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Relative Poisson bialgebras and Frobenius Jacobi algebras

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Abstract. Jacobi algebras, as the algebraic counterparts of Jacobi manifolds, are exactly the unital relative Poisson algebras. The direct approach of constructing Frobenius Jacobi algebras in terms of Manin triples is not available due to the existence of the units, and hence alternatively we replace it by studying Manin triples of relative Poisson algebras. Such structures are equivalent to certain bialgebra structures, namely, relative Poisson bialgebras. The study of coboundary cases leads to the introduction of the relative Poisson Yang–Baxter equation (RPYBE). Antisymmetric solutions of the RPYBE give coboundary relative Poisson bialgebras. The notions of $\mathcal{O}$-operators of relative Poisson algebras and relative pre-Poisson algebras are introduced to give antisymmetric solutions of the RPYBE. A direct application is that relative Poisson bialgebras can be used to construct Frobenius Jacobi algebras, and in particular, there is a construction of Frobenius Jacobi algebras from relative pre-Poisson algebras.

1. Introduction

The aim of this paper is to give a bialgebra theory for relative Poisson algebras, in which one of the motivations is to construct Frobenius Jacobi algebras.

1.1. Generalizations of Poisson algebras

Poisson algebras arose in the study of Poisson geometry ([9,30,39]) and are closely related to a lot of topics in mathematics and physics. Recall that a *Poisson algebra* is a vector space A equipped with two bilinear operations $\cdot, [-,-] : A \otimes A \to A$ such that $(A, \cdot)$ is a commutative associative algebra, $(A, [-,-])$ is a Lie algebra and they are compatible in the sense of the Leibniz rule,

$$[z, x \cdot y] = [z, x] \cdot y + x \cdot [z, y], \qquad \forall x, y, z \in A. \tag{1}$$

There are generalizations of Poisson algebras such as noncommutative Poisson algebras in which the commutativity of the associative algebras is canceled ([40]) or variations

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of algebraic structures by changing the compatible condition (1) (cf. [8, 15, 34]). Among the latter, Jacobi algebras are abstract algebraic counterparts of Jacobi manifolds ([26,31]), which are generalizations of symplectic or more generally Poisson manifolds. Recall that a Jacobi manifold is a smooth manifold endowed with a bivector Λ and a vector field E satisfying some compatible conditions, or equivalently, it is a smooth manifold M such that the commutative associative algebra $A := C^{\infty}(M)$ of real smooth functions on M is endowed with a Lie bracket $[-,-]$ satisfying

$$[z, x \cdot y] = [z, x] \cdot y + x \cdot [z, y] + x \cdot y \cdot [1_A, z], \qquad \forall x, y, z \in A, \tag{2}$$

where 1_A is the unit of A. Correspondingly, a *Jacobi algebra* is a triple $(A, \cdot, [-,-])$, such that $(A, \cdot)$ is a unital commutative associative algebra, $(A, [-,-])$ is a Lie algebra, and they satisfy equation (2).

Taking $z = 1_A$ in equation (2), one gets

$$[1_A, x \cdot y] = [1_A, x] \cdot y + x \cdot [1_A, y], \qquad \forall x, y \in A.$$

Hence $\mathrm{ad}(1_A)$ is a derivation of both $(A, \cdot)$ and $(A, [-,-])$, where $\mathrm{ad}(1_A)(x) = [1_A, x]$ for all $x \in A$. Therefore there is the following natural generalization of a Jacobi algebra.

Definition 1.1. A *relative Poisson algebra* is a quadruple $(A, \cdot, [-,-], P)$, where $(A, \cdot)$ is a commutative associative algebra, $(A, [-,-])$ is a Lie algebra, P is a derivation of both $(A, \cdot)$ and $(A, [-,-])$, that is, P satisfies the following conditions:

$$P(x \cdot y) = P(x) \cdot y + x \cdot P(y),$$
$$P([x, y]) = [P(x), y] + [x, P(y)],$$

for all $x, y \in A$, and the relative Leibniz rule is satisfied, that is,

$$[z, x \cdot y] = [z, x] \cdot y + x \cdot [z, y] + x \cdot y \cdot P(z), \qquad \forall x, y, z \in A. \tag{3}$$

We also denote it simply by (A, P).

A relative Poisson algebra $(A, \cdot, [-,-], P)$ is called unital if $(A, \cdot)$ is a unital commutative associative algebra. There is a one-to-one correspondence between Jacobi algebras and unital relative Poisson algebras: a Jacobi algebra $(A, \cdot, [-,-])$ is a unital relative Poisson algebra $(A, \cdot, [-,-], \mathrm{ad}(1_A))$, whereas a unital relative Poisson algebra $(A, \cdot, [-,-], P)$ is a Jacobi algebra since one gets $P(x) = [1_A, x]$ for all $x \in A$ by equation (3). Moreover, one can derive unital relative Poisson algebras (and thus Jacobi algebras) from not necessarily unital ones (see Lemma 5.5). That is, relative Poisson algebras not only generalize Jacobi algebras in a natural way, but also provide a good supplement for the latter. It is also natural to consider the possible geometry related to relative Poisson algebras, which is still little known and will be an interesting topic in the future study.

On the other hand, relative Poisson algebras are motivated by different approaches. In fact, they appeared first in a graded form under the name of generalized Poisson superalgebras ([12]), which arose in the classification of a class of simple Jordan superalgebras, named Kantor series ([21, 25]). See [13, 23] for more details and the related topics. Note that there are other different algebraic structures also called generalized Poisson algebras ([36, 38]). Hence we adopt the present notion of relative Poisson algebras.

Note that any Poisson algebra can be viewed as a relative Poisson algebra with the derivation $P = 0$. On the other hand, a relative Poisson algebra $(A, \cdot, [-,-], P)$ in which the commutative associative algebra $(A, \cdot)$ is trivial, that is, $x \cdot y = 0$ for all $x, y \in A$, is exactly the structure consisting of a Lie algebra $(A, [-,-])$ and a derivation P of it. A less trivial example of a relative Poisson algebra is given as follows.

Example 1.2. Let $(A, \cdot)$ be a commutative associative algebra with a derivation P. Define

$$[x, y] = x \cdot P(y) - P(x) \cdot y, \qquad \forall x, y \in A. \tag{4}$$

Then $(A, [-,-])$ is a Lie algebra and $(A, \cdot, [-,-], P)$ is a relative Poisson algebra.

1.2. Frobenius Jacobi algebras and relative Poisson bialgebras

It is important to study Frobenius algebras due to their close relationships with many areas such as topology, algebraic geometry, category theory, Hochschild cohomology and graph theory ([7, 19, 27, 29]). An associative algebra $(A, \cdot)$ is called Frobenius ([10, 18]) if there exists a nondegenerate bilinear form $\mathcal{B}$ which is invariant in the sense that

$$\mathcal{B}(x \cdot y, z) = \mathcal{B}(x, y \cdot z), \qquad \forall x, y, z \in A. \tag{5}$$

Note that if $(A, \cdot)$ is unital and commutative, then equation (5) requires $\mathcal{B}$ to be symmetric. Such bilinear forms on associative algebras reflect a certain natural symmetry, characterizing the so-called Frobenius property. In the context of Lie algebras, that is, a Lie algebra $(\mathfrak{g}, [-,-])$ equipped with a nondegenerate bilinear form $\mathcal{B}$ which is invariant in the sense that

$$\mathcal{B}([x, y], z) = \mathcal{B}(x, [y, z]), \qquad \forall x, y, z \in \mathfrak{g}, \tag{6}$$

is called self-dual ([17]). When $\mathcal{B}$ is symmetric, such a structure is also called a quadratic ([35]) or a metric Lie algebra ([22]).

In order to study the Frobenius property of Jacobi algebras, the notion of Frobenius Jacobi algebras was introduced in [1]. That is, a Jacobi algebra $(A, \cdot, [-,-])$ is called Frobenius if there exists a nondegenerate symmetric bilinear form $\mathcal{B}$ on A such that equations (5) and (6) hold. To get more examples of Frobenius Jacobi algebras is obviously an important problem.

An important class of quadratic Lie algebras is given by the following construction of (standard) Manin triples of Lie algebras ([14]). Suppose that $\mathfrak{g}$ is a Lie algebra and there exists a Lie algebra structure on the linear dual space $\mathfrak{g}^*$, such that there is a Lie algebra structure on the direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$ of vector spaces including $\mathfrak{g}$ and $\mathfrak{g}^*$ as Lie subalgebras and the bilinear form

$$\mathcal{B}_d(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \qquad \forall x, y \in \mathfrak{g},\ a^*, b^* \in \mathfrak{g}^*, \tag{7}$$

on $\mathfrak{g} \oplus \mathfrak{g}^*$ is invariant. Then, $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ is called a (standard) Manin triple of Lie algebras and obviously, the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ with the symmetric nondegenerate bilinear form $\mathcal{B}_d$ given by equation (7) is a quadratic Lie algebra. The above construction in the context of associative algebras is called a double construction of a Frobenius algebra ([7]), whereas in the context of Poisson algebras is called a Manin triple of Poisson algebras ([37]).

On the other hand, the above constructions also correspond to certain bialgebra structures with different motivations and applications. A bialgebra is a coupling of an algebra and a coalgebra satisfying certain compatible conditions. In fact, a Manin triple of Lie algebras is equivalent to a Lie bialgebra ([14, 16]) which serves as the algebraic structure of a Poisson-Lie group and plays an important role in the infinitesimalization of a quantum group. A double construction of a Frobenius algebra is equivalent to an antisymmetric infinitesimal bialgebra ([2, 4, 5, 7]), which is a special infinitesimal bialgebra introduced by Joni and Rota in order to provide an algebraic framework for the calculus of divided differences ([20]). A Manin triple of Poisson algebras is equivalent to a Poisson bialgebra ([37]), which naturally fits into a framework to construct compatible Poisson brackets in integrable systems.

Therefore in order to get Frobenius Jacobi algebras, it is natural to consider generalizing the above construction to the case of Jacobi algebras, that is, to consider the analogue structures like "Manin triples of Jacobi algebras" or "double constructions of Frobenius Jacobi algebras" and the corresponding bialgebra structures. Unfortunately, such an approach is unavailable, since in general a unital commutative associative algebra cannot be decomposed into the direct sum of the underlying vector spaces of two unital commutative associative subalgebras. That is, the structure of Manin triples does not make sense for Jacobi algebras due to the existence of the units in the commutative associative algebras. On the other hand, alternatively, the approach is still available for relative Poisson algebras, that is, "Manin triples of relative Poisson algebras" make sense for relative Poisson algebras without "the constraint of the units". More interestingly, we still may get examples of Frobenius Jacobi algebras from Manin triples of relative Poisson algebras due to the fact that a unital commutative associative algebra might be decomposed into the direct sum of the underlying vector spaces of two commutative associative subalgebras, in which one of them is unital.

In this paper, we realize the above approach for relative Poisson algebras to construct Frobenius Jacobi algebras. Explicitly, we introduce the notions of Manin triples of relative Poisson algebras and the corresponding bialgebra structures, namely, relative Poisson bialgebras. The equivalence between them is interpreted in terms of matched pairs of relative Poisson algebras. Relative Poisson bialgebras share some similar properties of Lie bialgebras or antisymmetric infinitesimal bialgebras. In particular, there are also coboundary cases which lead to the introduction of an analogue of the classical Yang–Baxter equation in a Lie algebra, called relative Poisson Yang–Baxter equation (RPYBE) in a relative Poisson algebra. Antisymmetric solutions of the RPYBE in relative Poisson algebras give rise to coboundary relative Poisson bialgebras, whereas the notions of $\mathcal{O}$-operators of relative Poisson algebras and relative pre-Poisson algebras are introduced to construct such solutions. Moreover, one can get Frobenius Jacobi algebras from relative Poisson bialgebras satisfying some additional conditions, and in particular, there is a construction of Frobenius Jacobi algebras from relative pre-Poisson algebras.

1.3. Layout of the paper

The paper is organized as follows.

In Section 2, we introduce the notions of representations and matched pairs of relative Poisson algebras. The condition that a relative Poisson algebra is dually represented is considered in order to construct a reasonable representation on the dual space. There is a relationship between representations of relative Poisson algebras and representations of Jacobi algebras. We also explain why there is not a “matched pair theory” for Jacobi algebras.

In Section 3, we introduce the notion of Manin triples of relative Poisson algebras, which are equivalent to certain matched pairs of relative Poisson algebras. Then we introduce the notion of relative Poisson bialgebras as the equivalent structures to the two former structures.

In Section 4, the coboundary relative Poisson bialgebras are studied, leading to the introduction of the RPYBE in a relative Poisson algebra. Antisymmetric solutions of the RPYBE in relative Poisson algebras give rise to coboundary relative Poisson bialgebras. The notions of $\mathcal{O}$-operators of relative Poisson algebras and relative pre-Poisson algebras are introduced to construct antisymmetric solutions of the RPYBE and hence relative Poisson bialgebras.

In Section 5, we give a construction of Frobenius Jacobi algebras from relative Poisson bialgebras. In particular, Frobenius Jacobi algebras can be obtained from relative pre-Poisson algebras.

Throughout this paper, unless otherwise specified, all the vector spaces and algebras are finite-dimensional over an algebraically closed field $\mathbb{K}$ of characteristic zero, although many results and notions remain valid in the infinite-dimensional case.

2. Representations and matched pairs of relative Poisson algebras

In Section 2.1, we introduce the notion of a representation of a relative Poisson algebra. The case when a relative Poisson algebra is dually represented is then studied, which gives a reasonable representation on the dual space. We also give a relationship between representations of relative Poisson algebras and representations of Jacobi algebras given in [1]. In Section 2.2, we introduce the notion of matched pairs of relative Poisson algebras and explain that a “matched pair theory” is not available for Jacobi algebras.

2.1. Representations of relative Poisson algebras and Jacobi algebras

Let $(A, \cdot)$ be a commutative associative algebra. A pair (μ, V) is called a *representation* of $(A, \cdot)$ if V is a vector space and $\mu : A \to \mathrm{End}(V)$ is a linear map satisfying

$$\mu(x \cdot y) = \mu(x)\mu(y), \qquad \forall x, y \in A.$$

Moreover, if $(A, \cdot)$ is unital with the unit 1_A, then the representation is assumed to be unital, that is, we assume $\mu(1_A)v = v$, $\forall v \in V$. In fact, (μ, V) is a representation of a (unital) commutative associative algebra $(A, \cdot)$ if and only if the direct sum $A \oplus V$ of vector spaces is a (unital) commutative associative algebra (the *semi-direct product*) by defining the multiplication on $A \oplus V$ by

$$(x + u) \cdot (y + v) = x \cdot y + \mu(x)v + \mu(y)u, \qquad \forall x, y \in A,\ u, v \in V. \tag{8}$$

We denote it by $A \ltimes_\mu V$. If 1_A is the unit of $(A, \cdot)$, then it is also the unit of $A \ltimes_\mu V$.

Let $\mathcal{L} : A \to \mathrm{End}(A)$ be a linear map defined by $\mathcal{L}(x)(y) = x \cdot y$ for all $x, y \in A$. Then $(\mathcal{L}, A)$ is a representation of $(A, \cdot)$, called the *adjoint representation* of $(A, \cdot)$.

Let V be a vector space. For a linear map $\varphi : A \to \mathrm{End}(V)$, define a linear map $\varphi^* : A \to \mathrm{End}(V^*)$ by

$$\langle \varphi^*(x)u^*, v\rangle = -\langle u^*, \varphi(x)v\rangle, \qquad \forall x \in A,\ u^* \in V^*,\ v \in V,$$

where $\langle -, -\rangle$ is the ordinary pair between V and V^*. If (μ, V) is a representation of a commutative associative algebra $(A, \cdot)$, then $(-\mu^*, V^*)$ is also a representation of $(A, \cdot)$. In particular, $(-\mathcal{L}^*, A^*)$ is a representation of $(A, \cdot)$.

Similarly, let $(A, [-, -])$ be a Lie algebra. A pair (ρ, V) is called a *representation* of $(A, [-, -])$ if V is a vector space and $\rho : A \to \mathrm{End}(V)$ is a linear map satisfying

$$\rho([x, y]) = [\rho(x), \rho(y)], \qquad \forall x, y \in A.$$

In fact, (ρ, V) is a representation of a Lie algebra $(A, [-, -])$ if and only if $A \oplus V$ is a Lie algebra (the *semi-direct product*) by defining the multiplication on $A \oplus V$ by

$$[x + u, y + v] = [x, y] + \rho(x)v - \rho(y)u, \qquad \forall x, y \in A,\ u, v \in V. \tag{9}$$

We denote it by $A \ltimes_\rho V$.

Let $\mathrm{ad} : A \to \mathrm{End}(A)$ be a linear map defined by $\mathrm{ad}(x)(y) = [x, y]$ for all $x, y \in A$. Then (ad, A) is a representation of $(A, [-, -])$, called the *adjoint representation* of $(A, [-, -])$. Moreover, if (ρ, V) is a representation of a Lie algebra $(A, [-, -])$, then (ρ^*, V^*) is also a representation of $(A, [-, -])$. In particular, (ad^*, A^*) is a representation of $(A, [-, -])$.

