J. Spectr. Theory 13 (2023), 841–902
DOI 10.4171/JST/474

Exponential moments for disk counting statistics at the hard edge of random normal matrices

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Abstract. We consider the multivariate moment generating function of the disk counting statistics of a model Mittag-Leffler ensemble in the presence of a hard wall. Let n be the number of points. We focus on two regimes: (a) the "hard edge regime" where all disk boundaries are at a distance of order $\frac{1}{n}$ from the hard wall, and (b) the "semi-hard edge regime" where all disk boundaries are at a distance of order $\frac{1}{\sqrt{n}}$ from the hard wall. As $n \to +\infty$, we prove that the moment generating function enjoys asymptotics of the form

$$\exp\Big(C_1 n + C_2 \ln n + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O}(n^{-\frac{3}{5}})\Big) \qquad \text{for the hard edge,}$$

$$\exp\Big(C_1 n + C_2 \sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O}\Big(\frac{(\ln n)^4}{n}\Big)\Big) \qquad \text{for the semi-hard edge.}$$

In both cases, we determine the constants $C_1, \dots, C_4$ explicitly. We also derive precise asymptotic formulas for all joint cumulants of the disk counting function, and establish several central limit theorems. Surprisingly, and in contrast to the "bulk", "soft edge", and "semi-hard edge" regimes, the second and higher order cumulants of the disk counting function in the "hard edge" regime are proportional to n and not to $\sqrt{n}$.

Contents

2020 Mathematics Subject Classification. Primary 60G55; Secondary 41A60, 60B20.
Keywords. Moment generating functions, random matrix theory, asymptotic analysis.

1. Introduction and statement of results

1.1. Hard wall constraints in random matrix theory

In this work we study random normal matrix eigenvalues on subsets of the plane which are obtained by imposing a hard wall constraint. These eigenvalues can also be seen as repelling Coulomb gas particles at the inverse temperature $\beta = 2$. While we shall soon specialize to a class of Mittag-Leffler ensembles, it is convenient to start out from a broader perspective.

Thus, we fix an arbitrary lower semi-continuous function $Q_0\colon \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$. Along with Q_0 we fix a suitable closed subset C of $\mathbb{C}$ and consider the modification ("external potential"):

$$Q(z) = \begin{cases} Q_0(z) & \text{if } z \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

The external potential is assumed to be finite on some set of positive capacity and to satisfy the basic growth constraint

$$Q(z) - \ln|z|^2 \to +\infty \quad \text{as } z \to \infty. \tag{1.1}$$

Observe that Q may satisfy the growth condition (1.1) even if Q_0 fails to do so. In particular, this is the case if Q_0 is a constant, or if Q_0 is an Elbau–Felder potential [13, 42, 52, 59]:

$$Q_0(z) = \frac{1}{t_0}(|z|^2 - 2\,\mathrm{Re}(t_1 z + \cdots + t_k z^k)).$$

Another basic class of hard walls is obtained by taking $C = \mathbb{R}$, which leads to the Hermitian random matrix theory.

Given a confining potential Q, we associate Coulomb gas ensembles in the following way (as mentioned, we will only consider the inverse temperature $\beta = 2$). We consider configurations of n points $\{\mathrm{z}_j\}_{j=1}^n \subset \mathbb{C}$. The total energy, or Hamiltonian of the configuration, is defined by

$$H_n = \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \ln \frac{1}{|\mathrm{z}_j - \mathrm{z}_k|} + n \sum_{j=1}^{n} Q(\mathrm{z}_j),$$

and the associated Boltzmann–Gibbs measure on $\mathbb{C}^n$ is

$$d\mathrm{P}_n = \frac{1}{\mathrm{Z}_n} e^{-H_n} \prod_{j=1}^{n} d^2\mathrm{z}_j,$$

where $d^2\mathrm{z}$ is the two-dimensional Lebesgue measure. The Coulomb gas ensemble (or "system") $\{\mathrm{z}_j\}_{j=1}^n$ corresponding to the external potential Q is a configuration picked randomly with respect to this measure.

To a first order approximation, the system tends to follow Frostman's equilibrium measure μ associated to the potential Q. This is the unique minimizer of the weighted logarithmic energy functional

$$I_Q[\nu] = \iint\limits_{\mathbb{C}^2} \ln \frac{1}{|z-w|}\, d\nu(z)d\nu(w) + \int\limits_{\mathbb{C}} Q(z)\, d\nu(z)$$

among all compactly supported Borel probability measures on $\mathbb{C}$. The support of μ is called the *droplet* and is denoted $S = S[Q]$. If the potential is C^2-smooth in a neighborhood of S, then the equilibrium measure is absolutely continuous with respect to the two-dimensional Lebesgue measure d^2z and takes the form (see [68])

$$d\mu(z) = \frac{1}{4\pi}\Delta Q(z)\chi_S(z)\, d^2z, \tag{1.2}$$

where χ_S is the indicator function of S and Δ is the standard Laplacian.

It is known that the system $\{\mathrm{z}_j\}_1^n$ tends to condensate on the droplet under quite general conditions [6, 24, 41, 51, 54, 55, 66], in the sense that as $n \to \infty$ the empirical measures $\frac{1}{n}\sum_{j=1}^n \delta_{\mathrm{z}_j}$ converge weakly to μ in probability.

Consider now a smooth confining potential Q_0 on the plane whose droplet is S_0. A case of some interest is obtained by placing the hard wall exactly along the edge of the droplet, i.e., we take $C = S_0$, where the equilibrium measure is still absolutely continuous and of the form (1.2). In this case, we obtain a so-called *local droplet* with a soft/hard edge. Such droplets have been studied in for example [12, 51, 59] and references therein. While the equilibrium measure is unchanged, the soft/hard edge produces some statistical effects near the edge. Interestingly, the concept of local droplets permits us to define some new and non-trivial ensembles, such as the "deltoid" – a droplet with three maximal cusps which arises for the cubic potential $|z|^2 + c\,\mathrm{Re}(z^3)$ for a certain critical value of the constant c, see e.g. [18].

However, the main case of interest for the present investigation is that of a hard wall in the bulk of the droplet. To study this case, we choose an external potential Q_0 giving rise to a well-defined droplet S_0 and a closed subset $C \subset \mathrm{Int}\, S_0$, and we modify Q_0 to a potential Q by defining it as $+\infty$ outside C. This has an effect even at the level of the equilibrium measure. Indeed, if the potential Q_0 is C^2-smooth in a neighborhood of S_0, then this effect is given by a balayage process which we briefly recall.

Let μ_0 be the equilibrium measure with respect to the potential Q_0, given in (1.2) (with "S" and "Q" replaced by "S_0" and "Q_0"). Assuming some regularity of the boundary ∂C, the equilibrium measure μ_h corresponding to the potential Q is then given by the formula (see [68, Theorem II.5.12])

$$\mu_h = \mu_0 \cdot \chi_C + \mathrm{Bal}(\mu_0|_{S_0 \setminus C}, \partial C), \tag{1.3}$$

where $\mathrm{Bal}(\mu_0|_{S_0 \setminus C}, \partial C)$ is the balayage of $\mu_0|_{S_0 \setminus C}$ onto the boundary ∂C. The formula (1.3) expresses the fact that the portion $\mu_0|_{S_0 \setminus C}$ is swept onto the boundary ∂C according to the balayage operation, which preserves (up to a constant) the exterior logarithmic potential in the exterior of the droplet S_0. See [68, Sections II.4 and II.5] as well as [35, 53, 70] for more details about the balayage.

The balayage part of (1.3) is a density on the curve ∂C, so this part is singular with respect to the two-dimensional Lebesgue measure. We think of this balayage as a first approximation of the density for the particles which would have occupied the forbidden region outside of C, were it not for the hard wall. On a statistical level, in the generic case where $\Delta Q(z) > 0$ for all $z \in \partial C$, the particles which are swept out of the forbidden region are expected to occupy a very narrow interface about the boundary ∂C of width of order $1/n$. We call this interface the "hard edge regime." The width $1/n$ is substantially smaller than the two-dimensional microscopic scale $1/\sqrt{n}$. We shall find below that on a $1/\sqrt{n}$-scale from ∂C, we obtain a transitional regime between hard edge and bulk statistics, which we call "semi-hard edge regime." The three regimes (bulk, semi-hard edge, and hard edge) each gives rise to different kinds of statistical behavior, which we study below for a class of radially symmetric potentials.

We remark that point-processes $\{z_j\}_1^n$ of the above type can be identified with the eigenvalues of an $n \times n$ random normal matrix M, picked randomly according to the probability measure proportional to $e^{-n \operatorname{tr} Q(M)} dM$, where "tr" is the trace and dM is the measure on the set of $n \times n$ normal matrices induced by the flat Euclidian metric of $\mathbb{C}^{n \times n}$ [32, 42, 63]. (Note that this makes precise the identification between eigenvalues and $\beta = 2$ Coulomb gas processes mentioned above.)

The process $\{z_j\}_1^n$ can be thought of as a conditional process where the eigenvalue process associated with Q_0 is conditioned on the event that none of the eigenvalues fall outside of the closed set C. If $C \subset \operatorname{Int} S$, we are conditioning on a rare event.

We mention in passing that for other conditional point processes, such as the zeros of Gaussian analytic functions conditioned on a hole event, the situation is drastically different because of the presence of a forbidden region around the singular part of the equilibrium measure [49, 65].

Remark 1.1. Hard wall ensembles from Hermitian random matrix theory have been well studied in the literature, see for example [27, 30, 36, 37, 40, 47, 62]; see also [34] for a soft/hard edge. We remark that imposing a hard wall in the interior of a one-dimensional droplet has a well-known global effect on the equilibrium measure, in contrast to (1.3) which just alters the measure locally at the edge. However, this apparent contradiction is quickly dispelled if we note that a one-dimensional droplet consists of only edge and no interior (regarded as a subset of $\mathbb{C}$).

1.2. Mittag-Leffler ensembles with a hard wall constraint

For what follows, we will restrict our attention to radially symmetric potentials of the form

$$Q_0(z) = |z|^{2b} - \frac{2\alpha}{n} \ln |z|, \tag{1.4}$$

where $b > 0$ and $\alpha > -1$ are fixed parameters. The unconstrained model Mittag-Leffler ensemble is a configuration $\{\zeta_j\}_1^n$ picked randomly with respect to the following joint probability density function:

$$\frac{1}{n!Z_n} \prod_{1 \le j < k \le n} |\zeta_k - \zeta_j|^2 \prod_{j=1}^{n} |\zeta_j|^{2\alpha} e^{-n|\zeta_j|^{2b}}, \quad \zeta_1, \dots, \zeta_n \in \mathbb{C}, \tag{1.5}$$

where Z_n is the normalization constant. It is well known that the droplet S_0 corresponding to the potential (1.4) is the disk of radius $b^{-\frac{1}{2b}}$ centered at 0; the density is given according to (1.2) by

$$d\mu_0(z) = \frac{b^2}{\pi} |z|^{2b-2} d^2 z.$$

Remark 1.2. The logarithmic and power-like singularities of (1.4) at the origin are not strong enough to affect the equilibrium measure. The term “Mittag-Leffler potential” is from [10] and refers to a much broader class of potentials having similar kinds of singularities at the origin. The motivation for the terminology is that, under some conditions, the local statistics near the origin can be described by a two-parametric Mittag-Leffler function [13].

We now fix a parameter ρ with $0 < \rho < b^{-\frac{1}{2b}}$ and place a hard wall outside the circle $|z| = \rho$. More precisely, we consider the probability density

$$\frac{1}{n!\mathcal{Z}_n} \prod_{1 \le j < k \le n} |z_k - z_j|^2 \prod_{j=1}^{n} e^{-nQ(z_j)}, \quad z_1, \dots, z_n \in \mathbb{C}, \tag{1.6}$$

where $\mathcal{Z}_n$ is the normalizing partition function and

$$Q(z) = \begin{cases} |z|^{2b} - \frac{2\alpha}{n} \ln |z| & \text{if } |z| \le \rho, \\ +\infty & \text{if } |z| > \rho. \end{cases} \tag{1.7}$$

This gives the hard-wall Mittag-Leffler process $\{z_j\}_1^n$, conditioned on the forbidden region $\{|z| > \rho\}$. For brevity, we shall in the sequel refer to $\{z_j\}_1^n$ corresponding to the potential (1.7) as the *restricted Mittag-Leffler process*.

The equilibrium measure μ_h corresponding to the potential (1.7) can be easily computed using standard balayage techniques [68] (see also [35, Section 4.1] or [70] for details) and is given by

$$\mu_h(d^2z) = \mu_{\mathrm{reg}}(d^2z) + \mu_{\mathrm{sing}}(d^2z),$$
$$\mu_{\mathrm{reg}}(d^2z) := 2b^2r^{2b-1}dr\frac{d\theta}{2\pi}, \quad \mu_{\mathrm{sing}}(d^2z) := c_\rho\delta_\rho(r)dr\frac{d\theta}{2\pi}, \tag{1.8}$$

where $z = re^{i\theta}$, $r > 0$, $\theta \in (-\pi, \pi]$ and

$$c_\rho := \int_\rho^{b^{-\frac{1}{2b}}} 2b^2r^{2b-1}dr = 1 - b\rho^{2b}. \tag{1.9}$$

Standard arguments [6,51,54] show that the empirical measures $\frac{1}{n}\sum\delta_{z_j}$ converge weakly in probability to μ_h as $n \to \infty$.

Clearly, the restricted Mittag-Leffler process is an example of a rotation invariant ensemble, i.e., the joint probability density function (1.6) remains unchanged if all z_j are multiplied by the same unimodular constant $e^{i\beta}$, $\beta \in \mathbb{R}$.

In this work we focus on the case $\rho < b^{-\frac{1}{2b}}$, which means that we are studying a hard wall in the bulk of the droplet S_0. The case of a soft/hard edge, i.e., $\rho = b^{-\frac{1}{2b}}$ could be included as well, but would require a somewhat different (and much simpler) analysis. We shall therefore omit this case.

Coulomb gas ensembles in the presence of a hard wall have previously been considered in the literature, but so far the focus has been on large gap probabilities (or partition functions) [1, 3–5, 29, 46, 48, 50, 53] and on the local statistics [64, 70, 77]. We refer to [11, 12, 23, 51, 59, 69] for studies of local droplets and local statistics near soft/hard edges.

In recent years, a lot of works dealing with the counting statistics of two-dimensional point processes have appeared [2, 25, 28, 31, 43, 45, 57, 58, 60, 72, 73], see also [71] for an earlier work. A common feature of these works is that they all deal exclusively with either “the bulk regime” or with “the soft edge regime.”

In this paper we study disk counting statistics of (1.6) near the hard edge $\{|z| = \rho\}$. To be specific, let $\mathrm{N}(y) := \#\{z_j : |z_j| < y\}$ be the random variable that counts the number of points of (1.6) in the disk of radius y centered at 0. Our main result is a precise asymptotic formula as $n \to +\infty$ for the multivariate moment generating

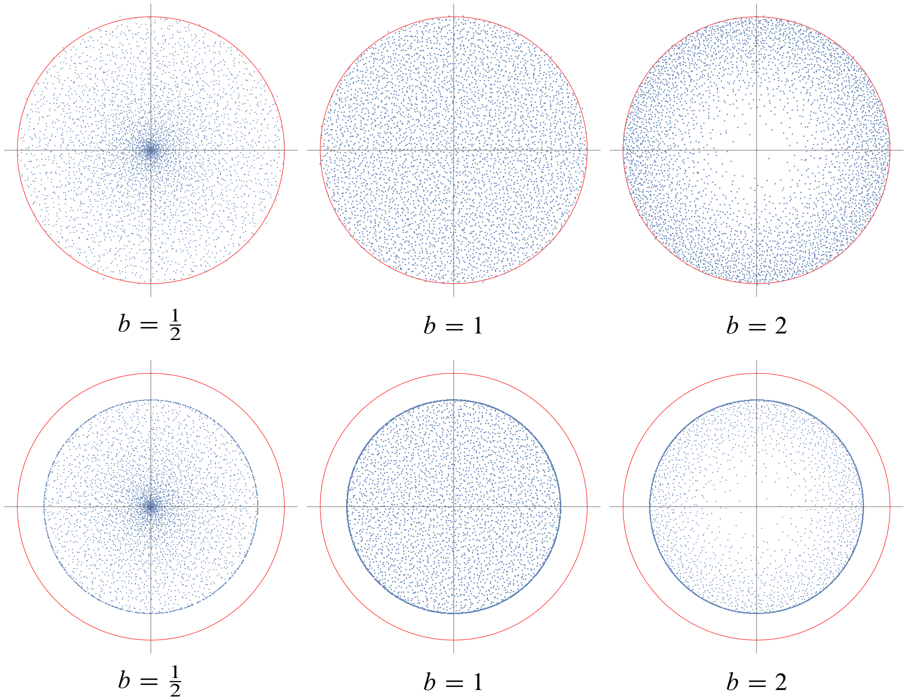

Figure 1. Illustration of the point processes corresponding to (1.5) (first row) and (1.6) (second row) with $n = 4096$, $\rho = \frac{4}{5} b^{-\frac{1}{2b}}$, $\alpha = 0$ and the indicated values of b. In each plot, the red circle is $\{z \in \mathbb{C} : |z| = b^{-\frac{1}{2b}}\}$. A narrow interface about the hard wall $|z| = \rho$, of width roughly $1/n$, accommodates the roughly $c_\rho n$ particles swept out from the forbidden region. The semi-hard regime of width roughly $1/\sqrt{n}$ is transitional between the hard edge and the bulk.

function (MGF)

$$\mathbb{E}\Big[\prod_{j=1}^{m} e^{u_j \, \mathrm{N}(r_j)}\Big] \tag{1.10}$$

where $m \in \mathbb{N}_{>0}$ is arbitrary (but fixed), $u_1, \ldots, u_m \in \mathbb{R}$, and the radii $r_1, \ldots, r_m$ are merging at a critical speed. We consider several regimes:

- hard edge,

$$0 < r_1 < \cdots < r_m, \quad r_\ell = \rho\Big(1 - \frac{t_\ell}{n}\Big)^{\frac{1}{2b}}, \qquad t_1 > \cdots > t_m \geq 0; \tag{1.11}$$

- semi-hard edge,

$$0 < r_1 < \cdots < r_m, \quad r_\ell = \rho\Big(1 - \frac{\sqrt{2}\mathfrak{s}_\ell}{\rho^b \sqrt{n}}\Big)^{\frac{1}{2b}}, \quad \mathfrak{s}_1 > \cdots > \mathfrak{s}_m > 0; \tag{1.12}$$

- bulk,

$$0 < r_1 < \cdots < r_m, \quad r_\ell = r\Big(1 + \frac{\sqrt{2}\mathfrak{s}_\ell}{r^b\sqrt{n}}\Big)^{\frac{1}{2b}}, \quad \mathfrak{s}_1 < \cdots < \mathfrak{s}_m \in \mathbb{R}, \tag{1.13}$$

  with $r < \rho$ in (1.13).

We emphasize that $\mathfrak{s}_m \neq 0$ in (1.12).

We shall prove that, as $n \to +\infty$, the joint MGF $\mathbb{E}[\prod_{j=1}^m e^{u_j \, \mathrm{N}(r_j)}]$ enjoys asymptotic expansions of the form

$$\exp\Big(C_1 n + C_2 \ln n + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O}(n^{-\frac{3}{5}})\Big) \qquad \text{for the hard edge,} \tag{1.14}$$

$$\exp\Big(C_1 n + C_2\sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O}\Big(\frac{(\ln n)^4}{n}\Big)\Big) \quad \text{for the semi-hard edge,} \tag{1.15}$$

$$\exp\Big(C_1 n + C_2\sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O}\Big(\frac{(\ln n)^2}{n}\Big)\Big) \quad \text{for the bulk.} \tag{1.16}$$

For each of these three regimes, we determine $C_1, \ldots, C_4$ explicitly.

As can be seen from (1.14)–(1.16), the counting statistics in the hard edge regime are drastically different from the counting statistics in the bulk and semi-hard edge regimes (and also very different from the counting statistics in the soft edge regime [28, 31]). Indeed, at the hard edge the subleading term is proportional to $\ln n$, while in all other regimes it is proportional to $\sqrt{n}$. Furthermore, in the hard edge regime, the leading coefficient C_1 will be shown to depend on the parameters $u_1, \ldots, u_m$ in a highly non-trivial non-linear way.

As we show below, the above asymptotic expansions have several interesting consequences; for example, $\mathrm{Var}[\mathrm{N}(r_j)] \asymp n$ in the hard edge regime, while $\mathrm{Var}[\mathrm{N}(r_j)] \asymp \sqrt{n}$ in the three other regimes (actually, a similar statement also holds for the higher order cumulants, as can be seen by comparing Corollary 1.5 with Corollary 1.8 and [31, Corollary 1.5]). This indicates that the counting statistics near a hard edge are considerably wilder than near a soft edge, in the bulk or near a semi-hard edge. From a technical point of view, we also found the hard edge regime to be significantly harder to analyze than the three other regimes. For example, our control of the error term in (1.14) is less precise than in (1.15) and (1.16).

In contrast to earlier works on smooth and non-smooth linear statistics in the soft edge and bulk regime, the leading coefficient C_1 in the hard edge regime is *not* given by the integral of the test function (in our case $\sum_{j=1}^m u_j \chi_{(0,r_j)}(z)$) against the equilibrium measure μ_h, and in fact it depends in a non-linear way on the parameters u_j. In a sense this behavior becomes less surprising if we recall that we are not considering fixed test functions, but rather increasing sequences corresponding to characteristic functions of expanding disks, and it is known due to Seo [70] that the 1-point function varies rather dramatically in the hard edge regime. On the other hand, the fact that the relationship becomes non-linear might be less clear on this intuitive level. See also Remark 1.4 below for more about this.

The transition from the hard edge regime to the bulk regime is very subtle. The semi-hard edge regime lies in between, i.e., it is genuinely different from the hard edge and the bulk regimes. To the best of our knowledge, it seems that this regime has been unnoticed (or at least unexplored) in the literature so far.[1] Our results for this regime can be seen as a first step towards understanding the hard-edge-to-bulk transition. However, the fact that the subleading terms in the hard edge and semi-hard edge regimes are of different orders indicates that there is still (at least) one intermediate regime where a critical transition takes place. We will return to this issue in a follow-up work.

As corollaries of our various results on the generating function (1.10), we also provide central limit theorems for the joint fluctuations of $\mathrm{N}(r_1),\dots,\mathrm{N}(r_m)$, and precise asymptotic formulas for all cumulants of these random variables (both at the hard edge and at the semi-hard edge). Our results for the hard edge and semi-hard edge regimes seem to be new, even for $m = 1$. Our results about the bulk regime are less novel. Indeed, in this regime the asymptotics of the MGF have been investigated in various settings [25, 28, 31, 45, 57]: see [25, Proposition 8.1] for second order asymptotics of the one-point MGF of counting statistics of general domains in Ginibre-type ensembles; see [57] for second order asymptotics of the one-point MGF of the disk counting statistics of rotation-invariant ensembles with a general potential; see [45] for third order asymptotics for the one-point MGF of disk counting statistics of Ginibre-type ensembles; and see [28, 31] for fourth order asymptotics for the m-point MGF of disk counting statistics in the Mittag-Leffler ensemble (1.5). Both the bulk and the soft edge regimes were investigated in [28, 31]; however in [28] the radii of the disks were taken fixed, while in [31] all radii were assumed to merge at the critical speed $\sim \frac{1}{\sqrt{n}}$ (in this critical regime one observes non-trivial correlations in the disk counting statistics). As it turns out, the bulk statistics of (1.5) and (1.6) are identical up to exponentially small errors (in other words, the points in the bulk almost do not feel the hard wall). Our formulas for the bulk regime (1.13) are in fact *identical* to the corresponding formulas in [31] (the proof is also almost identical, we only have to handle some additional exponentially small error terms). We have nevertheless decided to include a very short section in this paper on the bulk regime for

[1] In a different but somewhat related context, namely in the study of the statistics of the largest modulus of the complex Ginibre ensemble, a new intermediate regime was also recently discovered in [56].

completeness. We also point out that for C^2-smooth test functions f on the plane, the asymptotic normality of fluctuations was worked out quite generally in [9], for potentials having a connected droplet. In this case the asymptotic variance of fluctuations is given by a Dirichlet norm $\frac{1}{4\pi}\int|\nabla f^S(z)|^2\,d^2z$, where f^S equals f in S and is the bounded harmonic extension of $f|_S$ outside of S.

