalent, by a conjugation with $\tau_{F(\delta y)}$, with the operator denoted from now on by M_δ given by the distribution kernel

$$\mathfrak{M}_\delta(u,v) = f(\sqrt{\delta}v)e^{\frac{\delta v^2}{16}}\mathfrak{K}_0(u,v).$$

These operators have the same spectrum, for all $y \in \mathbb{R}^d$, thus from (4.1) and Lemma 4.1 we have

$$\langle \psi, \widetilde{K}_\delta \psi\rangle \leq E_+(M_\delta)\|\psi\|^2 + C\delta\|\psi\|^2. \tag{4.2}$$

Finally, we see that the operator $M_\delta - \mathring{K}_\delta$ has the integral kernel

$$(\delta/16)\int_0^1 dr\, f(\sqrt{\delta}v)e^{\frac{r\delta v^2}{16}}v^2\mathfrak{K}_0(u,v).$$

The factor v^2 multiplied by $\mathfrak{K}_0$ will generate (by the usual integration by parts procedure for oscillatory integrals) some second order derivatives in ξ of the symbol $a(x,\xi)$, while the Gaussian is just a smooth function depending on $\sqrt{\delta}v$, which on the support of f remains bounded. Hence, the operator $M_\delta - \mathring{K}_\delta$ has a norm bounded by δ, which together with (4.2) implies

$$\langle \psi, \widetilde{K}_\delta \psi\rangle \leq E_+(\mathring{K}_\delta)\|\psi\|^2 + C\delta\|\psi\|^2,$$

i.e., $E_+(\widetilde{K}_\delta) \leq E_+(\mathring{K}_\delta) + C\delta$. The inequality where $\widetilde{K}_\delta$ and $\mathring{K}_\delta$ exchange places can be proved in a similar way.

5. Proof of Corollaries 1.4 and 1.6

Corollary 1.4 is just a direct consequence of the definition of the Hausdorff distance. For Corollary 1.6, we use a similar trick with the one used in [11]. Let

$$\gamma_0 = (2\lambda_0 + \mu_0)/3$$

and let us define

$$T_\delta := (K_\delta - \gamma_0\mathbb{1})^2 = K_\delta^2 - 2\gamma_0 K_\delta + \gamma_0^2\mathbb{1}. \tag{5.1}$$

Let us assume for the moment that $E_\pm(T_\delta)$ are Lipschitz at $\delta = 0$, a fact which we will prove later. If $\delta \geq 0$ is small enough, then by using the spectral theorem, the fact that γ_0 is closer to λ_0 than to μ_0, and the a priori estimate from Theorem 1.3 which says that $|\lambda_\delta - \lambda_0| \leq C\sqrt{\delta}$, we have that

$$E_-(T_\delta) = (\lambda_\delta - \gamma_0)^2.$$

The Lipschitzianity of $E_-(T_\delta)$ implies the existence of a constant $C_1 > 0$ such that

$$|(\lambda_\delta - \gamma_0)^2 - (\lambda_0 - \gamma_0)^2| \leq C_1\delta, \quad \text{for all } 0 \leq \delta < \delta_0.$$

Writing

$$(\lambda_\delta - \gamma_0)^2 - (\lambda_0 - \gamma_0)^2 = (\lambda_\delta - \lambda_0)(\lambda_\delta + \lambda_0 - 2\gamma_0)$$

we have some small enough $\delta_1 < \delta_0$ such that

$$|\lambda_\delta - \lambda_0| \leq \frac{C_1\delta}{|\lambda_\delta + \lambda_0 - 2\gamma_0|} \leq \frac{C_1\delta}{\gamma_0 - \lambda_0} = \frac{3C_1\delta}{\mu_0 - \lambda_0}, \quad \text{for all } 0 \leq \delta < \delta_1$$

and we are done.

The only thing which remains to be proved is that $E_\pm(T_\delta)$ are Lipschitz at $\delta = 0$. We start with a lemma.