Definition 2.1. Let $(A, \cdot, [-, -], P)$ be a relative Poisson algebra, (μ, V) be a representation of the commutative associative algebra $(A, \cdot)$ and (ρ, V) be a representation of the

Lie algebra $(A, [-,-])$. We say the two representations (μ, V) and (ρ, V) are *compatible*, or (μ, ρ, V) is a *compatible structure* on (A, P) if the following equation holds:

$$\rho(y)\mu(x)v - \mu(x)\rho(y)v + \mu([x,y])v - \mu(x \cdot P(y))v = 0, \qquad \forall x, y \in A, v \in V. \tag{10}$$

If in addition, there is a linear map $\alpha : V \to V$ satisfying

$$\alpha(\mu(x)v) - \mu(P(x))v - \mu(x)\alpha(v) = 0, \tag{11}$$

$$\alpha(\rho(x)v) - \rho(P(x))v - \rho(x)\alpha(v) = 0, \tag{12}$$

$$\rho(x \cdot y)v - \mu(x)\rho(y)v - \mu(y)\rho(x)v + \mu(x \cdot y)\alpha(v) = 0, \tag{13}$$

for all $x, y \in A, v \in V$, we say the quadruple (μ, ρ, α, V) is a *representation* of (A, P). Two representations $(\mu_1, \rho_1, \alpha_1, V_1)$ and $(\mu_2, \rho_2, \alpha_2, V_2)$ of a relative Poisson algebra (A, P) are called *equivalent* if there exists a linear isomorphism $\varphi : V_1 \to V_2$ such that

$$\begin{aligned} \varphi(\mu_1(x)v) &= \mu_2(x)\varphi(v), \\ \varphi(\rho_1(x)v) &= \rho_2(x)\varphi(v), \qquad \forall x \in A, v \in V. \\ \varphi(\alpha_1(v)) &= \alpha_2(\varphi(v)), \end{aligned} \tag{14}$$

For vector spaces V_1 and V_2, and linear maps $\varphi_1 : V_1 \to V_1$ and $\varphi_2 : V_2 \to V_2$, we abbreviate $\varphi_1 + \varphi_2$ for the linear map

$$\begin{aligned} &\varphi_{V_1 \oplus V_2} : V_1 \oplus V_2 \to V_1 \oplus V_2, \\ &\varphi_{V_1 \oplus V_2}(v_1 + v_2) := \varphi_1(v_1) + \varphi_2(v_2), \end{aligned} \qquad \forall v_1 \in V_1, v_2 \in V_2.$$

Proposition 2.2. *Let (A, P) be a relative Poisson algebra, V be a vector space, and $\mu, \rho : A \to \mathrm{End}(V)$ and $\alpha : V \to V$ be linear maps. Define two bilinear operations $\cdot, [-,-]$ on $A \oplus V$ by equations (8) and (9) respectively. Then $(A \oplus V, \cdot, [-,-], P + \alpha)$ is a relative Poisson algebra if and only if (μ, ρ, α, V) is a representation of (A, P). We denote this relative Poisson algebra structure (*semi-direct product*) on $A \oplus V$ by $(A \ltimes_{\mu,\rho} V, P + \alpha)$ or simply $(A \ltimes V, P + \alpha)$.*

The proof is omitted since this result is a special case of the matched pairs of relative Poisson algebras in Proposition 2.17, when $A_2 = V$ is equipped with the zero multiplication.

Example 2.3. Let (A, P) be a relative Poisson algebra. Then $(\mathcal{L}, \mathrm{ad}, P, A)$ is a representation of (A, P), called the *adjoint representation* of (A, P).

Lemma 2.4. *Let (A, P) be a relative Poisson algebra. If (μ, ρ, V) is a compatible structure on (A, P), then $(-\mu^*, \rho^*, V^*)$ is also a compatible structure on (A, P).*

Proof. For all $x, y \in A, u^* \in V^*, v \in V$, we have

$$\begin{aligned} &\langle (-\rho^*(y)\mu^*(x) + \mu^*(x)\rho^*(y) - \mu^*([x,y]) + \mu^*(x \cdot P(y))u^*, v \rangle \\ &\quad = \langle u^*, \rho(y)\mu(x)v - \mu(x)\rho(y)v + \mu([x,y])v - \mu(x \cdot P(y))v \rangle = 0. \end{aligned}$$

Hence the conclusion holds. ∎

Let V_1, V_2 be two vector spaces and $T : V_1 \to V_2$ be a linear map. Denote the dual map $T^* : V_2^* \to V_1^*$ by

$$\langle v_1, T^*(v_2^*)\rangle = \langle T(v_1), v_2^*\rangle, \qquad \forall v_1 \in V_1,\ v_2^* \in V_2^*.$$

Proposition 2.5. *Let (A, P) be a relative Poisson algebra and (μ, ρ, V) be a compatible structure on (A, P). Let $\beta : V \to V$ be a linear map. Then $(-\mu^*, \rho^*, \beta^*, V^*)$ is a representation of (A, P) if and only if the following equations are satisfied:*

$$\mu(x)\beta(v) - \mu(P(x))v - \beta(\mu(x)v) = 0, \tag{15}$$

$$\rho(x)\beta(v) - \rho(P(x))v - \beta(\rho(x)v) = 0, \tag{16}$$

$$-\rho(x \cdot y)v + \rho(y)\mu(x)v + \rho(x)\mu(y)v + \beta(\mu(x \cdot y)v) = 0, \tag{17}$$

for all $x, y \in A, v \in V$.

Proof. For all $x, y \in A, u^* \in V^*, v \in V$, we have

$$\begin{aligned}
&\langle \mu^*(P(x))u^* + \mu^*(x)\beta^*(u^*) - \beta^*(\mu^*(x)u^*), v\rangle \\
&\qquad = \langle u^*, -\mu(P(x))v - \beta(\mu(x)v) + \mu(x)\beta(v)\rangle, \\
&\langle \rho^*(P(x))u^* + \rho^*(x)\beta^*(u^*) - \beta^*(\rho^*(x)u^*), v\rangle \\
&\qquad = \langle u^*, -\rho(P(x))v - \beta(\rho(x)v) + \rho(x)\beta(v)\rangle, \\
&\langle \rho^*(x \cdot y)u^* + \mu^*(x)\rho^*(y)u^* + \mu^*(y)\rho^*(x)u^* - \mu^*(x \cdot y)\beta^*(u^*), v\rangle \\
&\qquad = \langle u^*, -\rho(x \cdot y)v + \rho(y)\mu(x)v + \rho(x)\mu(y)v + \beta(\mu(x \cdot y)v)\rangle.
\end{aligned}$$

Then by Definition 2.1, $(-\mu^*, \rho^*, \beta^*, V^*)$ is a representation of (A, P) if and only if equations (15)–(17) hold. ∎

Definition 2.6. Let (A, P) be a relative Poisson algebra and (μ, ρ, V) be a compatible structure on (A, P). Let $\beta : V \to V$ be a linear map. If $(-\mu^*, \rho^*, \beta^*, V^*)$ is a representation of (A, P), that is, equations (15)–(17) hold, then we say that β *dually represents* (A, P) *on* (μ, ρ, V). When (μ, ρ, V) is taken to be $(\mathcal{L}, \mathrm{ad}, A)$ which is the compatible structure of the relative Poisson algebra (A, P) composing the adjoint representation $(\mathcal{L}, \mathrm{ad}, P, A)$, we simply say that β *dually represents* (A, P).

By Proposition 2.5, we have the following conclusion.

Corollary 2.7. *Let (A, P) be a relative Poisson algebra. A linear map $Q : A \to A$ dually represents (A, P) if and only if the following equations are satisfied:*

$$x \cdot Q(y) - P(x) \cdot y - Q(x \cdot y) = 0, \tag{18}$$

$$[x, Q(y)] - [P(x), y] - Q([x, y]) = 0, \tag{19}$$

$$[x, y \cdot z] + [y, z \cdot x] + [z, x \cdot y] + Q(x \cdot y \cdot z) = 0, \tag{20}$$

for all $x, y, z \in A$.

Proposition 2.8. *Let (A, P) be a relative Poisson algebra, (μ, ρ, α, V) be a representation of (A, P) and $\beta : V \to V$ be a linear map.*

(1) *Equation (15) holds if and only if the following condition holds:*

$$(\alpha + \beta)(\mu(x)v) - \mu(x)(\alpha + \beta)(v) = 0, \qquad \forall x \in A,\ v \in V. \tag{21}$$

(2) *Equation (16) holds if and only if the following condition holds:*

$$(\alpha + \beta)(\rho(x)v) - \rho(x)(\alpha + \beta)(v) = 0, \qquad \forall x \in A,\ v \in V. \tag{22}$$

(3) *Equation (17) holds if and only if the following equation holds:*

$$(\alpha + \beta)(\mu(x \cdot y)v) = 0, \qquad \forall x, y \in A,\ v \in V. \tag{23}$$

(4) *If equation (15) holds, then equation (23) holds if and only if the following equation holds:*

$$\mu(x \cdot y)(\alpha + \beta)v = 0, \qquad \forall x, y \in A,\ v \in V. \tag{24}$$

Proof. Let $x, y \in A$ and $v \in V$.

(1) By equation (11), we have

$$\mu(x)\beta(v) - \mu(P(x))v - \beta(\mu(x)v) = -(\alpha + \beta)(\mu(x)v) + \mu(x)(\alpha + \beta)(v).$$

Hence equation (15) holds if and only if equation (21) holds.

(2) By equation (12), we have

$$\rho(x)\beta(v) - \rho(P(x))v - \beta(\rho(x)v) = -(\alpha + \beta)(\rho(x)v) + \rho(x)(\alpha + \beta)(v).$$

Hence equation (16) holds if and only if equation (22) holds.

(3) We have

$$\begin{aligned}
&-\rho(x \cdot y)v + \rho(y)\mu(x)v + \rho(x)\mu(y)v + \beta(\mu(x \cdot y)v)\\
&\overset{(13)}{=} -\mu(x)\rho(y)v - \mu(y)\rho(x)v + \mu(x \cdot y)\alpha(v) + \rho(y)\mu(x)v\\
&\qquad + \rho(x)\mu(y)v + \beta(\mu(x \cdot y)v)\\
&\overset{(10)}{=} \mu(x \cdot P(y)) - \mu([x, y])v - \rho(y)\mu(x)v + \mu(y \cdot P(x)) - \mu([y, x])v\\
&\qquad - \rho(x)\mu(y)v + \mu(x \cdot y)\alpha(v) + \rho(y)\mu(x)v + \rho(x)\mu(y)v + \beta(\mu(x \cdot y)v)\\
&= \mu(P(x \cdot y))v + \mu(x \cdot y)\alpha(v) + \beta(\mu(x \cdot y)v)\\
&\overset{(11)}{=} (\alpha + \beta)\mu(x \cdot y)v.
\end{aligned}$$

Thus equation (17) holds if and only if equation (23) holds.

(4) If equation (15) holds, then by equation (11) again, we have

$$\begin{aligned}
(\alpha + \beta)\mu(x \cdot y)v &= \mu(x \cdot y)\beta(v) - \mu(P(x \cdot y))v + \mu(P(x \cdot y))v + \mu(x \cdot y)\alpha(v)\\
&= \mu(x \cdot y)(\alpha + \beta)v.
\end{aligned}$$

Hence in this case, equation (23) holds if and only if equation (24) holds. ■

Corollary 2.9. *Let (μ, ρ, α, V) be a representation of a relative Poisson algebra (A, P). Then $-\alpha$ dually represents (A, P) on (μ, ρ, V) automatically, that is, $(-\mu^*, \rho^*, -\alpha^*, V^*)$ is a representation of (A, P). In particular, $-P$ dually represents (A, P).*

Proof. The first part of the conclusion follows from Proposition 2.8 since $\beta = -\alpha$ satisfies equations (21)–(23). The second part of the conclusion follows immediately when (μ, ρ, α, V) is taken to be the adjoint representation $(\mathcal{L}, \mathrm{ad}, P, A)$. ∎

Corollary 2.10. *Let (A, P) be a relative Poisson algebra and $Q : A \to A$ be a linear map. Then Q dually represents (A, P) if and only if equations (18), (19) and the following equation hold:*

$$(P + Q)(x \cdot y \cdot z) = 0, \qquad \forall x, y, z \in A. \tag{25}$$

In particular, if Q dually represents (A, P), then the following equation holds:

$$x \cdot y \cdot (P + Q)(z) = 0, \qquad \forall x, y, z \in A. \tag{26}$$

Proof. It follows from Proposition 2.8 when the representation (μ, ρ, α, V) is taken to be the adjoint representation $(\mathcal{L}, \mathrm{ad}, P, A)$ of (A, P). ∎

At the end of the subsection, we give a relationship between representations of Jacobi algebras and representations of (unital) relative Poisson algebras.

Definition 2.11 ([1]). Let $(A, \cdot, [-, -])$ be a Jacobi algebra. A triple (μ, ρ, V) is called a *representation* of $(A, \cdot, [-, -])$ if (μ, V) is a representation of the unital commutative associative algebra $(A, \cdot)$, (ρ, V) is a representation of the Lie algebra $(A, [-, -])$, and they satisfy the following conditions:

$$\rho(x \cdot y)v - \mu(x)\rho(y)v - \mu(y)\rho(x)v + \mu(x \cdot y)\rho(1_A)v = 0, \tag{27}$$

$$\rho(y)\mu(x)v - \mu(x)\rho(y)v + \mu([x, y])v - \mu(x \cdot [1_A, y])v = 0, \tag{28}$$

for all $x, y \in A$, $v \in V$.

The following conclusion is a direct consequence of [1, Theorem 3.2].

Proposition 2.12. *Suppose that $(A, \cdot, [-, -])$ is a Jacobi algebra, V is a vector space and $\mu, \rho : A \to \mathrm{End}(V)$ are linear maps. Then (μ, ρ, V) is a representation of $(A, \cdot, [-, -])$ if and only if there is a Jacobi algebra structure on the direct sum $A \oplus V$ of vector spaces, where the bilinear operations $\cdot$ and $[-, -]$ on $A \oplus V$ are given by equations (8) and (9) respectively. We denote this Jacobi algebra structure on $A \oplus V$ by $A \ltimes_{\mu,\rho} V$.*

Proposition 2.13. *Let $(A, \cdot, [-, -])$ be a Jacobi algebra and $(A, \cdot, [-, -], \mathrm{ad}(1_A))$ be the corresponding unital relative Poisson algebra. Then (μ, ρ, V) is a representation of the Jacobi algebra $(A, \cdot, [-, -])$ if and only if $(\mu, \rho, \rho(1_A), V)$ is a representation of the unital relative Poisson algebra $(A, \cdot, [-, -], \mathrm{ad}(1_A))$.*

Proof. Suppose that (μ, ρ, V) is a representation of the Jacobi algebra $(A, \cdot, [-,-])$. Then $A \ltimes_{\mu,\rho} V$ is also a Jacobi algebra with the unit 1_A. Since

$$[1_A, x+u] = [1_A, x] + \rho(1_A)u = (\mathrm{ad}(1_A) + \rho(1_A))(x+u), \qquad \forall x \in A,\ v \in V,$$

we have that $(A \ltimes_{\mu,\rho} V, \mathrm{ad}(1_A) + \rho(1_A))$ is a unital relative Poisson algebra. Hence by Proposition 2.2, $(\mu, \rho, \rho(1_A), V)$ is a representation of the unital relative Poisson algebra $(A, \cdot, [-,-], \mathrm{ad}(1_A))$. Conversely, by a similar proof, the conclusion is obtained. ■

Corollary 2.14. *Let (μ, ρ, V) be a representation of a Jacobi algebra $(A, \cdot, [-,-])$. Then $(-\mu^*, \rho^*, V^*)$ is also a representation of $(A, \cdot, [-,-])$.*

Proof. Since

$$\langle \rho^*(1_A)u^*, v\rangle = \langle u^*, -\rho(1_A)v\rangle = \langle -(\rho(1_A))^* u^*, v\rangle, \qquad \forall u^* \in V^*,\ v \in V,$$

we have $\rho^*(1_A) = -(\rho(1_A))^*$. If (μ, ρ, V) is a representation of a Jacobi algebra $(A, \cdot, [-,-])$, then by Proposition 2.13, $(\mu, \rho, \rho(1_A), V)$ is a representation of the unital relative Poisson algebra $(A, \cdot, [-,-], \mathrm{ad}(1_A))$. By Corollary 2.9,

$$(-\mu^*, \rho^*, -(\rho(1_A))^* = \rho^*(1_A), V^*)$$

is a representation of the unital relative Poisson algebra $(A, \cdot, [-,-], \mathrm{ad}(1_A))$. By Proposition 2.13 again, $(-\mu^*, \rho^*, V^*)$ is a representation of the Jacobi algebra $(A, \cdot, [-,-])$. ■

Remark 2.15. Note that for a representation (μ, ρ, V) of a Jacobi algebra $(A, \cdot, [-,-])$, $(-\mu^*, \rho^*, V^*)$ is automatically a representation of $(A, \cdot, [-,-])$ without any additional condition, whereas in [1], there is the following additional equation for $(-\mu^*, \rho^*, V^*)$ to be a representation of the Jacobi algebra $(A, \cdot, [-,-])$:

$$-\rho(x \cdot y)v + \rho(y)\mu(x)v + \rho(x)\mu(y)v - \rho(1_A)\mu(x \cdot y)v = 0, \qquad \forall x, y \in A,\ v \in V. \tag{29}$$

In fact, equation (29) is not needed, that is, by a direct proof, equation (29) can be obtained by equations (27) and (28).

2.2. Matched pairs of relative Poisson algebras

We first recall the notions of matched pairs of commutative associative algebras ([7]) and Lie algebras ([33]) respectively.

Let $(A_1, \cdot_1)$ and $(A_2, \cdot_2)$ be two commutative associative algebras, and (μ_1, A_2) and (μ_2, A_1) be representations of $(A_1, \cdot_1)$ and $(A_2, \cdot_2)$ respectively. If the following equations are satisfied:

$$\mu_1(x)(a \cdot_2 b) = (\mu_1(x)a) \cdot_2 b + \mu_1(\mu_2(a)x)b,$$
$$\mu_2(a)(x \cdot_1 y) = (\mu_2(a)x) \cdot_1 y + \mu_2(\mu_1(x)a)y,$$

for all $x, y \in A_1, a, b \in A_2$, then (A_1, A_2, μ_1, μ_2) is called a *matched pair of commutative associative algebras*. In this case, there exists a commutative associative algebra structure on the vector space $A_1 \oplus A_2$ given by

$$(x + a) \cdot (y + b) = x \cdot_1 y + \mu_2(a)y + \mu_2(b)x + a \cdot_2 b + \mu_1(x)b + \mu_1(y)a, \qquad \forall x, y \in A_1,\ a, b \in A_2. \quad (30)$$

Moreover, every commutative associative algebra which is the direct sum of the underlying vector spaces of two subalgebras can be obtained from a matched pair of commutative associative algebras.