The presentation of our results is organized as follows: Section 1.3 treats the hard edge regime, Section 1.4 the semi-hard edge regime, and Section 1.5 the bulk regime.

1.3. Results for the hard edge regime

Let $r_1,\dots,r_m$ be as in (1.11), let $\vec{t} := (t_1,\dots,t_m)$ be such that $t_1 > \cdots > t_m \geq 0$, let $\vec{u} := (u_1,\dots,u_m) \in \mathbb{R}^m$, and define

$$f(x;\vec{t},\vec{u}) = -\Big(\frac{b\rho^{2b}}{x-b\rho^{2b}} + \frac{\alpha}{b}\Big)\frac{\mathsf{T}_1(x;\vec{t},\vec{u})}{1+\mathsf{T}_0(x;\vec{t},\vec{u})} - \frac{x}{2b}\frac{\mathsf{T}_2(x;\vec{t},\vec{u})}{1+\mathsf{T}_0(x;\vec{t},\vec{u})}, \tag{1.17}$$

$$\mathsf{T}_j(x;\vec{t},\vec{u}) = \sum_{\ell=1}^{m} \omega_\ell t_\ell^j e^{-\frac{t_\ell}{b}(x-b\rho^{2b})}, \quad j \geq 0, \tag{1.18}$$

$$\Omega(\vec{u}) = 1 + \mathsf{T}_0(b\rho^{2b};\vec{t},\vec{u}) = e^{u_1+\cdots+u_m},$$

where

$$\omega_\ell = \omega_\ell(\vec{u}) = \begin{cases} e^{u_\ell+\cdots+u_m} - e^{u_{\ell+1}+\cdots+u_m} & \text{if } \ell < m, \\ e^{u_m}-1 & \text{if } \ell = m, \\ 1 & \text{if } \ell = m+1. \end{cases}$$

Recall that the complementary error function is defined by

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}\,dx. \tag{1.19}$$

Throughout the paper $\ln(\cdot)$ denotes the principal branch of the logarithm and

$$D_\delta(z_0) = \{z \in \mathbb{C} : |z-z_0| < \delta\}$$

denotes an open disk of radius δ centered at $z_0 \in \mathbb{C}$.

Theorem 1.3 (Merging radii at the hard edge). *Let $m \in \mathbb{N}_{>0}$, $b > 0$, $\rho \in (0, b^{-\frac{1}{2b}})$, $t_1 > \cdots > t_m \geq 0$, and $\alpha > -1$ be fixed parameters, and for $n \in \mathbb{N}_{>0}$, define*

$$r_\ell = \rho\Big(1-\frac{t_\ell}{n}\Big)^{\frac{1}{2b}}, \quad \ell = 1,\dots,m. \tag{1.20}$$

For any fixed $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\mathbb{E}\Big[\prod_{j=1}^{m} e^{u_j \,\mathrm{N}(r_j)}\Big] = \exp\Big(C_1 n + C_2 \ln n + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O}(n^{-\frac{3}{5}})\Big) \quad \text{as } n \to +\infty \tag{1.21}$$

uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$, where $\{C_j = C_j(\vec{u})\}_{j=1}^{4}$ are given by

$$\begin{aligned}
C_1 &= b\rho^{2b} \sum_{j=1}^{m} u_j + \int_{b\rho^{2b}}^{1} \ln(1 + \mathsf{T}_0(x;\vec{t},\vec{u}))dx, \\
C_2 &= -\frac{b\rho^{2b}}{2} \frac{\mathsf{T}_1(b\rho^{2b};\vec{t},\vec{u})}{\Omega(\vec{u})} = -\frac{b\rho^{2b}}{2} \frac{\sum_{\ell=1}^{m} t_\ell \omega_\ell}{e^{u_1+\cdots+u_m}}, \\
C_3 &= -\frac{1}{2}\sum_{j=1}^{m} u_j + \frac{1}{2}\ln(1 + \mathsf{T}_0(1;\vec{t},\vec{u})) \\
&\quad + \int_{b\rho^{2b}}^{1} \Big\{ f(x;\vec{t},\vec{u}) + \frac{b\rho^{2b}\mathsf{T}_1(b\rho^{2b};\vec{t},\vec{u})}{\Omega(\vec{u})(x - b\rho^{2b})} \Big\} dx \\
&\quad + b\rho^{2b} \frac{\mathsf{T}_1(b\rho^{2b};\vec{t},\vec{u})}{\Omega(\vec{u})} \ln\Big(\frac{b\rho^{b}}{\sqrt{2\pi}(1 - b\rho^{2b})}\Big), \\
C_4 &= \sqrt{2}\mathcal{I} b\rho^{b} \Big(\rho^{2b} \frac{\mathsf{T}_2(b\rho^{2b};\vec{t},\vec{u})}{\Omega(\vec{u})} - \frac{\mathsf{T}_1(b\rho^{2b};\vec{t},\vec{u})}{\Omega(\vec{u})} - \rho^{2b} \frac{\mathsf{T}_1(b\rho^{2b};\vec{t},\vec{u})^2}{\Omega(\vec{u})^2} \Big),
\end{aligned}$$

and the real number $\mathcal{I} \in \mathbb{R}$ is given by

$$\mathcal{I} = \int_{-\infty}^{+\infty} \Big\{ \frac{y e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^2 + \frac{1}{2}\Big] \Big\} dy \approx -0.81367. \tag{1.22}$$

In particular, since $\mathbb{E}[\prod_{j=1}^{m} e^{u_j \,\mathrm{N}(r_j)}]$ depends analytically on $u_1, \dots, u_m \in \mathbb{C}$ and is strictly positive for $u_1, \dots, u_m \in \mathbb{R}$, the asymptotic formula (1.21) together with Cauchy's formula shows that

$$\partial_{u_1}^{k_1} \dots \partial_{u_m}^{k_m} \Big\{ \ln \mathbb{E}\Big[\prod_{j=1}^{m} e^{u_j \,\mathrm{N}(r_j)}\Big] - \Big(C_1 n + C_2 \ln n + C_3 + \frac{C_4}{\sqrt{n}}\Big) \Big\} = \mathcal{O}(n^{-\frac{3}{5}}) \quad \text{as } n \to +\infty, \tag{1.23}$$

for any $k_1, \dots, k_m \in \mathbb{N}$, and $u_1, \dots, u_m \in \mathbb{R}$.

Remark 1.4. The leading coefficient in the asymptotics of moment generating functions of linear statistics with respect to a fixed, bounded continuous test function g is of course given by the integral of g against the relevant equilibrium measure. However, in the hard edge regime of Theorem 1.3, we rather use a sequence $g = g_n$ of test-functions, given in terms of characteristic functions of expanding disks of radii (1.20) by $g_n(z) = \sum_{j=1}^{m} u_j \chi_{(0,r_j)}(z)$.

A direct computation using (1.2) shows that, as $n \to +\infty$,

$$\int g_n(x)d\mu_h(x) = \begin{cases} \sum_{j=1}^{m} u_j \int_0^{r_j} 2b^2 r^{2b-1} dr = b\rho^{2b} \sum_{j=1}^{m} u_j + o(1), \\ \sum_{j=1}^{m} u_j \int_0^{r_j} 2b^2 r^{2b-1} dr + u_m c_\rho = b\rho^{2b} \sum_{j=1}^{m} u_j + u_m c_\rho + o(1), \end{cases}$$

where the first line reads for $t_m > 0$ and the second one for $t_m = 0$, and where c_ρ is given by (1.9).

Since $b\rho^{2b} \sum_{j=1}^{m} u_j \neq C_1 \neq b\rho^{2b} \sum_{j=1}^{m} u_j + u_m c_\rho$, we see that in the hard edge regime, even the leading coefficient C_1 cannot straightforwardly be obtained from the equilibrium measure, which might be surprising at first sight. What is even more surprising is that C_1 is not even linear in $u_1, \ldots, u_m$ (this contrasts with all previously studied regimes, and also with the semi-hard edge regime).

For $\vec{j} \in (\mathbb{N}^m)_{>0} := \{\vec{j} = (j_1, \ldots, j_m) \in \mathbb{N} : j_1 + \cdots + j_m \geq 1\}$, the joint cumulant $\kappa_{\vec{j}} = \kappa_{\vec{j}}(r_1, \ldots, r_m; n, b, \alpha)$ of $\mathrm{N}(r_1), \ldots, \mathrm{N}(r_m)$ is defined by

$$\kappa_{\vec{j}} = \kappa_{j_1,\ldots,j_m} := \partial_{\vec{u}}^{\vec{j}} \ln \mathbb{E}[e^{u_1 \mathrm{N}(r_1) + \cdots + u_m \mathrm{N}(r_m)}]|_{\vec{u}=\vec{0}},$$

where $\partial_{\vec{u}}^{\vec{j}} := \partial_{u_1}^{j_1} \ldots \partial_{u_m}^{j_m}$. In particular,

$$\begin{aligned} \mathbb{E}[\mathrm{N}(r)] &= \kappa_1(r), \\ \mathrm{Var}[\mathrm{N}(r)] &= \kappa_2(r) = \kappa_{(1,1)}(r,r), \\ \mathrm{Cov}[\mathrm{N}(r_1), \mathrm{N}(r_2)] &= \kappa_{(1,1)}(r_1, r_2). \end{aligned}$$

Recall from (1.2)–(1.9) that $c_\rho = 1 - b\rho^{2b} = \int \mu_{\mathrm{sing}}(d^2 z)$, i.e. c_ρ is the density of particles accumulating near the hard-edge as $n \to +\infty$. It turns out that the asymptotics of $\mathbb{E}[\mathrm{N}(r_\ell)]$ and $\mathrm{Cov}(\mathrm{N}(r_\ell), \mathrm{N}(r_k))$, which are obtained in Corollary 1.5 below, are more elegantly described in terms of c_ρ, as well as the new parameter

$$\begin{aligned} s_\ell &:= \frac{t_\ell}{b}(1 - b\rho^{2b}) \\ &= \frac{c_\rho n}{b}\left(1 - \left(\frac{r_\ell}{\rho}\right)^{2b}\right) = 2 \cdot \frac{c_\rho n}{2\pi\rho} \cdot 2\pi(\rho - r_\ell)(1 + \mathcal{O}(n^{-1})). \end{aligned} \tag{1.24}$$

Corollary 1.5 (Hard edge). *Let $m \in \mathbb{N}_{>0}$, $b > 0$, $\rho \in (0, b^{-\frac{1}{2b}})$, $\vec{j} \in (\mathbb{N}^m)_{>0}$, $\alpha > -1$, and $t_1 > \cdots > t_m > 0$ be fixed. Define $s_1, \ldots, s_m$ as in (1.24). For $n \in \mathbb{N}_{>0}$, define $\{r_\ell\}_{\ell=1}^m$ by (1.20).*

(a) *The joint cumulant $\kappa_{\vec{j}}$ satisfies*

$$\kappa_{\vec{j}} = \partial_{\vec{u}}^{\vec{j}} C_1|_{\vec{u}=\vec{0}} n + \partial_{\vec{u}}^{\vec{j}} C_2|_{\vec{u}=\vec{0}} \ln n + \partial_{\vec{u}}^{\vec{j}} C_3|_{\vec{u}=\vec{0}} + \frac{\partial_{\vec{u}}^{\vec{j}} C_4|_{\vec{u}=\vec{0}}}{\sqrt{n}} + \mathcal{O}(n^{-\frac{3}{5}}), \quad n \to +\infty,$$

where $C_1, \ldots, C_4$ are as in Theorem 1.3. In particular, for any $1 \leq \ell < k \leq m$,

$$\begin{aligned}
\mathbb{E}[\mathrm{N}(r_\ell)] &= b_1(s_\ell)n + c_1(s_\ell)\ln n + d_1(s_\ell) + e_1(s_\ell)n^{-\frac{1}{2}} + \mathcal{O}(n^{-\frac{3}{5}}), \\
\mathrm{Var}[\mathrm{N}(r_\ell)] &= b_{(1,1)}(s_\ell, s_\ell)n + c_{(1,1)}(s_\ell, s_\ell)\ln n + d_{(1,1)}(s_\ell, s_\ell) + e_{(1,1)}(s_\ell, s_\ell)n^{-\frac{1}{2}} + \mathcal{O}(n^{-\frac{3}{5}}), \\
\mathrm{Cov}(\mathrm{N}(r_\ell), \mathrm{N}(r_k)) &= b_{(1,1)}(s_\ell, s_k)n + c_{(1,1)}(s_\ell, s_k)\ln n + d_{(1,1)}(s_\ell, s_k) + e_{(1,1)}(s_\ell, s_k)n^{-\frac{1}{2}} + \mathcal{O}(n^{-\frac{3}{5}})
\end{aligned}$$

as $n \to +\infty$, where

$$\begin{aligned}
b_1(s_\ell) &= 1 - c_\rho + c_\rho \frac{1 - e^{-s_\ell}}{s_\ell}, \\
c_1(s_\ell) &= -\frac{1 - c_\rho}{c_\rho}\frac{bs_\ell}{2}, \\
d_1(s_\ell) &= -\frac{1 - e^{-s_\ell}}{2} + \frac{1 - c_\rho}{c_\rho}\frac{bs_\ell}{2}\ln\Big(\frac{b(1 - c_\rho)}{2\pi c_\rho^2}\Big) \\
&\quad - s_\ell \int_0^1 \frac{e^{-s_\ell y}(yc_\rho(bs_\ell y + 2\alpha) + (1 - c_\rho)b(2 + s_\ell y))}{2c_\rho y} - \frac{2(1 - c_\rho)b}{2c_\rho y} dy, \\
e_1(s_\ell) &= \sqrt{2}\mathcal{I} b\rho^{-b}\frac{1 - c_\rho}{c_\rho}s_\ell\Big(\frac{1 - c_\rho}{c_\rho}s_\ell - 1\Big),
\end{aligned}$$

and, for $l \leq k$,

$$b_{(1,1)}(s_\ell, s_k) = c_\rho \frac{1 - e^{-s_\ell}}{s_\ell} - c_\rho \frac{1 - e^{-s_\ell - s_k}}{s_\ell + s_k}, \tag{1.25}$$

$$c_{(1,1)}(s_\ell, s_k) = \frac{1 - c_\rho}{c_\rho}\frac{bs_k}{2},$$

$$
\begin{aligned}
d_{(1,1)}(s_\ell, s_k) = {} & \frac{e^{-s_\ell}(1-e^{-s_k})}{2} - \frac{1-c_\rho}{c_\rho}\frac{bs_k}{2}\ln\Big(\frac{b(1-c_\rho)}{2\pi c_\rho^2}\Big) \\
& - \int_0^1 \frac{1}{y}\Big\{bs_k\frac{1-c_\rho}{c_\rho} \\
& \qquad + s_\ell e^{-s_\ell y}\Big(b\frac{1-c_\rho}{c_\rho} + \alpha y + \frac{bs_\ell}{2}y\Big(y+\frac{1-c_\rho}{c_\rho}\Big)\Big) \\
& \qquad - e^{-(s_\ell+s_k)y}\Big(\Big(\frac{1-c_\rho}{c_\rho}b+\alpha y\Big)(s_\ell+s_k) \\
& \qquad\qquad + \frac{by}{2}\Big(y+\frac{1-c_\rho}{c_\rho}\Big)(s_\ell^2+s_k^2)\Big)\Big\}dy, \\
e_{(1,1)}(s_\ell, s_k) = {} & \sqrt{2}\mathcal{I}b\rho^{-b}\frac{1-c_\rho}{c_\rho}s_k\Big(1-\frac{1-c_\rho}{c_\rho}(2s_\ell+s_k)\Big).
\end{aligned}
$$

(b) *As $n \to +\infty$, the random variable $(\mathcal{N}_1, \ldots, \mathcal{N}_m)$, where*

$$
\mathcal{N}_\ell := \frac{\mathrm{N}(r_\ell) - b_1(s_\ell)n}{\sqrt{b_{(1,1)}(s_\ell, s_\ell)n}}, \qquad \ell = 1, \ldots, m, \tag{1.26}
$$

convergences in distribution to a multivariate normal random variable of mean $(0, \ldots, 0)$ whose covariance matrix Σ is defined by

$$
\Sigma_{\ell,k} = \Sigma_{k,\ell} = \frac{b_{(1,1)}(s_\ell, s_k)}{\sqrt{b_{(1,1)}(s_\ell, s_\ell)b_{(1,1)}(s_k, s_k)}}, \qquad 1 \le \ell \le k \le m,
$$

where $b_{(1,1)}$ is given by (1.25).

Remark 1.6. Corollary 1.5 is stated for $t_1 > \cdots > t_m > 0$. It is important for Corollary 1.5 (b) that $t_m > 0$; note however that Corollary 1.5 (a) in fact also holds for $t_1 > \cdots > t_m \ge 0$. In the case when $t_m = 0 = s_m$, one finds $b_1(s_m) = n$ and $c_1(s_m) = d_1(s_m) = e_1(s_m) = 0$, which is consistent with the fact that $\mathrm{N}(r_m) = n$ with probability 1.

The central limit theorem of Corollary 1.5 (b), even though it only uses $b_1(s)$ and $b_{(1,1)}(s, s)$, is a non-trivial result because to determine just the leading term C_1 in Theorem 1.3 one already needs quite subtle asymptotics of the incomplete gamma function.

Proof of Corollary 1.5. Assertion (a) follows from (1.23) and the expressions for the C_j given in Theorem 1.3. By Lévy's continuity theorem, assertion (b) will follow if we can show that the characteristic function $\mathbb{E}[e^{i\sum_{\ell=1}^m v_\ell \mathcal{N}_\ell}]$ converges pointwise to $e^{-\frac{1}{2}\sum_{\ell,k=1}^m v_\ell \Sigma_{\ell,k} v_k}$ for every $v_\ell \in \mathbb{R}^m$ as $n \to +\infty$. Letting

$$
u_\ell = \frac{iv_\ell}{\sqrt{b_{(1,1)}(s_\ell, s_\ell)n}},
$$

(1.26) and (1.21) show that

$$\begin{aligned}\mathbb{E}[e^{i\sum_{\ell=1}^{m} v_\ell \mathcal{N}_\ell}] &= \mathbb{E}[e^{\sum_{\ell=1}^{m} u_\ell \mathrm{N}(r_\ell)}]e^{-\sum_{\ell=1}^{m} u_\ell b_1(s_\ell)n} \\ &= e^{C_1(\vec{u})n + C_2(\vec{u})\ln n + C_3(\vec{u}) + \mathcal{O}(n^{-\frac{1}{2}})}e^{-\sum_{\ell=1}^{m} u_\ell \partial_{u_\ell} C_1|_{\vec{u}=\vec{0}}n}\end{aligned}$$

as $n \to +\infty$ for any fixed $v_\ell \in \mathbb{R}^m$. Since $C_j|_{\vec{u}=\vec{0}} = 0$ for $j = 1, 2, 3$ and $u_\ell = \mathcal{O}(n^{-1/2})$, we obtain

$$\begin{aligned}\mathbb{E}[e^{i\sum_{\ell=1}^{m} v_\ell \mathcal{N}_\ell}] &= e^{\frac{1}{2}\sum_{\ell,k=1}^{m} u_\ell u_k \partial_{u_\ell}\partial_{u_k} C_1|_{\vec{u}=\vec{0}}n + \mathcal{O}(|\vec{u}|^3 n + |\vec{u}|\ln n + |\vec{u}| + n^{-1/2})} \\ &= e^{\frac{1}{2}\sum_{\ell,k=1}^{m} \frac{i v_\ell}{\sqrt{b_{(1,1)}(s_\ell,s_\ell)}}\frac{i v_k}{\sqrt{b_{(1,1)}(s_k,s_k)}} b_{(1,1)}(s_{\min(\ell,k)}, s_{\max(\ell,k)}) + \mathcal{O}(\frac{\ln n}{\sqrt{n}})} \\ &\to e^{-\frac{1}{2}\sum_{\ell,k=1}^{m} v_\ell \Sigma_{\ell,k} v_k}\end{aligned}$$

as $n \to +\infty$, which proves (b). ■

Let us analyze the leading coefficient $b_{(1,1)}(s,s)$ of Var[N(r)], where

$$r := \rho\Big(1 - \frac{t}{n}\Big)^{\frac{1}{2b}} \quad \text{and} \quad s := \frac{t}{b}c_\rho.$$

By (1.25),

$$b_{(1,1)}(s,s) = c_\rho \frac{1 - e^{-s}}{s} - c_\rho \frac{1 - e^{-2s}}{2s}. \tag{1.27}$$

Note that $b_{(1,1)}(0,0) := \lim_{s\to 0_+} b_{(1,1)}(s,s) = 0$, which, as mentioned in Remark 1.6, is consistent with the fact that N$(\rho) = n$ with probability 1. On the other hand, $b_{(1,1)}(s,s) = \frac{c_\rho}{2s} + \mathcal{O}(e^{-s})$ as $s \to +\infty$. It is therefore interesting to investigate where the maximum of $b_{(1,1)}(s,s)$ is achieved. It is possible to compute the unique maximum of $s \mapsto b_{(1,1)}(s,s)$ explicitly in terms of the Lambert function $W_{-1}(x)$, which for $-\frac{1}{e} \le x < 0$ is defined as the unique solution to

$$W_{-1}(x)e^{W_{-1}(x)} = x, \quad W_{-1}(x) \le -1.$$

Indeed, taking the derivative of (1.27) yields

$$\frac{d}{ds}b_{(1,1)}(s,s) = -\frac{c_\rho}{2s^2}(1 - e^{-s})(1 - (1 + 2s)e^{-s}), \quad s > 0,$$

and a direct inspection shows that $\frac{d}{ds}b_{(1,1)}(s,s) = 0$ if and only if $s = s_\star$, where

$$s_\star = -\Big(W_{-1}\Big(\frac{-1}{2\sqrt{e}}\Big) + \frac{1}{2}\Big) \approx 1.2564.$$

Furthermore,

$$b_{(1,1)}(s_\star, s_\star) = \frac{-2W_{-1}\left(\frac{-1}{2\sqrt{e}}\right) - 1}{4W_{-1}\left(\frac{-1}{2\sqrt{e}}\right)^2} c_\rho \approx 0.20363 c_\rho.$$

As ρ decreases, the hard wall gets stronger (in the sense that the mass c_ρ of μ_{sing} increases), and we observe that $b_{(1,1)}(s_\star, s_\star)$ increases. The graphs of $b_1(s)$ and $b_{(1,1)}(s,s)$ are displayed in Figure 2 for certain values of ρ and b.

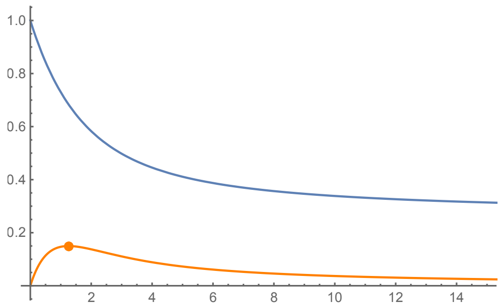

Figure 2. The coefficients $s \mapsto b_1(s)$ (blue) and $s \mapsto b_{(1,1)}(s,s)$ (orange) for $\rho = 0.6b^{-\frac{1}{2b}}$ and $b = \frac{13}{10}$. The orange dot has coordinates $(s_\star, b_{(1,1)}(s_\star, s_\star))$.

