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# The Ergodicity of the Convolution $\mu*\nu$ on a Vector Space

By

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## Abstract

Let G be a subgroup of a vector space $X$ and $\mu, \nu$ be two probability measures on $X$. If $\mu$ and $\nu$ are $G$-quasi-invariant and $G$-ergodic, then the convolution $\mu*\nu$ is also $G$-ergodic.

## § 1. Introduction

Let $X$ be a vector space, $\mathscr{R}$ be a subspace of $X^a$ (the algebraical dual of $X$) and $\mathscr{B}_{\mathscr{R}}$ be the smallest $\sigma$-algebra on $X$ which makes each $x'\in\mathscr{R}$ measurable. For probability measures $\mu$ and $\nu$ on $\mathscr{B}_{\mathscr{R}}$, $\mu$ is said to be absolutely continuous with respect to $\nu$ (denoted by $\mu<\nu$) if $\nu(A)=0$, $A\in\mathscr{B}_{\mathscr{R}}$, implies that $\mu(A)=0$. $\mu$ and $\nu$ are equivalent (denoted by $\mu\sim\nu$) if $\mu<\nu$ and $\nu<\mu$. Denote by $A\ominus B=(A\cap B^c)\cup(A^c\cap B)$ the symmetric difference.

Denote by $\tau_x(x\in X)$ the translation $\tau_x(z)=z+x$. $\tau_x$: $(X, \mathscr{B}_{\mathscr{R}})\rightarrow(X, \mathscr{B}_{\mathscr{R}})$ is measurable and $\tau_x\mathscr{B}_{\mathscr{R}}=\mathscr{B}_{\mathscr{R}}$ for every $x\in X$. We put for $x\in X$,

$$\mu_x(A)=\tau_x(\mu)(A)=\mu(A-x), A\in\mathscr{B}_{\mathscr{R}}.$$

Let $A_\mu=\{x\in X;\ \mu_x\sim\mu\}$ be the set of all admissible translates of $\mu$. $A_\mu$ is an additive subgroup of $X$. For a subset $G\subset X$, $\mu$ is called $G$-quasi-invariant if $G\subset A_\mu$, and $\mu$ is called $G$-ergodic if $\mu(A\ominus(A-x))=0$ for every $x\in G$ implies that $\mu(A)=0$ or 1.

Let $\Phi(x, y)=x+y$. Then $\Phi$: $(E\times E, \mathscr{B}_{\mathscr{R}}\otimes\mathscr{B}_{\mathscr{R}})\rightarrow(E, \mathscr{B}_{\mathscr{R}})$ is measurable, where $\mathscr{B}_{\mathscr{R}}\otimes\mathscr{B}_{\mathscr{R}}$ denotes the product $\sigma$-algebra. So the

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convolution $\mu*\nu$ of two probability measures $\mu$ and $\nu$ is defined as follows:

$$\mu*\nu(A)=\int_X \mu(A-x)\,d\nu(x).$$

$\mu*\nu$ coincides with the image measure $\Phi(\mu\times\nu)$, where $\mu\times\nu$ is the product measure on $(E\times E, \mathscr{B}_{\mathscr{R}}\otimes\mathscr{B}_{\mathscr{R}})$.

It holds that $A_{\mu*\nu}\supset A_\mu+A_\nu$, see Yamasaki [2], p. 170, [3], Theorem 13. 1. By this result, for a subgroup $G$ of $X$, it follows that if $\mu$ or $\nu$ is $G$-quasi-invariant, then $\mu*\nu$ is also $G$-quasi-invariant.

Concerning the ergodicity of $\mu*\nu$, Yamasaki [2], p. 170, raised the following problem.

**Problem.** *Let $G$ be a subgroup of $X$ and let $\mu$, $\nu$ be probability measures on $(X, \mathscr{B}_{\mathscr{R}})$ which are $G$-quasi-invariant and $G$-ergodic. Then is the convolution $\mu*\nu$ $G$-ergodic?*

Yamasaki [2], Theorem 23. 2, proved that if $G$ is either

(1) *$G$ is a linear subspace of algebraically countable dimension, or*

(2) *$G$ is a complete separable metrizable topological vector subspace of $X$ such that the identity $G\to X_{\sigma(X,\mathscr{R})}$ is continuous, where $\sigma(X,\mathscr{R})$ is the weak topology determined by $\mathscr{R}$,*

then the answer is affirmative.