Let $(A_1, [-,-]_1)$ and $(A_2, [-,-]_2)$ be two Lie algebras, and (ρ_1, A_2) and (ρ_2, A_1) be representations of $(A_1, [-,-]_1)$ and $(A_2, [-,-]_2)$ respectively. If the following equations are satisfied:

$$\rho_1(x)[a,b]_2 - [\rho_1(x)a, b]_2 - [a, \rho_1(x)b]_2 + \rho_1(\rho_2(a)x)b - \rho_1(\rho_2(b)x)a = 0,$$
$$\rho_2(a)[x,y]_1 - [\rho_2(a)x, y]_1 - [x, \rho_2(a)y]_1 + \rho_2(\rho_1(x)a)y - \rho_2(\rho_1(y)a)x = 0,$$

for all $x, y \in A_1, a, b \in A_2$, then $(A_1, A_2, \rho_1, \rho_2)$ is called a *matched pair of Lie algebras*. In this case, there is a Lie algebra structure on the vector space $A_1 \oplus A_2$ given by

$$[x + a, y + b] = [x, y]_1 + \rho_2(a)y - \rho_2(b)x + [a, b]_2 + \rho_1(x)b - \rho_1(y)a, \qquad \forall x, y \in A_1, a, b \in A_2. \quad (31)$$

Moreover, every Lie algebra which is the direct sum of the underlying vector spaces of two subalgebras can be obtained from a matched pair of Lie algebras.

Now we consider the case of relative Poisson algebras.

Definition 2.16. Let $(A_1, \cdot_1, [-,-]_1, P_1)$ and $(A_2, \cdot_2, [-,-]_2, P_2)$ be two relative Poisson algebras. Suppose that $(\mu_1, \rho_1, P_2, A_2)$ is a representation of $(A_1, \cdot_1, [-,-]_1, P_1)$ and $(\mu_2, \rho_2, P_1, A_1)$ is a representation of $(A_2, \cdot_2, [-,-]_2, P_2)$, such that (A_1, A_2, μ_1, μ_2) is a matched pair of commutative associative algebras and $(A_1, A_2, \rho_1, \rho_2)$ is a matched pair of Lie algebras. Suppose that the following compatible conditions are satisfied:

$$\rho_2(a)(x \cdot_1 y) + \mu_2(\rho_1(y)a)x - x \cdot_1 \rho_2(a)y + \mu_2(\rho_1(x)a)y - y \cdot_1 \rho_2(a)x - \mu_2(P_2(a))(x \cdot_1 y) = 0, \quad (32)$$
$$\rho_1(x)(a \cdot_2 b) + \mu_1(\rho_2(b)x)a - a \cdot_2 \rho_1(x)b + \mu_1(\rho_2(a)x)b - b \cdot_2 \rho_1(x)a - \mu_1(P_1(x))(a \cdot_2 b) = 0, \quad (33)$$
$$\rho_2(\mu_1(x)a)y + [\mu_2(a)x, y]_1 - x \cdot_1 \rho_2(a)y + \mu_2(\rho_1(y)a)x - \mu_2(a)([x, y]_1) + \mu_2(a)(x \cdot_1 P_1(y)) = 0, \quad (34)$$
$$\rho_1(\mu_2(a)x)b + [\mu_1(x)a, b]_2 - a \cdot_2 \rho_1(x)b + \mu_1(\rho_2(b)x)a - \mu_1(x)([a, b]_2) + \mu_1(x)(a \cdot_2 P_2(b)) = 0, \quad (35)$$

for all $x, y \in A_1, a, b \in A_2$. Such a structure is called a *matched pair of relative Poisson algebras* (A_1, P_1) and (A_2, P_2). We denote it by $((A_1, P_1), (A_2, P_2), \mu_1, \rho_1, \mu_2, \rho_2)$.

Proposition 2.17. *Suppose that* (A_1, P_1) *and* (A_2, P_2) *are relative Poisson algebras. For linear maps* $\mu_1, \rho_1 : A_1 \to \mathrm{End}(A_2)$ *and* $\mu_2, \rho_2 : A_2 \to \mathrm{End}(A_1)$*, define two bilinear operations* $\cdot$ *and* $[-,-]$ *on* $A_1 \oplus A_2$ *by equations* (30) *and* (31) *respectively. Then* $(A_1 \oplus A_2, \cdot, [-,-], P_1 + P_2)$ *is a relative Poisson algebra if and only if* $((A_1, P_1), (A_2, P_2), \mu_1, \rho_1, \mu_2, \rho_2)$ *is a matched pair of relative Poisson algebras. In this case, we denote this relative Poisson algebra by* $(A_1 \bowtie A_2, P_1 + P_2)$*. Moreover, every relative Poisson algebra which is the direct sum of the underlying vector spaces of two subalgebras can be obtained from a matched pair of relative Poisson algebras.*

Proof. It is known that $(A_1 \oplus A_2, \cdot)$ is a commutative associative algebra if and only if (A_1, A_2, μ_1, μ_2) is a matched pair of commutative associative algebras and $(A_1 \oplus A_2, [-,-])$ is a Lie algebra if and only if $(A_1, A_2, \rho_1, \rho_2)$ is a matched pair of Lie algebras.

Let $x, y \in A_1, a, b \in A_2$. By equation (31), we have

$$\begin{aligned}(P_1 + P_2)([x + a, y + b]) &= P_1([x, y]_1 + \rho_2(a)y - \rho_2(b)x)\\ &\quad + P_2([a, b]_2 + \rho_1(x)b - \rho_1(y)a),\\ [(P_1 + P_2)(x + a), y + b] &= [P_1(x), y]_1 + \rho_2(P_2(a)y) - \rho_2(b)P_1(x)\\ &\quad + [P_2(a), b]_2 + \rho_1(P_1(x))b - \rho_1(y)P_2(a),\\ [x + a, (P_1 + P_2)(y + b)] &= [x, P_1(y)]_1 + \rho_2(a)P_1(y) - \rho_2(P_2(b))x\\ &\quad + [a, P_2(b)]_2 + \rho_1(x)P_2(b) - \rho_1(P_1(y))a.\end{aligned}$$

If $P_1 + P_2$ is a derivation of the Lie algebra $(A_1 \oplus A_2, [-,-])$, then P_1 and P_2 are derivations of the Lie algebras $(A_1, [-,-]_1)$ and $(A_2, [-,-]_2)$ by taking $a = b = 0$ and $x = y = 0$ respectively. Moreover, equation (12) holds for $(\mu_1, \rho_1, P_2, A_2)$ and $(\mu_2, \rho_2, P_1, A_1)$ as the representations of (A_1, P_1) and (A_2, P_2) by taking $a = y = 0$ and $x = b = 0$ respectively. Conversely, if P_1 and P_2 are derivations of the Lie algebras $(A_1, [-,-]_1)$ and $(A_2, [-,-])$ respectively and $(\mu_1, \rho_1, P_2, A_2)$ and $(\mu_2, \rho_2, P_1, A_1)$ are representations of (A_1, P_1) and (A_2, P_2) respectively, then by equation (12), $P_1 + P_2$ is a derivation of the Lie algebra $(A_1 \oplus A_2, [-,-])$. Similarly, $P_1 + P_2$ is a derivation of the commutative associative algebra $(A_1 \oplus A_2, \cdot)$ if and only if P_1 and P_2 are derivations of the commutative associative algebras $(A_1, \cdot_1)$ and $(A_2, \cdot_2)$ respectively and equation (11) holds for $(\mu_1, \rho_1, P_2, A_2)$ and $(\mu_2, \rho_2, P_1, A_1)$ as the representations of (A_1, P_1) and (A_2, P_2) respectively.

Let $x, y, z \in A_1, a, b, c \in A_2$. We consider the relative Leibniz rule (3) on $A_1 \oplus A_2$,

$$\begin{aligned}[z + c, (x + a) \cdot (y + b)] &= (x + a) \cdot [z + c, y + b] + [z + c, x + a] \cdot (y + b)\\ &\quad + (x + a) \cdot (y + b) \cdot (P_1 + P_2)(z + c). \end{aligned} \tag{36}$$

If equation (36) holds, then equation (3) holds for (A_1, P_1) and (A_2, P_2) as relative Poisson algebras and the following equations hold:

$$[c, x \cdot_1 y] = x \cdot [c, y] + [c, x] \cdot y + (x \cdot_1 y) \cdot P_2(c), \tag{37}$$

$$[z, a \cdot_2 b] = a \cdot [z, b] + [z, a] \cdot b + (a \cdot_2 b) \cdot P_1(z), \tag{38}$$

$$[z, x \cdot b] = x \cdot [z, b] + [z, x]_1 \cdot b + b \cdot (x \cdot_1 P_1(z)), \tag{39}$$
$$[c, x \cdot b] = b \cdot [c, x] + [c, b]_2 \cdot x + x \cdot (b \cdot_2 P_2(c)), \tag{40}$$

by taking $a = b = c = 0, x = y = z = 0, a = b = z = 0, x = y = c = 0, a = y = c = 0,$ $a = y = z = 0$ respectively in equation (36). Conversely, since (A_1, P_1) and (A_2, P_2) are relative Poisson algebras, if equations (37)–(40) hold, then it is straightforward to show that equation (36) holds on $A_1 \oplus A_2$. Moreover, we have

$$\begin{aligned}
[c, x \cdot_1 y] &= \rho_2(c)(x \cdot_1 y) - \rho_1(x \cdot_1 y)c,\\
x \cdot [c, y] &= x \cdot (\rho_2(c)y - \rho_1(y)c) = x \cdot_1 \rho_2(c)y - \mu_1(x)\rho_1(y)c - \mu_2(\rho_1(y)c)x,\\
[c, x] \cdot y &= (\rho_2(c)x - \rho_1(x)c) \cdot y = \rho_2(c)x \cdot_1 y - \mu_1(y)\rho_1(x)c - \mu_2(\rho_1(x)c)y,\\
(x \cdot_1 y) \cdot P_2(c) &= \mu_1(x \cdot_1 y)P_2(c) + \mu_2(P_2(c))(x \cdot_1 y).
\end{aligned}$$

Therefore equation (37) holds if and only if equation (32) by replacing a by c, and equation (13) for $(\mu_1, \rho_1, P_2, A_2)$ as a representation of (A_1, P_1) hold. Similarly, we have

(1) equation (38) holds if and only if equation (33) by replacing x by z, and equation (13) for $(\mu_2, \rho_2, P_1, A_1)$ as a representation of (A_2, P_2) hold;
(2) equation (39) holds if and only if equation (34) by replacing a by b, y by z, and equation (10) for $(\mu_1, \rho_1, P_2, A_2)$ as a representation of (A_1, P_1) hold;
(3) equation (40) holds if and only if equation (35) by replacing a by b, b by c, and equation (10) for $(\mu_2, \rho_2, P_1, A_1)$ as a representation of (A_2, P_2) hold.

Hence the conclusion holds. ■

Example 2.18. Let $(A_1, \cdot_1)$ and $(A_2, \cdot_2)$ be two commutative associative algebras, and P_1 and P_2 be their derivations respectively. Let $(A_1, [-,-]_1)$ and $(A_2, [-,-]_2)$ be the Lie algebras defined by equation (4) respectively. Hence $(A_1, \cdot_1, [-,-]_1, P_1)$ and $(A_2, \cdot_2, [-,-]_2, P_2)$ are relative Poisson algebras. Suppose that there are linear maps $\mu_1 : A_1 \to \mathrm{End}(A_2)$ and $\mu_2 : A_2 \to \mathrm{End}(A_1)$, such that (A_1, A_2, μ_1, μ_2) is a matched pair of commutative associative algebras and the following conditions hold:

$$P_2(\mu_1(x)a) - \mu_1(P_1(x))a - \mu_1(x)P_2(a) = 0, \tag{41}$$
$$P_1(\mu_2(a)x) - \mu_2(P_2(a))x - \mu_2(a)P_1(x) = 0, \tag{42}$$

for all $x \in A_1, a \in A_2$. Then $P_1 + P_2$ is a derivation of the resulting commutative associative algebra $(A_1 \oplus A_2, \cdot)$ defined by equation (30). Moreover, there is a Lie bracket $[-,-]$ on $A_1 \oplus A_2$ given by

$$\begin{aligned}
[x + a, y + b] &= (x + a) \cdot (P_1(y) + P_2(b)) - (P_1(x) + P_2(a)) \cdot (y + b)\\
&= x \cdot_1 P_1(y) + \mu_2(a)P_1(y) + \mu_2(P_2(b))x + a \cdot_2 P_2(b)\\
&\quad + \mu_1(x)P_2(b) + \mu_1(P_1(y))a - \big(P_1(x) \cdot y + \mu_2(P_2(a))y\\
&\quad + \mu_2(b)P_1(x) + P_2(a) \cdot_2 b + \mu_1(P_1(x))b + \mu_1(y)P_2(a)\big)\\
&= [x, y]_1 + \rho_2(a)y - \rho_2(b)x + [a, b]_2 + \rho_1(x)b - \rho_1(y)a,
\end{aligned}$$

for all $x, y \in A_1, a, b \in A_2$. Here $\rho_1 : A_1 \to \mathrm{End}(A_2)$ and $\rho_2 : A_2 \to \mathrm{End}(A_1)$ are linear maps given by

$$\rho_1(x)a = \mu_1(x)P_2(a) - \mu_1(P_1(x))a,$$
$$\rho_2(a)x = \mu_2(a)P_1(x) - \mu_2(P_2(a))x,$$

for all $x \in A_1$, $a \in A_2$. Then by Example 1.2, $(A_1 \oplus A_2, \cdot, [-,-], P_1 + P_2)$ is a relative Poisson algebra. Hence by Proposition 2.17, $((A_1, P_1), (A_2, P_2), \mu_1, \rho_1, \mu_2, \rho_2)$ is a matched pair of relative Poisson algebras.

Remark 2.19. We would like to point out that there is not a "matched pair theory" for Jacobi algebras or unital commutative associative algebras due to the appearance of the units. In fact, if a unital commutative associative algebra $(A \oplus B, 1_{A\oplus B})$ is decomposed into the direct sum of the underlying spaces of two unital commutative associative algebras $(A, 1_A)$ and $(B, 1_B)$, then there are representations (μ_B, A) and (μ_A, B) of the commutative associative algebras B and A respectively such that equation (30) gives the commutative associative algebra structure on $A \oplus B$ due to the matched pairs of commutative associative algebras. Suppose that $1_{A\oplus B} = a + b$, where $a \in A, b \in B$. Then

$$1_A = 1_A \cdot 1_{A\oplus B} = 1_A \cdot (a + b) = a + \mu_A(1_A)b + \mu_B(b)1_A = a + b + \mu_B(b)1_A.$$

Therefore $b = 0$ and $a = 1_A$. Thus $1_{A\oplus B} = 1_A$. Similarly, $1_{A\oplus B} = 1_B$. Hence $1_{A\oplus B} \in A \cap B = \{0\}$, which is a contradiction.

3. Relative Poisson bialgebras

We introduce the notions of Manin triples of relative Poisson algebras and relative Poisson bialgebras. The equivalence between them is interpreted in terms of certain matched pairs of relative Poisson algebras.

3.1. Frobenius relative Poisson algebras and Manin triples of relative Poisson algebras

We generalize the notion of Frobenius Jacobi algebras ([1]) to the following notion of Frobenius relative Poisson algebras.

Definition 3.1. A bilinear form $\mathcal{B}$ on a relative Poisson algebra (A, P) is called *invariant* if it satisfies the following equations:

$$\mathcal{B}(x \cdot y, z) = \mathcal{B}(x, y \cdot z), \tag{43}$$
$$\mathcal{B}([x, y], z) = \mathcal{B}(x, [y, z]), \tag{44}$$

for all $x, y, z \in A$. A relative Poisson algebra (A, P) is called *Frobenius* if there is a nondegenerate invariant bilinear form $\mathcal{B}$ on (A, P), which is denoted by $(A, P, \mathcal{B})$.

Let $\mathcal{B}$ be a nondegenerate bilinear form on a relative Poisson algebra (A, P). Then there is a unique map $\widehat{P} : A \to A$ given by

$$\mathcal{B}(P(x), y) = \mathcal{B}(x, \widehat{P}(y)), \qquad \forall x, y \in A,$$

that is, $\widehat{P}$ is the adjoint linear transformation of P under the nondegenerate bilinear form $\mathcal{B}$.

We have the following characterization of Frobenius relative Poisson algebras.

Proposition 3.2. *Let $(A, P, \mathcal{B})$ be a Frobenius relative Poisson algebra. Let $\widehat{P}$ be the adjoint map of P with respect to $\mathcal{B}$. Then $\widehat{P}$ dually represents (A, P), that is, $(-\mathcal{L}^*, \mathrm{ad}^*, \widehat{P}^*, A^*)$ is a representation of (A, P). Furthermore, as representations of (A, P), $(-\mathcal{L}^*, \mathrm{ad}^*, \widehat{P}^*, A^*)$ and $(\mathcal{L}, \mathrm{ad}, P, A)$ are equivalent. Conversely, let (A, P) be a relative Poisson algebra and $Q : A \to A$ be a linear map. If $(-\mathcal{L}^*, \mathrm{ad}^*, Q^*, A^*)$ is a representation which is equivalent to $(\mathcal{L}, \mathrm{ad}, P, A)$, then there exists a nondegenerate bilinear form $\mathcal{B}$ such that $(A, P, \mathcal{B})$ is a Frobenius relative Poisson algebra and $Q = \widehat{P}$.*

Proof. Let $x, y, z, w \in A$. Since P is a derivation of the Lie algebra $(A, [-,-])$, we have

$$\begin{aligned} 0 &= \mathcal{B}(P([x, y]) - [P(x), y] - [x, P(y)], z) \\ &= \mathcal{B}([x, y], \widehat{P}(z)) - \mathcal{B}(P(x), [y, z]) - \mathcal{B}(x, [P(y), z]) \\ &= \mathcal{B}(x, [y, \widehat{P}(z)] - \widehat{P}([y, z]) - [P(y), z]). \end{aligned}$$

Thus by the nondegeneracy of $\mathcal{B}$, equation (19) holds. Similarly, equation (18) holds since P is a derivation of the commutative associative algebra $(A, \cdot)$. Moreover, by equation (3), we have

$$\begin{aligned} 0 &= \mathcal{B}([x \cdot y, z] - x \cdot [y, z] - [x, z] \cdot y + x \cdot y \cdot P(z), w) \\ &= \mathcal{B}(z, [w, x \cdot y] + [x, y \cdot w] + [y, w \cdot x] + \widehat{P}(w \cdot x \cdot y)). \end{aligned}$$

Thus equation (20) holds. Hence $\widehat{P}$ dually represents (A, P). Define a linear map $\varphi : A \to A^*$ by

$$\langle \varphi(x), y \rangle = \mathcal{B}(x, y), \qquad \forall x, y \in A. \tag{45}$$

By the nondegeneracy of $\mathcal{B}$, φ is a linear isomorphism. Moreover, we have

$$\begin{aligned} \langle \varphi(\mathrm{ad}(x)y), z \rangle = \langle \varphi([x, y]), z \rangle = \mathcal{B}([x, y], z) &= -\mathcal{B}(y, [x, z]) \\ = -\langle \varphi(y), [x, z] \rangle &= \langle \mathrm{ad}^*(x)\varphi(y), z \rangle. \end{aligned}$$

Thus $\varphi \mathrm{ad}(x) = \mathrm{ad}^*(x)\varphi$ for all $x \in A$. Similarly $\varphi\mathcal{L}(x) = -\mathcal{L}^*(x)\varphi$ for all $x \in A$. Moreover, we have

$$\langle \varphi(P(x)), y \rangle = \mathcal{B}(P(x), y) = \mathcal{B}(x, \widehat{P}(y)) = \langle \varphi(x), \widehat{P}(y) \rangle = \langle \widehat{P}^*(\varphi(x)), y \rangle.$$

Hence $\varphi P = \widehat{P}^*\varphi$. Therefore as representations of (A, P), $(-\mathcal{L}^*, \mathrm{ad}^*, \widehat{P}^*, A^*)$ and $(\mathcal{L}, \mathrm{ad}, P, A)$ are equivalent.