1.4. Results for the semi-hard edge

Theorem 1.7 (Merging radii at the semi-hard edge). *Let $m \in \mathbb{N}_{>0}$, $b > 0$, $\rho \in (0, b^{-\frac{1}{2b}})$, $\mathfrak{s}_1 > \cdots > \mathfrak{s}_m > 0$, and $\alpha > -1$ be fixed parameters, and for $n \in \mathbb{N}_{>0}$, define*

$$r_\ell = \rho\left(1 - \frac{\sqrt{2}\mathfrak{s}_\ell}{\rho^b\sqrt{n}}\right)^{\frac{1}{2b}}, \quad \ell = 1, \ldots, m. \tag{1.28}$$

For any fixed $x_1, \ldots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\mathbb{E}\Big[\prod_{j=1}^m e^{u_j \mathrm{N}(r_j)}\Big] = \exp\Big(C_1 n + C_2\sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O}\Big(\frac{(\ln n)^4}{n}\Big)\Big), \quad \text{as } n \to +\infty$$

uniformly for $u_1 \in D_\delta(x_1), \ldots, u_m \in D_\delta(x_m)$, where

$$C_1 = b\rho^{2b}\sum_{j=1}^m u_j,$$

$$C_2 = \sqrt{2}b\rho^b \int_{-\infty}^{+\infty} \Big(h_0(y) - \chi_{(-\infty,0)}(y) \sum_{j=1}^m u_j\Big) dy,$$

$$C_3 = -\Big(\frac{1}{2} + \alpha\Big) \sum_{j=1}^m u_j + b \int_{-\infty}^{+\infty} \Big(4y\Big(h_0(y) - \chi_{(-\infty,0)}(y) \sum_{j=1}^m u_j\Big) + \sqrt{2}h_1(y)\Big) dy,$$

$$C_4 = b\rho^{-b} \int_{-\infty}^{+\infty} \Big[6\sqrt{2}y^2\Big(h_0(y) - \chi_{(-\infty,0)}(y) \sum_{j=1}^m u_j\Big) + 4yh_1(y) + \sqrt{2}h_2(y)\Big] dy,$$

where

$$h_0(y) = \ln(g_0(y)), \quad h_1(y) = \frac{g_1(y)}{g_0(y)}, \quad h_2(y) = \frac{g_2(y)}{g_0(y)} - \frac{1}{2}\Big(\frac{g_1(y)}{g_0(y)}\Big)^2,$$

and

$$g_0(y) = 1 + \sum_{\ell=1}^m \omega_\ell \frac{\operatorname{erfc}(y + \mathfrak{s}_\ell)}{\operatorname{erfc}(y)},$$

$$\begin{aligned} g_1(y) = \sum_{\ell=1}^m \frac{\sqrt{2}}{3\sqrt{\pi}} \omega_\ell \Big\{ &(5y^2 - 1) \frac{e^{-y^2}}{\operatorname{erfc}(y)} \frac{\operatorname{erfc}(y + \mathfrak{s}_\ell)}{\operatorname{erfc}(y)} \\ &- (5y^2 + \mathfrak{s}_\ell y + 2\mathfrak{s}_\ell^2 - 1) \frac{e^{-(y+\mathfrak{s}_\ell)^2}}{\operatorname{erfc}(y)} \Big\}, \end{aligned}$$

$$\begin{aligned} g_2(y) = \sum_{\ell=1}^m \omega_\ell \Big\{ &\frac{1}{18\sqrt{\pi}} \Big[50y^5 + 70y^4\mathfrak{s}_\ell + y^3(62\mathfrak{s}_\ell^2 - 73) + y^2\mathfrak{s}_\ell(50\mathfrak{s}_\ell^2 - 33) \\ &- y(3 + 18\mathfrak{s}_\ell^2 - 16\mathfrak{s}_\ell^4) + \mathfrak{s}_\ell(3 - 22\mathfrak{s}_\ell^2 + 8\mathfrak{s}_\ell^4)\Big] \frac{e^{-(y+\mathfrak{s}_\ell)^2}}{\operatorname{erfc}(y)} \\ &+ \frac{2(1 - 5y^2)(5y^2 + y\mathfrak{s}_\ell - 1 + 2\mathfrak{s}_\ell^2)}{9\pi} \frac{e^{-y^2}}{\operatorname{erfc}(y)} \frac{e^{-(y+\mathfrak{s}_\ell)^2}}{\operatorname{erfc}(y)} \\ &+ \frac{y(3 + 73y^2 - 50y^4)}{18\sqrt{\pi}} \frac{e^{-y^2}}{\operatorname{erfc}(y)} \frac{\operatorname{erfc}(y + \mathfrak{s}_\ell)}{\operatorname{erfc}(y)} \\ &+ \frac{2(1 - 5y^2)^2}{9\pi} \Big(\frac{e^{-y^2}}{\operatorname{erfc}(y)}\Big)^2 \frac{\operatorname{erfc}(y + \mathfrak{s}_\ell)}{\operatorname{erfc}(y)} \Big\}. \end{aligned}$$

In particular, since $\mathbb{E}[\prod_{j=1}^m e^{u_j \,\mathrm{N}(r_j)}]$ *depends analytically on* $u_1, \ldots, u_m \in \mathbb{C}$ *and is strictly positive for* $u_1, \ldots, u_m \in \mathbb{R}$*, the asymptotic formula* (1.31) *together with Cauchy's formula shows that*

$$\partial_{u_1}^{k_1} \ldots \partial_{u_m}^{k_m} \Big\{ \ln \mathbb{E}\Big[\prod_{j=1}^m e^{u_j \,\mathrm{N}(r_j)}\Big] - \Big(C_1 n + C_2\sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}}\Big)\Big\} = \mathcal{O}\Big(\frac{(\ln n)^4}{n}\Big)$$

as $n \to +\infty$*, for any* $k_1, \ldots, k_m \in \mathbb{N}$ *and* $u_1, \ldots, u_m \in \mathbb{R}$*.*

The proof of the following corollary is similar to that of Corollary 1.5 and is omitted.

Corollary 1.8 (Semi-hard edge). *Let $m \in \mathbb{N}_{>0}$, $b > 0$, $\rho \in (0, b^{-\frac{1}{2b}})$, $\vec{j} \in (\mathbb{N}^m)_{>0}$, $\alpha > -1$, and $\mathfrak{s}_1 > \cdots > \mathfrak{s}_m > 0$ be fixed. For $n \in \mathbb{N}_{>0}$, define $\{r_\ell\}_{\ell=1}^m$ by (1.28).*

(a) *The joint cumulant $\kappa_{\vec{j}}$ satisfies*

- *for $\vec{j} = 1$,*

$$\kappa_{\vec{j}} = \partial_{\vec{u}}^{\vec{j}} C_1|_{\vec{u}=\vec{0}} n + \partial_{\vec{u}}^{\vec{j}} C_2|_{\vec{u}=\vec{0}} \sqrt{n} + \partial_{\vec{u}}^{\vec{j}} C_3|_{\vec{u}=\vec{0}} + \partial_{\vec{u}}^{\vec{j}} C_4|_{\vec{u}=\vec{0}} \frac{1}{\sqrt{n}} + \mathcal{O}\Big(\frac{(\ln n)^4}{n}\Big),$$

- *for $\vec{j} \neq 1$,*

$$\kappa_{\vec{j}} = \partial_{\vec{u}}^{\vec{j}} C_2|_{\vec{u}=\vec{0}} \sqrt{n} + \partial_{\vec{u}}^{\vec{j}} C_3|_{\vec{u}=\vec{0}} + \partial_{\vec{u}}^{\vec{j}} C_4|_{\vec{u}=\vec{0}} \frac{1}{\sqrt{n}} + \mathcal{O}\Big(\frac{(\ln n)^4}{n}\Big),$$

as $n \to +\infty$, where $C_1, \ldots, C_4$ are as in Theorem 1.7.
In particular, for any $1 \le \ell < k \le m$,

$$\mathbb{E}[\mathrm{N}(r_\ell)] = b_1(\mathfrak{s}_\ell) n + c_1(\mathfrak{s}_\ell)\sqrt{n} + d_1(\mathfrak{s}_\ell) + e_1(\mathfrak{s}_\ell) n^{-\frac{1}{2}} + \mathcal{O}((\ln n)^4 n^{-1}),$$

$$\mathrm{Var}[\mathrm{N}(r_\ell)] = c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_\ell)\sqrt{n} + d_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_\ell) + e_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_\ell) n^{-\frac{1}{2}} + \mathcal{O}((\ln n)^4 n^{-1}),$$

$$\mathrm{Cov}(\mathrm{N}(r_\ell), \mathrm{N}(r_k)) = c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k)\sqrt{n} + d_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) + e_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) n^{-\frac{1}{2}} + \mathcal{O}((\ln n)^4 n^{-1})$$

as $n \to +\infty$, where

$$b_1(\mathfrak{s}_\ell) = b\rho^{2b},$$

$$c_1(\mathfrak{s}_\ell) = \sqrt{2} b\rho^b \int_{-\infty}^{+\infty} \Big(\frac{\mathrm{erfc}(y + \mathfrak{s}_\ell)}{\mathrm{erfc}(y)} - \chi_{(-\infty,0)}(y)\Big)\, dy,$$

$$d_1(\mathfrak{s}_\ell) = -\Big(\frac{1}{2} + \alpha\Big) + 2b \int_{-\infty}^{+\infty} \Big\{ 2y \Big(\frac{\mathrm{erfc}(y + \mathfrak{s}_\ell)}{\mathrm{erfc}(y)} - \chi_{(-\infty,0)}(y)\Big) + \frac{5y^2 - 1}{3\sqrt{\pi}} \frac{e^{-y^2}}{\mathrm{erfc}(y)} \frac{\mathrm{erfc}(y + \mathfrak{s}_\ell)}{\mathrm{erfc}(y)} + \frac{1 - 5y^2 - y\mathfrak{s}_\ell - 2\mathfrak{s}_\ell^2}{3\sqrt{\pi}} \frac{e^{-(y+\mathfrak{s}_\ell)^2}}{\mathrm{erfc}(y)} \Big\} dy,$$

$$e_1(\mathfrak{s}_\ell) = \frac{b\rho^{-b}}{9\sqrt{2}\pi} \int_{-\infty}^{\infty} \frac{1}{\operatorname{erfc}(y)^3} \mathcal{M}\, dy,$$

where

$$\begin{aligned}
\mathcal{M} := {} & 108\pi y^2 \operatorname{erfc}(y)^2 \operatorname{erfc}(y+\mathfrak{s}_\ell) \\
& + \sqrt{\pi}\operatorname{erfc}(y)^2 e^{-(y+\mathfrak{s}_\ell)^2} (2\mathfrak{s}_\ell^3(25y^2-11) + 2\mathfrak{s}_\ell^2 y(31y^2-33) \\
& \qquad + \mathfrak{s}_\ell(70y^4 - 57y^2 + 3) + 16\mathfrak{s}_\ell^4 y + 8\mathfrak{s}_\ell^5 \\
& \qquad + y(50y^4 - 193y^2 + 21)) \\
& + \operatorname{erfc}(y)(-e^{-y^2}\sqrt{\pi}\, y(50y^4 - 193y^2 + 21)\operatorname{erfc}(y+\mathfrak{s}_\ell) \\
& \qquad - 4e^{-(y+\mathfrak{s}_\ell)^2 - y^2}(5y^2-1)(\mathfrak{s}_\ell y + 2\mathfrak{s}_\ell^2 + 5y^2 - 1)) \\
& + 4e^{-2y^2}(1-5y^2)^2 \operatorname{erfc}(y+\mathfrak{s}_\ell) - 108\pi\chi_{(-\infty,0)}(y) y^2 \operatorname{erfc}(y)^3,
\end{aligned}$$

and, for $l \le k$,

$$\begin{aligned}
& c_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) \\
& \quad = \sqrt{2} b\rho^b \int_{-\infty}^{\infty} \frac{\operatorname{erfc}(y+\mathfrak{s}_\ell)(\operatorname{erfc}(y) - \operatorname{erfc}(y+\mathfrak{s}_k))}{\operatorname{erfc}(y)^2}\, dy,
\end{aligned} \tag{1.29}$$

$$d_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) = \frac{2b}{3\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{1}{\operatorname{erfc}(y)^3} \mathcal{M}_1\, dy,$$

$$e_{(1,1)}(\mathfrak{s}_\ell, \mathfrak{s}_k) = \frac{b\rho^{-b}}{9\sqrt{2}\pi} \int_{-\infty}^{+\infty} \frac{e^{-(y+\mathfrak{s}_\ell)^2 - (y+\mathfrak{s}_k)^2}}{\operatorname{erfc}(y)^4} \mathcal{M}_2\, dy,$$

where

$$\begin{aligned}
\mathcal{M}_1 := {} & \operatorname{erfc}(y)^2(6\sqrt{\pi}\, y \operatorname{erfc}(y+\mathfrak{s}_\ell) - e^{-(y+\mathfrak{s}_\ell)^2}(\mathfrak{s}_\ell y + 2\mathfrak{s}_\ell^2 + 5y^2 - 1)) \\
& + \operatorname{erfc}(y)(e^{-(y+\mathfrak{s}_\ell)^2} \operatorname{erfc}(y+\mathfrak{s}_k)(\mathfrak{s}_\ell y + 2\mathfrak{s}_\ell^2 + 5y^2 - 1) \\
& \qquad - 6\sqrt{\pi}\, y \operatorname{erfc}(y+\mathfrak{s}_\ell)\operatorname{erfc}(y+\mathfrak{s}_k) \\
& \qquad + (e^{-y^2} + e^{-(y+\mathfrak{s}_k)^2})\operatorname{erfc}(y+\mathfrak{s}_\ell)(5y^2-1) \\
& \qquad + e^{-(y+\mathfrak{s}_k)^2}\operatorname{erfc}(y+\mathfrak{s}_\ell)\mathfrak{s}_k(2\mathfrak{s}_k + y)) \\
& + 2e^{-y^2}(1-5y^2)\operatorname{erfc}(y+\mathfrak{s}_\ell)\operatorname{erfc}(y+\mathfrak{s}_k)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}_2 := {} & -\operatorname{erfc}(y)^2 \mathcal{M}_{2,1} + \sqrt{\pi}\operatorname{erfc}(y)^3 e^{(y+\mathfrak{s}_k)^2}\mathcal{M}_{2,2} + 2\operatorname{erfc}(y)\mathcal{M}_{2,3} \\
& - 12(1-5y^2)^2 e^{2(\mathfrak{s}_\ell+\mathfrak{s}_k)y + \mathfrak{s}_\ell^2 + \mathfrak{s}_k^2}\operatorname{erfc}(y+\mathfrak{s}_\ell)\operatorname{erfc}(y+\mathfrak{s}_k),
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{M}_{2,1} := {} & \sqrt{\pi}\,\mathrm{erfc}(y+\mathfrak{s}_\ell) \\
& \times \big(108\sqrt{\pi}\,y^2\,\mathrm{erfc}(y+\mathfrak{s}_k)e^{2(\mathfrak{s}_\ell+\mathfrak{s}_k)y+\mathfrak{s}_\ell^2+\mathfrak{s}_k^2+2y^2} \\
& \quad + (50y^4-193y^2+21)ye^{(y+\mathfrak{s}_\ell)^2}(e^{\mathfrak{s}_k(\mathfrak{s}_k+2y)}+1) \\
& \quad + \mathfrak{s}_k e^{(y+\mathfrak{s}_\ell)^2}(62\mathfrak{s}_k y^3 + (50\mathfrak{s}_k^2-57)y^2 + 2\mathfrak{s}_k(8\mathfrak{s}_k^2-33)y \\
& \qquad + 8\mathfrak{s}_k^4 - 22\mathfrak{s}_k^2 + 70y^4 + 3)\big) \\
& + \sqrt{\pi}e^{(y+\mathfrak{s}_k)^2}\big(2\mathfrak{s}_\ell^3(25y^2-11) + 2\mathfrak{s}_\ell^2 y(31y^2-33) \\
& \qquad + \mathfrak{s}_\ell(70y^4-57y^2+3) + 16\mathfrak{s}_\ell^4 y \\
& \qquad + 8\mathfrak{s}_\ell^5 + y(50y^4-193y^2+21)\big)\,\mathrm{erfc}(y+\mathfrak{s}_k) \\
& + 4(\mathfrak{s}_\ell y + 2\mathfrak{s}_\ell^2 + 5y^2 - 1)\big((5y^2-1)e^{\mathfrak{s}_k(\mathfrak{s}_k+2y)} \\
& \qquad + \mathfrak{s}_k(2\mathfrak{s}_k + y) + 5y^2 - 1\big), \\
\mathcal{M}_{2,2} := {} & 108\sqrt{\pi}\,y^2 e^{(y+\mathfrak{s}_\ell)^2}\,\mathrm{erfc}(y+\mathfrak{s}_\ell) + 2\mathfrak{s}_\ell^3(25y^2-11) \\
& + 2\mathfrak{s}_\ell^2 y(31y^2-33) + \mathfrak{s}_\ell(70y^4-57y^2+3) \\
& + 16\mathfrak{s}_\ell^4 y + 8\mathfrak{s}_\ell^5 + y(50y^4-193y^2+21), \\
\mathcal{M}_{2,3} := {} & 4(5y^2-1)e^{\mathfrak{s}_k(\mathfrak{s}_k+2y)}(\mathfrak{s}_\ell y + 2\mathfrak{s}_\ell^2 + 5y^2 - 1)\,\mathrm{erfc}(y+\mathfrak{s}_k) \\
& + e^{\mathfrak{s}_\ell(\mathfrak{s}_\ell+2y)}\,\mathrm{erfc}(y+\mathfrak{s}_\ell) \\
& \quad \times \big(\sqrt{\pi}\,y(50y^4-193y^2+21)e^{(y+\mathfrak{s}_k)^2}\,\mathrm{erfc}(y+\mathfrak{s}_k) \\
& \qquad + 2(1-5y^2)^2(e^{\mathfrak{s}_k(\mathfrak{s}_k+2y)}+2) + 4\mathfrak{s}_k(5y^2-1)(2\mathfrak{s}_k+y)\big).
\end{aligned}$$

(b) *As $n \to +\infty$, the random variable $(\mathcal{N}_1, \dots, \mathcal{N}_m)$, where*

$$\mathcal{N}_\ell := \frac{\mathrm{N}(r_\ell) - (b_1(\mathfrak{s}_\ell)n + c_1(\mathfrak{s}_\ell)\sqrt{n})}{\sqrt{c_{(1,1)}(\mathfrak{s}_\ell,\mathfrak{s}_\ell)}\,n^{1/4}}, \qquad \ell = 1, \dots, m,$$

convergences in distribution to a multivariate normal random variable of mean $(0, \dots, 0)$ whose covariance matrix Σ is defined by

$$\Sigma_{\ell,\ell} = 1,$$

$$\Sigma_{\ell,k} = \Sigma_{k,\ell} = \frac{c_{(1,1)}(\mathfrak{s}_\ell,\mathfrak{s}_k)}{\sqrt{c_{(1,1)}(\mathfrak{s}_\ell,\mathfrak{s}_\ell)c_{(1,1)}(\mathfrak{s}_k,\mathfrak{s}_k)}}, \qquad 1 \le \ell < k \le m,$$

where $c_{(1,1)}$ is given by (1.29).

1.5. Results for the bulk

It turns out that the points in the bulk only feel the hard wall via exponentially small corrections. Consequently, the formulas for the bulk regime presented in our next theorem are *identical* to the corresponding formulas for the case without a hard edge presented in [31]. Moreover, the proof is almost identical to the proof of the analogous theorem in [31] and is therefore omitted (the only difference between the proofs is that a number of exponentially small error terms stemming from the hard wall appear in the proof of Theorem 1.9).

Theorem 1.9 (Merging radii in the bulk). *Let $m \in \mathbb{N}_{>0}$, $b > 0$, $r \in (0, b^{-\frac{1}{2b}})$, $\mathfrak{s}_1 < \cdots < \mathfrak{s}_m$, and $\alpha > -1$ be fixed parameters, and, for $n \in \mathbb{N}_{>0}$, define*

$$r_\ell = r\Big(1 + \frac{\sqrt{2}\mathfrak{s}_\ell}{r^b\sqrt{n}}\Big)^{\frac{1}{2b}}, \quad \ell = 1, \dots, m. \tag{1.30}$$

For any fixed $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{j=1}^{m} e^{u_j \mathrm{N}(r_j)}\Big] \\
&= \exp\Big(C_1 n + C_2\sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O}\Big(\frac{(\ln n)^2}{n}\Big)\Big), \quad \textit{as } n \to +\infty
\end{aligned} \tag{1.31}$$

uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$, where

$$\begin{aligned}
C_1 &= b r^{2b} \sum_{j=1}^{m} u_j, \\
C_2 &= \sqrt{2} b r^b \int_0^{+\infty} (\ln \mathcal{H}_1(t; \vec{u}, \vec{\mathfrak{s}}) + \ln \mathcal{H}_2(t; \vec{u}, \vec{\mathfrak{s}}))dt, \\
C_3 &= -\Big(\frac{1}{2} + \alpha\Big) \sum_{j=1}^{m} u_j + 4b \int_0^{+\infty} t(\ln \mathcal{H}_1(t; \vec{u}, \vec{\mathfrak{s}}) - \ln \mathcal{H}_2(t; \vec{u}, \vec{\mathfrak{s}}))dt \\
&\quad + \sqrt{2} b \int_{-\infty}^{+\infty} \mathcal{G}_1(t; \vec{u}, \vec{\mathfrak{s}})dt, \\
C_4 &= \frac{6\sqrt{2}b}{r^b} \int_0^{+\infty} t^2(\ln \mathcal{H}_1(t; \vec{u}, \vec{\mathfrak{s}}) + \ln \mathcal{H}_2(t; \vec{u}, \vec{\mathfrak{s}}))dt \\
&\quad + \frac{b}{r^b} \int_{-\infty}^{+\infty} \Big(4t\mathcal{G}_1(t; \vec{u}, \vec{\mathfrak{s}}) - \frac{\mathcal{G}_1(t; \vec{u}, \vec{\mathfrak{s}})^2}{\sqrt{2}} + \mathcal{G}_2(t; \vec{u}, \vec{\mathfrak{s}})\Big)dt,
\end{aligned}$$

where

$$\mathcal{H}_1(t; \vec{u}, \vec{\mathfrak{s}}) := 1 + \sum_{\ell=1}^{m} \frac{e^{u_\ell} - 1}{2} \exp\Big[\sum_{j=\ell+1}^{m} u_j\Big] \operatorname{erfc}(t - \mathfrak{s}_\ell),$$

$$\mathcal{H}_2(t;\vec{u},\vec{\mathfrak{s}}) := 1 + \sum_{\ell=1}^{m} \frac{e^{-u_\ell}-1}{2} \exp\Big[-\sum_{j=1}^{\ell-1} u_j\Big] \operatorname{erfc}(t+\mathfrak{s}_\ell),$$

and

$$\begin{aligned}&\mathcal{G}_1(t;\vec{u},\vec{\mathfrak{s}})\\ &\quad := \frac{1}{\mathcal{H}_1(t;\vec{u},\vec{\mathfrak{s}})} \sum_{\ell=1}^{m} (e^{u_\ell}-1) \exp\Big[\sum_{j=\ell+1}^{m} u_j\Big] \frac{e^{-(t-\mathfrak{s}_\ell)^2}}{\sqrt{2\pi}} \frac{1-2\mathfrak{s}_\ell^2+t\mathfrak{s}_\ell-5t^2}{3},\\ &\mathcal{G}_2(t;\vec{u},\vec{\mathfrak{s}})\\ &\quad := \frac{1}{\mathcal{H}_1(t;\vec{u},\vec{\mathfrak{s}})} \sum_{\ell=1}^{m} (e^{u_\ell}-1) \exp\Big[\sum_{j=\ell+1}^{m} u_j\Big] \frac{e^{-(t-\mathfrak{s}_\ell)^2}}{18\sqrt{2\pi}} \mathcal{M}_3,\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}_3 := {}& 50t^5 - 70t^4\mathfrak{s}_\ell - t^3(73-62\mathfrak{s}_\ell^2) + t^2\mathfrak{s}_\ell(33-50\mathfrak{s}_\ell^2) - t(3+18\mathfrak{s}_\ell^2-16\mathfrak{s}_\ell^4)\\ &- \mathfrak{s}_\ell(3-22\mathfrak{s}_\ell^2+8\mathfrak{s}_\ell^4).\end{aligned}$$

In particular, since $\mathbb{E}[\prod_{j=1}^{m} e^{u_j \mathrm{N}(r_j)}]$ *depends analytically on* $u_1,\dots,u_m \in \mathbb{C}$ *and is strictly positive for* $u_1,\dots,u_m \in \mathbb{R}$*, the asymptotic formula* (1.31) *together with Cauchy's formula shows that*

$$\partial_{u_1}^{k_1} \dots \partial_{u_m}^{k_m} \Big\{ \ln \mathbb{E}\Big[\prod_{j=1}^{m} e^{u_j \mathrm{N}(r_j)}\Big] - \Big(C_1 n + C_2\sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}}\Big)\Big\} = \mathcal{O}\Big(\frac{(\ln n)^2}{n}\Big),$$

as $n \to +\infty$*, for any* $k_1,\dots,k_m \in \mathbb{N}$*, and* $u_1,\dots,u_m \in \mathbb{R}$*.*

Remark 1.10. In the above expressions for C_2, C_3, C_4, the functions $\mathcal{H}_1$, $\mathcal{H}_2$ appear inside logarithms. It was proved in [31, Lemma 1.1] that one has $\mathcal{H}_1(t;\vec{u},\vec{\mathfrak{s}}) > 0$ and $\mathcal{H}_2(t;\vec{u},\vec{\mathfrak{s}}) > 0$ for all $t \in \mathbb{R}$, $\vec{u} = (u_1,\dots,u_m) \in \mathbb{R}^m$ and $\mathfrak{s}_1 < \dots < \mathfrak{s}_m$. This ensures that C_2, C_3, C_4 are well defined and real valued for $\vec{u} = (u_1,\dots,u_m) \in \mathbb{R}^m$, $\mathfrak{s}_1 < \dots < \mathfrak{s}_m$.