Lemma 5.1. *Given $B = \mathfrak{Op}^w(b)$ with $b \in S^0_{0,0}(\mathbb{R}^{2d})$, we denote by $\mathcal{M}_\delta(B)$ the Weyl quantization of the perturbed symbol $b(x + F(\delta x), \xi)$. If $a \in S^0_{0,0}(\mathbb{R}^{2d})$ then $K_\delta = \mathcal{M}_\delta(K_0)$ and*

$$\|\mathcal{M}_\delta(K_0^2) - K_\delta^2\| \leq C\delta.$$

Proof. Denote by χ the indicator function of the unit hypercube $\Omega := [-1/2, 1/2]^d$. The operator K_δ can be seen as an operator in $\bigoplus_{\gamma\in\mathbb{Z}^d} L^2(\Omega)$ given by the operator valued matrix

$$A_{\gamma,\gamma'}(\delta)(\underline{x}, \underline{x}') := \mathfrak{H}_0((\underline{x} + \underline{x}' + \gamma + \gamma')/2 + F(\delta(\underline{x} + \underline{x}')/2 + \delta(\gamma + \gamma')/2), \underline{x} + \gamma - \underline{x}' - \gamma'),$$

where $(\underline{x}, \underline{x}') \in \Omega \times \Omega$ and $(\gamma, \gamma') \in \mathbb{Z}^d \times \mathbb{Z}^d$.

Due to the strong localization of $\mathfrak{H}_0$ with respect to $v = x - x'$, one can prove that for every $N \geq 1$ there exists $C_N > 0$ such that

$$\|A_{\gamma,\gamma'}(\delta)\|_{L^2(\Omega)} \leq C_N \langle \gamma - \gamma' \rangle^{-N}.$$

For $(\underline{x}, \underline{x}') \in \Omega \times \Omega$ let us define

$$\widetilde{A}_{\gamma,\gamma'}(\delta)(\underline{x}, \underline{x}') := \mathfrak{H}_0((\underline{x} + \underline{x}' + \gamma + \gamma')/2 + F(\delta(\gamma + \gamma')/2), \underline{x} + \gamma - \underline{x}' - \gamma').$$

Using a Taylor expansion for F together with the strong localization of $\mathfrak{H}_0$ in the v variable, one may also show that for every $N \geq 1$ there exists $C_N > 0$ such that

$$\|A_{\gamma,\gamma'}(\delta) - \widetilde{A}_{\gamma,\gamma'}(\delta)\|_{L^2(\Omega)} \leq C_N \delta \langle \gamma - \gamma' \rangle^{-N}.$$

Up to a use of the Schur test in $\bigoplus_{\gamma\in\mathbb{Z}^d} L^2(\Omega)$, we get that

$$K_\delta - \sum_{\gamma,\gamma'\in\mathbb{Z}^d} \chi(\cdot + \gamma)\widetilde{A}_{\gamma,\gamma'}(\delta)\chi(\cdot + \gamma') = \mathcal{O}(\delta).$$

Thus, up to an error of order δ in operator norm, we have that K_δ^2 is given by

$$\sum_{\gamma,\gamma'\in\mathbb{Z}^d}\sum_{\gamma''\in\mathbb{Z}^d}\chi(\cdot+\gamma)\widetilde{A}_{\gamma,\gamma''}(\delta)\widetilde{A}_{\gamma'',\gamma'}(\delta)\chi(\cdot+\gamma').$$

By replacing $F(\delta(\gamma+\gamma'')/2)$ and $F(\delta(\gamma''+\gamma')/2)$ with $F(\delta(\gamma+\gamma')/2)$, we produce an error of the type

$$\delta\langle\gamma'-\gamma''\rangle\langle\gamma-\gamma''\rangle,$$

that is controlled by the strong off-diagonal decay in both $|\gamma-\gamma''|$ and $|\gamma''-\gamma'|$ induced by $\mathfrak{K}_0$. Hence, K_δ^2 is up to an error of order δ given by the operator valued matrix:

$$\begin{aligned}B_{\gamma,\gamma'}(\delta)(\underline{x},\underline{x}') := (\text{Integral kernel of } K_0^2)((\underline{x}+\underline{x}'+\gamma+\gamma')/2+F(\delta(\gamma+\gamma')/2),\\ \underline{x}+\gamma-\underline{x}'-\gamma').\end{aligned}$$

Finally, by again using a Taylor expansion and a Schur test, one shows that this operator and $\mathcal{M}_\delta(K_0^2)$ differ from each other by something of order δ in the operator topology, and the proof is finished. ■

Going back to (5.1), we notice that Lemma 5.1 implies that modulo an error of order δ we can replace T_b by

$$\mathcal{M}_\delta(K_0^2)-2\gamma_0\mathcal{M}_\delta(K_0)+\gamma_0^2\mathbb{1}=\mathcal{M}_\delta(K_0^2-2\gamma_0K_0+\gamma_0^2\mathbb{1}),$$

which is the same type as K_δ and thus the Lipschitzianity of $E_-(T_b)$ will follow from Theorem 1.5.

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