Conversely, suppose that $\varphi : A \to A^*$ is the linear isomorphism giving the equivalence between the two representations $(-\mathcal{L}^*, \mathrm{ad}^*, Q^*, A^*)$ and $(\mathcal{L}, \mathrm{ad}, P, A)$. Define a bilinear form $\mathcal{B}$ on A by equation (45). Then by a similar proof as above, $\mathcal{B}$ is a nondegenerate invariant bilinear form on (A, P) such that $Q = \widehat{P}$. ■

Definition 3.3. Let (A, P) be a relative Poisson algebra. Suppose that (A^*, Q^*) is a relative Poisson algebra. If there is a relative Poisson algebra structure on the direct sum $A \oplus A^*$ of vector spaces such that $(A \oplus A^*, P + Q^*, \mathcal{B}_d)$ is a Frobenius relative Poisson algebra, where $\mathcal{B}_d$ is given by

$$\mathcal{B}_d(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \qquad \forall x, y \in A,\ a^*, b^* \in A^*, \tag{46}$$

and both (A, P) and (A^*, Q^*) are relative Poisson subalgebras, then $((A \oplus A^*, P + Q^*, \mathcal{B}_d), (A, P), (A^*, Q^*))$ is called a *Manin triple of relative Poisson algebras*. We denote it by $((A \bowtie A^*, P + Q^*, \mathcal{B}_d), (A, P), (A^*, Q^*))$.

The notation $A \bowtie A^*$ is justified since the relative Poisson algebra structure on $A \oplus A^*$ comes from a matched pair of relative Poisson algebras (A, P) and (A^*, Q^*) in Proposition 2.17.

Lemma 3.4. *Let $((A \bowtie A^*, P + Q^*, \mathcal{B}_d), (A, P), (A^*, Q^*))$ be a Manin triple of relative Poisson algebras.*

1. *The adjoint $\widehat{P + Q^*}$ of $P + Q^*$ with respect to $\mathcal{B}_d$ is $Q + P^*$. Further $Q + P^*$ dually represents $(A \bowtie A^*, P + Q^*)$.*
2. *Q dually represents (A, P).*
3. *P^* dually represents (A^*, Q^*).*

Proof. Let $x, y \in A, a^*, b^* \in A^*$. Then we have

$$\begin{aligned}
\mathcal{B}_d((P + Q^*)(x + a^*), y + b^*) &= \mathcal{B}_d(P(x) + Q^*(a^*), y + b^*) \\
&= \langle P(x), b^* \rangle + \langle Q^*(a^*), y \rangle \\
&= \langle x, P^*(b^*) \rangle + \langle a^*, Q(y) \rangle \\
&= \mathcal{B}_d(x + a^*, (Q + P^*)(y + b^*)).
\end{aligned}$$

Hence the adjoint $\widehat{P + Q^*}$ of $P + Q^*$ with respect to $\mathcal{B}_d$ is $Q + P^*$. Then by Proposition 3.2, $Q + P^*$ dually represents $(A \bowtie A^*, P + Q^*)$. By Corollary 2.10, it holds if and only if for all $x, y, z \in A, a^*, b^*, c^* \in A^*$,

$$\begin{aligned}
&(x + a^*) \cdot (Q + P^*)(y + b^*) - (P + Q^*)(x + a^*) \cdot (y + b^*) \\
&\qquad - (Q + P^*)((x + a^*) \cdot (y + b^*)) = 0, \\
&[x + a^*, (Q + P^*)(y + b^*)] - [(P + Q^*)(x + a^*), y + b^*] \\
&\qquad - (Q + P^*)[(x + a^*) \cdot (y + b^*)] = 0, \\
&(P + Q^* + Q + P^*)((x + a^*) \cdot (y + b^*) \cdot (z + c^*)) = 0.
\end{aligned}$$

Now taking $a^* = b^* = c^* = 0$ in the above equations, we get equations (18), (19) and (25). Hence Q dually represents (A, P). Similarly, P^* dually represents (A^*, Q^*) by taking $x = y = z = 0$. ∎

Theorem 3.5. *Let $(A, \cdot_A, [-,-]_A, P)$ be a relative Poisson algebra. Suppose that there is a relative Poisson algebra structure $(A^*, \cdot_{A^*}, [-,-]_{A^*}, Q^*)$ on the dual space A^*. Then there is a Manin triple of relative Poisson algebras $((A \bowtie A^*, P + Q^*, \mathcal{B}_d), (A, P), (A^*, Q^*))$ if and only if $((A, P), (A^*, Q^*), -\mathcal{L}_A^*, \mathrm{ad}_A^*, -\mathcal{L}_{A^*}^*, \mathrm{ad}_{A^*}^*)$ is a matched pair of relative Poisson algebras.*

Proof. It is known in [14] that there is a Lie algebra structure on the direct sum $A \oplus A^*$ of vector spaces such that both $(A, [-,-]_A)$ and $(A^*, [-,-]_{A^*})$ are Lie subalgebras and the bilinear form $\mathcal{B}_d$ on $A \oplus A^*$ given by equation (46) satisfies equation (44) if and only if $(A, A^*, \mathrm{ad}_A^*, \mathrm{ad}_{A^*}^*)$ is a matched pair of Lie algebras. Similarly, by [7], there is a commutative associative algebra structure on $A \oplus A^*$ such that both $(A, \cdot_A)$ and $(A^*, \cdot_{A^*})$ are commutative associative subalgebras and $\mathcal{B}_d$ satisfies equation (43) if and only if $(A, A^*, -\mathcal{L}_A^*, -\mathcal{L}_{A^*}^*)$ is a matched pair of commutative associative algebras. Hence if $((A \bowtie A^*, P + Q^*, \mathcal{B}_d), (A, P), (A^*, Q^*))$ is a Manin triple of relative Poisson algebras, then by Proposition 2.17 with

$$
\begin{aligned}
&A_1 = A, && P_1 = P, && A_2 = A^*, && P_2 = Q^*, \\
&\mu_1 = -\mathcal{L}_A^*, && \rho_1 = \mathrm{ad}_A^*, && \mu_2 = -\mathcal{L}_{A^*}^*, && \rho_2 = \mathrm{ad}_{A^*}^*,
\end{aligned}
$$

$((A, P), (A^*, Q^*), -\mathcal{L}_A^*, \mathrm{ad}_A^*, -\mathcal{L}_{A^*}^*, \mathrm{ad}_{A^*}^*)$ is a matched pair of relative Poisson algebras. Conversely, if $((A, P), (A^*, Q^*), -\mathcal{L}_A^*, \mathrm{ad}_A^*, -\mathcal{L}_{A^*}^*, \mathrm{ad}_{A^*}^*)$ is a matched pair of relative Poisson algebras, then by Proposition 2.17 again, there is a relative Poisson algebra $(A \bowtie A^*, P + Q^*)$ obtained from the matched pair with both (A, P) and (A^*, Q^*) as relative Poisson subalgebras. Moreover, the bilinear form $\mathcal{B}_d$ is invariant on $(A \bowtie A^*, P + Q^*)$. Hence $((A \bowtie A^*, P + Q^*, \mathcal{B}_d), (A, P), (A^*, Q^*))$ is a Manin triple of relative Poisson algebras. ∎

3.2. Relative Poisson bialgebras

We recall the notions of (commutative and cocommutative) infinitesimal bialgebras ([7]) and Lie bialgebras ([14]) before we introduce the notion of relative Poisson bialgebras.

Definition 3.6. A *cocommutative coassociative coalgebra* is a pair (A, Δ), such that A is a vector space and $\Delta : A \to A \otimes A$ is a linear map satisfying

$$
\begin{aligned}
\tau\Delta &= \Delta, \\
(\mathrm{id} \otimes \Delta)\Delta &= (\Delta \otimes \mathrm{id})\Delta,
\end{aligned}
$$

where $\tau : A \otimes A \to A \otimes A$ is the exchanging operator defined as $\tau(x \otimes y) = y \otimes x$, for all $x, y \in A$.

Definition 3.7. A *commutative and cocommutative infinitesimal bialgebra* is a triple $(A, \cdot, \Delta)$ such that

(1) $(A, \cdot)$ is a commutative associative algebra;

(2) (A, Δ) is a cocommutative coassociative coalgebra;

(3) Δ satisfies the following condition:

$$\Delta(x \cdot y) = (\mathcal{L}(x) \otimes \mathrm{id})\Delta(y) + (\mathrm{id} \otimes \mathcal{L}(y))\Delta(x), \qquad \forall x, y \in A. \tag{47}$$

Definition 3.8. A *Lie coalgebra* is a pair (A, δ), such that A is a vector space and $\delta : A \to A \otimes A$ is a linear map satisfying

$$\tau\delta = -\delta,$$
$$(\mathrm{id} + \xi + \xi^2)(\mathrm{id} \otimes \delta)\delta = 0,$$

where $\xi : A \otimes A \otimes A \to A \otimes A \otimes A$ is the linear map defined as $\xi(x \otimes y \otimes z) = y \otimes z \otimes x$, for all $x, y, z \in A$.

Definition 3.9. A *Lie bialgebra* is a triple $(A, [-,-], \delta)$, such that

(1) $(A, [-,-])$ is a Lie algebra;

(2) (A, δ) is a Lie coalgebra;

(3) δ is a 1-cocycle of $(A, [-,-])$ with values in $A \otimes A$, that is,

$$\begin{aligned}\delta([x, y]) = {} & (\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))\delta(y) \\ & - (\mathrm{ad}(y) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(y))\delta(x), \qquad \forall x, y \in A.\end{aligned} \tag{48}$$

Now we give the definitions of a relative Poisson coalgebra and a relative Poisson bialgebra.

Definition 3.10. Let A be a vector space, and $\Delta, \delta : A \to A \otimes A$ and $Q : A \to A$ be linear maps. Then (A, Δ, δ, Q) is called a *relative Poisson coalgebra* if (A, Δ) is a cocommutative coassociative coalgebra, (A, δ) is a Lie coalgebra, and the following conditions are satisfied:

$$\Delta Q = (Q \otimes \mathrm{id} + \mathrm{id} \otimes Q)\Delta, \tag{49}$$

$$\delta Q = (Q \otimes \mathrm{id} + \mathrm{id} \otimes Q)\delta, \tag{50}$$

$$\begin{aligned}&(\mathrm{id} \otimes \Delta)\delta(x) - (\delta \otimes \mathrm{id})\Delta(x) - (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \delta)\Delta(x) \\ &\quad - (Q \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta(x) = 0, \qquad \forall x \in A.\end{aligned} \tag{51}$$

Proposition 3.11. *Under the finite-dimensional assumption,* (A, Δ, δ, Q) *is a relative Poisson coalgebra if and only if* $(A^*, \Delta^*, \delta^*, Q^*)$ *is a relative Poisson algebra.*

Proof. Clearly, (A, Δ) is a cocommutative coassociative coalgebra if and only if (A^*, Δ^*) is a commutative associative algebra and (A, δ) is a Lie coalgebra if and only if (A^*, δ^*) is a Lie algebra.

For all $a^*, b^* \in A^*$, set $\Delta^*(a^* \otimes b^*) = a^* \cdot b^*$, $\delta^*(a^* \otimes b^*) = [a^*, b^*]$. Then we have

$$\begin{aligned}
\langle (\mathrm{id} \otimes \Delta)\delta(x), a^* \otimes b^* \otimes c^* \rangle &= \langle x, \delta^*(\mathrm{id} \otimes \Delta^*)(a^* \otimes b^* \otimes c^*) \rangle \\
&= \langle x, [a^*, b^* \cdot c^*] \rangle, \\
\langle (\delta \otimes \mathrm{id})\Delta(x), a^* \otimes b^* \otimes c^* \rangle &= \langle x, \Delta^*(\delta^* \otimes \mathrm{id})(a^* \otimes b^* \otimes c^*) \rangle \\
&= \langle x, [a^*, b^*] \cdot c^* \rangle, \\
\langle (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \delta)\Delta(x), a^* \otimes b^* \otimes c^* \rangle & \\
&= \langle x, \Delta^*(\mathrm{id} \otimes \delta^*)(\tau \otimes \mathrm{id})(a^* \otimes b^* \otimes c^*) \rangle \\
&= \langle x, b^* \cdot [a^*, c^*] \rangle, \\
\langle (Q \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta(x), a^* \otimes b^* \otimes c^* \rangle & \\
&= \langle x, \Delta^*(\Delta^* \otimes \mathrm{id})(Q^* \otimes \mathrm{id} \otimes \mathrm{id})(a^* \otimes b^* \otimes c^*) \rangle \\
&= \langle x, Q^*(a^*) \cdot b^* \cdot c^* \rangle,
\end{aligned}$$

for all $x \in A$ and $a^*, b^*, c^* \in A^*$. Thus equation (3) holds for $(A^*, \Delta^*, \delta^*, Q^*)$ as a relative Poisson algebra if and only if equation (51) holds. Similarly, Q^* is a derivation of (A^*, Δ^*) as a commutative associative algebra if and only if equation (49) holds and Q^* is a derivation of (A^*, δ^*) as a Lie algebra if and only if equation (50) holds. ∎

Definition 3.12. A *relative Poisson bialgebra* is a collection $(A, \cdot, [-,-], \Delta, \delta, P, Q)$ satisfying the following conditions:

(1) $(A, \cdot, [-,-], P)$ is a relative Poisson algebra;

(2) (A, Δ, δ, Q) is a relative Poisson coalgebra;

(3) Δ satisfies equation (47) and hence $(A, \cdot, \Delta)$ is a commutative and cocommutative infinitesimal bialgebra;

(4) δ satisfies equation (48) and hence $(A, [-,-], \delta)$ is a Lie bialgebra;

(5) Q dually represents (A, P), that is, equations (18), (19) and (25) hold;

(6) P^* dually represents (A^*, Q^*), that is, the following equations hold:

$$\Delta P = (P \otimes \mathrm{id} - \mathrm{id} \otimes Q)\Delta, \tag{52}$$

$$\delta P = (P \otimes \mathrm{id} - \mathrm{id} \otimes Q)\delta, \tag{53}$$

$$(\Delta \otimes \mathrm{id})\Delta(P + Q) = 0; \tag{54}$$

(7) the following equations hold:

$$\begin{aligned}
&\delta(x \cdot y) - (\mathrm{id} \otimes \mathrm{ad}(y))\Delta(x) - (\mathcal{L}(x) \otimes \mathrm{id})\delta(y) - (\mathrm{id} \otimes \mathrm{ad}(x))\Delta(y) \\
&\quad - (\mathcal{L}(y) \otimes \mathrm{id})\delta(x) - (\mathrm{id} \otimes Q)\Delta(x \cdot y) = 0,
\end{aligned} \tag{55}$$

$$\begin{aligned}
&\Delta([x, y]) - (\mathcal{L}(y) \otimes \mathrm{id})\delta(x) - (\mathrm{id} \otimes \mathrm{ad}(x))\Delta(y) + (\mathrm{id} \otimes \mathcal{L}(y))\delta(x) \\
&\quad - (\mathrm{ad}(x) \otimes \mathrm{id})\Delta(y) + \Delta(P(x) \cdot y) = 0,
\end{aligned} \tag{56}$$

for all $x, y \in A$.

Remark 3.13. The notion of a Poisson bialgebra given in [37] is recovered from the above notion of a relative Poisson bialgebra by letting $P = Q = 0$.

Remark 3.14. Let $(A, \cdot_A, [-,-]_A, \Delta_A, \delta_A, P, Q)$ be a relative Poisson bialgebra. It is straightforward to show that $(A^*, \cdot_{A^*}, [-,-]_{A^*}, \Delta_{A^*}, \delta_{A^*}, Q^*, P^*)$ is also a relative Poisson bialgebra, where $\cdot_{A^*}, [-,-]_{A^*} : A^* \otimes A^* \to A^*$ and $\Delta_{A^*}, \delta_{A^*} : A^* \to A^* \otimes A^*$ are respectively given by

$$a^* \cdot_{A^*} b^* = \Delta_A^*(a^* \otimes b^*), \qquad [a^*, b^*]_{A^*} = \delta_A^*(a^* \otimes b^*),$$
$$\langle \Delta_{A^*}(a^*), x \otimes y\rangle = -\langle a^*, x \cdot_A y\rangle, \qquad \langle \delta_{A^*}(a^*), x \otimes y\rangle = -\langle a^*, [x, y]_A\rangle,$$

for all $x, y \in A$, $a^*, b^* \in A^*$.