In a similar way as in Sections 1.3 and 1.4, one could derive from Theorem 1.9 asymptotic formulas for the joint cumulants of $\mathrm{N}(r_1),\dots,\mathrm{N}(r_m)$ in the bulk regime. For example, with r_ℓ as in (1.30), i.e. $r_\ell = r(1+\frac{\sqrt{2}\mathfrak{s}_\ell}{r^b\sqrt{n}})^{\frac{1}{2b}}$ with $\mathfrak{s}_\ell \in \mathbb{R}$, we have

$$\mathbb{E}[\mathrm{N}(r_\ell)] = br^{2b}n + \sqrt{2}br^b\mathfrak{s}_\ell\sqrt{n} + \frac{b-1-2\alpha}{2} + \mathcal{O}\Big(\frac{(\ln n)^2}{n}\Big), \qquad \text{as } n \to +\infty. \tag{1.32}$$

We do not write down the formulas for the other cumulants as they are identical to the corresponding formulas in [31, Corollary 1.5].

It is interesting to compare (1.32) with the corresponding formula for the semi-hard edge regime of Corollary 1.8. To ease the comparison, it is convenient to replace $\mathfrak{s}_\ell$ by $-\mathfrak{s}_\ell$ in (1.12), i.e., here we take $r_\ell = \rho\big(1 + \frac{\sqrt{2}\mathfrak{s}_\ell}{\rho^b\sqrt{n}}\big)^{\frac{1}{2b}}$ with $\mathfrak{s}_\ell < 0$. Then it follows from Corollary 1.8 that

$$\mathbb{E}[\mathrm{N}(r_\ell)] = b\rho^{2b}n + c_1(-\mathfrak{s}_\ell)\sqrt{n} + d_1(-\mathfrak{s}_\ell) + \mathcal{O}(n^{-\frac{1}{2}}), \quad \text{as } n \to +\infty. \quad (1.33)$$

Furthermore, by a long but direct analysis, we obtain as $\mathfrak{s}_\ell \to -\infty$ that

$$c_1(-\mathfrak{s}_\ell) = \sqrt{2}b\rho^b\mathfrak{s}_\ell + \mathcal{O}(e^{-c\mathfrak{s}_\ell^2}), \quad d_1(-\mathfrak{s}_\ell) = \frac{b-1-2\alpha}{2} + \mathcal{O}(e^{-c\mathfrak{s}_\ell^2}), \quad (1.34)$$

for a small but fixed $c > 0$. Recall that the asymptotic formula (1.33) is proved for fixed $s_\ell < 0$. However, if we formally replace $c_1(-\mathfrak{s}_\ell)$ by $\sqrt{2}b\rho^b\mathfrak{s}_\ell$ and $d_1(-\mathfrak{s}_\ell)$ by $\frac{b-1-2\alpha}{2}$ in (1.33), then the terms of order $\sqrt{n}$ and 1 in (1.32) and (1.34) are identical. Thus, the above computation suggests that (i) the asymptotic formula (1.33) probably holds as $n \to +\infty$ and simultaneously as $\mathfrak{s}_\ell \to -\infty$ at a sufficiently slow speed, and (ii) that the transition between the semi-hard edge regime and the bulk regime does not contain an intermediate regime.

Outline of proof. Relying on the determinantal structure of (1.6), we can rewrite $\mathbb{E}\big[\prod_{\ell=1}^m e^{u_\ell \mathrm{N}(r_\ell)}\big]$ as a ratio of two determinants using e.g. [76, Lemma 2.1] or [28, Lemma 1.9] (see also [22]),

$$\begin{aligned}\mathbb{E}\Big[\prod_{\ell=1}^m e^{u_\ell \mathrm{N}(r_\ell)}\Big] &= \frac{1}{n!\mathcal{Z}_n}\int_{\mathbb{C}}\cdots\int_{\mathbb{C}} \prod_{1\le j<k\le n}|z_k - z_j|^2 \prod_{j=1}^n w(z_j)d^2z_j \\ &= \frac{1}{\mathcal{Z}_n}\det\Big(\int_{\mathbb{C}} z^j\bar{z}^k w(z)\,d^2z\Big)_{j,k=0}^{n-1} \\ &= \frac{1}{\mathcal{Z}_n}(2\pi)^n\prod_{j=0}^{n-1}\int_0^\rho u^{2j+1}w(u)\,du, \end{aligned} \quad (1.35)$$

where

$$w(z) := |z|^{2\alpha}e^{-n|z|^{2b}}\omega(|z|), \quad \omega(x) := \prod_{\ell=1}^m \begin{cases} e^{u_\ell} & \text{if } x < r_\ell, \\ 1 & \text{if } x \ge r_\ell. \end{cases} \quad (1.36)$$

For $x < \rho$, let us write

$$\omega(x) = \sum_{\ell=1}^{m+1}\omega_\ell \mathbf{1}_{[0,r_\ell)}(x), \quad \omega_\ell := \begin{cases} e^{u_\ell+\cdots+u_m} - e^{u_{\ell+1}+\cdots+u_m} & \text{if } \ell < m, \\ e^{u_m} - 1 & \text{if } \ell = m, \\ 1 & \text{if } \ell = m+1, \end{cases} \quad (1.37)$$

where $r_{m+1} := \rho$. Note also that $\Omega := e^{u_1+\cdots+u_m} = \sum_{j=1}^{m+1} \omega_j$. By (1.36)–(1.37),

$$\begin{aligned}
&\int_0^\rho u^{2j+1} w(u)du \\
&\quad = \int_0^\rho u^{2j+1} u^{2\alpha} e^{-nu^{2b}} du + \sum_{\ell=1}^m \omega_\ell \int_0^{r_\ell} u^{2j+1} u^{2\alpha} e^{-nu^{2b}} du \\
&\quad = \int_0^{n\rho^{2b}} \Big(\frac{y}{n}\Big)^{\frac{j+1+\alpha}{b}} \frac{e^{-y}}{2by} dy + \sum_{\ell=1}^m \omega_\ell \int_0^{nr_\ell^{2b}} \Big(\frac{y}{n}\Big)^{\frac{j+1+\alpha}{b}} \frac{e^{-y}}{2by} dy \\
&\quad = \frac{n^{-\frac{j+1+\alpha}{b}}}{2b} \Big(\gamma\Big(\frac{j+1+\alpha}{b}, n\rho^{2b}\Big) + \sum_{\ell=1}^m \omega_\ell \gamma\Big(\frac{j+1+\alpha}{b}, nr_\ell^{2b}\Big)\Big),
\end{aligned}$$

where $\gamma(a,z)$ is the incomplete gamma function

$$\gamma(a,z) = \int_0^z t^{a-1} e^{-t} dt.$$

Hence,

$$\begin{aligned}
&(2\pi)^n \prod_{j=0}^{n-1} \int_0^\rho u^{2j+1} w(u)du \\
&\quad = n^{-\frac{n^2}{2b}} n^{-\frac{1+2\alpha}{2b}n} \frac{\pi^n}{b^n} \prod_{j=1}^n \Big(\gamma\Big(\frac{j+\alpha}{b}, n\rho^{2b}\Big) + \sum_{\ell=1}^m \omega_\ell \gamma\Big(\frac{j+\alpha}{b}, nr_\ell^{2b}\Big)\Big).
\end{aligned}$$

An expression for $\mathcal{Z}_n$ in terms of γ can be found by setting $\omega_1 = \cdots = \omega_m = 0$ above:

$$\mathcal{Z}_n = n^{-\frac{n^2}{2b}} n^{-\frac{1+2\alpha}{2b}n} \frac{\pi^n}{b^n} \prod_{j=1}^n \gamma\Big(\frac{j+\alpha}{b}, n\rho^{2b}\Big),$$

and therefore, by (1.35),

$$\ln \mathcal{E}_n = \sum_{j=1}^n \ln\Big(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma\big(\frac{j+\alpha}{b}, nr_\ell^{2b}\big)}{\gamma\big(\frac{j+\alpha}{b}, n\rho^{2b}\big)}\Big), \tag{1.38}$$

where $\mathcal{E}_n := \mathbb{E}[\prod_{\ell=1}^m e^{u_\ell \mathrm{N}(r_\ell)}]$. The above formula is the starting point of the proofs of Theorems 1.3, 1.7 and 1.9. We infer from (1.38) that, to obtain the large n asymptotics of $\mathcal{E}_n$, we need the asymptotics of $\gamma(a,z)$ as a, z tend to $+\infty$ at various relative speeds. The uniform asymptotics of γ are actually well known, and we recall them in Appendix A.

The approach considered here shows similarities with [21, 28, 29, 31]. The large n behavior of $\gamma(\frac{j+\alpha}{b}, n\rho^{2b})$ depends crucially on whether $\frac{j+\alpha}{b} \ll n\rho^{2b}$, $\frac{j+\alpha}{b} \approx n\rho^{2b}$ or $\frac{j+\alpha}{b} \gg n\rho^{2b}$. Hence, for the proofs of both Theorem 1.3 and Theorem 1.7, we will split the sum in (1.38) into four parts,

$$\ln \mathcal{E}_n = S_0 + S_1 + S_2 + S_3,$$

where $S_0, \ldots, S_3$ are defined in (2.4)–(2.5). The sum S_0 involves a large but fixed number of j's; the sum S_1 corresponds to those j's that are "large" and for which $\frac{j+\alpha}{b} \ll n\rho^{2b}$; and the sum S_3 involves the j's for which $\frac{j+\alpha}{b} \gg n\rho^{2b}$. For both theorems, the most delicate sum is S_2: this sum involves the j-terms in (1.38) for which $\frac{j+\alpha}{b} \approx n\rho^{2b}$, and therefore critical transitions occur in the asymptotic behavior of the functions $\{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})\}_{\ell=1}^m$ and $\gamma(\frac{j+\alpha}{b}, n\rho^{2b})$ when performing the sum S_2.

For the two novel regimes considered in this work, namely the hard edge regime (1.11) and the semi-hard edge regime (1.12), the proofs require precise Riemann sum approximations for functions with singularities (the singularities are more difficult to handle in the hard edge regime). In comparison, the bulk regime of Theorem 1.9 (whose proof is omitted here as it is essentially identical to [31]) is simpler as the corresponding Riemann sum approximations involve more well-behaved functions.

Related works. By (1.35)–(1.36), we have $\mathcal{E}_n = D_n/\mathcal{Z}_n$ where D_n is an $n \times n$ determinant with a rotation-invariant weight supported on $\mathbb{C}$ and with m merging discontinuities: for Theorem 1.3, the discontinuities are merging near the hard edge at speed $1/n$; for Theorem 1.7, the discontinuities are merging near the hard edge at speed $1/\sqrt{n}$; and for Theorem 1.9, the discontinuities are merging in the bulk at speed $1/\sqrt{n}$.

The problem of determining asymptotics of structured determinants with discontinuities has a long history. When the weight is supported on the unit circle or on the real line, this problem was studied by many authors, including Lenard, Fisher, Hartwig, Widom, Basor, Böttcher, Silbermann, Ehrhardt, Deift, Its, and Krasovsky, see e.g. [16, 26, 39] for some historical background, [27, 30, 36, 37, 62] for structured determinants with discontinuities near a hard edge, and [33, 44] for merging discontinuities in the bulk.

A central theme in normal random matrix theories concerns the asymptotic distribution of linear statistics $\sum_1^n f(z_j)$ where f is a given test-function on the plane. The analytical situation depends crucially on whether or not f belongs to the Sobolev class $W^{1,2}$, since this is believed to be the right condition under which we obtain a well-defined limiting normal distribution (say, after subtracting the expectation). This is rigorously verified in the Ginibre case in [67] and if the test-function is C^2-smooth for more general ensembles in [9]. However, the class $W^{1,2}$ excludes certain natural test-functions, or the logarithm $l_z(w) = \ln|z-w|$ (or close relatives like Green's functions) which is used in connection with the Gaussian free field, and characteristic functions $\chi_E(z)$ which define counting statistics.

The works [25, 28, 31, 45, 57] were already mentioned earlier in the introduction and deal with determinants with discontinuities in dimension two. Determinants corresponding to the logarithmic test-function l_z, for some special ensembles, have attracted considerable attention in recent years [20, 21, 38, 76], see also e.g. [13–15, 17, 61].

2. Proof of Theorem 1.3

In this section, the r_ℓ's are as in (1.11). Our proof strategy follows [21, 28, 29, 31].

Let us define

$$j_- := \left\lceil \frac{bn\rho^{2b}}{1+\varepsilon} - \alpha \right\rceil, \quad j_+ := \left\lfloor \frac{bn\rho^{2b}}{1-\varepsilon} - \alpha \right\rfloor, \tag{2.1}$$

where $\varepsilon > 0$ is independent of n. We assume that ε is sufficiently small such that

$$\frac{b\rho^{2b}}{1-\varepsilon} < 1, \tag{2.2}$$

so that, recalling the formula (1.38) for $\ln \mathcal{E}_n$, we can write

$$\ln \mathcal{E}_n = S_0 + S_1 + S_2 + S_3, \tag{2.3}$$

where

$$S_0 = \sum_{j=1}^{M'} \ln\Big(1+\sum_{\ell=1}^{m} \omega_\ell \frac{\gamma\big(\frac{j+\alpha}{b}, nr_\ell^{2b}\big)}{\gamma\big(\frac{j+\alpha}{b}, n\rho^{2b}\big)}\Big), \qquad S_1 = \sum_{j=M'+1}^{j_--1} \ln\Big(1+\sum_{\ell=1}^{m} \omega_\ell \frac{\gamma(\frac{j+\alpha}{b}, nr_\ell^{2b})}{\gamma(\frac{j+\alpha}{b}, n\rho^{2b})}\Big), \tag{2.4}$$

$$S_2 = \sum_{j=j_-}^{j_+} \ln\Big(1+\sum_{\ell=1}^{m} \omega_\ell \frac{\gamma\big(\frac{j+\alpha}{b}, nr_\ell^{2b}\big)}{\gamma\big(\frac{j+\alpha}{b}, n\rho^{2b}\big)}\Big), \quad S_3 = \sum_{j=j_++1}^{n} \ln\Big(1+\sum_{\ell=1}^{m} \omega_\ell \frac{\gamma\big(\frac{j+\alpha}{b}, nr_\ell^{2b}\big)}{\gamma\big(\frac{j+\alpha}{b}, n\rho^{2b}\big)}\Big). \tag{2.5}$$

In the above, $M' > 0$ is an integer independent of n. For $j = 1, \ldots, n$ and $k = 1, \ldots, m$, we also define $a_j := \frac{j+\alpha}{b}$, and

$$\lambda_{j,k} := \frac{bnr_k^{2b}}{j+\alpha}, \qquad \eta_{j,k} := (\lambda_{j,k}-1)\sqrt{\frac{2(\lambda_{j,k}-1-\ln\lambda_{j,k})}{(\lambda_{j,k}-1)^2}}, \tag{2.6a}$$

$$\lambda_j := \frac{bn\rho^{2b}}{j+\alpha}, \qquad \eta_j := (\lambda_j - 1)\sqrt{\frac{2(\lambda_j - 1 - \ln\lambda_j)}{(\lambda_j - 1)^2}}. \tag{2.6b}$$

With this notation, the summand in (2.4)–(2.5) can be rewritten as

$$\ln\Big(1 + \sum_{\ell=1}^{m} \omega_\ell \frac{\gamma(a_j, a_j\lambda_{j,\ell})}{\gamma(a_j, a_j\lambda_j)}\Big).$$

The notation η_j and $\eta_{j,k}$ in (2.4)–(2.5) is introduced in the same spirit as the notation η of Lemma A.2. Recall also that $\Omega := e^{u_1+\cdots+u_m} = \sum_{j=1}^{m+1} \omega_j$.

Lemma 2.1. *For any $x_1, \ldots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_0 = M' \ln\Omega + \mathcal{O}(e^{-cn}), \quad \text{as } n \to +\infty,$$

uniformly for $u_1 \in D_\delta(x_1), \ldots, u_m \in D_\delta(x_m)$.

Proof. We infer from (2.4) and Lemma A.1 that

$$S_0 = \sum_{j=1}^{M'} \ln\Big(\sum_{\ell=1}^{m+1} \omega_\ell[1 + \mathcal{O}(e^{-cn})]\Big) = \sum_{j=1}^{M'} \ln\Omega + \mathcal{O}(e^{-cn}), \quad \text{as } n \to +\infty.$$

In the above, the error terms before the second equality are independent of $u_1, \ldots, u_m$, so the claim follows. ∎

Lemma 2.2. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \ldots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_1 = (j_- - M' - 1)\ln\Omega + \mathcal{O}(e^{-cn}),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \ldots, u_m \in D_\delta(x_m)$.

Proof. According to (2.4) and (2.6), we have

$$S_1 = \sum_{j=M'+1}^{j_- -1} \ln\Big(1 + \sum_{\ell=1}^{m} \omega_\ell \frac{\gamma(a_j, a_j\lambda_{j,\ell})}{\gamma(a_j, a_j\lambda_j)}\Big).$$

There is a $\delta > 0$ such that $\lambda_j > 1 + \delta$ and $\lambda_{j,\ell} = \lambda_j(1 - t_\ell/n) > 1 + \delta$ for all $j \in \{M'+1, \ldots, j_- - 1\}$ and $\ell \in \{1, \ldots, m\}$. Hence, by Lemma A.2 (i) we can choose M' such that

$$S_1 = \sum_{j=M'+1}^{j_- -1} \ln\Big(1 + \sum_{\ell=1}^{m} \omega_\ell \frac{1 + \mathcal{O}(e^{-\frac{a_j\eta_{j,\ell}^2}{2}})}{1 + \mathcal{O}(e^{-\frac{a_j\eta_j^2}{2}})}\Big),$$

where the error terms are uniform with respect to j and ℓ. The functions $j \mapsto a_j\eta_j^2$ and $j \mapsto a_j\eta_{j,\ell}^2$ are decreasing, because

$$\partial_j(a_j\eta_j^2) = -\frac{2}{b}\ln\lambda_j < 0, \quad \partial_j(a_j\eta_{j,\ell}^2) = -\frac{2}{b}\ln\lambda_{j,\ell} < 0.$$

Moreover, we have $a_{j_-}\eta_{j_-}^2 > 2cn$ and hence $a_{j_-}\eta_{j_-,\ell}^2 = a_{j_-}\eta_{j_-}^2 + \mathcal{O}(1) > cn$ for all sufficiently large n for some $c > 0$. It follows that

$$S_1 = \sum_{j=M'+1}^{j_--1}\ln\Big(1+\sum_{\ell=1}^{m}\omega_\ell\frac{1+\mathcal{O}(e^{-cn})}{1+\mathcal{O}(e^{-cn})}\Big) = \sum_{j=M'+1}^{j_--1}\ln\Big(1+\sum_{\ell=1}^{m}\omega_\ell\Big)+\mathcal{O}(e^{-cn}),$$

from which the desired conclusion follows. ■

To obtain the large n asymptotics of S_3, we will rely on the following lemma.

Lemma 2.3 (Adapted from [29, Lemma 3.4]). *Let $A = A(n), a_0 = a_0(n)$, $B = B(n)$, $b_0 = b_0(n)$ be bounded functions of $n \in \{1, 2, \dots\}$, such that*

$$a_n := An + a_0 \quad \textit{and} \quad b_n := Bn + b_0$$

are integers. Assume also that $B - A$ is positive and remains bounded away from 0. Let f be a function independent of n, which is $C^2([\min\{\frac{a_n}{n}, A\}, \max\{\frac{b_n}{n}, B\}])$ for all $n \in \{1, 2, \dots\}$. Then as $n \to +\infty$, we have

$$\sum_{j=a_n}^{b_n} f\Big(\frac{j}{n}\Big) = n\int_A^B f(x)dx + \frac{(1-2a_0)f(A)+(1+2b_0)f(B)}{2} + \mathcal{O}\Big(\frac{\mathfrak{m}_{A,n}(f')+\mathfrak{m}_{B,n}(f')}{n} + \sum_{j=a_n}^{b_n-1}\frac{\mathfrak{m}_{j,n}(f'')}{n^2}\Big),$$

where, for a given function g continuous on $\big[\min\{\frac{a_n}{n}, A\}, \max\{\frac{b_n}{n}, B\}\big]$,

$$\mathfrak{m}_{A,n}(g) := \max_{x\in[\min\{\frac{a_n}{n},A\},\max\{\frac{a_n}{n},A\}]}|g(x)|,$$

$$\mathfrak{m}_{B,n}(g) := \max_{x\in[\min\{\frac{b_n}{n},B\},\max\{\frac{b_n}{n},B\}]}|g(x)|,$$

and for $j \in \{a_n, \dots, b_n - 1\}$, $\mathfrak{m}_{j,n}(g) := \max_{x\in[\frac{j}{n},\frac{j+1}{n}]}|g(x)|$.

Following the approach of [28, 29], we define

$$\theta_+^{(n,\varepsilon)} = \Big(\frac{bn\rho^{2b}}{1-\varepsilon}-\alpha\Big) - \Big\lfloor\frac{bn\rho^{2b}}{1-\varepsilon}-\alpha\Big\rfloor, \quad \theta_-^{(n,\varepsilon)} = \Big\lceil\frac{bn\rho^{2b}}{1+\varepsilon}-\alpha\Big\rceil - \Big(\frac{bn\rho^{2b}}{1+\varepsilon}-\alpha\Big).$$

Lemma 2.4. *For any $x_1, \ldots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_3 = n \int_{\frac{b\rho^{2b}}{1-\varepsilon}}^{1} f_1(x)\,dx + \int_{\frac{b\rho^{2b}}{1-\varepsilon}}^{1} f(x)\,dx + \Big(\alpha + \theta_+^{(n,\varepsilon)} - \frac{1}{2}\Big) f_1\Big(\frac{b\rho^{2b}}{1-\varepsilon}\Big) + \frac{1}{2} f_1(1) + \mathcal{O}(n^{-1}),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \ldots, u_m \in D_\delta(x_m)$, where

$$f_1(x) := \ln(1 + \mathsf{T}_0(x))$$

and f and T_j are defined in (1.17) *and* (1.18).