In this paper, we shall prove that the answer to the above problem is affirmative without any assumption on $G$.

## § 2. Main Result

**Theorem.** *Let $\mu$, $\nu$ be probability measures on $(X, \mathscr{B}_{\mathscr{R}})$ and $G_\mu$, $G_\nu$ be two subgroups of $X$. Suppose that $\mu$, $\nu$ be $G_\mu$, $G_\nu$-quasi-invariant and $G_\mu$, $G_\nu$-ergodic, respectively. Suppose also that $G_\nu\subset A_\mu$, that is, $\mu$ is $G_\nu$-quasi-invariant. Then $\mu*\nu$ is $G_\mu$-ergodic.*

**Corollary.** *Let $\mu$, $\nu$ be probability measures on $(X, \mathscr{B}_{\mathscr{R}})$ and $G$ be a* *subgroup of X. If $\mu$ and $\nu$ are G-quasi-invariant and G-ergodic, then $\mu*\nu$ is G-ergodic.*

To prove the theorem, we use the following lemma due to Yamasaki [2], Theorem 25. 6, p. 182. The proof given here is a modification of Shimomura [1], p. 706-707.

**Lemma.** *Let $\{x_\alpha\}\subset A_\mu$ be a net satisfying that $\int |(d\mu_{x_\alpha}/d\mu)(z)-1|\, d\mu(z)\to 0$. Then it follows that $\mu(B\ominus(B-x_\alpha))\to 0$ for every $B\in\mathscr{B}_{\mathscr{R}}$.*

*Proof.* First we show that $x_\alpha\to 0$ in $\sigma(X,\mathscr{R})$. For every $x'\in\mathscr{R}$, take $\delta$ so that $|t|<\delta$ implies that $|\int \exp(it\langle z,x'\rangle)\, d\mu(z)|>1/2$. By

$$|1-\exp(it\langle x_\alpha,x'\rangle)|\,|\int \exp(it\langle z,x'\rangle)d\mu(z)|$$
$$=|\int \exp(it\langle z,x'\rangle)((d\mu_{x_\alpha}/d\mu)(z)-1)d\mu(z)|,$$

we have

$$|1-\exp(it\langle x_\alpha,x'\rangle)|<2\int |(d\mu_{x_\alpha}/d\mu)(z)-1|d\mu(z)$$

for every $t$ with $|t|<\delta$. Thus $\langle x_\alpha,x'\rangle\to 0$.

Next we claim that $\int |(d\mu_{x_\alpha}/d\mu)(z)f(z-x_\alpha)-f(z)|d\mu(z)\to 0$ for every $f\in L^1(X,\mathscr{B}_{\mathscr{R}})$. In fact, for each function of the form $f(z)=\sum_{j=1}^n c_j\exp(i\langle z,x_j'\rangle)$, $c_j$ are real numbers and $x_j'\in\mathscr{R}$, the assertion holds since $\langle x_\alpha,x_j'\rangle\to 0$ for every $j$. Since these functions are dense in $L^1(X,\mathscr{B}_{\mathscr{R}})$, we get the claim.

For every $B\in\mathscr{B}_{\mathscr{R}}$, we have

$$\mu(B\ominus(B-x_\alpha))=\int |\chi_{B-x_\alpha}(z)-\chi_B(z)|d\mu(z)\leqq\int |(d\mu_{-x_\alpha}/d\mu)(z)$$
$$-1|d\mu(z)+\int |(d\mu_{-x_\alpha}/d\mu)(z)\chi_B(z+x_\alpha)-\chi_B(z)|d\mu(z)\to 0$$

remarking that

$$\int |(d\mu_x/d\mu)(z)-1|d\mu(z)=\int |(d\mu_{-x}/d\mu)(z)-1|d\mu(z),$$

where $\chi_B$ is the characteristic function of $B$. This proves the Lemma.