Theorem 3.15. *Let $(A, \cdot_A, [-,-]_A, P)$ be a relative Poisson algebra. Suppose that there is a relative Poisson algebra structure $(A^*, \cdot_{A^*}, [-,-]_{A^*}, Q^*)$ on the dual space A^* which is given by a relative Poisson coalgebra (A, Δ, δ, Q). Then $(A, \cdot_A, [-,-]_A, \Delta, \delta, P, Q)$ is a relative Poisson bialgebra if and only if $((A, P), (A^*, Q^*), -\mathcal{L}_A^*, \mathrm{ad}_A^*, -\mathcal{L}_{A^*}^*, \mathrm{ad}_{A^*}^*)$ is a matched pair of relative Poisson algebras.*

Proof. By [14], δ satisfies equation (48) if and only if $(A, A^*, \mathrm{ad}_A^*, \mathrm{ad}_{A^*}^*)$ is a matched pair of Lie algebras and by [7], Δ satisfies equation (47) if and only if $(A, A^*, -\mathcal{L}_A^*, -\mathcal{L}_{A^*}^*)$ is a matched pair of commutative associative algebras. Moreover, by Definition 2.6, Q dually represents (A, P) if and only if $(-\mathcal{L}_A^*, \mathrm{ad}_A^*, Q^*, A^*)$ is a representation of (A, P), and P^* dually represents (A^*, Q^*) if and only if $(-\mathcal{L}_{A^*}^*, \mathrm{ad}_{A^*}^*, P, A)$ is a representation of (A^*, Q^*). Next we show that

$$(32) \Leftrightarrow (55) \Leftrightarrow (35) \text{ and } (33) \Leftrightarrow (56) \Leftrightarrow (34)$$

in the case that $P_1 = P$, $P_2 = Q^*$, $\mu_1 = -\mathcal{L}_A^*$, $\mu_2 = -\mathcal{L}_{A^*}^*$, $\rho_1 = \mathrm{ad}_A^*$, $\rho_2 = \mathrm{ad}_{A^*}^*$. As an example, we give an explicit proof for the fact that equation (32) $\Leftrightarrow$ equation (55). The proof for the other equivalences is similar. Let $x, y \in A$, $a^*, b^* \in A^*$. Then we have

$$\begin{aligned}
\langle \mathrm{ad}_{A^*}^*(a^*)(x \cdot_A y), b^*\rangle &= \langle x \cdot_A y, [b^*, a^*]_{A^*}\rangle = \langle \delta(x \cdot_A y), b^* \otimes a^*\rangle,\\
-\langle \mathcal{L}_{A^*}^*(\mathrm{ad}_A^*(y)a^*)x, b^*\rangle &= \langle x, b^* \cdot_{A^*} (\mathrm{ad}_A^*(y)a^*)\rangle\\
&= \langle x, \Delta^*(\mathrm{id} \otimes \mathrm{ad}_A^*(y))(b^* \otimes a^*)\rangle\\
&= -\langle (\mathrm{id} \otimes \mathrm{ad}_A(y))\Delta(x), b^* \otimes a^*\rangle,\\
-\langle x \cdot \mathrm{ad}_{A^*}^*(a^*)y, b^*\rangle &= \langle \mathrm{ad}_{A^*}^*(a^*)y, \mathcal{L}_A^*(x)b^*\rangle = \langle y, [\mathcal{L}_A^*(x)b^*, a^*]_{A^*}\rangle\\
&= \langle y, \delta^*(\mathcal{L}_A^*(x) \otimes \mathrm{id})(b^* \otimes a^*)\rangle\\
&= -\langle (\mathcal{L}_A(x) \otimes \mathrm{id})\delta(y), b^* \otimes a^*\rangle,\\
\langle \mathcal{L}_{A^*}^*(Q^*(a^*))(x \cdot_A y), b^*\rangle &= -\langle x \cdot_A y, b^* \cdot_{A^*} Q^*(a^*)\rangle\\
&= -\langle x \cdot_A y, \Delta^*(\mathrm{id} \otimes Q^*)(b^* \otimes a^*)\rangle\\
&= -\langle (\mathrm{id} \otimes Q)\Delta(x \cdot_A y), b^* \otimes a^*\rangle.
\end{aligned}$$

Thus equation (32) holds if and only if equation (55) holds. Therefore the conclusion holds. ∎

Combining Theorems 3.5 and 3.15, we have the following conclusion.

Corollary 3.16. *Let $(A, \cdot_A, [-,-]_A, P)$ be a relative Poisson algebra. Suppose that there is a relative Poisson algebra structure $(A^*, \cdot_{A^*}, [-,-]_{A^*}, Q^*)$ on the dual space A^* which is given by a relative Poisson coalgebra (A, Δ, δ, Q). Then the following conditions are equivalent:*

(1) *there is a Manin triple of relative Poisson algebras $((A \bowtie A^*, P + Q^*, \mathcal{B}_d), (A, P), (A^*, Q^*))$;*

(2) *$((A, P), (A^*, Q^*), -\mathcal{L}_A^*, \mathrm{ad}_A^*, -\mathcal{L}_{A^*}^*, \mathrm{ad}_{A^*}^*)$ is a matched pair of relative Poisson algebras;*

(3) *$(A, \cdot_A, [-,-]_A, \Delta, \delta, P, Q)$ is a relative Poisson bialgebra.*

4. Coboundary relative Poisson bialgebras

We study the coboundary relative Poisson bialgebras, which lead to the introduction of the relative Poisson Yang–Baxter equation (RPYBE) in a relative Poisson algebra. In particular, an antisymmetric solution of the RPYBE in a relative Poisson algebra gives a coboundary relative Poisson bialgebra. We also introduce the notion of $\mathcal{O}$-operators of relative Poisson algebras to interpret the RPYBE and an $\mathcal{O}$-operator gives an antisymmetric solution of the RPYBE in a semi-direct product relative Poisson algebra. Finally, the notion of relative pre-Poisson algebras is introduced to construct $\mathcal{O}$-operators of their sub-adjacent relative Poisson algebras.

4.1. Coboundary relative Poisson bialgebras

Recall that a commutative and cocommutative infinitesimal bialgebra $(A, \cdot, \Delta)$ is called *coboundary* if there exists an $r \in A \otimes A$ such that

$$\Delta(x) = (\mathrm{id} \otimes \mathcal{L}(x) - \mathcal{L}(x) \otimes \mathrm{id})r, \qquad \forall x \in A. \tag{57}$$

A Lie bialgebra $(A, [-,-], \delta)$ is called *coboundary* if there exists an $r \in A \otimes A$ such that

$$\delta(x) = (\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))r, \qquad \forall x \in A. \tag{58}$$

Therefore they motivate us to give the following notion.

Definition 4.1. A relative Poisson bialgebra $(A, \cdot, [-,-], \Delta, \delta, P, Q)$ is called *coboundary* if there exists an $r \in A \otimes A$ such that equations (57) and (58) hold.

Let A be a vector space with a bilinear operation $\diamond : A \otimes A \to A$. Let $r = \sum_i a_i \otimes b_i \in A \otimes A$. Set

$$r_{12} \diamond r_{13} = \sum_{i,j} a_i \diamond a_j \otimes b_i \otimes b_j,$$
$$r_{12} \diamond r_{23} = \sum_{i,j} a_i \otimes b_i \diamond a_j \otimes b_j,$$
$$r_{13} \diamond r_{23} = \sum_{i,j} a_i \otimes a_j \otimes b_i \diamond b_j.$$

Let $(A, \cdot)$ be a commutative associative algebra and $\Delta : A \to A \otimes A$ be a linear map defined by equation (57). Then Δ satisfies equation (47) automatically. Moreover, by [7], Δ makes (A, Δ) into a cocommutative coassociative coalgebra such that $(A, \cdot, \Delta)$ is a commutative and cocommutative infinitesimal bialgebra if and only if for all $x \in A$,

$$(\mathrm{id} \otimes \mathcal{L}(x) - \mathcal{L}(x) \otimes \mathrm{id})(r + \tau(r)) = 0, \tag{59}$$
$$(\mathrm{id} \otimes \mathrm{id} \otimes \mathcal{L}(x) - \mathcal{L}(x) \otimes \mathrm{id} \otimes \mathrm{id})\mathbf{A}(r) = 0, \tag{60}$$

where

$$\mathbf{A}(r) = r_{12} \cdot r_{13} - r_{12} \cdot r_{23} + r_{13} \cdot r_{23}. \tag{61}$$

The equation $\mathbf{A}(r) = 0$ is called *associative Yang–Baxter equation (AYBE)* in $(A, \cdot)$.

Let $(A, [-,-])$ be a Lie algebra and $\delta : A \to A \otimes A$ be a linear map defined by equation (58). Then δ satisfies equation (48) automatically. Moreover, by [14], δ makes (A, δ) into a Lie coalgebra such that $(A, [-,-], \delta)$ is a Lie bialgebra if and only if for all $x \in A$,

$$(\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))(r + \tau(r)) = 0, \tag{62}$$
$$(\mathrm{ad}(x) \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ad}(x))\mathbf{C}(r) = 0, \tag{63}$$

where

$$\mathbf{C}(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]. \tag{64}$$

The equation $\mathbf{C}(r) = 0$ is called *classical Yang–Baxter equation (CYBE)* in $(A, [-,-])$.

Proposition 4.2. *Let $(A, \cdot, [-,-], P)$ be a relative Poisson algebra which is dually represented by Q. Let $r = \sum_i a_i \otimes b_i \in A \otimes A$ and $\Delta, \delta : A \to A \otimes A$ be linear maps defined by equations* (57) *and* (58) *respectively. Define $\mathbf{A}(r)$ and $\mathbf{C}(r)$ by equations* (61) *and* (64) *respectively.*

(1) *Equation* (49) *holds if and only if the following equation holds:*

$$(\mathrm{id} \otimes \mathcal{L}(x))(\mathrm{id} \otimes P - Q \otimes \mathrm{id})r + (\mathcal{L}(x) \otimes \mathrm{id})(\mathrm{id} \otimes Q - P \otimes \mathrm{id})r = 0, \quad \forall x \in A. \tag{65}$$

(2) *Equation* (50) *holds if and only if the following equation holds:*

$$(\mathrm{id} \otimes \mathrm{ad}(x))(\mathrm{id} \otimes P - Q \otimes \mathrm{id})r - (\mathrm{ad}(x) \otimes \mathrm{id})(\mathrm{id} \otimes Q - P \otimes \mathrm{id})r = 0, \quad \forall x \in A. \tag{66}$$

(3) *Equation* (51) *holds if and only if the following equation holds:*

$$\begin{aligned}&(\mathrm{ad}(x)\otimes\mathrm{id}\otimes\mathrm{id}+Q\otimes\mathrm{id}\otimes\mathcal{L}(x))\mathbf{A}(r)\\&\quad+(\mathrm{id}\otimes\mathrm{id}\otimes\mathcal{L}(x)-\mathrm{id}\otimes\mathcal{L}(x)\otimes\mathrm{id})\mathbf{C}(r)\\&\quad+\sum_j(\mathrm{ad}(a_j)\otimes\mathrm{id})(\mathcal{L}(x)\otimes\mathrm{id}-\mathrm{id}\otimes\mathcal{L}(x))(r+\tau(r))\otimes b_j\\&\quad+\sum_j(\mathrm{id}\otimes\mathcal{L}(x\cdot a_j))(Q\otimes\mathrm{id}-\mathrm{id}\otimes P)r\otimes b_j\\&\quad+\sum_j(\mathrm{id}\otimes\mathrm{id}\otimes\mathcal{L}(x\cdot b_j))\\&\qquad\cdot(\mathrm{id}\otimes\tau)((\mathrm{id}\otimes P-Q\otimes\mathrm{id})r\otimes a_j)=0,\qquad\forall x\in A.\end{aligned}\tag{67}$$

(4) *Equation* (52) *holds if and only if the following equation holds:*

$$(\mathrm{id}\otimes\mathcal{L}(x)-\mathcal{L}(x)\otimes\mathrm{id})(\mathrm{id}\otimes Q-P\otimes\mathrm{id})r=0,\qquad\forall x\in A.\tag{68}$$

(5) *Equation* (53) *holds if and only if the following equation holds:*

$$(\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x))(\mathrm{id}\otimes Q-P\otimes\mathrm{id})r=0,\qquad\forall x\in A.\tag{69}$$

(6) *Equation* (54) *holds if and only if the following equation holds:*

$$(\mathrm{id}\otimes\mathrm{id}\otimes\mathcal{L}((P+Q)x))\mathbf{A}(r)=0,\qquad\forall x\in A.\tag{70}$$

(7) *Equation* (55) *holds if and only if the following equation holds:*

$$(\mathcal{L}(x\cdot y)\otimes\mathrm{id})(\mathrm{id}\otimes Q-P\otimes\mathrm{id})r=0,\qquad\forall x,y\in A.\tag{71}$$

(8) *Equation* (56) *holds automatically.*

Proof. Let $x, y\in A$. (1) Substituting equation (57) into equation (65), we have

$$\begin{aligned}&\Delta Q(x)-(Q\otimes\mathrm{id}+\mathrm{id}\otimes Q)\Delta(x)\\&\quad=\sum_i\big(a_i\otimes Q(x)\cdot b_i-Q(x)\cdot a_i\otimes b_i-Q(a_i)\otimes x\cdot b_i+Q(x\cdot a_i)\otimes b_i\\&\qquad-a_i\otimes Q(x\cdot b_i)+x\cdot a_i\otimes Q(b_i)\big)\\&\quad\overset{(18)}{=}\sum_i\big(a_i\otimes P(b_i)\cdot x-P(a_i)\cdot x\otimes b_i-Q(a_i)\otimes x\cdot b_i+x\cdot a_i\otimes Q(b_i)\big)\\&\quad=(\mathrm{id}\otimes\mathcal{L}(x))(\mathrm{id}\otimes P-Q\otimes\mathrm{id})r+(\mathcal{L}(x)\otimes\mathrm{id})(\mathrm{id}\otimes Q-P\otimes\mathrm{id})r.\end{aligned}$$

Thus equation (49) holds if and only if equation (65) holds.

(2) By a similar proof as the one of (1), equation (50) holds if and only if equation (65) holds.

(3) Substituting equations (57) and (58) into equation (51), we have

$$
\begin{aligned}
&(\mathrm{id}\otimes\Delta)\delta(x)-(\delta\otimes\mathrm{id})\Delta(x)-(\tau\otimes\mathrm{id})(\mathrm{id}\otimes\delta)\Delta(x)-(Q\otimes\mathrm{id}\otimes\mathrm{id})(\Delta\otimes\mathrm{id})\Delta(x)\\
&\quad=\sum_{i,j}\big([x,a_i]\otimes a_j\otimes b_i\cdot b_j-[x,a_i]\otimes b_i\cdot a_j\otimes b_j+a_i\otimes a_j\otimes[x,b_i]\cdot b_j\\
&\qquad-a_i\otimes[x,b_i]\cdot a_j\otimes b_j-[a_i,a_j]\otimes b_j\otimes x\cdot b_i-a_j\otimes[a_i,b_j]\otimes x\cdot b_i\\
&\qquad+[x\cdot a_i,a_j]\otimes b_j\otimes b_i+a_j\otimes[x\cdot a_i,b_j]\otimes b_i-[x\cdot b_i,a_j]\otimes a_i\otimes b_j\\
&\qquad-a_j\otimes a_i\otimes[x\cdot b_i,b_j]+[b_i,a_j]\otimes x\cdot a_i\otimes b_j+a_j\otimes x\cdot a_i\otimes[b_i,b_j]\\
&\qquad-Q(a_j)\otimes a_i\cdot b_j\otimes x\cdot b_i+Q(a_i\cdot a_j)\otimes b_j\otimes x\cdot b_i\\
&\qquad+Q(a_j)\otimes x\cdot a_i\cdot b_j\otimes b_i-Q(x\cdot a_i\cdot a_j)\otimes b_j\otimes b_i\big)\\
&\quad=W(1)+W(2)+W(3),
\end{aligned}
$$

where

$$
\begin{aligned}
W(1):=&\sum_{i,j}\big([x,a_i]\otimes a_j\otimes b_i\cdot b_j-[x,a_i]\otimes b_i\cdot a_j\otimes b_j+[x\cdot a_i,a_j]\otimes b_j\otimes b_i\\
&-[x\cdot b_i,a_j]\otimes a_i\otimes b_j-Q(x\cdot a_i\cdot a_j)\otimes b_j\otimes b_i\big)\\
\overset{(61)}{=}&(\mathrm{ad}(x)\otimes\mathrm{id}\otimes\mathrm{id})\mathbf{A}(r)+\sum_{i,j}\big(-[x,a_i\cdot a_j]\otimes b_i\otimes b_j+[x\cdot a_i,a_j]\otimes b_j\otimes b_i\\
&-[x\cdot b_i,a_j]\otimes a_i\otimes b_j-Q(x\cdot a_i\cdot a_j)\otimes b_j\otimes b_i\big)\\
\overset{(20)}{=}&(\mathrm{ad}(x)\otimes\mathrm{id}\otimes\mathrm{id})\mathbf{A}(r)+\sum_{i,j}\big([a_j,x\cdot a_i]\otimes b_i\otimes b_j+[a_j,x\cdot b_i]\otimes a_i\otimes b_j\big)\\
=&(\mathrm{ad}(x)\otimes\mathrm{id}\otimes\mathrm{id})\mathbf{A}(r)+\sum_{j}(\mathrm{ad}(a_j)\otimes\mathrm{id})(\mathcal{L}(x)\otimes\mathrm{id})(r+\tau(r))\otimes b_j,\\
W(2):=&\sum_{i,j}\big(-a_i\otimes[x,b_i]\cdot a_j\otimes b_j+a_j\otimes[x\cdot a_i,b_j]\otimes b_i+[b_i,a_j]\otimes x\cdot a_i\otimes b_j\\
&+a_j\otimes x\cdot a_i\otimes[b_i,b_j]+Q(a_j)\otimes x\cdot a_i\cdot b_j\otimes b_i\big)\\
=&\sum_{i,j}\big(-a_i\otimes[x,b_i]\cdot a_j\otimes b_j+a_j\otimes[x\cdot a_i,b_j]\otimes b_i+a_j\otimes x\cdot a_i\otimes[b_i,b_j]\\
&+[a_j,a_i]\otimes x\cdot b_i\otimes b_j+Q(a_j)\otimes x\cdot a_i\cdot b_j\otimes b_i\big)\\
&-\sum_{j}(\mathrm{ad}(a_j)\otimes\mathcal{L}(x))(r+\tau(r))\otimes b_j\\
\overset{(64)}{=}&-(\mathrm{id}\otimes\mathcal{L}(x)\otimes\mathrm{id})\mathbf{C}(r)+\sum_{i,j}\big(-a_i\otimes[x,b_i]\cdot a_j\otimes b_j\\
&+a_i\otimes[x\cdot a_j,b_i]\otimes b_j+a_i\otimes x\cdot[b_i,a_j]\otimes b_j+Q(a_j)\otimes x\cdot a_i\cdot b_j\otimes b_i\big)\\
&-\sum_{j}(\mathrm{ad}(a_j)\otimes\mathcal{L}(x))(r+\tau(r))\otimes b_j
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(3)}{=} -(\mathrm{id}\otimes\mathcal{L}(x)\otimes\mathrm{id})\mathbf{C}(r)+\sum_{i,j}\big(-a_i\otimes x\cdot a_j\cdot P(b_i)\otimes b_j\\
&\quad + Q(a_j)\otimes x\cdot a_i\cdot b_j\otimes b_i\big)-\sum_j(\mathrm{ad}(a_j)\otimes\mathcal{L}(x))(r+\tau(r))\otimes b_j\\
&= -(\mathrm{id}\otimes\mathcal{L}(x)\otimes\mathrm{id})\mathbf{C}(r)+\sum_j(\mathrm{id}\otimes\mathcal{L}(x\cdot a_j)\otimes\mathrm{id})\\
&\quad \cdot((Q\otimes\mathrm{id}-\mathrm{id}\otimes P)r\otimes b_j)-\sum_j(\mathrm{ad}(a_j)\otimes\mathcal{L}(x))(r+\tau(r))\otimes b_j,
\end{aligned}
$$