Proof. Recall that a_j, λ_j, $\lambda_{j,\ell}$, η_j and $\eta_{j,\ell}$ are defined in (2.6). By (2.5), we have

$$S_3 = \sum_{j=j_++1}^{n} \ln(1 + X_j), \quad \text{where } X_j := \frac{\sum_{\ell=1}^m \omega_\ell \gamma(a_j, a_j\lambda_{j,\ell})}{\gamma(a_j, a_j\lambda_j)}. \tag{2.7}$$

For $j \geq j_+ + 1$ and $k \in \{1, \ldots, m\}$, $1 - \lambda_{j,k}$ and $1 - \lambda_j$ are positive and bounded away from 0. Hence, using Lemma A.4 (ii), we obtain

$$\begin{aligned} X_j &= \frac{\sum_{\ell=1}^m \omega_\ell \frac{e^{-\frac{a_j}{2}\eta_{j,\ell}^2}}{\sqrt{2\pi}} \Big\{\sum_{k=0}^1 \frac{S(\varphi_k(\lambda_{j,\ell}))}{a_j^{k+1/2}} + \mathcal{O}\big(\frac{1}{a_j^{5/2}}\big) + \mathcal{O}\big(\frac{1}{(a_j\eta_{j,\ell}^2)^{5/2}}\big)\Big\}}{\frac{e^{-\frac{a_j}{2}\eta_j^2}}{\sqrt{2\pi}} \Big\{\sum_{k=0}^1 \frac{S(\varphi_k(\lambda_j))}{a_j^{k+1/2}} + \mathcal{O}\big(\frac{1}{a_j^{5/2}}\big) + \mathcal{O}\big(\frac{1}{(a_j\eta_j^2)^{5/2}}\big)\Big\}} \\ &= \sum_{\ell=1}^m \omega_\ell \frac{e^{-\frac{a_j\eta_{j,\ell}^2}{2}} \Big(\frac{-1}{\lambda_{j,\ell}-1}\frac{1}{\sqrt{a_j}} + \frac{1+10\lambda_{j,\ell}+\lambda_{j,\ell}^2}{12(\lambda_{j,\ell}-1)^3}\frac{1}{a_j^{3/2}} + \mathcal{O}(n^{-5/2})\Big)}{e^{-\frac{a_j\eta_j^2}{2}} \Big(\frac{-1}{\lambda_j-1}\frac{1}{\sqrt{a_j}} + \frac{1+10\lambda_j+\lambda_j^2}{12(\lambda_j-1)^3}\frac{1}{a_j^{3/2}} + \mathcal{O}(n^{-5/2})\Big)}, \end{aligned} \tag{2.8}$$

where the above $\mathcal{O}$-terms are uniform for $j \in \{j_+ + 1, \ldots, n\}$. Let $x := j/n$. As $n \to +\infty$ we have

$$x \in \Big[\frac{b\rho^{2b}}{1-\varepsilon} + \mathcal{O}(n^{-1}), 1\Big], \quad a_j = \frac{nx}{b} + \mathcal{O}(1),$$

uniformly for $j_+ + 1 \leq j \leq n$. Thus, multiplying both the numerator and denominator on the right-hand side of (2.8) by $-a_j^{1/2}(\lambda_j - 1)$, we get

$$X_j = \sum_{\ell=1}^m \omega_\ell e^{-\frac{a_j}{2}(\eta_{j,\ell}^2 - \eta_j^2)} Y_{j,\ell}, \tag{2.9}$$

where

$$Y_{j,\ell} := \frac{\frac{\lambda_j - 1}{\lambda_{j,\ell}-1} - (\lambda_j - 1)\frac{1+10\lambda_{j,\ell}+\lambda_{j,\ell}^2}{12(\lambda_{j,\ell}-1)^3}\frac{1}{a_j} + \mathcal{O}(n^{-2})}{1 - (\lambda_j - 1)\frac{1+10\lambda_j+\lambda_j^2}{12(\lambda_j-1)^3}\frac{1}{a_j} + \mathcal{O}(n^{-2})},$$

and where the above $\mathcal{O}$-terms are uniform for $j \in \{j_+ + 1, \ldots, n\}$. Using that $a_j = \frac{nx+\alpha}{b}$, we get

$$e^{-\frac{a_j}{2}(\eta_{j,\ell}^2 - \eta_j^2)} = e^{a_j \ln(1-\frac{t_\ell}{n}) + a_j \frac{b\rho^{2b}t_\ell}{nx+\alpha}} = e^{-\frac{t_\ell}{b}(x - b\rho^{2b})}\Big(1 - \frac{t_\ell^2 x + 2t_\ell \alpha}{2bn} + \mathcal{O}\Big(\frac{1}{n^2}\Big)\Big),$$

$$\lambda_{j,\ell} = \frac{b\rho^{2b}}{x}\Big(1 - \frac{\alpha + xt_\ell}{xn} + \frac{\alpha(\alpha + xt_\ell)}{x^2n^2} + \mathcal{O}\Big(\frac{1}{n^3}\Big)\Big),$$

$$\lambda_j = \frac{b\rho^{2b}}{x}\Big(1 - \frac{\alpha}{xn} + \frac{\alpha^2}{x^2n^2} + \mathcal{O}\Big(\frac{1}{n^3}\Big)\Big),$$

uniformly for $j_+ + 1 \leq j \leq n$. Substituting these expansions into the expression for $Y_{j,\ell}$ in (2.9), a calculation gives $\ln(1 + X_j) = f_1(j/n) + \frac{1}{n} f(j/n) + \mathcal{O}(n^{-2})$ as $n \to \infty$ uniformly for $j_+ + 1 \leq j \leq n$. In view of (2.7), we thus have

$$S_3 = \sum_{j=j_++1}^{n} \Big(f_1\Big(\frac{j}{n}\Big) + \frac{1}{n} f\Big(\frac{j}{n}\Big) + \mathcal{O}(n^{-2})\Big), \quad \text{as } n \to +\infty.$$

The claim then follows after a computation using Lemma 2.3 (with $A = \frac{b\rho^{2b}}{1-\varepsilon}$, $a_0 = 1 - \alpha - \theta_+^{(n,\varepsilon)}$, $B = 1$ and $b_0 = 0$). ■

We now focus on S_2. Let $M := n^{\frac{1}{10}}$. We split S_2 in three pieces as follows:

$$S_2 = S_2^{(1)} + S_2^{(2)} + S_2^{(3)}, \tag{2.10}$$

where

$$S_2^{(v)} := \sum_{j:\lambda_j \in I_v} \ln\Big(1 + \sum_{\ell=1}^{m} \omega_\ell \frac{\gamma(a_j, a_j\lambda_{j,\ell})}{\gamma(a_j, a_j\lambda_j)}\Big), \quad v = 1, 2, 3,$$

and where

$$I_1 = \Big[1 - \varepsilon, 1 - \frac{M}{\sqrt{n}}\Big), \quad I_2 = \Big[1 - \frac{M}{\sqrt{n}}, 1 + \frac{M}{\sqrt{n}}\Big], \quad I_3 = \Big(1 + \frac{M}{\sqrt{n}}, 1 + \varepsilon\Big]. \tag{2.11}$$

From (2.10), we see that the large n asymptotics of $\{S_2^{(v)}\}_{v=1,2,3}$ involve the asymptotics of $\gamma(a, z)$ when $a \to +\infty$, $z \to +\infty$ with $\lambda = \frac{z}{a} \in [1 - \varepsilon, 1 + \varepsilon]$. These sums can also be rewritten using

$$\sum_{j:\lambda_j \in I_3} = \sum_{j=j_-}^{g_- - 1}, \qquad \sum_{j:\lambda_j \in I_2} = \sum_{j=g_-}^{g_+}, \qquad \sum_{j:\lambda_j \in I_1} = \sum_{j=g_+ + 1}^{j_+},$$

where $g_- := \lceil \frac{bn\rho^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha \rceil$ and $g_+ := \lfloor \frac{bn\rho^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha \rfloor$. Let us also define

$$\theta_-^{(n,M)} := g_- - \Big(\frac{bn\rho^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha\Big) = \Big\lceil \frac{bn\rho^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha \Big\rceil - \Big(\frac{bn\rho^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha\Big),$$
$$\theta_+^{(n,M)} := \Big(\frac{bn\rho^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha\Big) - g_+ = \Big(\frac{bn\rho^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha\Big) - \Big\lfloor \frac{bn\rho^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha \Big\rfloor.$$

Clearly, $\theta_-^{(n,M)}, \theta_+^{(n,M)} \in [0,1)$. Note that the individual sums $S_2^{(1)}, S_2^{(2)}, S_2^{(3)}$ depend on M, although $S_2 = S_2^{(1)} + S_2^{(2)} + S_2^{(3)}$ is independent of M. Below, we will first obtain large n asymptotics of $S_2^{(1)}, S_2^{(2)}, S_2^{(3)}$. After adding the asymptotic formulas of $S_2^{(1)}, S_2^{(2)}, S_2^{(3)}$, we will find that all M-dependent terms cancel, as they must. For this reason, below we will not replace M by $n^{1/10}$ until the last step of the proof. The reason why we choose $M = n^{1/10}$ is technical. In the various asymptotic formulas below, there will be different types of error terms, such as $\mathcal{O}(\frac{M^4}{\sqrt{n}})$, $\mathcal{O}(\frac{\sqrt{n}}{M^{11}})$, etc., and in the last step of the proof we will find that $M = n^{1/10}$ is the choice that produces the best control over the total error.

Lemma 2.5. *For any $x_1, \ldots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2^{(3)} = (b\rho^{2b}n - j_- - bM\rho^{2b}\sqrt{n} + bM^2\rho^{2b} - \alpha + \theta_-^{(n,M)} - bM^3\rho^{2b}n^{-\frac{1}{2}})\ln\Omega + \mathcal{O}(M^4 n^{-1}),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \ldots, u_m \in D_\delta(x_m)$.

Proof. Recall that $a_j, \lambda_j, \lambda_{j,k}, \eta_j, \eta_{j,k}$ are defined in (2.6). By (2.10), we have

$$S_2^{(3)} = \sum_{j:\lambda_j \in I_3} \ln\Big(1 + \sum_{\ell=1}^m \omega_\ell \frac{\gamma(a_j, a_j\lambda_{j,\ell})}{\gamma(a_j, a_j\lambda_j)}\Big).$$

If $\lambda_j \in I_3$, then $\lambda_j > 1 + \frac{M}{\sqrt{n}}$ and $\lambda_{j,\ell} = \lambda_j(1 - \frac{t_\ell}{n}) > 1 + \frac{M}{\sqrt{n}} + \mathcal{O}(n^{-1})$. So, there exists a constant $c > 0$ such that

$$\eta_j \geq c\frac{M}{\sqrt{n}}, \quad -\eta_j\sqrt{\frac{a_j}{2}} \leq -cM, \quad \eta_{j,\ell} \geq c\frac{M}{\sqrt{n}}, \quad -\eta_{j,\ell}\sqrt{\frac{a_j}{2}} \leq -cM,$$

for all sufficiently large n, $\ell \in \{1, \dots, m\}$ and $j \in \{j : \lambda_j \in I_3\}$. By Lemma A.4 (i),

$$S_2^{(3)} = \sum_{j:\lambda_j \in I_3} \ln\Big(1 + \sum_{\ell=1}^{m} \omega_\ell \frac{1 + \mathcal{O}(e^{-\frac{a_j \eta_{j,\ell}^2}{2}})}{1 + \mathcal{O}(e^{-\frac{a_j \eta_j^2}{2}})}\Big) = \sum_{j=j_-}^{g_- -1} \ln \Omega + \mathcal{O}(e^{-c^2 M^2})$$
$$= (g_- - j_-) \ln \Omega + \mathcal{O}(e^{-c^2 M^2})$$

as $n \to +\infty$. Since

$$g_- - j_- = \Big(\frac{bn\rho^{2b}}{1 + \frac{M}{\sqrt{n}}} - \alpha\Big) + \theta_-^{(n,M)} - j_-$$
$$= b\rho^{2b} n - j_- - bM\rho^{2b}\sqrt{n} + bM^2\rho^{2b} - \alpha + \theta_-^{(n,M)} - bM^3\rho^{2b} n^{-\frac{1}{2}} + \mathcal{O}(M^4 n^{-1})$$

as $n \to +\infty$, the desired conclusion follows. ■

Lemma 2.6. *For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2^{(1)} = D_1^{(\varepsilon)} n + D_2^{(M)} \sqrt{n} + D_3 \ln n + D_4^{(n,\varepsilon,M)} + \frac{D_5^{(n,M)}}{\sqrt{n}} + \mathcal{O}\Big(\frac{M^4}{n} + \frac{1}{\sqrt{n}M} + \frac{1}{M^6} + \frac{\sqrt{n}}{M^{11}}\Big),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$, where

$$D_1^{(\varepsilon)} = \int_{b\rho^{2b}}^{\frac{b\rho^{2b}}{1-\varepsilon}} f_1(x)\, dx,$$
$$D_2^{(M)} = -b\rho^{2b} f_1(b\rho^{2b}) M,$$
$$D_3 = -\frac{b\rho^{2b} \mathsf{T}_1(b\rho^{2b})}{2(1 + \mathsf{T}_0(b\rho^{2b}))},$$
$$D_4^{(n,\varepsilon,M)} = -b\rho^{2b} M^2 \Big(f_1(b\rho^{2b}) + \frac{b\rho^{2b}}{2} f_1'(b\rho^{2b})\Big) - \frac{b\rho^{2b}\mathsf{T}_1(b\rho^{2b})}{1 + \mathsf{T}_0(b\rho^{2b})} \ln\Big(\frac{\varepsilon}{M(1-\varepsilon)}\Big)$$
$$+ \int_{b\rho^{2b}}^{\frac{b\rho^{2b}}{1-\varepsilon}} \Big\{ f(x) + \frac{b\rho^{2b}\mathsf{T}_1(b\rho^{2b})}{(1 + \mathsf{T}_0(b\rho^{2b}))(x - b\rho^{2b})} \Big\} dx$$
$$+ \Big(\alpha - \frac{1}{2} + \theta_+^{(n,M)}\Big) f_1(b\rho^{2b})$$

$$+\Big(\frac{1}{2}-\alpha-\theta_+^{(n,\varepsilon)}\Big)f_1\Big(\frac{b\rho^{2b}}{1-\varepsilon}\Big)$$
$$+\frac{b\mathsf{T}_1(b\rho^{2b})}{M^2(1+\mathsf{T}_0(b\rho^{2b}))}+\frac{-5b\mathsf{T}_1(b\rho^{2b})}{2\rho^{2b}M^4(1+\mathsf{T}_0(b\rho^{2b}))},$$

$$D_5^{(n,M)} = -M^3 b\rho^{2b}\Big(f_1(b\rho^{2b})+b\rho^{2b}f_1'(b\rho^{2b})+\frac{(b\rho^{2b})^2}{6}f_1''(b\rho^{2b})\Big)$$
$$+Mb\rho^{2b}f_1'(b\rho^{2b})\Big(\alpha-\frac{1}{2}+\theta_+^{(n,M)}\Big)$$
$$+M\Big(\frac{(b+\alpha)\rho^{2b}\mathsf{T}_1(b\rho^{2b})}{1+\mathsf{T}_0(b\rho^{2b})}-\frac{b\rho^{4b}\mathsf{T}_2(b\rho^{2b})}{2(1+\mathsf{T}_0(b\rho^{2b}))}+\frac{b\rho^{4b}\mathsf{T}_1(b\rho^{2b})^2}{(1+\mathsf{T}_0(b\rho^{2b}))^2}\Big),$$

where f_1 and f are as in the statement of Lemma 2.4.

Proof. We have

$$S_2^{(1)}=\sum_{j=g_++1}^{j_+}\ln(1+X_j),\quad \text{where } X_j:=\frac{\sum_{\ell=1}^m \omega_\ell\gamma(a_j,a_j\lambda_{j,\ell})}{\gamma(a_j,a_j\lambda_j)}. \tag{2.12}$$

Since $\lambda_j\in\big[1-\varepsilon,1-\frac{M}{\sqrt{n}}\big)$ for $g_++1\le j\le j_+$ and $\lambda_{j,\ell}=\lambda_j(1-\frac{t_\ell}{n})$, we can apply Lemma A.4 (ii) to find, for each $N\ge 0$,

$$X_j=\frac{\sum_{\ell=1}^m\omega_\ell\frac{e^{-\frac{a_j}{2}\eta_{j,\ell}^2}}{\sqrt{2\pi}}\Big\{\sum_{k=0}^{N-1}\frac{S(\varphi_k(\lambda_{j,\ell}))}{a_j^{k+1/2}}+\mathcal{O}\Big(\frac{1}{a_j^{N+1/2}}\Big)+\mathcal{O}\Big(\frac{1}{(a_j\eta_{j,\ell}^2)^{N+1/2}}\Big)\Big\}}{\frac{e^{-\frac{a_j}{2}\eta_j^2}}{\sqrt{2\pi}}\Big\{\sum_{k=0}^{N-1}\frac{S(\varphi_k(\lambda_j))}{a_j^{k+1/2}}+\mathcal{O}\Big(\frac{1}{a_j^{N+1/2}}\Big)+\mathcal{O}\Big(\frac{1}{(a_j\eta_j^2)^{N+1/2}}\Big)\Big\}}. \tag{2.13}$$

Let $x:=j/n$. For all sufficiently large n we have $\eta_j\asymp\lambda_j-1$,[2] $\eta_{j,\ell}\asymp\lambda_{j,\ell}-1\asymp\lambda_j-1$, and

$$x\in\Big[\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}+\mathcal{O}(n^{-1}),\frac{b\rho^{2b}}{1-\varepsilon}+\mathcal{O}(n^{-1})\Big],\quad a_j=\frac{xn}{b}+\mathcal{O}(1),$$

uniformly for $g_++1\le j\le j_+$. Thus, multiplying both the numerator and denominator on the right-hand side of (2.13) by $-a_j^{1/2}(\lambda_j-1)$ and using that $S(\varphi_0(\lambda))=-\frac{1}{\lambda-1}$, we find

$$X_j=\sum_{\ell=1}^m\omega_\ell e^{-\frac{a_j}{2}(\eta_{j,\ell}^2-\eta_j^2)}Y_{j,\ell}, \tag{2.14}$$

[2] More precisely, this means that η_j and λ_j-1 are of the same order in the sense that there exist constants $c_1,c_2>0$ such that $c_1\le\eta_j/(\lambda_j-1)\le c_2$ for all sufficiently large n and all $g_++1\le j\le j_+$.

$$Y_{j,\ell} := \frac{\frac{\lambda_j - 1}{\lambda_{j,\ell}-1} - (\lambda_j - 1)\sum_{k=1}^{N-1} \frac{S(\varphi_k(\lambda_{j,\ell}))}{a_j^k} + \mathcal{O}\big(\frac{1}{(n(\lambda_j-1)^2)^N}\big)}{1 - (\lambda_j - 1)\sum_{k=1}^{N-1} \frac{S(\varphi_k(\lambda_j))}{a_j^k} + \mathcal{O}\big(\frac{1}{(n(\lambda_j-1)^2)^N}\big)}.$$

Using that $a_j = \frac{xn+\alpha}{b}$, we can expand the exponential as $n \to +\infty$:

$$\begin{aligned} e^{-\frac{a_j}{2}(\eta_{j,\ell}^2 - \eta_j^2)} &= e^{a_j \ln(1-\frac{t_\ell}{n}) + a_j \frac{b\rho^{2b} t_\ell}{nx+\alpha}} \\ &= e^{-\frac{t_\ell}{b}(x - b\rho^{2b})}\Big(1 - \frac{t_\ell^2 x + 2t_\ell \alpha}{2bn} + \mathcal{O}\Big(\frac{1}{n^2}\Big)\Big) \end{aligned} \tag{2.15}$$

uniformly for $g_+ + 1 \leq j \leq j_+$. On the other hand, as $n \to +\infty$,

$$\begin{aligned} \lambda_{j,\ell} &= \frac{b\rho^{2b}}{x}\Big(1 - \frac{\alpha + xt_\ell}{xn} + \frac{\alpha(\alpha + xt_\ell)}{x^2n^2} + \mathcal{O}\Big(\frac{1}{n^3}\Big)\Big), \\ \lambda_j &= \frac{b\rho^{2b}}{x}\Big(1 - \frac{\alpha}{xn} + \frac{\alpha^2}{x^2n^2} + \mathcal{O}\Big(\frac{1}{n^3}\Big)\Big), \end{aligned}$$

uniformly for $g_+ + 1 \leq j \leq j_+$. Substituting these expansions into the expression for $Y_{j,\ell}$ in (2.14) with $N = 6$, a calculation gives

$$\begin{aligned} Y_{j,\ell} =& 1 - \frac{b\rho^{2b} t_\ell}{n(x - b\rho^{2b})} + \frac{2b^3\rho^{4b} t_\ell}{n^2(x - b\rho^{2b})^3} + \mathcal{O}\Big(\frac{1}{n^2(x - b\rho^{2b})^2}\Big) - \frac{10b^5\rho^{6b} t_\ell}{n^3(x - b\rho^{2b})^5} \\ &+ \mathcal{O}\Big(\frac{1}{n^3(x - b\rho^{2b})^4}\Big) + \mathcal{O}\Big(\frac{1}{n^4(x - b\rho^{2b})^7}\Big) + \mathcal{O}\Big(\frac{1}{(n(x - b\rho^{2b})^2)^6}\Big) \end{aligned} \tag{2.16}$$

uniformly for $g_+ + 1 \leq j \leq j_+$. The asymptotic formulas (2.15) and (2.16) imply that

$$\begin{aligned} X_j =& \mathsf{T}_0(x) - \frac{b\mathsf{T}_1(x)\rho^{2b}}{n(x - b\rho^{2b})} - \frac{x\mathsf{T}_2(x)}{2bn} - \frac{\alpha\mathsf{T}_1(x)}{bn} + \frac{2b^3\mathsf{T}_1(x)\rho^{4b}}{n^2(x - b\rho^{2b})^3} - \frac{10b^5\mathsf{T}_1(x)\rho^{6b}}{n^3(x - b\rho^{2b})^5} \\ &+ \mathcal{O}\Big(\frac{1}{n^2(x - b\rho^{2b})^2} + \frac{1}{n^3(x - b\rho^{2b})^4} + \frac{1}{n^4(x - b\rho^{2b})^7} + \frac{1}{n^6(x - b\rho^{2b})^{12}}\Big). \end{aligned} \tag{2.17}$$

If $A, B > 1$, then

$$\begin{aligned} \sum_{j=g_++1}^{j_+} \mathcal{O}\Big(\frac{1}{n^A(x - b\rho^{2b})^B}\Big) &= \mathcal{O}\Big(\int_{g_+}^{j_+} \frac{1}{n^A(j/n - b\rho^{2b})^B} dj\Big) \\ &= \mathcal{O}\Big(\int_{g_+/n}^{j_+/n} \frac{1}{n^{A-1}(x - b\rho^{2b})^B} dx\Big) \end{aligned}$$

$$= \mathcal{O}\Big(\frac{1}{n^{A-1}(M/\sqrt{n})^{B-1}}\Big)$$
$$= \mathcal{O}\Big(\frac{1}{n^{A-(B+1)/2}M^{B-1}}\Big),$$

so substitution of (2.17) into (2.12) yields

$$S_2^{(1)} = \sum_{j=g_+ +1}^{j_+} \Big(f_1(x) + \frac{1}{n}f(x) + \frac{1}{n^2}\frac{2b^3\rho^{4b}\mathsf{T}_1(x)}{(1+\mathsf{T}_0(x))(x-b\rho^{2b})^3} + \frac{1}{n^3}\frac{-10b^5\rho^{6b}\mathsf{T}_1(x)}{(1+\mathsf{T}_0(x))(x-b\rho^{2b})^5}\Big) + \mathcal{O}\Big(\frac{1}{M\sqrt{n}} + \frac{1}{M^3\sqrt{n}} + \frac{1}{M^6} + \frac{\sqrt{n}}{M^{11}}\Big). \tag{2.18}$$