*Proof of the Theorem.* Suppose that for $A\in\mathscr{B}_{\mathscr{R}}$, $\mu*\nu(A\ominus(A-x))$

$=0$ for every $x\in G_\mu$. By the definition of the $\sigma$-algebra $\mathscr{B}_{\mathscr{R}}$, there exists a countable subset $\Gamma=\{x_i'\}_{i=1}^\infty\subset\mathscr{R}$ such that $A\in\mathscr{B}_\Gamma$; $\mathscr{B}_\Gamma$ is the minimal $\sigma$-algebra on $X$ which makes each $x_i'(i=1, 2, \dots)$ measurable. The measures $\mu$, $\nu$ are also $G_\mu$, $G_\nu$-quasi-invariant and $G_\mu$, $G_\nu$-ergodic on the sub-$\sigma$-algebra $\mathscr{B}_\Gamma\subset\mathscr{B}_{\mathscr{R}}$. Consequently, in order to show $\mu*\nu(A)=0$ or 1, we can suppose in advance that the $\sigma$-algebra $\mathscr{B}_{\mathscr{R}}$ is countably generated. In particular, $L^1(X, \mathscr{B}_{\mathscr{R}})$ is separable.

Take a countable dense subset $\{d\mu_{x_n}/d\mu\}_{n=1}^\infty$ of $\{d\mu_x/d\mu;\ x\in G_\mu\}$ in $L^1(X, \mathscr{B}_{\mathscr{R}})$. We claim that for each $A\in\mathscr{B}_{\mathscr{R}}$, if $\mu(A\ominus(A-x_n))=0$ for every $n$, then $\mu(A\ominus(A-x))=0$ for every $x\in G_\mu$. Let $x\in G_\mu$ be arbitrary. By $\mu(A\ominus(A-x_n))=0$ for every $n$, it follows that $\mu(A\ominus(A-x))=\mu((A-x_n)\ominus(A-x))$ for every $n$. By the preceding Lemma (putting $B=A-x$), for every $\varepsilon>0$, there exists $\delta=\delta(A, x, \varepsilon)$ such that $\int|(d\mu_y/d\mu)(z)-1|d\mu(z)<\delta$ implies $\mu((A-x)\ominus(A-x-y))<\varepsilon$. By the definition of the sequence $\{x_n\}$, there exists $n=n(\delta)$ such that $\int|(d\mu_x/d\mu)(z)-(d\mu_{x_n}/d\mu)(z)|d\mu(z)=\int|(d\mu_{-x+x_n}/d\mu)(z)-1|d\mu(z)<\delta$. Thus we have $\mu((A-x)\ominus(A-x_n))=\mu((A-x)\ominus(A-x-(-x+x_n)))<\varepsilon$, that is, $\mu(A\ominus(A-x))<\varepsilon$ for every $\varepsilon>0$ which proves $\mu(A\ominus(A-x))=0$.

By $\mu*\nu(A\ominus(A-x_n))=\int_X\mu((A-z)\ominus(A-z-x_n))d\nu(z)=0$ for every $n$, there exists a subset $\Omega\in\mathscr{B}_{\mathscr{R}}$ satisfying that $\nu(\Omega)=1$ and $\mu((A-z)\ominus(A-z-x_n))=0$ for every $n$ and for every $z\in\Omega$. By the second step of this proof and by the ergodicity of $\mu$, it follows that $\mu(A-z)=0$ or 1 for every $z\in\Omega$. We set $B=\{z\in X;\ \mu(A-z)=1\}$. Since $G_\nu\subset A_\mu$, $B$ is a $G_\nu$-invariant subset. By the $G_\nu$-ergodicity of $\nu$, it follows that $\nu(B)=0$ or 1. If $\nu(B)=1$, then we have $\mu*\nu(A)=1$ and if $\nu(B)=0$ then $\mu*\nu(A)=0$. Thus $\mu*\nu(A)=0$ or 1. This completes the proof.

## References

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