$$
\begin{aligned}
W(3) :=& \sum_{i,j}\big(a_i\otimes a_j\otimes[x,b_i]\cdot b_j-[a_i,a_j]\otimes b_j\otimes x\cdot b_i-a_j\otimes[a_i,b_j]\otimes x\cdot b_i\\
&-a_j\otimes a_i\otimes[x\cdot b_i,b_j]-Q(a_j)\otimes a_i\cdot b_j\otimes x\cdot b_i+Q(a_i\cdot a_j)\otimes b_j\otimes x\cdot b_i\big)\\
\overset{(64)}{=}& (\mathrm{id}\otimes\mathrm{id}\otimes\mathcal{L}(x))\mathbf{C}(r)+\sum_{i,j}\big(-a_i\otimes a_j\otimes x\cdot[b_i,b_j]+a_i\otimes a_j\otimes[x,b_i]\cdot b_j\\
&-a_i\otimes a_j\otimes[x\cdot b_j,b_i]-Q(a_j)\otimes a_i\cdot b_j\otimes x\cdot b_i+Q(a_i\cdot a_j)\otimes b_j\otimes x\cdot b_i\big)\\
\overset{(3)}{=}& (\mathrm{id}\otimes\mathrm{id}\otimes\mathcal{L}(x))\mathbf{C}(r)+\sum_{i,j}\big(a_i\otimes a_j\otimes x\cdot b_j\cdot P(b_i)\\
&-Q(a_j)\otimes a_i\cdot b_j\otimes x\cdot b_i+Q(a_i\cdot a_j)\otimes b_j\otimes x\cdot b_i\big)\\
\overset{(61)}{=}& (\mathrm{id}\otimes\mathrm{id}\otimes\mathcal{L}(x))\mathbf{C}(r)+\sum_{i,j}\big(a_i\otimes a_j\otimes x\cdot b_j\cdot P(b_i)\\
&-Q(a_i)\otimes a_j\otimes x\cdot b_i\cdot b_j\big)+(Q\otimes\mathrm{id}\otimes\mathcal{L}(x))\mathbf{A}(r)\\
=& (\mathrm{id}\otimes\mathrm{id}\otimes\mathcal{L}(x))\mathbf{C}(r)+(Q\otimes\mathrm{id}\otimes\mathcal{L}(x))\mathbf{A}(r)\\
&+\sum_j(\mathrm{id}\otimes\mathrm{id}\otimes\mathcal{L}(x\cdot b_j))(\mathrm{id}\otimes\tau)((\mathrm{id}\otimes P-Q\otimes\mathrm{id})r\otimes a_j).
\end{aligned}
$$

Thus equation (51) holds if and only if equation (67) holds.

(4) By a similar proof as the one of (1), equation (52) holds if and only if equation (68) holds.

(5) By a similar proof as of the one (1), equation (53) holds if and only if equation (69) holds.

(6) Substituting equation (57) into equation (54), we have

$$
\begin{aligned}
&(\Delta\otimes\mathrm{id})\Delta(P+Q)(x)\\
&\quad=\sum_{i,j}\big(a_j\otimes a_i\cdot b_j\otimes((P+Q)x)\cdot b_i-a_i\cdot a_j\otimes b_j\otimes((P+Q)x)\cdot b_i\\
&\qquad+(((P+Q)x)\cdot a_i)\cdot a_j\otimes b_j\otimes b_i-a_j\otimes(((P+Q)x)\cdot a_i)\cdot b_j\otimes b_i\big)\\
&\quad\overset{(26)}{=}\sum_{i,j}\big(a_j\otimes a_i\cdot b_j\otimes((P+Q)x)\cdot b_i-a_i\cdot a_j\otimes b_j\otimes((P+Q)x)\cdot b_i
\end{aligned}
$$

$$- a_i \otimes a_j \otimes ((P+Q)x) \cdot b_i \cdot b_j\big)$$
$$= -(\mathrm{id} \otimes \mathrm{id} \otimes \mathcal{L}((P+Q)x))\mathbf{A}(r).$$

Thus equation (54) holds if and only if equation (70) holds.

(7) Substituting equations (57) and (58) into equation (55), we have

$$\begin{aligned}
&\delta(x \cdot y) - (\mathrm{id} \otimes \mathrm{ad}(y))\Delta(x) - (\mathcal{L}(x) \otimes \mathrm{id})\delta(y) - (\mathrm{id} \otimes \mathrm{ad}(x))\Delta(y) \\
&\quad - (\mathcal{L}(y) \otimes \mathrm{id})\delta(x) - (\mathrm{id} \otimes Q)\Delta(x \cdot y) \\
&= \sum_i \big([x \cdot y, a_i] \otimes b_i + a_i \otimes [x \cdot y, b_i] - a_i \otimes [y, x \cdot b_i] + x \cdot a_i \otimes [y, b_i] \\
&\quad - a_i \otimes [x, y \cdot b_i] + y \cdot a_i \otimes [x, b_i] - x \cdot [y, a_i] \otimes b_i - x \cdot a_i \otimes [y, b_i] \\
&\quad - y \cdot [x, a_i] \otimes b_i - y \cdot a_i \otimes [x, b_i] - a_i \otimes Q(x \cdot y \cdot b_i) + x \cdot y \cdot a_i \otimes Q(b_i)\big) \\
&= \sum_i \big(a_i \otimes [x \cdot y, b_i] - a_i \otimes [y, x \cdot b_i] - a_i \otimes [x, y \cdot b_i] - a_i \otimes Q(x \cdot y \cdot b_i)\big) \\
&\quad + \sum_i \big([x \cdot y, a_i] \otimes b_i - x \cdot [y, a_i] \otimes b_i - y \cdot [x, a_i] \otimes b_i + x \cdot y \cdot a_i \otimes Q(b_i)\big) \\
&\overset{(20)}{=} \sum_i \big([x \cdot y, a_i] \otimes b_i - x \cdot [y, a_i] \otimes b_i - y \cdot [x, a_i] \otimes b_i + x \cdot y \cdot a_i \otimes Q(b_i)\big) \\
&\overset{(3)}{=} \sum_i \big(-x \cdot y \cdot P(a_i) \otimes b_i + x \cdot y \cdot a_i \otimes Q(b_i)\big) \\
&= (\mathcal{L}(x \cdot y) \otimes \mathrm{id})(\mathrm{id} \otimes Q - P \otimes \mathrm{id})r.
\end{aligned}$$

Thus equation (55) holds if and only if equation (71) holds.

(8) Substituting equations (57) and (58) into equation (56), we have

$$\begin{aligned}
&\Delta([x, y]) - (\mathcal{L}(y) \otimes \mathrm{id})\delta(x) + (\mathrm{id} \otimes \mathcal{L}(y))\delta(x) - (\mathrm{id} \otimes \mathrm{ad}(x))\Delta(y) \\
&\quad - (\mathrm{ad}(x) \otimes \mathrm{id})\Delta(y) + \Delta(P(x) \cdot y) \\
&= \sum_i \big(a_i \otimes [x, y] \cdot b_i - [x, y] \cdot a_i \otimes b_i - y \cdot [x, a_i] \otimes b_i - y \cdot a_i \otimes [x, b_i] \\
&\quad + [x, a_i] \otimes y \cdot b_i + a_i \otimes y \cdot [x, b_i] - a_i \otimes [x, y \cdot b_i] + y \cdot a_i \otimes [x, b_i] \\
&\quad - [x, a_i] \otimes y \cdot b_i + [x, y \cdot a_i] \otimes b_i + a_i \otimes P(x) \cdot y \cdot b_i - P(x) \cdot y \cdot a_i \otimes b_i\big) \\
&= \sum_i \big(a_i \otimes [x, y] \cdot b_i + a_i \otimes y \cdot [x, b_i] - a_i \otimes [x, y \cdot b_i] + a_i \otimes P(x) \cdot y \cdot b_i\big) \\
&\quad + \sum_i \big(-[x, y] \cdot a_i \otimes b_i - y \cdot [x, a_i] \otimes b_i + [x, y \cdot a_i] \otimes b_i \\
&\quad - P(x) \cdot y \cdot a_i \otimes b_i\big) \\
&\overset{(3)}{=} 0.
\end{aligned}$$

Thus equation (56) holds automatically. ∎

Summarizing the above study, we have the following conclusion.

Theorem 4.3. *Let* $(A, \cdot, [-,-], P)$ *be a relative Poisson algebra which is dually represented by* Q. *Let* $r = \sum_i a_i \otimes b_i \in A \otimes A$ *and* $\Delta, \delta : A \to A \otimes A$ *be linear maps defined by* (57) *and* (58) *respectively. Then* $(A, \cdot, [-,-], \Delta, \delta, P, Q)$ *is a coboundary relative Poisson bialgebra if and only if equations* (59), (60), (62), (63) *and* (65)–(71) *hold.*

Let A be a vector space and $r = \sum_i a_i \otimes b_i \in A \otimes A$; r can be identified with a linear map from A^* to A as follows:

$$r(a^*) = \sum_i \langle a^*, b_i \rangle a_i, \qquad \forall a^* \in A^*. \tag{72}$$

There is an analogue of the Drinfeld classical double ([14]) for a relative Poisson bialgebra.

Theorem 4.4. *Let* $(A, \cdot_A, [-,-]_A, \Delta_A, \delta_A, P, Q)$ *be a relative Poisson bialgebra. Let* $(A^*, \cdot_{A^*}, [-,-]_{A^*}, \Delta_{A^*}, \delta_{A^*}, Q^*, P^*)$ *be the relative Poisson bialgebra given in Remark* 3.14. *Then there is a coboundary relative Poisson bialgebra structure on the direct sum* $A \oplus A^*$ *of vector spaces which contains these two relative Poisson bialgebra structures on* A *and* A^* *respectively as relative Poisson sub-bialgebras.*

Proof. Let $r \in A \otimes A^* \subset (A \oplus A^*) \otimes (A \oplus A^*)$ correspond to the identity map $\mathrm{id} : A \to A$. Let $\{e_1, \dots, e_n\}$ be a basis of A and $\{e_1^*, \dots, e_n^*\}$ be the dual basis. Then by equation (72), $r = \sum_i e_i \otimes e_i^*$. Since $(A, \cdot_A, [-,-]_A, \Delta_A, \delta_A, P, Q)$ is a relative Poisson bialgebra, there is a relative Poisson algebra $(A \bowtie A^*, P + Q^*)$ given by the matched pair $((A, P), (A^*, Q^*), -\mathcal{L}_A^*, -\mathcal{L}_{A^*}^*, \mathrm{ad}_A^*, \mathrm{ad}_{A^*}^*)$, that is,

$$\begin{aligned} x \cdot_{A \bowtie A^*} y &= x \cdot_A y, & x \cdot_{A \bowtie A^*} a^* &= -\mathcal{L}_A^*(x) a^* - \mathcal{L}_{A^*}^*(a^*) x, \\ a^* \cdot_{A \bowtie A^*} b^* &= a^* \cdot_{A^*} b^*, & [x, y]_{A \bowtie A^*} &= [x, y]_A, \\ [x, a^*]_{A \bowtie A^*} &= \mathrm{ad}_A^*(x) a^* - \mathrm{ad}_{A^*}^*(a^*) x, & [a^*, b^*]_{A \bowtie A^*} &= [a^*, b^*]_{A^*}, \end{aligned}$$

for all $x, y \in A, a^*, b^* \in A^*$. By Lemma 3.4, $Q + P^*$ dually represents $(A \bowtie A^*, P + Q^*)$. Define two linear maps $\Delta_{A \bowtie A^*}, \delta_{A \bowtie A^*} : A \bowtie A^* \to (A \bowtie A^*) \otimes (A \bowtie A^*)$ respectively by

$$\begin{aligned} \Delta_{A \bowtie A^*}(u) &= (\mathrm{id} \otimes \mathcal{L}_{A \bowtie A^*}(u) - \mathcal{L}_{A \bowtie A^*}(u) \otimes \mathrm{id}) r, \\ \delta_{A \bowtie A^*}(u) &= (\mathrm{ad}_{A \bowtie A^*}(u) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{A \bowtie A^*}(u)) r, \end{aligned}$$

for all $u \in A \bowtie A^*$. Then by [7, Theorem 2.3.6], r satisfies the AYBE in the commutative associative algebra $(A \bowtie A^*, \cdot_{A \bowtie A^*})$, and the following equation holds:

$$(\mathrm{id} \otimes \mathcal{L}_{A \bowtie A^*}(u) - \mathcal{L}_{A \bowtie A^*}(u) \otimes \mathrm{id})(r + \tau(r)) = 0, \qquad \forall u \in A \bowtie A^*.$$

By [14, Propositions 1.4.2 and 2.1.11], r satisfies the CYBE in the Lie algebra $(A \bowtie A^*, [-,-]_{A \bowtie A^*})$, and the following equation holds:

$$(\mathrm{ad}_{A \bowtie A^*}(u) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{A \bowtie A^*}(u))(r + \tau(r)) = 0, \qquad \forall u \in A \bowtie A^*.$$

Moreover, we have

$$\begin{aligned}
&((P+Q^*)\otimes \mathrm{id}-\mathrm{id}\otimes (Q+P^*))r\\
&\quad=\sum_{i=1}^{n}((P+Q^*)\otimes \mathrm{id}-\mathrm{id}\otimes (Q+P^*))(e_i\otimes e_i^*)\\
&\quad=\sum_{i=1}^{n}(P(e_i)\otimes e_i^*-e_i\otimes P^*(e_i^*))=0,\\
&((Q+P^*)\otimes \mathrm{id}-\mathrm{id}\otimes (P+Q^*))r\\
&\quad=\sum_{i=1}^{n}((Q+P^*)\otimes \mathrm{id}-\mathrm{id}\otimes (P+Q^*))(e_i\otimes e_i^*)\\
&\quad=\sum_{i=1}^{n}(Q(e_i)\otimes e_i^*-e_i\otimes Q^*(e_i^*))=0.
\end{aligned}$$

Then by Theorem 4.3, $(A\bowtie A^*, \cdot_{A\bowtie A^*}, [-,-]_{A\bowtie A^*}, \Delta_{A\bowtie A^*}, \delta_{A\bowtie A^*}, P+Q^*, Q+P^*)$ is a coboundary relative Poisson bialgebra. Moreover, again by [7, Theorem 2.3.6] and [14, Propositions 1.4.2 and 2.1.11], we have

$$\Delta_{A\bowtie A^*}(x)=\Delta_A(x),\qquad \delta_{A\bowtie A^*}(x)=\delta_A(x),\qquad \forall x\in A.$$

Thus $(A\bowtie A^*, \cdot_{A\bowtie A^*}, [-,-]_{A\bowtie A^*}, \Delta_{A\bowtie A^*}, \delta_{A\bowtie A^*}, P+Q^*, Q+P^*)$ contains $(A, \cdot_A, [-,-]_A, \Delta_A, \delta_A, P, Q)$ as a relative Poisson sub-bialgebra. Similarly, $(A\bowtie A^*, \cdot_{A\bowtie A^*}, [-,-]_{A\bowtie A^*}, \Delta_{A\bowtie A^*}, \delta_{A\bowtie A^*}, P+Q^*, Q+P^*)$ contains $(A^*, \cdot_{A^*}, [-,-]_{A^*}, \Delta_{A^*}, \delta_{A^*}, Q^*, P^*)$ as a relative Poisson sub-bialgebra. ■

A direct consequence of Theorem 4.3 is given as follows.

Corollary 4.5. *Let $(A, \cdot, [-,-], P)$ be a relative Poisson algebra which is dually represented by Q. Let $r\in A\otimes A$ and $\Delta, \delta : A\to A\otimes A$ be linear maps defined by equations (57) and (58) respectively. If r is antisymmetric in the sense that $r+\tau(r)=0$ and satisfies the AYBE in the commutative associative algebra $(A,\cdot)$, the CYBE in the Lie algebra $(A, [-,-])$, and the following equations:*

$$(P\otimes \mathrm{id}-\mathrm{id}\otimes Q)r=0, \tag{73}$$

$$(Q\otimes \mathrm{id}-\mathrm{id}\otimes P)r=0, \tag{74}$$

then $(A, \cdot, [-,-], \Delta, \delta, P, Q)$ is a coboundary relative Poisson bialgebra.

It motivates us to give the following notion.

Definition 4.6. Let $(A, \cdot, [-,-], P)$ be a relative Poisson algebra, $Q : A\to A$ be a linear map and $r\in A\otimes A$. r is called a solution of the *relative Poisson Yang–Baxter equation (RPYBE) associated to Q (Q-RPYBE)* in (A, P) if r satisfies the AYBE in the commutative associative algebra $(A,\cdot)$, the CYBE in the Lie algebra $(A, [-,-])$ and equations (73)–(74).

Remark 4.7. If $r \in A \otimes A$ is antisymmetric, then equation (73) holds if and only if equation (74) holds. On the other hand, the notion of the *Poisson Yang–Baxter equation (PYBE)* in a Poisson algebra was given in [37] whose solutions are exactly the solutions of both the AYBE and the CYBE. So from the form, the Q-RPYBE (defined in relative Poisson algebras) is exactly the PYBE (defined in Poisson algebras) satisfying the additional equations (73)–(74).