Employing Lemma 2.3 with

$$A = \frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}, \quad a_0 = 1-\alpha-\theta_+^{(n,M)}, \quad B = \frac{b\rho^{2b}}{1-\varepsilon}, \quad b_0 = -\alpha-\theta_+^{(n,\varepsilon)},$$

and using that $f^{(k)}(A) = \mathcal{O}(n^{(k+1)/2}M^{-(k+1)})$ for $k \geq 0$, we get

$$\sum_{j=g_+ +1}^{j_+} f_1(x) = n\int_{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho^{2b}}{1-\varepsilon}} f_1(x)\,dx + \Big(\alpha - \frac{1}{2} + \theta_+^{(n,M)}\Big) f_1\Big(\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}\Big) + \Big(\frac{1}{2} - \alpha - \theta_+^{(n,\varepsilon)}\Big) f_1\Big(\frac{b\rho^{2b}}{1-\varepsilon}\Big) + \mathcal{O}(n^{-1}),$$

$$\frac{1}{n}\sum_{j=g_+ +1}^{j_+} f(x) = \int_{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho^{2b}}{1-\varepsilon}} f(x)\,dx + \mathcal{O}\Big(\frac{1}{M\sqrt{n}}\Big),$$

$$\frac{1}{n^2}\sum_{j=g_+ +1}^{j_+} \frac{2b^3\rho^{4b}\mathsf{T}_1(x)}{(1+\mathsf{T}_0(x))(x-b\rho^{2b})^3} = \frac{1}{n}\int_{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho^{2b}}{1-\varepsilon}} \frac{2b^3\rho^{4b}\mathsf{T}_1(x)}{(1+\mathsf{T}_0(x))(x-b\rho^{2b})^3}\,dx + \mathcal{O}\Big(\frac{1}{M^3\sqrt{n}}\Big),$$

$$\frac{1}{n^3}\sum_{j=g_++1}^{j_+}\frac{-10b^5\rho^{6b}\mathsf{T}_1(x)}{(1+\mathsf{T}_0(x))(x-b\rho^{2b})^5}$$
$$=\frac{1}{n^2}\int_{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho^{2b}}{1-\varepsilon}}\frac{-10b^5\rho^{6b}\mathsf{T}_1(x)}{(1+\mathsf{T}_0(x))(x-b\rho^{2b})^5}\,dx+\mathcal{O}\Big(\frac{1}{M^5\sqrt{n}}\Big). \tag{2.19}$$

The large n behavior of the integrals in (2.19) can be determined as follows. Let us write

$$n\int_{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho^{2b}}{1-\varepsilon}} f_1(x)\,dx = n\int_{b\rho^{2b}}^{\frac{b\rho^{2b}}{1-\varepsilon}} f_1(x)\,dx - n\int_{b\rho^{2b}}^{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}} f_1(x)\,dx. \tag{2.20}$$

Using the integration by parts formula

$$\int_A^B f_1(x)\,dx = \Big((x-A)f_1(x)-\frac{(x-A)^2}{2!}f_1'(x)+\frac{(x-A)^3}{3!}f_1''(x)\Big)\Big|_A^B - \int_A^B\frac{(x-A)^3}{3!}f_1'''(x)\,dx$$

with

$$A=b\rho^{2b}\quad\text{and}\quad B=\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}$$

in the second integral in (2.20), and then expanding as $n\to+\infty$, we obtain

$$\begin{aligned} n\int_{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho^{2b}}{1-\varepsilon}} f_1(x)\,dx &= n\int_{b\rho^{2b}}^{\frac{b\rho^{2b}}{1-\varepsilon}} f_1(x)dx - b\rho^{2b}f_1(b\rho^{2b})M\sqrt{n} \\ &\quad - M^2 b\rho^{2b}\Big(f_1(b\rho^{2b})+\frac{b\rho^{2b}}{2}f_1'(b\rho^{2b})\Big) \\ &\quad - \frac{M^3}{\sqrt{n}}b\rho^{2b}\Big(f_1(b\rho^{2b})+b\rho^{2b}f_1'(b\rho^{2b})+\frac{(b\rho^{2b})^2}{6}f_1''(b\rho^{2b})\Big) \\ &\quad + \mathcal{O}\Big(\frac{M^4}{n}\Big), \end{aligned}$$

where we have used that

$$n\int_A^B \frac{(x-A)^3}{3!} f_1'''(x)dx = \mathcal{O}(n(B-A)^4) = \mathcal{O}(M^4/n).$$

Similar calculations using that

$$\mathsf{T}_j^{(k)}(x) = \Big(-\frac{1}{b}\Big)^k \mathsf{T}_{j+k}(x)$$

for $j,k \geq 0$ give

$$\begin{aligned}\int_{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho^{2b}}{1-\varepsilon}} f(x)\,dx &= \int_{b\rho^{2b}}^{\frac{b\rho^{2b}}{1-\varepsilon}} \Big\{f(x) + \frac{b\rho^{2b}\mathsf{T}_1(b\rho^{2b})}{(1+\mathsf{T}_0(b\rho^{2b}))(x-b\rho^{2b})}\Big\}\,dx \\ &\quad - \frac{b\rho^{2b}\mathsf{T}_1(b\rho^{2b})}{2(1+\mathsf{T}_0(b\rho^{2b}))}\ln n - \frac{b\rho^{2b}\mathsf{T}_1(b\rho^{2b})}{1+\mathsf{T}_0(b\rho^{2b})}\ln\frac{\varepsilon}{M(1-\varepsilon)} \\ &\quad + \frac{M}{\sqrt{n}}\Big\{\frac{(b+\alpha)\rho^{2b}\mathsf{T}_1(b\rho^{2b})}{1+\mathsf{T}_0(b\rho^{2b})} - \frac{b\rho^{4b}\mathsf{T}_2(b\rho^{2b})}{2(1+\mathsf{T}_0(b\rho^{2b}))} \\ &\qquad + \frac{b\rho^{4b}\mathsf{T}_1(b\rho^{2b})^2}{(1+\mathsf{T}_0(b\rho^{2b}))^2}\Big\} + \mathcal{O}\Big(\frac{M^2}{n}\Big).\end{aligned}$$

Furthermore,

$$\begin{aligned}&\frac{1}{n}\int_{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho^{2b}}{1-\varepsilon}} \frac{2b^3\rho^{4b}\mathsf{T}_1(x)}{(1+\mathsf{T}_0(x))(x-b\rho^{2b})^3}\,dx \\ &\quad = \frac{1}{n}\int_{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho^{2b}}{1-\varepsilon}} \Big(\frac{2b^3\rho^{4b}\mathsf{T}_1(b\rho^{2b})}{(1+\mathsf{T}_0(b\rho^{2b}))(x-b\rho^{2b})^3} + \mathcal{O}\Big(\frac{1}{(x-b\rho^{2b})^2}\Big)\Big)\,dx \\ &\quad = \frac{b\mathsf{T}_1(b\rho^{2b})}{M^2(1+\mathsf{T}_0(b\rho^{2b}))} + \mathcal{O}\Big(\frac{1}{M\sqrt{n}}\Big),\end{aligned}$$

and a similar calculation yields

$$\frac{1}{n^2}\int_{\frac{b\rho^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{b\rho^{2b}}{1-\varepsilon}} \frac{-10b^5\rho^{6b}\mathsf{T}_1(x)}{(1+\mathsf{T}_0(x))(x-b\rho^{2b})^5}dx = \frac{-5b\mathsf{T}_1(b\rho^{2b})}{2\rho^{2b}M^4(1+\mathsf{T}_0(b\rho^{2b}))} + \mathcal{O}\Big(\frac{1}{M^3\sqrt{n}}\Big).$$

Substituting the above expansions into (2.19), the claim follows from (2.18). ∎

For $k \in \{1, \dots, m\}$ and $j \in \{j : \lambda_j \in I_2\} = \{g_-, \dots, g_+\}$, we define $M_{j,k} := \sqrt{n}(\lambda_{j,k} - 1)$ and $M_j := \sqrt{n}(\lambda_j - 1)$. For the large n asymptotics of $S_2^{(2)}$ we will need the following lemma.

Lemma 2.7 (Taken from [29, Lemma 3.11]). *Let $h \in C^3(\mathbb{R})$. As $n \to +\infty$, we have*

$$\begin{aligned}\sum_{j=g_-}^{g_+} h(M_j) = b\rho^{2b} \int_{-M}^{M} h(t)dt\sqrt{n} - 2b\rho^{2b} \int_{-M}^{M} th(t)\,dt + \Big(\frac{1}{2} - \theta_-^{(n,M)}\Big)h(M) \\ + \Big(\frac{1}{2} - \theta_+^{(n,M)}\Big)h(-M) \\ + \frac{1}{\sqrt{n}}\Big[3b\rho^{2b}\int_{-M}^{M} t^2 h(t)dt \\ + \Big(\frac{1}{12} + \frac{\theta_-^{(n,M)}(\theta_-^{(n,M)} - 1)}{2}\Big)\frac{h'(M)}{b\rho^{2b}} \\ - \Big(\frac{1}{12} + \frac{\theta_+^{(n,M)}(\theta_+^{(n,M)} - 1)}{2}\Big)\frac{h'(-M)}{b\rho^{2b}}\Big] \\ + \mathcal{O}\Big(\frac{1}{n^{3/2}}\sum_{j=g_-+1}^{g_+}\big((1 + |M_j|^3)\tilde{\mathfrak{m}}_{j,n}(h) + (1 + M_j^2)\tilde{\mathfrak{m}}_{j,n}(h') \\ + (1 + |M_j|)\tilde{\mathfrak{m}}_{j,n}(h'') + \tilde{\mathfrak{m}}_{j,n}(h''')\big)\Big),\end{aligned}$$

where, for $\tilde{h} \in C(\mathbb{R})$ and $j \in \{g_- + 1, \dots, g_+\}$, we define

$$\tilde{\mathfrak{m}}_{j,n}(\tilde{h}) := \max_{x \in [M_j, M_{j-1}]} |\tilde{h}(x)|.$$

Lemma 2.8. *For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\begin{aligned}S_2^{(2)} &= E_2^{(M)}\sqrt{n} + E_4^{(M)} + \frac{E_5^{(M)}}{\sqrt{n}} + \mathcal{O}\Big(\frac{M^4}{n} + \frac{M^{14}}{n^2}\Big), \\ E_2^{(M)} &= 2b\rho^{2b} M \ln(1 + \mathsf{T}_0(b\rho^{2b})), \\ E_4^{(M)} &= \ln(1 + \mathsf{T}_0(b\rho^{2b}))(1 - \theta_-^{(n,M)} - \theta_+^{(n,M)}) + b\rho^{2b}\int_{-M}^{M} h_1(t)\,dt, \\ E_5^{(M)} &= 2b\rho^{2b} M^3 \ln(1 + \mathsf{T}_0(b\rho^{2b})) + \Big(\frac{1}{2} - \theta_-^{(n,M)}\Big)h_1(M) \\ &\quad + \Big(\frac{1}{2} - \theta_+^{(n,M)}\Big)h_1(-M) + b\rho^{2b}\int_{-M}^{M}(h_2(t) - 2th_1(t))\,dt,\end{aligned}$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$, where h_1, h_2 are given by

$$h_1(x) = -\frac{2\rho^b \mathsf{T}_1(b\rho^{2b})}{1+\mathsf{T}_0(b\rho^{2b})} \frac{e^{-\frac{1}{2}x^2\rho^{2b}}}{\sqrt{2\pi}\,\mathrm{erfc}\big(-\frac{x\rho^b}{\sqrt{2}}\big)}, \tag{2.21}$$

$$\begin{aligned} h_2(x) = &-\frac{h_1(x)^2}{2} + \frac{1}{1+\mathsf{T}_0(b\rho^{2b})} \frac{e^{-\frac{1}{2}x^2\rho^{2b}}}{\sqrt{2\pi}\,\mathrm{erfc}\big(-\frac{x\rho^b}{\sqrt{2}}\big)} \bigg\{ \Big(\rho^b x - \frac{5}{3}\rho^{3b}x^3\Big)\mathsf{T}_1(b\rho^{2b}) \\ &- \rho^{3b} x \mathsf{T}_2(b\rho^{2b}) + \frac{4-10\rho^{2b}x^2}{3}\mathsf{T}_1(b\rho^{2b}) \frac{e^{-\frac{1}{2}x^2\rho^{2b}}}{\sqrt{2\pi}\,\mathrm{erfc}\big(-\frac{x\rho^b}{\sqrt{2}}\big)} \bigg\}. \end{aligned}$$

Proof. Using (2.10) and Lemma A.2, we obtain

$$S_2^{(2)} = \sum_{j:\lambda_j \in I_2} \ln\Bigg(1 + \sum_{\ell=1}^m \omega_\ell \frac{\frac{1}{2}\mathrm{erfc}\big(-\eta_{j,\ell}\sqrt{\frac{a_j}{2}}\big) - R_{a_j}(\eta_{j,\ell})}{\frac{1}{2}\mathrm{erfc}\big(-\eta_j\sqrt{\frac{a_j}{2}}\big) - R_{a_j}(\eta_j)}\Bigg). \tag{2.22}$$

For $j \in \{j : \lambda_j \in I_2\}$, we have

$$1 - \frac{M}{\sqrt{n}} \le \lambda_j = \frac{bn\rho^{2b}}{j+\alpha} \le 1 + \frac{M}{\sqrt{n}},$$

$-M \le M_j \le M$, and

$$M_{j,k} = M_j - \frac{t_k}{\sqrt{n}} - \frac{t_k M_j}{n}, \quad k = 1, \dots, m.$$

Furthermore, as $n \to +\infty$ we have

$$\begin{aligned} \eta_{j,\ell} = &\frac{M_j}{\sqrt{n}} - \frac{M_j^2 + 3t_\ell}{3n} + \frac{7M_j^3 - 12t_\ell M_j}{36n^{3/2}} \\ &- \frac{73M_j^4 - 45M_j^2 t_\ell + 180t_\ell^2}{540n^2} \\ &+ \frac{1331M_j^5 - 552M_j^3 t_\ell - 1080M_j t_\ell^2}{12960n^{5/2}} + \mathcal{O}\Big(\frac{1+M_j^6}{n^3}\Big) \end{aligned}$$

$$\begin{aligned} -\eta_{j,\ell}\sqrt{a_j/2} = &-\frac{M_j\rho^b}{\sqrt{2}} + \frac{(5M_j^2 + 6t_\ell)\rho^b}{6\sqrt{2}\sqrt{n}} - \frac{\rho^b M_j(53M_j^2 + 12t_\ell)}{72\sqrt{2}n} \\ &+ \frac{\rho^b(270M_j^2 t_\ell + 1447M_j^4 + 720t_\ell^2)}{2160\sqrt{2}n^{3/2}} \\ &- \frac{M_j\rho^b(5352M_j^2 t_\ell + 32183M_j^4 + 4320t_\ell^2)}{51840\sqrt{2}n^2} + \mathcal{O}\Big(\frac{1+M_j^6}{n^{5/2}}\Big) \end{aligned}$$

uniformly for $j \in \{j : \lambda_j \in I_2\}$. Hence, by (A.1), as $n \to +\infty$ we have

$$R_{a_j}(\eta_{j,\ell}) = \frac{e^{-\frac{M_j^2 \rho^{2b}}{2}}}{\sqrt{2\pi}} \mathcal{M}_4 \tag{2.23}$$

where

$$\begin{aligned}
\mathcal{M}_4 := &\frac{-1}{3\rho^b \sqrt{n}} - \frac{M_j(3 + 10M_j^2\rho^{2b} + 12t_\ell \rho^{2b})}{36\rho^b n} \\
&+ \frac{1}{1080\rho^{3b}n^{3/2}}(45\rho^{4b}(6M_j^2 t_\ell + 7M_j^4 + 4t_\ell^2) + 2\rho^{2b}(22M_j^2 - 45t_\ell) \\
&\qquad - 5\rho^{6b}(5M_j^3 + 6M_j t_\ell)^2 - 2) \\
&+ \frac{M_j \rho^{-3b}}{38880n^2}(-6\rho^{4b}(1806M_j^2 t_\ell + 1967M_j^4 + 1350t_\ell^2) \\
&\qquad + 45\rho^{6b}(5M_j^2 + 6t_\ell)(42M_j^2 t_\ell + 47M_j^4 + 24t_\ell^2) \\
&\qquad - 36\rho^{2b}(29M_j^2 + 45t_\ell) - 10M_j^2\rho^{8b}(5M_j^2 + 6t_\ell)^3 - 243) \\
&\qquad + \mathcal{O}((1 + M_j^{12})n^{-\frac{5}{2}})
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\operatorname{erfc}\Big(-\eta_{j,\ell}\sqrt{\frac{a_j}{2}}\Big) \\
&\quad = \frac{1}{2}\operatorname{erfc}\Big(-\frac{\rho^b M_j}{\sqrt{2}}\Big) - \frac{e^{-\frac{M_j^2\rho^{2b}}{2}}\rho^b(5M_j^2 - 6t_\ell)}{6\sqrt{2\pi}\sqrt{n}} \\
&\qquad + \frac{e^{-\frac{M_j^2\rho^{2b}}{2}} M_j \rho^b}{72\sqrt{2\pi}n}(53M_j^2 + 12t_\ell - 25M_j^4\rho^{2b} - 60M_j^2 t_\ell \rho^{2b} - 36t_\ell^2\rho^{2b}) \\
&\qquad + \frac{e^{-\frac{M_j^2\rho^{2b}}{2}} P_8(M_j, t_\ell)}{n^{3/2}} + \frac{e^{-\frac{M_j^2\rho^{2b}}{2}} P_{11}(M_j, t_\ell)}{n^2} + \mathcal{O}\Big(e^{-\frac{M_j^2\rho^{2b}}{2}} \frac{1 + M_j^{14}}{n^{5/2}}\Big),
\end{aligned} \tag{2.24}$$

uniformly for $j \in \{j : \lambda_j \in I_2\}$, where $P_8(M_j, t_\ell)$ and $P_{11}(M_j, t_\ell)$ are polynomials in M_j of order 8 and 11, respectively. If $t_\ell = 0$, then $\lambda_{j,\ell} = \lambda_j$ and $\eta_{j,\ell} = \eta_j$; hence analogous expansions of $R_{a_j}(\eta_j)$ and $\frac{1}{2}\operatorname{erfc}(-\eta_j\sqrt{a_j/2})$ can be obtained by setting $t_\ell = 0$ in (2.23) and (2.24). Substituting the above asymptotics into (2.22), we obtain

$$\begin{aligned}
&1 + \sum_{\ell=1}^{m} \omega_\ell \frac{\frac{1}{2}\operatorname{erfc}\big(-\eta_{j,\ell}\sqrt{\frac{a_j}{2}}\big) - R_{a_j}(\eta_{j,\ell})}{\frac{1}{2}\operatorname{erfc}\big(-\eta_j\sqrt{\frac{a_j}{2}}\big) - R_{a_j}(\eta_j)} \\
&\quad = g_1(M_j) + \frac{g_2(M_j)}{\sqrt{n}} + \frac{g_3(M_j)}{n} + \frac{g_4(M_j)}{n^{3/2}} + \frac{g_5(M_j)}{n^2} + \mathcal{O}\Big(\frac{1 + |M_j|^{13}}{n^{5/2}}\Big),
\end{aligned} \tag{2.25}$$

as $n \to +\infty$, where

$$g_1(x) = 1 + \mathsf{T}_0(b\rho^{2b}),$$
$$g_2(x) = -\frac{e^{-\frac{1}{2}x^2\rho^{2b}} 2\rho^b \mathsf{T}_1(b\rho^{2b})}{\sqrt{2\pi}\,\mathrm{erfc}\big(-\frac{x\rho^b}{\sqrt{2}}\big)},$$
$$g_3(x) = \frac{e^{-\frac{1}{2}x^2\rho^{2b}}}{3\sqrt{2\pi}\,\mathrm{erfc}\big(-\frac{x\rho^b}{\sqrt{2}}\big)} \Big\{ \frac{e^{-\frac{1}{2}x^2\rho^{2b}} \mathsf{T}_1(b\rho^{2b})}{\sqrt{2\pi}\,\mathrm{erfc}\big(-\frac{x\rho^b}{\sqrt{2}}\big)} (4 - 10x^2\rho^{2b})$$
$$+ \mathsf{T}_1(b\rho^{2b})(3x\rho^b - 5x^3\rho^{3b}) - 3\rho^{3b} x \mathsf{T}_2(b\rho^{2b}) \Big\}.$$

The functions g_4 and g_5 can also be computed explicitly, but we do not write them down. The functions $g_j(x)$, $j = 2, \dots, 5$, have exponential decay as $x \to +\infty$. Also, since

$$\frac{e^{-\frac{1}{2}x^2\rho^{2b}}}{\sqrt{2\pi}\,\mathrm{erfc}\big(-\frac{x\rho^b}{\sqrt{2}}\big)} = -\frac{\rho^b x}{2} + \mathcal{O}(x^{-1}), \quad \text{as } x \to -\infty, \tag{2.26}$$

$g_2(x) = \mathcal{O}(x)$ as $x \to -\infty$. It appears at first sight that $g_3(x) = \mathcal{O}(x^4)$ as $x \to -\infty$. However, a direct computation using (2.26) shows that some cancellations occur and in fact $g_3(x) = \mathcal{O}(x^2)$ as $x \to -\infty$. Similarly, the exact expressions for g_4 and g_5 suggest at first sight that $g_4(x) = \mathcal{O}(x^7)$ and $g_5(x) = \mathcal{O}(x^{10})$ as $x \to -\infty$, but here too, cancellations occur and in fact we have $g_4(x) = \mathcal{O}(x^3)$ and $g_5(x) = \mathcal{O}(x^4)$ as $x \to -\infty$. Thus, after a computation using (2.25), we obtain

$$S_2^{(2)} = \sum_{j=g_-}^{g_+} \Big\{ \ln(1 + \mathsf{T}_0(b\rho^{2b})) + \frac{h_1(M_j)}{\sqrt{n}} + \frac{h_2(M_j)}{n}$$
$$+ \mathcal{O}\Big(\frac{1 + |M_j|^3}{n^{3/2}} + \frac{1 + |M_j|^{13}}{n^{5/2}}\Big)\Big\}$$

as $n \to +\infty$, where $h_1 = g_2/g_1$ and $h_2 = -h_1^2/2 + g_3/g_1$. Note that

$$\sum_{j=g_-}^{g_+} \mathcal{O}\Big(\frac{1 + |M_j|^3}{n^{3/2}} + \frac{1 + |M_j|^{13}}{n^{5/2}}\Big) = \mathcal{O}\Big(\frac{M^4}{n} + \frac{M^{14}}{n^2}\Big), \quad \text{as } n \to +\infty.$$

Using Lemma 2.7, we find the claim. ■

Let us define

$$\mathcal{I}_1 = \int_{-\infty}^{+\infty} \Big\{ \frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y) \Big[y + \frac{y}{2(1+y^2)} \Big] \Big\}\, dy, \tag{2.27}$$

$$\mathcal{I}_2 = \int_{-\infty}^{+\infty} \Big\{ \frac{y^3 e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^4 + \frac{y^2}{2} - \frac{1}{2}\Big]\Big\}\, dy, \tag{2.28}$$

$$\mathcal{I}_3 = \int_{-\infty}^{+\infty} \Big\{ \Big(\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 - \chi_{(0,+\infty)}(y)[y^2 + 1]\Big\}\, dy, \tag{2.29}$$

$$\mathcal{I}_4 = \int_{-\infty}^{+\infty} \Big\{ \Big(\frac{y e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 - \chi_{(0,+\infty)}(y)\Big[y^4 + y^2 - \frac{3}{4}\Big]\Big\}\, dy, \tag{2.30}$$

and recall that $\mathcal{I}$ is defined in (1.22).

Lemma 2.9. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \ldots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\begin{aligned} S_2 = &- j_- \ln \Omega + C_1^{(\varepsilon)} n + C_2 \ln n + C_3^{(n,\varepsilon)} + \frac{\widehat{C}_4}{\sqrt{n}} \\ &+ \mathcal{O}\Big(\frac{\sqrt{n}}{M^{11}} + \frac{1}{M^6} + \frac{1}{\sqrt{n}M} + \frac{M^4}{n} + \frac{M^{14}}{n^2}\Big), \end{aligned}$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \ldots, u_m \in D_\delta(x_m)$, where C_2 is as in the statement of Theorem 1.3 and

$$C_1^{(\varepsilon)} = b\rho^{2b} \ln \Omega + \int_{b\rho^{2b}}^{\frac{b\rho^{2b}}{1-\varepsilon}} f_1(x)\, dx,$$

$$\begin{aligned} C_3^{(n,\varepsilon)} = &\frac{1}{2}\ln \Omega + \int_{b\rho^{2b}}^{\frac{b\rho^{2b}}{1-\varepsilon}} \Big\{ f(x) + \frac{b\rho^{2b}\mathsf{T}_1(b\rho^{2b})}{\Omega(x - b\rho^{2b})}\Big\}\, dx + \Big(\frac{1}{2} - \alpha - \theta_+^{(n,\varepsilon)}\Big) f_1\Big(\frac{b\rho^{2b}}{1-\varepsilon}\Big) \\ &- \frac{2b\rho^{2b}}{\Omega}\mathsf{T}_1(b\rho^{2b})\mathcal{I}_1 + \frac{b\rho^{2b}}{2\Omega}\mathsf{T}_1(b\rho^{2b})(\ln 2 - 2b\ln(\rho)) \\ &- \frac{\mathsf{T}_1(b\rho^{2b})}{\Omega} b\rho^{2b} \ln\Big(\frac{\varepsilon}{1-\varepsilon}\Big), \end{aligned}$$

$$\begin{aligned} \widehat{C}_4 = &\sqrt{2}b\rho^b \frac{\rho^{2b}\mathsf{T}_2(b\rho^{2b}) - 5\mathsf{T}_1(b\rho^{2b})}{\Omega}\mathcal{I} + \frac{10\sqrt{2}b\rho^b}{3}\frac{\mathsf{T}_1(b\rho^{2b})}{\Omega}\mathcal{I}_2 \\ &+ \sqrt{2}b\rho^{2b}\frac{\mathsf{T}_1(b\rho^{2b})}{\Omega}\Big(\frac{2}{3\rho^b} - \rho^b\frac{\mathsf{T}_1(b\rho^{2b})}{\Omega}\Big)\mathcal{I}_3 - \frac{10\sqrt{2}b\rho^b}{3}\frac{\mathsf{T}_1(b\rho^{2b})}{\Omega}\mathcal{I}_4, \end{aligned}$$

and f_1 and f are as in the statement of Lemma 2.4.

Proof. By combining Lemmas 2.5, 2.6, and 2.8, we have

$$S_2 = -j_- \ln\Omega + C_1^{(\varepsilon)} n + \widetilde{C}_2 \sqrt{n} + C_2 \ln n + C_3^{(n,\varepsilon,M)} + \frac{C_4^{(M)}}{\sqrt{n}} + \mathcal{O}\Big(\frac{\sqrt{n}}{M^{11}} + \frac{1}{M^6} + \frac{1}{\sqrt{n}M} + \frac{M^4}{n} + \frac{M^{14}}{n^2}\Big),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \ldots, u_m \in D_\delta(x_m)$, where $C_1^{(\varepsilon)}$ is as in the statement, and

$$\begin{aligned}
\widetilde{C}_2 &= -bM\rho^{2b} \ln\Omega + D_2^{(M)} + E_2^{(M)}, \\
C_3^{(n,\varepsilon,M)} &= (bM^2\rho^{2b} - \alpha + \theta_-^{(n,M)}) \ln\Omega + D_4^{(n,\varepsilon,M)} + E_4^{(M)}, \\
C_4^{(n,M)} &= -bM^3\rho^{2b} \ln\Omega + D_5^{(n,M)} + E_5^{(M)}.
\end{aligned}$$

Using that

$$f_1(b\rho^{2b}) = \ln(1 + \mathsf{T}_0(b\rho^{2b})) = \ln\Omega,$$

we readily verify that $\widetilde{C}_2 = 0$. Furthermore, by rearranging the terms and using

$$f_1'(b\rho^{2b}) = \frac{\frac{-1}{b}\mathsf{T}_1(b\rho^{2b})}{1 + \mathsf{T}_0(b\rho^{2b})},$$

we obtain

$$\begin{aligned}
C_3^{(n,\varepsilon,M)} = \frac{1}{2} \ln\Omega + \widetilde{C}_3^{(\varepsilon,M)} &+ \int_{b\rho^{2b}}^{\frac{b\rho^{2b}}{1-\varepsilon}} \Big\{ f(x) + \frac{b\rho^{2b}\mathsf{T}_1(b\rho^{2b})}{(1 + \mathsf{T}_0(b\rho^{2b}))(x - b\rho^{2b})} \Big\} dx \\
&+ \Big(\frac{1}{2} - \alpha - \theta_+^{(n,\varepsilon)}\Big) f_1\Big(\frac{b\rho^{2b}}{1-\varepsilon}\Big),
\end{aligned}$$

where

$$\widetilde{C}_3^{(\varepsilon,M)} := b\rho^{2b} \int_{-M}^{M} h_1(t)\, dt + \frac{\mathsf{T}_1(b\rho^{2b})}{1 + \mathsf{T}_0(b\rho^{2b})} \Big(M^2 \frac{b\rho^{4b}}{2} - b\rho^{2b} \ln\Big(\frac{\varepsilon}{M(1-\varepsilon)}\Big) + \frac{b}{M^2} + \frac{-5b}{2\rho^{2b}M^4} \Big).$$

Using the definition (2.21) of h_1 and a change of variables, we rewrite $\widetilde{C}_3^{(\varepsilon,M)}$ as

$$\begin{aligned}\widetilde{C}_3^{(\varepsilon,M)} = -2b\rho^{2b}\frac{\mathsf{T}_1(b\rho^{2b})}{1+\mathsf{T}_0(b\rho^{2b})}\int_{-\frac{M\rho^b}{\sqrt{2}}}^{\frac{M\rho^b}{\sqrt{2}}}\Big\{\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} \\ -\chi_{(0,+\infty)}(y)\Big[y+\frac{y}{2(1+y^2)}+\frac{3y}{4(1+y^6)}\Big]\Big\}dy \\ +\frac{\mathsf{T}_1(b\rho^{2b})}{1+\mathsf{T}_0(b\rho^{2b})}\Big\{-2b\rho^{2b}\int_0^{\frac{M\rho^b}{\sqrt{2}}}\Big(y+\frac{y}{2(1+y^2)}+\frac{3y}{4(1+y^6)}\Big)dy \\ +M^2\frac{b\rho^{4b}}{2}+b\rho^{2b}\ln M+\frac{b}{M^2}+\frac{-5b}{2\rho^{2b}M^4}\Big\} \\ -\frac{\mathsf{T}_1(b\rho^{2b})}{1+\mathsf{T}_0(b\rho^{2b})}b\rho^{2b}\ln\frac{\varepsilon}{1-\varepsilon}.\end{aligned}$$

The reason for the above rewriting stems from the following asymptotics:

$$\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}-\Big[y+\frac{y}{2(1+y^2)}+\frac{3y}{4(1+y^6)}\Big]=\mathcal{O}(y^{-7}),\quad \text{as } y\to+\infty,$$

which implies

$$\begin{aligned}&\int_{-\frac{M\rho^b}{\sqrt{2}}}^{\frac{M\rho^b}{\sqrt{2}}}\Big\{\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}-\chi_{(0,+\infty)}(y)\Big[y+\frac{y}{2(1+y^2)}+\frac{3y}{4(1+y^6)}\Big]\Big\}dy \\ &=\int_{-\infty}^{\infty}\Big\{\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}-\chi_{(0,+\infty)}(y)\Big[y+\frac{y}{2(1+y^2)}+\frac{3y}{4(1+y^6)}\Big]\Big\}dy+\mathcal{O}(M^{-6}) \\ &=\int_{-\infty}^{\infty}\Big\{\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}-\chi_{(0,+\infty)}(y)\Big[y+\frac{y}{2(1+y^2)}\Big]\Big\}dy-\frac{\pi}{4\sqrt{3}}+\mathcal{O}(M^{-6}),\end{aligned}$$

as $n\to+\infty$. Furthermore, using a primitive and then expanding yields

$$\begin{aligned}&-2b\rho^{2b}\int_0^{\frac{M\rho^b}{\sqrt{2}}}\Big(y+\frac{y}{2(1+y^2)}+\frac{3y}{4(1+y^6)}\Big)dy \\ &+M^2\frac{b\rho^{4b}}{2}+b\rho^{2b}\ln M+\frac{b}{M^2}+\frac{-5b}{2\rho^{2b}M^4} \\ &\quad=-\frac{b\rho^{2b}}{6}(\sqrt{3}\pi-3\ln 2+6b\ln\rho)+\mathcal{O}(M^{-6})\quad \text{as } n\to+\infty.\end{aligned}$$

It follows from the above and some further simplifications that

$$C_3^{(n,\varepsilon,M)} = C_3^{(n,\varepsilon)} + \mathcal{O}(M^{-6}) \quad \text{as } n \to +\infty,$$

where $C_3^{(n,\varepsilon)}$ is as in the statement. Similar (but longer) computation, using among other things that

$$f_1''(b\rho^{2b}) = -\Big(\frac{\frac{-1}{b}\mathsf{T}_1(b\rho^{2b})}{\Omega}\Big)^2 + \frac{\big(-\frac{1}{b}\big)^2\mathsf{T}_2(b\rho^{2b})}{\Omega},$$

show that $C_4^{(n,M)}$ can be rewritten as

$$C_4^{(n,M)} = Q_1^{(n,M)} + Q_2^{(n,M)} + Q_3^{(M)} + Q_4^{(M)} + Q_5^{(M)} + Q_6^{(M)}, \tag{2.31}$$

where

$$\begin{aligned}
Q_1^{(n,M)} &= -\frac{2\rho^b\mathsf{T}_1(b\rho^{2b})}{\Omega}\Big(\frac{1}{2} - \theta_-^{(n,M)}\Big)\frac{e^{-\frac{M^2\rho^{2b}}{2}}}{\sqrt{2\pi}\,\mathrm{erfc}\big(-\frac{M\rho^b}{\sqrt{2}}\big)},\\
Q_2^{(n,M)} &= -\frac{2\rho^b\mathsf{T}_1(b\rho^{2b})}{\Omega}\Big(\frac{1}{2} - \theta_+^{(n,M)}\Big)\Big(\frac{e^{-\frac{M^2\rho^{2b}}{2}}}{\sqrt{2\pi}\,\mathrm{erfc}\big(\frac{M\rho^b}{\sqrt{2}}\big)} - \frac{M\rho^b}{2}\Big),\\
Q_3^{(M)} &= \frac{\sqrt{2}b\rho^b}{\Omega}(-5\mathsf{T}_1(b\rho^{2b}) + \rho^{2b}\mathsf{T}_2(b\rho^{2b}))\\
&\quad\times\int_{-\frac{M\rho^b}{\sqrt{2}}}^{\frac{M\rho^b}{\sqrt{2}}}\Big\{\frac{ye^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^2 + \frac{1}{2}\Big]\Big\}dy,\\
Q_4^{(M)} &= \frac{10\sqrt{2}b\rho^b}{3\Omega}\mathsf{T}_1(b\rho^{2b})\int_{-\frac{M\rho^b}{\sqrt{2}}}^{\frac{M\rho^b}{\sqrt{2}}}\Big\{\frac{y^3e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^4 + \frac{y^2}{2} - \frac{1}{2}\Big]\Big\}dy,\\
Q_5^{(M)} &= \sqrt{2}b\rho^b\frac{\mathsf{T}_1(b\rho^{2b})}{\Omega}\Big(\frac{2}{3} - \rho^{2b}\frac{\mathsf{T}_1(b\rho^{2b})}{\Omega}\Big)\\
&\quad\times\int_{-\frac{M\rho^b}{\sqrt{2}}}^{\frac{M\rho^b}{\sqrt{2}}}\Big\{\Big(\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 - \chi_{(0,+\infty)}(y)\Big[y^2 + 1\Big]\Big\}dy,\\
Q_6^{(M)} &= -\frac{10\sqrt{2}b\rho^b}{3}\frac{\mathsf{T}_1(b\rho^{2b})}{\Omega}\int_{-\frac{M\rho^b}{\sqrt{2}}}^{\frac{M\rho^b}{\sqrt{2}}}\Big\{\Big(\frac{ye^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 - \chi_{(0,+\infty)}(y)\Big[y^4 + y^2 - \frac{3}{4}\Big]\Big\}dy.
\end{aligned}$$

Furthermore, using the asymptotics of $\mathrm{erfc}(y)$ as $y \to \pm\infty$, we infer that

$$
\begin{aligned}
Q_1^{(n,M)} &= \mathcal{O}(e^{-\frac{M^2\rho^{2b}}{2}}), \\
Q_2^{(n,M)} &= \mathcal{O}(M^{-1}), \\
Q_3^{(M)} &= \frac{\sqrt{2}b\rho^b}{\Omega}(\rho^{2b}\mathsf{T}_2(b\rho^{2b}) - 5\mathsf{T}_1(b\rho^{2b})) \\
&\quad \times \int_{-\infty}^{\infty}\Big\{\frac{ye^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^2+\frac{1}{2}\Big]\Big\}dy + \mathcal{O}(M^{-1}), \\
Q_4^{(M)} &= \frac{10\sqrt{2}b\rho^b}{3\Omega}\mathsf{T}_1(b\rho^{2b})\int_{-\infty}^{\infty}\Big\{\frac{y^3e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^4+\frac{y^2}{2}-\frac{1}{2}\Big]\Big\}dy \\
&\quad + \mathcal{O}(M^{-1}), \\
Q_5^{(M)} &= \sqrt{2}b\rho^b\frac{\mathsf{T}_1(b\rho^{2b})}{\Omega}\Big(\frac{2}{3} - \rho^{2b}\frac{\mathsf{T}_1(b\rho^{2b})}{\Omega}\Big) \\
&\quad \times \int_{-\infty}^{\infty}\Big\{\Big(\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 - \chi_{(0,+\infty)}(y)[y^2+1]\Big\}dy + \mathcal{O}(M^{-1}), \\
Q_6^{(M)} &= -\frac{10\sqrt{2}b\rho^b}{3}\frac{\mathsf{T}_1(b\rho^{2b})}{\Omega}\int_{-\infty}^{\infty}\Big\{\Big(\frac{ye^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 \\
&\qquad\qquad - \chi_{(0,+\infty)}(y)\Big[y^4+y^2-\frac{3}{4}\Big]\Big\}dy + \mathcal{O}(M^{-1}),
\end{aligned}
$$

as $n \to +\infty$. Substituting the above asymptotics in (2.31) yields

$$C_4^{(n,M)} = \widehat{C}_4 + \mathcal{O}(M^{-1}),$$

and the claim follows. ■

Recall that $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$ are defined in (2.27)–(2.30), and that $\mathcal{I}$ is defined in (1.22).

Lemma 2.10. *The following relations hold:*

$$\mathcal{I}_1 = \frac{\ln(2\sqrt{\pi})}{2}, \quad \mathcal{I}_3 = \mathcal{I}, \quad \mathcal{I}_4 = \mathcal{I}_2 - \mathcal{I}. \tag{2.32}$$

In particular, $\widehat{C}_4 = C_4$, where C_4 is as in the statement of Theorem 1.3.

Proof. The first identity in (2.32) follows from a direct calculation using the primitive

$$\int \frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}dy = -\frac{1}{2}\ln(\mathrm{erfc}(y)) + \text{const.}$$

Integration by parts gives

$$\int \Big(\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 dy = \frac{e^{-y^2}}{2\sqrt{\pi}\,\mathrm{erfc}(y)} + \int \frac{y e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} dy + \text{const},$$

$$\int \Big(\frac{y e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 dy = \frac{y^2 e^{-y^2}}{2\sqrt{\pi}\,\mathrm{erfc}(y)} + \int \frac{(y^3 - y) e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} dy + \text{const}.$$

Hence, for any $N > 0$,

$$\begin{aligned}
&\int_{-N}^{N} \Big\{\Big(\frac{e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 - \chi_{(0,+\infty)}(y)[y^2+1]\Big\} dy \\
&\quad = \Big(\frac{e^{-N^2}}{2\sqrt{\pi}\,\mathrm{erfc}(N)} - \frac{N}{2}\Big) - \frac{e^{-N^2}}{2\sqrt{\pi}\,\mathrm{erfc}(-N)} \\
&\qquad + \int_{-N}^{N} \Big\{\frac{y e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^2 + \frac{1}{2}\Big]\Big\} dy,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-N}^{N} \Big\{\Big(\frac{y e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)}\Big)^2 - \chi_{(0,+\infty)}(y)\Big[y^4 + y^2 - \frac{3}{4}\Big]\Big\} dy \\
&\quad = \Big(\frac{N^2 e^{-N^2}}{2\sqrt{\pi}\,\mathrm{erfc}(N)} - \frac{N^3}{2} - \frac{N}{4}\Big) - \frac{N^2 e^{-N^2}}{2\sqrt{\pi}\,\mathrm{erfc}(-N)} \\
&\qquad + \int_{-N}^{N} \Big\{\frac{y^3 e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^4 + \frac{y^2}{2} - \frac{1}{2}\Big]\Big\} dy \\
&\qquad - \int_{-N}^{N} \Big\{\frac{y e^{-y^2}}{\sqrt{\pi}\,\mathrm{erfc}(y)} - \chi_{(0,+\infty)}(y)\Big[y^2 + \frac{1}{2}\Big]\Big\} dy.
\end{aligned}$$

The second and third identities in (2.32) are obtained by letting $N \to +\infty$ in the above two formulas. We then find $\widehat{C}_4 = C_4$ after a direct computation. ∎

End of the proof of Theorem 1.3. Let $M' > 0$ be sufficiently large such that Lemmas 2.2 and 2.9 hold. Using (2.3) and Lemmas 2.1, 2.2, 2.4, and 2.9, we conclude that for any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\begin{aligned}
\ln \mathcal{E}_n &= S_0 + S_1 + S_2 + S_3 \\
&= M' \ln \Omega + (j_- - M' - 1) \ln \Omega - j_- \ln \Omega + C_1^{(\varepsilon)} n
\end{aligned}$$

$$+ n \int_{\frac{b\rho^{2b}}{1-\varepsilon}}^{1} f_1(x)\,dx + C_2 \ln n + C_3^{(n,\varepsilon)} + \frac{C_4}{\sqrt{n}} + \int_{\frac{b\rho^{2b}}{1-\varepsilon}}^{1} f(x)\,dx$$
$$+ \Big(\alpha + \theta_+^{(n,\varepsilon)} - \frac{1}{2}\Big) f_1\Big(\frac{b\rho^{2b}}{1-\varepsilon}\Big)$$
$$+ \frac{1}{2} f_1(1) + \mathcal{O}\Big(\frac{\sqrt{n}}{M^{11}} + \frac{1}{M^6} + \frac{1}{\sqrt{n}M} + \frac{M^4}{n} + \frac{M^{14}}{n^2}\Big),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$. Since $M = n^{1/10}$, the above error term is $\mathcal{O}(n^{-3/5})$. Furthermore, using Lemma 2.10, a computation shows that

$$C_1^{(\varepsilon)} + \int_{\frac{b\rho^{2b}}{1-\varepsilon}}^{1} f_1(x)\,dx = C_1,$$

and

$$-\ln\Omega + C_3^{(n,\varepsilon)} + \int_{\frac{b\rho^{2b}}{1-\varepsilon}}^{1} f(x)\,dx + \Big(\alpha + \theta_+^{(n,\varepsilon)} - \frac{1}{2}\Big) f_1\Big(\frac{b\rho^{2b}}{1-\varepsilon}\Big) + \frac{1}{2} f_1(1) = C_3,$$

where C_1 and C_3 are as in the statement of Theorem 1.3. This concludes the proof of Theorem 1.3. ■

3. Proof of Theorem 1.7

As in the proof of Theorem 1.3, our starting point is formula (2.3), where $M' > 0$ is an integer independent of n, $j_\pm$ are defined in (2.1), and $\varepsilon > 0$ is such that (2.2) holds. The variables $a_j, \lambda_j, \lambda_{j,k}, \eta_j, \eta_{j,k}$ are given by (2.6), where r_k is now defined by (1.12) (in contrast to Section 2 where r_k was given by (1.11)). The following two lemmas are analogous to Lemmas 2.1 and 2.2 and are proved in the same way.

Lemma 3.1. *For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_0 = M' \ln\Omega + \mathcal{O}(e^{-cn}), \quad \text{as } n \to +\infty,$$

uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$.

Lemma 3.2. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_1 = (j_- - M' - 1)\ln\Omega + \mathcal{O}(e^{-cn}),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$.

Lemma 3.3. *For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_3 = \mathcal{O}(e^{-c\sqrt{n}}),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$.

Proof. For $j \geq j_+ + 1$ and $k \in \{1, \dots, m\}$, $1 - \lambda_j$ and $1 - \lambda_{j,k}$ are positive and remain bounded away from 0. Hence, using Lemma A.4 (ii), we obtain

$$\begin{aligned} S_3 &= \sum_{j=j_+ + 1}^{n} \ln\Bigg\{1 + \sum_{\ell=1}^{m} \omega_\ell \frac{e^{-\frac{a_j \eta_{j,\ell}^2}{2}} \big(\frac{-1}{\lambda_{j,\ell}-1}\frac{1}{\sqrt{a_j}} + \mathcal{O}(n^{-\frac{3}{2}})\big)}{e^{-\frac{a_j \eta_j^2}{2}} \big(\frac{-1}{\lambda_j - 1}\frac{1}{\sqrt{a_j}} + \mathcal{O}(n^{-\frac{3}{2}})\big)}\Bigg\} \\ &= \sum_{j=j_+ + 1}^{n} \ln\Big\{1 + \sum_{\ell=1}^{m} \omega_\ell \mathcal{O}(e^{\frac{a_j}{2}(\eta_j^2 - \eta_{j,\ell}^2)})\Big\}, \end{aligned}$$

where the $\mathcal{O}$-terms are uniform for $j \in \{j_+ + 1, \dots, n\}$ and independent of $u_1, \dots, u_m$. Using that r_k is given by (1.12), we find, as $n \to +\infty$,

$$\frac{a_j}{2}(\eta_j^2 - \eta_{j,\ell}^2) = -\frac{\sqrt{2}\mathfrak{s}_\ell\big(\frac{j}{n} - b\rho^{2b}\big)\sqrt{n}}{b\rho^b} + \mathcal{O}(1) \tag{3.1}$$

and hence

$$S_3 = \sum_{j=j_+ + 1}^{n} \ln\Big(1 + \sum_{\ell=1}^{m} \omega_\ell \mathcal{O}(e^{-\frac{\sqrt{2}\mathfrak{s}_\ell(j/n - b\rho^{2b})\sqrt{n}}{b\rho^b}})\Big),$$

where the $\mathcal{O}$-terms are uniform for $j \in \{j_+ + 1, \dots, n\}$ and independent of $u_1, \dots, u_m$. Since $\mathfrak{s}_\ell > 0$ for all $\ell \in \{1, \dots, m\}$ and since $j/n - b\rho^{2b}$ is positive and bounded away from 0 as $n \to +\infty$ with $j \in \{j_+ + 1, \dots, n\}$, the claim follows. ■

We now focus on S_2. As in Section 2, we decompose S_2 into three pieces, $S_2 = S_2^{(1)} + S_2^{(2)} + S_2^{(3)}$, where the $S_2^{(v)}$ are given by (2.10). However, in contrast to Section 2, we let the intervals I_v be given by (2.11) with $M := M' \ln n$. Using this M, we define $g_\pm$ and $\theta_-^{(n,M)}, \theta_+^{(n,M)} \in [0, 1)$ as in Section 2. The following lemma is analogous to Lemma 2.5 and is proved in the same way.

Lemma 3.4. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\begin{aligned} S_2^{(3)} &= (b\rho^{2b} n - j_- - bM\rho^{2b}\sqrt{n} + bM^2\rho^{2b} - \alpha + \theta_-^{(n,M)} - bM^3\rho^{2b} n^{-\frac{1}{2}}) \ln \Omega \\ &\quad + \mathcal{O}(M^4 n^{-1}), \end{aligned}$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$.

In the case of the hard edge, we found that $S_2^{(1)}$ made important contributions to the asymptotic formula for large n (see Lemma 2.6). However, in the semi-hard regime, $S_2^{(1)}$ is small as the next lemma shows.

Lemma 3.5. *M' can be chosen sufficiently large such that the following holds. For any $x_1, \ldots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2^{(1)} = \mathcal{O}(n^{-100}),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \ldots, u_m \in D_\delta(x_m)$.