Theorem 4.8. *Let $(A, \cdot, [-,-], P)$ be a relative Poisson algebra and $r \in A \otimes A$ be antisymmetric. Let $Q : A \to A$ be a linear map. Then r is a solution of the Q-RPYBE in (A, P) if and only if r satisfies*

$$[r(a^*), r(b^*)] = r(\mathrm{ad}^*(r(a^*))b^* - \mathrm{ad}^*(r(b^*))a^*), \tag{75}$$

$$r(a^*) \cdot r(b^*) = -r(\mathcal{L}^*(r(a^*))b^* + \mathcal{L}^*(r(b^*))a^*), \tag{76}$$

$$Pr = rQ^*, \tag{77}$$

for all $a^*, b^* \in A^*$.

Proof. Let $r = \sum_i a_i \otimes b_i$. By [28], r is an antisymmetric solution of the CYBE in the Lie algebra $(A, [-,-])$ if and only if equation (75) holds. By [7], r is an antisymmetric solution of the AYBE in the commutative associative algebra $(A, \cdot)$ if and only if equation (76) holds. Moreover, for all $a^* \in A^*$, by equation (72), we have

$$r(Q^*(a^*)) = \sum_i \langle Q^*(a^*), b_i\rangle a_i = \sum_i \langle a^*, Q(b_i)\rangle a_i, \qquad P(r(a^*)) = \sum_i \langle a^*, b_i\rangle P(a_i).$$

So equation (73) holds if and only if equation (77) holds. This completes the proof. ∎

4.2. $\mathcal{O}$-operators of relative Poisson algebras

Definition 4.9. Let (A, P) be a relative Poisson algebra, (μ, ρ, V) be a compatible structure on (A, P) and $\alpha : V \to V$ be a linear map. A linear map $T : V \to A$ is called a *weak $\mathcal{O}$-operator of (A, P) associated to (μ, ρ, V) and α* if the following equations hold:

$$T(u) \cdot T(v) = T(\mu(T(u))v + \mu(T(v))u), \tag{78}$$

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \tag{79}$$

$$PT = T\alpha, \tag{80}$$

for all $u, v \in V$. If, in addition, (μ, ρ, α, V) is a representation of (A, P), then T is called an *$\mathcal{O}$-operator of (A, P) associated to (μ, ρ, α, V)*.

Remark 4.10. In fact, T is called *an $\mathcal{O}$-operator of the commutative associative algebra $(A, \cdot)$ associated to (μ, V)* ([7]) if T satisfies equation (78), and T is called an *$\mathcal{O}$-operator of the Lie algebra $(A, [-,-])$ associated to (ρ, V)* ([28]) if T satisfies equation (79).

Therefore Theorem 4.8 is rewritten in terms of $\mathcal{O}$-operators as follows.

Corollary 4.11. *Let (A, P) be a relative Poisson algebra and $r \in A \otimes A$ be antisymmetric. Let $Q : A \to A$ be a linear map. Then r is a solution of the Q-RPYBE in (A, P) if and only if r is a weak $\mathcal{O}$-operator of (A, P) associated to $(-\mathcal{L}^*, \mathrm{ad}^*, A^*)$ and Q^*. If, in addition, (A, P) is dually represented by Q, then r is a solution of the Q-RPYBE in (A, P) if and only if r is an $\mathcal{O}$-operator of (A, P) associated to the representation $(-\mathcal{L}^*, \mathrm{ad}^*, Q^*, A^*)$.*

We consider the semi-direct product relative Poisson algebras which are dually represented.

Theorem 4.12. *Let (A, P) be a relative Poisson algebra and (μ, ρ, V) be a compatible structure on (A, P). Let $Q : A \to A$ and $\alpha, \beta : V \to V$ be linear maps. Then the following conditions are equivalent.*

(1) *There is a relative Poisson algebra $(A \ltimes_{\mu,\rho} V, P + \alpha)$ which is dually represented by the linear operator $Q + \beta$.*
(2) *There is a relative Poisson algebra $(A \ltimes_{-\mu^*,\rho^*} V^*, P + \beta^*)$ which is dually represented by the linear operator $Q + \alpha^*$.*
(3) *The following conditions are satisfied:*
   (a) *(μ, ρ, α, V) is a representation of (A, P);*
   (b) *β dually represents (A, P) on (μ, ρ, V);*
   (c) *Q dually represents (A, P);*
   (d) *for all $x \in A$, $v \in V$, we have*

$$\mu(Q(x))v - \mu(x)\alpha(v) - \beta(\mu(x)v) = 0, \tag{81}$$
$$\rho(Q(x))v - \rho(x)\alpha(v) - \beta(\rho(x)v) = 0. \tag{82}$$

Proof. (1) $\Leftrightarrow$ (3). By Proposition 2.2, $(A \ltimes_{\mu,\rho} V, P + \alpha)$ is a relative Poisson algebra if and only if (μ, ρ, α, V) is a representation of the relative Poisson algebra (A, P). Let $x, y, z \in A, u, v, w \in V$. By Corollary 2.10, $Q + \beta$ dually represents $(A \ltimes_{\mu,\rho} V, P + \alpha)$ if and only if the following equations are satisfied:

$$\begin{aligned}
0 &= (x + u) \cdot (Q + \beta)(y + v) - (P + \alpha)(x + u) \cdot (y + v) \\
&\quad - (Q + \beta)((x + u) \cdot (y + v)) \\
&= x \cdot Q(y) - P(x) \cdot y - Q(x \cdot y) + \mu(x)\beta(v) - \mu(P(x))v - \beta(\mu(x))v \\
&\quad + \mu(Q(y))u - \mu(y)\alpha(u) - \beta(\mu(y)u); \\
0 &= [x + u, (Q + \beta)(y + v)] - [(P + \alpha)(x + u), y + v] - (Q + \beta)([x + u, y + v]) \\
&= [x, Q(y)] - [P(x), y] - Q([x, y]) + \rho(x)\beta(v) - \rho(P(x))v - \beta(\rho(x))v \\
&\quad + \rho(Q(y))u - \rho(y)\alpha(u) - \beta(\rho(y)u); \\
0 &= (P + \alpha + Q + \beta)((x + u) \cdot (y + v) \cdot (z + w)) \\
&= (P + Q)(x \cdot y \cdot z) + (\alpha + \beta)(\mu(x \cdot y)w + \mu(x \cdot z)v + \mu(y \cdot z)u).
\end{aligned}$$

If the above equations hold, then

(i) equations (15), (16) and (23) (where y is replaced by z) hold by letting $y = u = 0$;

(ii) equations (18), (19) and (25) hold by letting $u = v = w = 0$;

(iii) equations (81) and (82) (where x is replaced by y, and v by u) hold by letting $x = v = 0$.

Conversely, obviously, if equations (15), (16), (23), (18), (19), (25), (81) and (82) hold, then the above equations hold. Hence condition (1) holds if and only if condition (3) holds.

(2) $\Leftrightarrow$ (3). From the above equivalence between condition (1) and condition (3), we have that condition (2) holds if and only if the items (a)–(c) in condition (3) as well as the following two equations hold (for all $x \in A$, $u^* \in V^*$):

$$-\mu^*(Q(x))u^* + \mu^*(x)\beta^*(u^*) + \alpha^*(\mu^*(x)u^*) = 0, \tag{83}$$
$$\rho^*(Q(x))u^* - \rho^*(x)\beta^*(u^*) - \alpha^*(\rho^*(x)u^*) = 0. \tag{84}$$

For all $x \in A$, $u^* \in V^*$, $v \in V$, we have

$$\begin{aligned}&\langle -\mu^*(Q(x))u^* + \mu^*(x)\beta^*(u^*) + \alpha^*(\mu^*(x)u^*), v\rangle \\ &\quad = \langle u^*, -\mu(Q(x))v + \mu(x)\alpha(v) + \beta(\mu(x)v)\rangle.\end{aligned}$$

Hence equation (83) holds if and only if equation (81) holds. Similarly, equation (84) holds if and only if equation (82) holds. Hence condition (2) holds if and only if condition (3) holds. ■

Next we show that $\mathcal{O}$-operators give antisymmetric solutions of the RPYBE in semi-direct product relative Poisson algebras and hence give rise to relative Poisson bialgebras.

Theorem 4.13. *Let (A, P) be a relative Poisson algebra and (μ, ρ, V) be a compatible structure on (A, P). Let β dually represent (A, P) on (μ, ρ, V) and let $(-\mu^*, \rho^*, \beta^*, V^*)$ be the representation of (A, P) defined in Proposition 2.5. Let $Q : A \to A$ and $\alpha : V \to V$ be linear maps. Let $T : V \to A$ be a linear map which is identified as an element in $(A \ltimes_{-\mu^*,\rho^*} V^*) \otimes (A \ltimes_{-\mu^*,\rho^*} V^*)$.*

(1) *$r = T - \tau(T)$ is an antisymmetric solution of the $(Q + \alpha^*)$-RPYBE in the relative Poisson algebra $(A \ltimes_{-\mu^*,\rho^*} V^*, P + \beta^*)$ if and only if T is a weak $\mathcal{O}$-operator of (A, P) associated to (μ, ρ, V) and α, and satisfies $T\beta = QT$.*

(2) *Assume that (A, P) is dually represented by Q and (μ, ρ, α, V) is a representation of (A, P). If T is an $\mathcal{O}$-operator of (A, P) associated to (μ, ρ, α, V) satisfying $T\beta = QT$, then $r = T - \tau(T)$ is an antisymmetric solution of the $(Q + \alpha^*)$-RPYBE in the relative Poisson algebra $(A \ltimes_{-\mu^*,\rho^*} V^*, P + \beta^*)$. If, in addition, equations (81) and (82) hold, then $Q + \alpha^*$ dually represents the relative Poisson algebra $(A \ltimes_{-\mu^*,\rho^*} V^*, P + \beta^*)$. In this case, there is a relative Poisson bialgebra $(A \ltimes_{-\mu^*,\rho^*} V^*, \cdot, [-, -], \Delta, \delta, P + \beta^*, Q + \alpha^*)$, where the linear maps Δ and δ are defined respectively by equations (57) and (58) with $r = T - \tau(T)$.*

Proof. (1) By [6], $r = T - \tau(T)$ satisfies the CYBE in the Lie algebra $A \ltimes_{\rho^*} V^*$ if and only if equation (78) holds and by [7], r satisfies the AYBE in the commutative associative algebra $A \ltimes_{-\mu^*} V^*$ if and only if equation (79) holds.

Let $\{v_1, \dots, v_m\}$ be a basis of V and $\{v_1^*, \dots, v_m^*\}$ be the dual basis. Then $T = \sum_{i=1}^m T(v_i) \otimes v_i^* \in (A \ltimes_{-\mu^*,\rho^*} V^*) \otimes (A \ltimes_{-\mu^*,\rho^*} V^*)$. Hence

$$r = T - \tau(T) = \sum_{i=1}^m T(v_i) \otimes v_i^* - v_i^* \otimes T(v_i).$$

Note that

$$((P + \beta^*) \otimes \mathrm{id})r = \sum_{i=1}^m (P(T(v_i)) \otimes v_i^* - \beta^*(v_i^*) \otimes T(v_i)),$$
$$(\mathrm{id} \otimes (Q + \alpha^*))r = \sum_{i=1}^m (T(v_i) \otimes \alpha^*(v_i^*) - v_i^* \otimes Q(T(v_i))).$$

Further

$$\begin{aligned}\sum_{i=1}^m \beta^*(v_i^*) \otimes T(v_i) &= \sum_{i=1}^m \sum_{j=1}^m \langle \beta^*(v_i^*), v_j\rangle v_j^* \otimes T(v_i)\\ &= \sum_{i=1}^m \sum_{j=1}^m v_j^* \otimes \langle v_i^*, \beta(v_j)\rangle T(v_i)\\ &= \sum_{i=1}^m \sum_{j=1}^m v_i^* \otimes T(\langle \beta(v_i), v_j^*\rangle v_j) = \sum_{i=1}^m v_i^* \otimes T(\beta(v_i)),\end{aligned}$$

and similarly,

$$\sum_{i=1}^m T(v_i) \otimes \alpha^*(v_i^*) = \sum_{i=1}^m T(\alpha(v_i)) \otimes v_i^*.$$

Therefore $((P + \beta^*) \otimes \mathrm{id})r = (\mathrm{id} \otimes (Q + \alpha^*))r$ if and only if $PT = T\alpha$ and $T\beta = QT$. Thus the conclusion holds.

(2) It follows from item (1) and Theorem 4.12. ∎

Therefore starting from an $\mathcal{O}$-operator T of a relative Poisson algebra (A, P) associated to a representation (μ, ρ, α, V), one gets an antisymmetric solution of the $(Q + \alpha^*)$-RPYBE in the relative Poisson algebra $(A \ltimes_{-\mu^*,\rho^*} V^*, P + \beta^*)$ for suitable linear maps β and Q and hence gives rise to a relative Poisson bialgebra on the latter. There are some natural choices of Q and β. For example, assume that $\beta = \pm\alpha$, $Q = \pm P$ or $\beta = \theta\alpha^{-1}$, $Q = \theta P^{-1}$ for $0 \neq \theta \in K$ when α and P are invertible. Note that in these cases $T\beta = QT$ automatically and then one can get the other corresponding constraint conditions due to Theorem 4.13. In particular, when $\beta = -\alpha$ and $Q = -P$, there is not any constraint condition.

Corollary 4.14. *Let (μ,ρ,α,V) be a representation of a relative Poisson algebra (A,P) and $T: V\rightarrow A$ be an $\mathcal{O}$-operator of (A,P) associated to (μ,ρ,α,V). Then $r=T-\tau(T)$ is an antisymmetric solution of the $(-P+\alpha^*)$-RPYBE in the relative Poisson algebra $(A\ltimes_{-\mu^*,\rho^*}V^*, P-\alpha^*)$. Further the relative Poisson algebra $(A\ltimes_{-\mu^*,\rho^*}V^*, P-\alpha^*)$ is dually represented by $-P+\alpha^*$ and there is a relative Poisson bialgebra $(A\ltimes_{-\mu^*,\rho^*}V^*, \cdot,[-,-],\Delta,\delta,P-\alpha^*,-P+\alpha^*)$, where the linear maps Δ and δ are defined respectively by equations (57) and (58) through $r=T-\tau(T)$.*

Proof. By Corollary 2.9, $(-\mu^*,\rho^*,-\alpha^*,V^*)$ is a representation of (A,P) and $-P$ dually represents (A,P). Moreover, equations (81) and (82) hold when $\beta=-\alpha$, $Q=-P$. Hence the conclusion follows from Theorem 4.13 when $\beta=-\alpha$, $Q=-P$. ■

4.3. Relative pre-Poisson algebras

Recall the notions of Zinbiel algebras ([32]) and pre-Lie algebras ([11]).

Definition 4.15. A *Zinbiel algebra* is a vector space A equipped with a bilinear operation $\star: A\otimes A\rightarrow A$ such that

$$x\star(y\star z)=(y\star x)\star z+(x\star y)\star z,\qquad \forall x,y,z\in A. \tag{85}$$

Let $(A,\star)$ be a Zinbiel algebra. Define a new bilinear operation $\cdot$ on A by

$$x\cdot y=x\star y+y\star x,\qquad \forall x,y\in A. \tag{86}$$

Then $(A,\cdot)$ is a commutative associative algebra. Moreover, $(\mathcal{L}_\star, A)$ is a representation of the commutative associative algebra $(A,\cdot)$, where $\mathcal{L}_\star(x)y=x\star y$ for all $x,y\in A$.

Definition 4.16. A *pre-Lie algebra* is a vector space A equipped with a bilinear operation $\circ: A\otimes A\rightarrow A$ such that

$$(x\circ y)\circ z-x\circ(y\circ z)=(y\circ x)\circ z-y\circ(x\circ z),\qquad \forall x,y,z\in A. \tag{87}$$

Let $(A,\circ)$ be a pre-Lie algebra. Define a new bilinear operation $[-,-]$ on A by

$$[x,y]=x\circ y-y\circ x,\qquad \forall x,y\in A. \tag{88}$$

Then $(A,[-,-])$ is a Lie algebra. Moreover, $(\mathcal{L}_\circ, A)$ is a representation of the Lie algebra $(A,[-,-])$, where $\mathcal{L}_\circ(x)y=x\circ y$ for all $x,y\in A$.

Definition 4.17. A *relative pre-Poisson algebra* is a quadruple $(A,\star,\circ,P)$, where $(A,\star)$ is a Zinbiel algebra, $(A,\circ)$ is a pre-Lie algebra, $P: A\rightarrow A$ is a derivation of both $(A,\star)$ and $(A,\circ)$, that is, for all $x,y\in A$,

$$P(x\star y)=P(x)\star y+x\star P(y), \tag{89}$$
$$P(x\circ y)=P(x)\circ y+x\circ P(y), \tag{90}$$

and the following compatible conditions are satisfied:

$$(x \star y + y \star x) \circ z - x \star (y \circ z) - y \star (x \circ z) + (x \star y + y \star x) \star P(z) = 0, \tag{91}$$

$$y \circ (x \star z) - x \star (y \circ z) + (x \circ y - y \circ x) \star z - (x \star P(y) + P(y) \star x) \star z = 0, \tag{92}$$

for all $x, y, z \in A$.

Remark 4.18. Recall a pre-Poisson algebra ([3]) is a triple $(A, \star, \circ)$, where $(A, \star)$ is a Zinbiel algebra and $(A, \circ)$ is a pre-Lie algebra such that the following conditions hold:

$$(x \circ y - y \circ x) \star z = x \star (y \circ z) - y \circ (x \star z),$$
$$(x \star y + y \star x) \circ z = x \star (y \circ z) + y \star (x \circ z),$$

for all $x, y, z \in A$. Thus any pre-Poisson algebra is a relative pre-Poisson algebra with the derivation $P = 0$.

There is the following construction of relative pre-Poisson algebras from Zinbiel algebras with their derivations, which is an analogue of Example 1.2.

Proposition 4.19. *Let $(A, \star)$ be a Zinbiel algebra and P be a derivation of $(A, \star)$. Define a new bilinear operation $\circ : A \otimes A \to A$ by*

$$x \circ y = x \star P(y) - P(x) \star y, \qquad \forall x, y \in A.$$

Then $(A, \circ)$ is a pre-Lie algebra and $(A, \star, \circ, P)$ is a relative pre-Poisson algebra.