Proof. Since $\lambda_j \in \big[1-\varepsilon, 1-\frac{M}{\sqrt{n}}\big)$ for $g_+ + 1 \leq j \leq j_+$ and $\lambda_{j,\ell} = \lambda_j\big(1 - \frac{\sqrt{2}\mathfrak{s}_\ell}{\rho^b\sqrt{n}}\big)$, we have $\eta_j, \eta_{j,\ell} \leq -cM/\sqrt{n}$ for some $c > 0$, and so Lemma A.4 (ii) yields

$$\begin{aligned}
S_2^{(1)} &= \sum_{j=g_++1}^{j_+} \ln\Big(1 + \frac{\sum_{\ell=1}^m \omega_\ell \gamma(a_j, a_j\lambda_{j,\ell})}{\gamma(a_j, a_j\lambda_j)}\Big) \\
&= \sum_{j=g_++1}^{j_+} \ln\Big(1 + \sum_{\ell=1}^m \omega_\ell \frac{e^{-\frac{a_j\eta_{j,\ell}^2}{2}}\big(\frac{-1}{\lambda_{j,\ell}-1}\frac{1}{\sqrt{a_j}} + \mathcal{O}\big((a_j\frac{M^2}{n})^{-\frac{3}{2}}\big)\big)}{e^{-\frac{a_j\eta_j^2}{2}}\big(\frac{-1}{\lambda_j-1}\frac{1}{\sqrt{a_j}} + \mathcal{O}\big((a_j\frac{M^2}{n})^{-\frac{3}{2}}\big)\big)}\Big) \\
&= \sum_{j=g_++1}^{j_+} \ln\Big(1 + \sum_{\ell=1}^m \omega_\ell \mathcal{O}(e^{\frac{a_j}{2}(\eta_j^2 - \eta_{j,\ell}^2)})\Big) \\
&= \sum_{j=g_++1}^{j_+} \ln\Big(1 + \sum_{\ell=1}^m \omega_\ell \mathcal{O}(e^{-\frac{\sqrt{2}\mathfrak{s}_\ell(j/n - b\rho^{2b})\sqrt{n}}{b\rho^b}})\Big),
\end{aligned}$$

where we have used (3.1) in the last step. Since $M = M'\ln n$ and $\mathfrak{s}_\ell > 0$, the claim follows from the fact that

$$\frac{j}{n} - b\rho^{2b} \geq b\rho^{2b}\frac{M + \mathcal{O}(1)}{\sqrt{n}} \quad \text{as } n \to +\infty$$

for $j \in \{g_+ + 1, \ldots, j_+\}$. ∎

For $k \in \{1, \ldots, m\}$ and $j \in \{j : \lambda_j \in I_2\} = \{g_-, \ldots, g_+\}$, we define

$$M_{j,k} := \sqrt{n}(\lambda_{j,k} - 1) \quad \text{and} \quad M_j := \sqrt{n}(\lambda_j - 1).$$

Lemma 3.6. *For any $x_1, \ldots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2^{(2)} = E_2^{(M)}\sqrt{n} + E_3^{(M)} + \frac{E_4^{(M)}}{\sqrt{n}} + \mathcal{O}\Big(\frac{M^4}{n}\Big),$$

$$E_2^{(M)} = \sqrt{2} b \rho^b \int_{-\frac{M\rho^b}{\sqrt{2}}}^{\frac{M\rho^b}{\sqrt{2}}} h_0(y)\, dy,$$

$$\begin{aligned} E_3^{(M)} &= b \int_{-\frac{M\rho^b}{\sqrt{2}}}^{\frac{M\rho^b}{\sqrt{2}}} (4y h_0(y) + \sqrt{2} h_1(y))\, dy + \Big(\frac{1}{2} - \theta_-^{(n,M)}\Big) h_0\Big(-\frac{M\rho^b}{\sqrt{2}}\Big) \\ &\quad + \Big(\frac{1}{2} - \theta_+^{(n,M)}\Big) h_0\Big(\frac{M\rho^b}{\sqrt{2}}\Big), \end{aligned}$$

$$\begin{aligned} E_4^{(M)} &= b\rho^{-b} \int_{-\frac{M\rho^b}{\sqrt{2}}}^{\frac{M\rho^b}{\sqrt{2}}} (6\sqrt{2} y^2 h_0(y) + 4y h_1(y) + \sqrt{2} h_2(y))\, dy \\ &\quad - \Big(\frac{1}{12} + \frac{\theta_-^{(n,M)}(\theta_-^{(n,M)} - 1)}{2}\Big) \frac{h_0'\big(-\frac{M\rho^b}{\sqrt{2}}\big)}{\sqrt{2} b \rho^b} \\ &\quad + \Big(\frac{1}{12} + \frac{\theta_+^{(n,M)}(\theta_+^{(n,M)} - 1)}{2}\Big) \frac{h_0'\big(\frac{M\rho^b}{\sqrt{2}}\big)}{\sqrt{2} b \rho^b} + \Big(\frac{1}{2} - \theta_-^{(n,M)}\Big) \rho^{-b} h_1\Big(-\frac{M\rho^b}{\sqrt{2}}\Big) \\ &\quad + \Big(\frac{1}{2} - \theta_+^{(n,M)}\Big) \rho^{-b} h_1\Big(\frac{M\rho^b}{\sqrt{2}}\Big) \end{aligned}$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$, where h_0, h_1, h_2 are as in the statement of Theorem 1.7.

Proof. Using (2.10) and Lemma A.2, we obtain

$$S_2^{(2)} = \sum_{j:\lambda_j \in I_2} \ln\Big(1 + \sum_{\ell=1}^{m} \omega_\ell \frac{\frac{1}{2}\operatorname{erfc}\big(-\eta_{j,\ell}\sqrt{\frac{a_j}{2}}\big) - R_{a_j}(\eta_{j,\ell})}{\frac{1}{2}\operatorname{erfc}\big(-\eta_j\sqrt{\frac{a_j}{2}}\big) - R_{a_j}(\eta_j)}\Big).$$

For $j \in \{j : \lambda_j \in I_2\}$, we have

$$1 - \frac{M}{\sqrt{n}} \le \lambda_j = \frac{bn\rho^{2b}}{j + \alpha} \le 1 + \frac{M}{\sqrt{n}},$$

$-M \le M_j \le M$, and

$$M_{j,k} = M_j - \frac{\sqrt{2}\mathfrak{s}_k}{\rho^b} - \frac{\sqrt{2}\mathfrak{s}_k M_j}{\rho^b \sqrt{n}}, \quad k = 1, \dots, m.$$

Furthermore, as $n \to +\infty$ we have

$$\eta_{j,\ell} = \frac{M_j - \sqrt{2}\mathfrak{s}_\ell \rho^{-b}}{\sqrt{n}} - \frac{M_j^2 + \sqrt{2}M_j\mathfrak{s}_\ell\rho^{-b} + 2\mathfrak{s}_\ell^2\rho^{-2b}}{3n} + \frac{7M_j^3 + 3\sqrt{2}M_j^2\mathfrak{s}_\ell\rho^{-b} - 6M_j\mathfrak{s}_\ell^2\rho^{-2b} - 14\sqrt{2}\mathfrak{s}_\ell^3\rho^{-3b}}{36n^{3/2}} + \mathcal{O}\Big(\frac{1+M_j^4}{n^2}\Big),$$

$$-\eta_{j,\ell}\sqrt{a_j/2} = -\frac{M_j\rho^b}{\sqrt{2}} + \mathfrak{s}_\ell + \frac{5\sqrt{2}M_j^2\rho^b - 2M_j\mathfrak{s}_\ell + 4\sqrt{2}\mathfrak{s}_\ell^2\rho^{-b}}{12\sqrt{n}} - \frac{53\sqrt{2}M_j^3\rho^b - 18M_j^2\mathfrak{s}_\ell + 12\sqrt{2}M_j\mathfrak{s}_\ell^2\rho^{-b} - 56\mathfrak{s}_\ell^3\rho^{-2b}}{144n} + \mathcal{O}\Big(\frac{1+M_j^4}{n^{3/2}}\Big)$$

uniformly for $j \in \{j : \lambda_j \in I_2\}$. Hence, after a long computation using (A.1), we obtain

$$1 + \sum_{\ell=1}^{m} \omega_\ell \frac{\frac{1}{2}\operatorname{erfc}\big(-\eta_{j,\ell}\sqrt{\frac{a_j}{2}}\big) - R_{a_j}(\eta_{j,\ell})}{\frac{1}{2}\operatorname{erfc}\big(-\eta_j\sqrt{\frac{a_j}{2}}\big) - R_{a_j}(\eta_j)} = g_0\Big(-\frac{\rho^b M_j}{\sqrt{2}}\Big) + \frac{g_1\big(-\frac{\rho^b M_j}{\sqrt{2}}\big)}{\rho^b\sqrt{n}} + \frac{g_2\big(-\frac{\rho^b M_j}{\sqrt{2}}\big)}{\rho^{2b} n} + \mathcal{O}\Big(\frac{e^{-c|M_j|}}{n^{3/2}}\Big),$$

as $n \to +\infty$, where g_0, g_1 and g_2 are as in the statement of Theorem 1.7. For the above error term, we have used that $s_\ell > 0$, $\ell \in \{1, \ldots, m\}$. Thus

$$S_2^{(2)} = \sum_{j=g_-}^{g_+} \ln\Big(1 + \sum_{\ell=1}^{m} \omega_\ell \frac{\frac{1}{2}\operatorname{erfc}\big(-\eta_{j,\ell}\sqrt{\frac{a_j}{2}}\big) - R_{a_j}(\eta_{j,\ell})}{\frac{1}{2}\operatorname{erfc}\big(-\eta_j\sqrt{\frac{a_j}{2}}\big) - R_{a_j}(\eta_j)}\Big) = \sum_{j=g_-}^{g_+}\Big\{h_0\Big(-\frac{\rho^b M_j}{\sqrt{2}}\Big) + \frac{h_1\big(-\frac{\rho^b M_j}{\sqrt{2}}\big)}{\rho^b\sqrt{n}} + \frac{h_2\big(-\frac{\rho^b M_j}{\sqrt{2}}\big)}{\rho^{2b} n} + \mathcal{O}\Big(\frac{e^{-c|M_j|}}{n^{3/2}}\Big)\Big\}$$

as $n \to +\infty$.After a computation using Lemma 2.7, a change of variables and the fact that $g_1(y), g_2(y) = \mathcal{O}(e^{-c|y|})$ as $y \to \pm\infty$, we find the claim. ∎

Lemma 3.7. *The constant M' can be chosen sufficiently large such that the following holds. For any $x_1, \ldots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$S_2 = -j_-\ln\Omega + C_1 n + C_2\sqrt{n} + C_3 + \ln\Omega + \frac{C_4}{\sqrt{n}} + \mathcal{O}\Big(\frac{M^4}{n}\Big),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \ldots, u_m \in D_\delta(x_m)$, where $C_1, \ldots, C_4$ are as in the statement of Theorem 1.7.

Proof. By combining Lemmas 3.4, 3.5, and 3.6, we obtain

$$S_2 = -j_- \ln \Omega + C_1 n + C_2^{(M)} \sqrt{n} + C_3^{(M)} + \frac{C_4^{(M)}}{\sqrt{n}} + \mathcal{O}\Big(\frac{M^4}{n}\Big),$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$, where C_1 is as in the statement, and

$$\begin{aligned} C_2^{(M)} &= -bM\rho^{2b} \ln \Omega + E_2^{(M)}, \\ C_3^{(M)} &= (bM^2\rho^{2b} - \alpha + \theta_-^{(n,M)}) \ln \Omega + E_3^{(M)}, \\ C_4^{(M)} &= -bM^3\rho^{2b} \ln \Omega + E_4^{(M)}. \end{aligned}$$

A direct analysis shows that M' can be chosen sufficiently large such that

$$\begin{aligned} C_2^{(M)} &= C_2 + \mathcal{O}(n^{-100}), \\ C_3^{(M)} &= C_3 + \ln \Omega + \mathcal{O}(n^{-100}), \\ C_4^{(M)} &= C_4 + \mathcal{O}(n^{-100}), \end{aligned}$$

and the claim follows. ∎

End of the proof of Theorem 1.7. Let $M' > 0$ be sufficiently large such that Lemmas 3.2 and 3.7 hold. Using (2.3) and Lemmas 3.1, 3.2, 3.3, and 3.7, we conclude that for any $x_1, \dots, x_m \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\begin{aligned} \ln \mathcal{E}_n &= S_0 + S_1 + S_2 + S_3 \\ &= M' \ln \Omega + (j_- - M' - 1) \ln \Omega - j_- \ln \Omega + C_1 n + C_2 \sqrt{n} + C_3 + \ln \Omega \\ &\quad + \frac{C_4}{\sqrt{n}} + \mathcal{O}(M^4 n^{-1}) \\ &= C_1 n + C_2 \sqrt{n} + C_3 + \frac{C_4}{\sqrt{n}} + \mathcal{O}(M^4 n^{-1}), \end{aligned}$$

as $n \to +\infty$ uniformly for $u_1 \in D_\delta(x_1), \dots, u_m \in D_\delta(x_m)$. This concludes the proof of Theorem 1.7. ∎

A. Uniform asymptotics of the incomplete gamma function

Lemma A.1 (From [19, formula 8.11.2]). *Let $a > 0$ be fixed. As $z \to +\infty$,*

$$\gamma(a, z) = \Gamma(a) + \mathcal{O}(e^{-\frac{z}{2}}).$$

Lemma A.2 (From [74, Section 11.2.4]). *We have*

$$\frac{\gamma(a,z)}{\Gamma(a)} = \frac{1}{2}\operatorname{erfc}\Big(-\eta\sqrt{\frac{a}{2}}\Big) - R_a(\eta), \qquad R_a(\eta) = \frac{e^{-\frac{1}{2}a\eta^2}}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} g(u)\, du,$$

where erfc *is defined in* (1.19),

$$\lambda = \frac{z}{a}, \qquad \eta = (\lambda - 1)\sqrt{\frac{2(\lambda - 1 - \ln\lambda)}{(\lambda-1)^2}}, \qquad g(u) := \frac{dt}{du}\frac{1}{\lambda - t} + \frac{1}{u + i\eta},$$

with t and u being related by the bijection $t \mapsto u$ from $\mathcal{L} := \big\{\frac{\theta}{\sin\theta}e^{i\theta} : -\pi < \theta < \pi\big\}$ to $\mathbb{R}$ given by

$$u = -i(t-1)\sqrt{\frac{2(t - 1 - \ln t)}{(t-1)^2}}, \qquad t \in \mathcal{L},$$

and the principal branch is used for the roots. Furthermore, as $a \to +\infty$, uniformly for $z \in [0,\infty)$,

$$R_a(\eta) \sim \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}} \sum_{j=0}^{\infty} \frac{c_j(\eta)}{a^j}, \tag{A.1}$$

where all coefficients $c_j(\eta)$ are bounded functions of $\eta \in \mathbb{R}$ (i.e. bounded for $\lambda \in (0, +\infty)$). The first two coefficients are given by (see [74, p. 312]*)*

$$c_0(\eta) = \frac{1}{\lambda - 1} - \frac{1}{\eta}, \qquad c_1(\eta) = \frac{1}{\eta^3} - \frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda-1)^2} - \frac{1}{12(\lambda-1)}.$$

More generally, we have

$$c_j(\eta) = \frac{1}{\eta}\frac{d}{d\eta}c_{j-1}(\eta) + \frac{\gamma_j}{\lambda - 1}, \qquad j \geq 1, \tag{A.2}$$

where the γ_j are the Stirling coefficients

$$\gamma_j = \frac{(-1)^j}{2^j j!}\Big[\frac{d^{2j}}{dx^{2j}}\Big(\frac{1}{2}\frac{x^2}{x - \ln(1+x)}\Big)^{j+\frac{1}{2}}\Big]_{x=0}. \tag{A.3}$$

In particular, the following hold.

(i) *Let $z = \lambda a$ and let $\delta > 0$ be fixed. As $a \to +\infty$, uniformly for $\lambda \geq 1 + \delta$,*

$$\gamma(a,z) = \Gamma(a)(1 + \mathcal{O}(e^{-\frac{a\eta^2}{2}})).$$

(ii) *Let $z = \lambda a$. As $a \to +\infty$, uniformly for λ in compact subsets of* $(0,1)$*,*

$$\gamma(a,z) = \Gamma(a)\mathcal{O}(e^{-\frac{a\eta^2}{2}}).$$

The following lemma establishes a non-recursive formula for the coefficients c_j, which is new to our knowledge.

Lemma A.3. *For $j \geq 0$, the coefficients $c_j(\eta)$ in (A.1) can be expressed as*

$$c_j(\eta) = \varphi_j(\lambda) - S(\varphi_j(\lambda)), \quad \text{where } \varphi_j(\lambda) := \frac{(-1)^{j+1}(2j-1)!!}{\eta^{2j+1}} \tag{A.4}$$

and where $S(\varphi_j(\lambda))$ denotes the singular part of $\varphi_j(\lambda)$ at $\lambda = 1$, i.e., $S(\varphi_j(\lambda))$ is the sum of the singular terms in the Laurent expansion of $\varphi_j(\lambda)$ at $\lambda = 1$.

Proof. The formula (A.4) holds for $j = 0$. Suppose it holds for $j = k-1 \geq 0$. Then (A.2) yields

$$c_k(\eta) = \frac{1}{\eta}\frac{d}{d\eta}\varphi_{k-1}(\lambda) - \frac{1}{\eta}\frac{d}{d\eta}S(\varphi_{k-1}(\lambda)) + \frac{\gamma_k}{\lambda-1}.$$

We have $\partial_\eta \varphi_{k-1}(\lambda) = \eta\varphi_k(\lambda)$. Hence, using also that ∂_η commutes with S,

$$c_k(\eta) = \varphi_k(\lambda) - \frac{1}{\eta}S(\eta\varphi_k(\lambda)) + \frac{\gamma_k}{\lambda-1}.$$

On the other hand, φ_k has a pole of order $2k+1$ at $\lambda = 1$, so in light of the identity $(2k)! = (2k-1)!!2^k k!$ and (A.3), we obtain

$$\begin{aligned}\operatorname*{Res}_{\lambda=1} \varphi_k(\lambda) &= \frac{1}{(2k)!}\lim_{\lambda\to1}\frac{d^{2k}}{d\lambda^{2k}}((\lambda-1)^{2k+1}\varphi_k(\lambda)) \\ &= \frac{(-1)^{k+1}}{2^k k!}\lim_{\lambda\to1}\frac{d^{2k}}{d\lambda^{2k}}\left(\frac{(\lambda-1)^2}{2(\lambda-1-\ln\lambda)}\right)^{k+\frac{1}{2}} = -\gamma_k.\end{aligned}$$

It follows that (A.4) holds also for $j = k$, completing the proof. ∎

Note that $S(\varphi_j(\lambda))$ is a polynomial of order $2j+1$ in $(\lambda-1)^{-1}$ without constant term. The first $S(\varphi_j(\lambda))$ are given by

$$\begin{aligned}S(\varphi_0(\lambda)) &= -\frac{1}{\lambda-1}, \\ S(\varphi_1(\lambda)) &= \frac{1}{(\lambda-1)^3} + \frac{1}{(\lambda-1)^2} + \frac{1}{12(\lambda-1)}, \\ S(\varphi_2(\lambda)) &= -\frac{3}{(\lambda-1)^5} - \frac{5}{(\lambda-1)^4} - \frac{25}{12(\lambda-1)^3} - \frac{1}{12(\lambda-1)^2} - \frac{1}{288(\lambda-1)}.\end{aligned}$$

The following lemma follows from a result of Tricomi [75], see also [7]. However, in contrast to [7, 75], the coefficients appearing in Lemma A.4 below are written in a non-recursive way. Here we give a short proof relying on Lemmas A.2 and A.3.

Lemma A.4. *Let $N \geq 0$ be an integer and let η and $S(\varphi_j(\lambda))$ be as in (A.4).*

(i) *As $a \to +\infty$, uniformly for $\lambda \geq 1 + \frac{1}{\sqrt{a}}$,*

$$\frac{\gamma(a, \lambda a)}{\Gamma(a)} = 1 + \frac{e^{-\frac{a}{2}\eta^2}}{\sqrt{2\pi}} \Big\{ \sum_{j=0}^{N-1} \frac{S(\varphi_j(\lambda))}{a^{j+\frac{1}{2}}} + \mathcal{O}\Big(\frac{1}{a^{N+\frac{1}{2}}}\Big) + \mathcal{O}\Big(\frac{1}{(a\eta^2)^{N+\frac{1}{2}}}\Big)\Big\}.$$

(ii) *As $a \to +\infty$, uniformly for $\lambda \in \big[\varepsilon, 1 - \frac{1}{\sqrt{a}}\big]$ for any fixed $\varepsilon > 0$,*

$$\frac{\gamma(a, \lambda a)}{\Gamma(a)} = \frac{e^{-\frac{a}{2}\eta^2}}{\sqrt{2\pi}} \Big\{ \sum_{j=0}^{N-1} \frac{S(\varphi_j(\lambda))}{a^{j+\frac{1}{2}}} + \mathcal{O}\Big(\frac{1}{a^{N+\frac{1}{2}}}\Big) + \mathcal{O}\Big(\frac{1}{(a\eta^2)^{N+\frac{1}{2}}}\Big)\Big\}.$$

Proof. (i) The assumption $\lambda \geq 1 + \frac{1}{\sqrt{a}}$ implies that $-\eta\sqrt{a} \leq -c$ for some $c > 0$. In view of the identity $\operatorname{erfc}(-x) = 2 - \operatorname{erfc}(x)$ and the expansion

$$\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j \big(\frac{1}{2}\big)_j}{x^{2j+1}}, \qquad x \to +\infty, \tag{A.5}$$

where $\big(\frac{1}{2}\big)_j = \prod_{k=0}^{j-1}(\frac{1}{2} + k)$ is the rising factorial, Lemma A.2 implies that, for any $N \geq 0$,

$$\begin{aligned}\frac{\gamma(a, \lambda a)}{\Gamma(a)} &= 1 - \frac{e^{-\frac{a}{2}\eta^2}}{2\sqrt{\pi}} \sum_{j=0}^{N-1} \frac{(-1)^j \big(\frac{1}{2}\big)_j}{\big(\eta\sqrt{\frac{a}{2}}\big)^{2j+1}} + \mathcal{O}\Big(\frac{1}{(\eta\sqrt{a})^{2N+1}}\Big) \\ &\quad - \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}} \sum_{j=0}^{N-1} \frac{c_j(\eta)}{a^j} + \mathcal{O}\Big(\frac{1}{a^{N+\frac{1}{2}}}\Big) \\ &= 1 - \frac{e^{-\frac{a}{2}\eta^2}}{\sqrt{2\pi}} \sum_{j=0}^{N-1} \frac{1}{(\sqrt{a})^{2j+1}} \Big(\frac{(-1)^j \big(\frac{1}{2}\big)_j 2^j}{\eta^{2j+1}} + c_j(\eta)\Big) \\ &\quad + \mathcal{O}\Big(\frac{1}{(a\eta^2)^{N+\frac{1}{2}}}\Big) + \mathcal{O}\Big(\frac{1}{a^{N+\frac{1}{2}}}\Big).\end{aligned}$$

Since $\big(\frac{1}{2}\big)_j 2^j = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \frac{2j-1}{2} 2^j = (2j-1)!!$, the desired conclusion follows from (A.4).

(ii) The assumption $\lambda \leq 1 - \frac{1}{\sqrt{a}}$ implies that $-\eta\sqrt{a} \geq c$ for some $c > 0$. Using (A.5) and Lemma A.2, the desired conclusion now follows as in the proof of (i). ∎

Acknowledgments. We thank S.-S. Byun for help with Figure 1.

Funding. Christophe Charlier acknowledges support from the Swedish Research Council, Grant No. 2021-04626. Jonatan Lenells acknowledges support from the Swedish Research Council, Grant No. 2021-03877 and the Ruth and Nils-Erik Stenbäck Foundation.

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Received 10 August 2022; revised 15 July 2023.

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