Proof. Let $x, y, z \in A$. Then we have

$$\begin{aligned}(x \circ y) \circ z - x \circ (y \circ z) &= (x \star P(y)) \star P(z) - (P(x) \star y) \star P(z) - (x \star P^2(y)) \star z \\ &\quad + (P^2(x) \star y) \star z - x \star (y \star P^2(z)) + x \star (P^2(y) \star z) \\ &\quad + P(x) \star (y \star P(z)) - P(x) \star (P(y) \star z), \\ (y \circ x) \circ z - y \circ (x \circ z) &= (y \star P(x)) \star P(z) - (P(y) \star x) \star P(z) - (y \star P^2(x)) \star z \\ &\quad + (P^2(y) \star x) \star z - y \star (x \star P^2(z)) + y \star (P^2(x) \star z) \\ &\quad + P(y) \star (x \star P(z)) - P(y) \star (P(x) \star z).\end{aligned}$$

Note that by equation (85), we have

$$\begin{aligned}-x \star (y \star P^2(z)) &= -y \star (x \star P^2(z)), \\ -P(x) \star (P(y) \star z) &= -P(y) \star (P(x) \star z), \\ -(P(x) \star y) \star P(z) + P(x) \star (y \star P(z)) & \\ &= (y \star P(x)) \star P(z), \\ (x \star P(y)) \star P(z) &= -(P(y) \star x) \star P(z) + P(y) \star (x \star P(z)), \\ -(x \star P^2(y)) \star z + x \star (P^2(y) \star z) &= (P^2(y) \star x) \star z, \\ (P^2(x) \star y) \star z &= -(y \star P^2(x)) \star z + y \star (P^2(x) \star z).\end{aligned}$$

Thus equation (87) holds, that is, $(A, \circ)$ is a pre-Lie algebra. Similarly, we show that P is a derivation of $(A, \circ)$ and equations (91) and (92) hold. Hence $(A, \star, \circ, P)$ is a relative pre-Poisson algebra. ∎

Example 4.20. Let $(A, \star)$ be a 3-dimensional Zinbiel algebra with a basis $\{e_1, e_2, e_3\}$ whose non-zero products ([24]) are given as follows:

$$e_1 \star e_1 = e_3, \qquad e_1 \star e_2 = e_3.$$

Define a linear map $P : A \to A$ as

$$P(e_1) = e_1 + e_2, \qquad P(e_2) = 2e_2, \qquad P(e_3) = 3e_3.$$

Then P is a derivation of $(A, \star)$. Hence there is a relative pre-Poisson algebra $(A, \star, \circ, P)$ with the following non-zero products of the pre-Lie algebra $(A, \circ)$:

$$e_1 \circ e_1 = e_3, \qquad e_1 \circ e_2 = e_3.$$

Proposition 4.21. *Let $(A, \star, \circ, P)$ be a relative pre-Poisson algebra. Then $(A, \cdot, [-,-], P)$ is a relative Poisson algebra, where $\cdot, [-,-] : A \otimes A \to A$ are defined by equations (86) and (88) respectively. Moreover, $(\mathcal{L}_\star, \mathcal{L}_\circ, P, A)$ is a representation of the relative Poisson algebra $(A, \cdot, [-,-], P)$ and hence the identity map* id *is an $\mathcal{O}$-operator of $(A, \cdot, [-,-], P)$ associated to $(\mathcal{L}_\star, \mathcal{L}_\circ, P, A)$.*

Proof. Let $x, y, z \in A$. By equations (89) and (90), P is a derivation of both $(A, \cdot)$ and $(A, [-,-])$. Moreover, by equations (85), (91) and (92), we have

$$\begin{aligned}
&[z, x \cdot y] - x \cdot [z, y] - y \cdot [z, x] - x \cdot y \cdot P(z)\\
&= (z \circ (x \star y) + z \circ (y \star x) - (x \star y) \circ z - (y \star x) \circ z)\\
&\quad - (x \star (z \circ y) - x \star (y \circ z) + (z \circ y) \star x - (y \circ z) \star x)\\
&\quad - (y \star (z \circ x) - y \star (x \circ z) + (z \circ x) \star y - (x \circ z) \star y)\\
&\quad - ((x \star y + y \star x) \star P(z) + P(z) \star (x \star y + y \star x))\\
&= (x \star P(z) + P(z) \star x) \star y + (y \star P(z) + P(z) \star y) \star x\\
&\quad - P(z) \star (x \star y + y \star x) = 0.
\end{aligned}$$

Thus $(A, \cdot, [-,-], P)$ is a relative Poisson algebra. Moreover, by equations (89)–(92), we have

$$\begin{aligned}
&P(\mathcal{L}_\star(x)y) = \mathcal{L}_\star(P(x))y + \mathcal{L}_\star(x)P(y),\\
&P(\mathcal{L}_\circ(x)y) = \mathcal{L}_\circ(P(x))y + \mathcal{L}_\circ(x)P(y),\\
&\mathcal{L}_\circ(x \cdot y)z - \mathcal{L}_\star(x)\mathcal{L}_\circ(y)z - \mathcal{L}_\star(y)\mathcal{L}_\circ(x)z + \mathcal{L}_\star(x \cdot y)P(z) = 0,\\
&\mathcal{L}_\circ(y)\mathcal{L}_\star(x)z - \mathcal{L}_\star(x)\mathcal{L}_\circ(y)z + \mathcal{L}_\star([x, y])z - \mathcal{L}_\star(x \cdot P(y))z = 0.
\end{aligned}$$

Thus $(\mathcal{L}_\star, \mathcal{L}_\circ, P, A)$ is a representation of the relative Poisson algebra $(A, \cdot, [-,-], P)$. The rest of the conclusion is obvious. ∎

Definition 4.22. Let $(A, \star, \circ, P)$ be a relative pre-Poisson algebra. Define two bilinear operations $\cdot, [-,-] : A \otimes A \to A$ by equations (86) and (88) respectively. Then $(A, \cdot, [-,-], P)$ is called the *sub-adjacent relative Poisson algebra* of $(A, \star, \circ, P)$, and $(A, \star, \circ, P)$ is called a *compatible relative pre-Poisson algebra* structure on the relative Poisson algebra $(A, \cdot, [-,-], P)$.

By Corollary 4.14 and Proposition 4.21, we obtain the following construction of antisymmetric solutions of the RPYBE and hence relative Poisson bialgebras from relative pre-Poisson algebras.

Proposition 4.23. *Let $(A, \star, \circ, P)$ be a relative pre-Poisson algebra and $(A, \cdot, [-,-], P)$ be the sub-adjacent relative Poisson algebra. Let $\{e_1, \dots, e_n\}$ be a basis of A and $\{e_1^*, \dots, e_n^*\}$ be the dual basis. Then*

$$r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i)$$

is an antisymmetric solution of the $(-P + P^*)$-RPYBE in the relative Poisson algebra $(A \ltimes_{-\mathcal{L}_\star^*, \mathcal{L}_\circ^*} A^*, P - P^*)$. Further the relative Poisson algebra $(A \ltimes_{-\mathcal{L}_\star^*, \mathcal{L}_\circ^*} A^*, P - P^*)$ is dually represented by $(-P + P^*)$ and hence there is a relative Poisson bialgebra $(A \ltimes_{-\mathcal{L}_\star^*, \mathcal{L}_\circ^*} A^*, \cdot, [-,-], \Delta, \delta, P - P^*, -P + P^*)$, where the linear maps Δ and δ are defined respectively by equations (57) and (58) through r.

5. Relative Poisson bialgebras and Frobenius Jacobi algebras

We use relative Poisson bialgebras to construct Frobenius Jacobi algebras. In particular, there is a construction of Frobenius Jacobi algebras from relative pre-Poisson algebras. We give an example to illustrate this construction explicitly.

By Remark 2.19, the approach for the bialgebra theory of relative Poisson algebras in terms of matched pairs is not available for Jacobi algebras any more. That is, for a relative Poisson bialgebra $(A, \cdot_A, [-,-]_A, \Delta, \delta, P, Q)$, it is impossible for the commutative associative algebra structure on $A \oplus A^*$ to be unital such that both $(A, \cdot_A)$ and $(A^*, \cdot_{A^*})$ are unital, where $\cdot_{A^*}$ is given by the dual of Δ. However, it might be possible for the commutative associative algebra structure on $A \oplus A^*$ to be unital such that one of $(A, \cdot_A)$ and $(A^*, \cdot_{A^*})$ is unital. Note that in this case, the induced relative Poisson algebra $A \bowtie A^*$ is a Jacobi algebra and hence with the bilinear form $\mathcal{B}_d$ defined by equation (46), it is a Frobenius Jacobi algebra. Explicitly, we have the following proposition.

Proposition 5.1. *Let $(A, \cdot_A)$ be a unital commutative associative algebra with the unit 1_A and $\Delta : A \to A \otimes A$ be a linear map such that $(A, \cdot_A, \Delta)$ is a commutative and cocommutative infinitesimal bialgebra. If the induced commutative associative algebra structure on $A \oplus A^*$ is unital, then the unit is 1_A. Moreover, 1_A is the unit of the commutative associative algebra structure on $A \oplus A^*$ if and only if $\Delta(1_A) = 0$. In particular, if $(A, \cdot, \Delta)$ is coboundary, then $\Delta(1_A) = 0$ automatically.*

Proof. By Remark 2.19, if the induced commutative associative algebra structure on $A \oplus A^*$ is unital, then the unit is 1_A. Note that the commutative associative algebra structure on $A \oplus A^*$ is given by

$$(x + a^*) \cdot (y + b^*) = x \cdot_A y - \mathcal{L}_{A^*}^*(a^*)y - \mathcal{L}_{A^*}^*(b^*)x + a^* \cdot_{A^*} b^* \\ - \mathcal{L}_A^*(x)b^* - \mathcal{L}_A^*(y)a^*, \qquad \forall x, y \in A,\ a^*, b^* \in A^*, \tag{93}$$

where $\cdot_{A^*}$ is given by the dual of Δ. Then 1_A is the unit of the commutative associative algebra structure on $A \oplus A^*$ if and only if

$$\begin{aligned} & 1_A \cdot a^* = -\mathcal{L}_A^*(1_A)a^* - \mathcal{L}_{A^*}^*(a^*)1_A = a^*, && \forall a^* \in A^*, \\ \Leftrightarrow\ & -\mathcal{L}_{A^*}^*(a^*)1_A = 0, && \forall a^* \in A^*, \\ \Leftrightarrow\ & \langle \Delta(1_A), a^* \otimes b^* \rangle = 0, && \forall a^*, b^* \in A^*, \\ \Leftrightarrow\ & \Delta(1_A) = 0. \end{aligned}$$

In particular, if $(A, \cdot, \Delta)$ is coboundary, that is, there exists $r = \sum_i a_i \otimes b_i \in A \otimes A$ such that equation (57) holds, then we have

$$\Delta(1_A) = (\mathrm{id} \otimes \mathcal{L}(1_A) - \mathcal{L}(1_A) \otimes \mathrm{id})r = \sum_i (a_i \otimes b_i - a_i \otimes b_i) = 0.$$

Hence the conclusion holds. ■

Let $(A, \cdot, [-,-])$ be a Jacobi algebra and $(A, \cdot, [-,-], \mathrm{ad}(1_A))$ be the corresponding unital relative Poisson algebra. By Corollary 2.10, Q dually represents $(A, \cdot, [-,-], \mathrm{ad}(1_A))$ if and only if $Q = -\mathrm{ad}(1_A)$. Then we have the following conclusion at the level of Jacobi algebras.

Corollary 5.2. *Let $(A, \cdot, [-,-])$ be a Jacobi algebra and $(A, \cdot, [-,-], \mathrm{ad}(1_A))$ be the corresponding unital relative Poisson algebra. Suppose that $(A, \cdot, [-,-], \Delta, \delta, \mathrm{ad}(1_A), -\mathrm{ad}(1_A))$ is a relative Poisson bialgebra. Then the induced relative Poisson algebra $(A \bowtie A^*, \mathrm{ad}(1_A) - (\mathrm{ad}(1_A))^*)$ is unital, that is, it is a Jacobi algebra, if and only if $\Delta(1_A) = 0$. In this case, with the bilinear form $\mathcal{B}_d$ defined by equation (46), it is a Frobenius Jacobi algebra. In particular, if $(A, \cdot, [-,-], \Delta, \delta, \mathrm{ad}(1_A), -\mathrm{ad}(1_A))$ is coboundary, then $\Delta(1_A) = 0$ automatically.*

Combining Corollaries 4.5 and 5.2 together, we have the following construction of Frobenius Jacobi algebras from antisymmetric solutions of the RPYBE in Jacobi algebras.

Corollary 5.3. *Let $(A, \cdot, [-,-])$ be a Jacobi algebra and $(A, \cdot, [-,-], \mathrm{ad}(1_A))$ be the corresponding unital relative Poisson algebra. Let $r \in A \otimes A$ and $\Delta, \delta : A \to A \otimes A$ be linear maps defined by equations (57) and (58) respectively. If r is an antisymmetric solution of the $(-\mathrm{ad}(1_A))$-RPYBE in the relative Poisson algebra $(A, \cdot, [-,-], \mathrm{ad}(1_A))$, then $(A, \cdot, [-,-], \Delta, \delta, \mathrm{ad}(1_A), -\mathrm{ad}(1_A))$ is a coboundary relative Poisson bialgebra and hence the induced relative Poisson algebra $(A \bowtie A^*, \mathrm{ad}(1_A) - (\mathrm{ad}(1_A))^*)$ with the bilinear form $\mathcal{B}_d$ defined by equation (46) is a Frobenius Jacobi algebra.*

Remark 5.4. Unfortunately, the construction of antisymmetric solutions of the RPYBE in relative Poisson algebras from relative pre-Poisson algebras given in Proposition 4.23 cannot be applied directly to the above conclusion since the sub-adjacent relative Poisson algebra $(A, \cdot, [-,-], P)$ of a relative pre-Poisson algebra $(A, \star, \circ, P)$ is not a Jacobi algebra (hence the semi-direct product relative Poisson algebra $(A \ltimes_{-\mathcal{L}^*_\star, \mathcal{L}^*_\circ} A^*, P - P^*)$ is not a Jacobi algebra, either). In fact, let $(A, \star)$ be a Zinbiel algebra. Suppose that the commutative associative algebra $(A, \cdot)$ defined by equation (86) has the unit 1_A. Then we have

$$x = x \star 1_A + 1_A \star x, \qquad \forall x \in A.$$

Thus $1_A = 2(1_A \star 1_A)$. On the other hand, we have

$$x \star (1_A \star 1_A) = (x \cdot 1_A) \star 1_A = x \star 1_A, \qquad \forall x \in A.$$

Hence $x \star 1_A = 0$ and thus $1_A \star x = x$ for all $x \in A$. Taking $x = 1_A$, we have $x = 0$, which is a contradiction.

Next we give an approach in which the construction given by Corollary 5.3 can be applied.

Lemma 5.5. *Let $(A, \cdot, [-,-], P)$ be a relative Poisson algebra. Extend the vector space A to be a $(\dim A + 1)$-dimensional vector space $\widetilde{A} = A \oplus \mathbb{K}e$. Define two bilinear operations $\cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}} : \widetilde{A} \otimes \widetilde{A} \to \widetilde{A}$ and a linear map $\widetilde{P} : \widetilde{A} \to \widetilde{A}$ as*

$$\begin{aligned}
&x \cdot_{\widetilde{A}} y = x \cdot y, \qquad && e \cdot_{\widetilde{A}} x = x \cdot_{\widetilde{A}} e = x, \qquad && e \cdot_{\widetilde{A}} e = e,\\
&[x, y]_{\widetilde{A}} = [x, y], && [e, x]_{\widetilde{A}} = -[x, e]_{\widetilde{A}} = P(x), && [e, e]_{\widetilde{A}} = 0,\\
&\widetilde{P}(x) = P(x), && \widetilde{P}(e) = 0,
\end{aligned}$$

for all $x, y \in A$. Then $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}}, \widetilde{P})$ is a unital relative Poisson algebra. Consequently, there is a corresponding Jacobi algebra $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}})$, which is called the extended Jacobi algebra *of the relative Poisson algebra $(A, \cdot, [-,-], P)$.*

Proof. It is obvious that $(\widetilde{A}, \cdot_{\widetilde{A}})$ is a commutative associative algebra and e is the unit. Let $x, y, z \in A$. Then we have

$$[e, [x, y]_{\widetilde{A}}]_{\widetilde{A}} + [x, [y, e]_{\widetilde{A}}]_{\widetilde{A}} + [y, [e, x]_{\widetilde{A}}]_{\widetilde{A}} = P([x, y]) - [x, P(y)] + [y, P(x)] = 0.$$

Hence $(\widetilde{A}, [-,-]_{\widetilde{A}})$ is a Lie algebra and $\widetilde{P} = \mathrm{ad}_{\widetilde{A}}(e)$. Moreover, we have

$$\begin{aligned}
[e, x]_{\widetilde{A}} \cdot_{\widetilde{A}} y + x \cdot_{\widetilde{A}} [e, y]_{\widetilde{A}} + x \cdot_{\widetilde{A}} y \cdot_{\widetilde{A}} \widetilde{P}(e) &= P(x) \cdot y + x \cdot P(y)\\
&= P(x \cdot y) = [e, x \cdot_{\widetilde{A}} y]_{\widetilde{A}},\\
[z, x]_{\widetilde{A}} \cdot_{\widetilde{A}} e + x \cdot_{\widetilde{A}} [z, e]_{\widetilde{A}} + x \cdot_{\widetilde{A}} e \cdot_{\widetilde{A}} \widetilde{P}(z) &= [z, x] - x \cdot P(z) + x \cdot P(z)\\
&= [z, x \cdot_{\widetilde{A}} e]_{\widetilde{A}}.
\end{aligned}$$

Therefore $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}}, \widetilde{P})$ is a unital relative Poisson algebra. ■

Proposition 5.6. *Let (μ, ρ, α, V) be a representation of a relative Poisson algebra $(A, \cdot, [-,-], P)$. Then $(\widetilde{\mu}, \widetilde{\rho}, V)$ is a representation of the extended Jacobi algebra $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}})$ with the linear maps $\widetilde{\mu}, \widetilde{\rho} : \widetilde{A} \to \mathrm{End}(V)$ defined as*

$$\widetilde{\mu}(x) = \mu(x), \qquad \widetilde{\mu}(e) = \mathrm{id}_V, \qquad \widetilde{\rho}(x) = \rho(x), \qquad \widetilde{\rho}(e) = \alpha, \qquad \forall x \in A. \tag{94}$$

Moreover, $(\widetilde{\mu}, \widetilde{\rho}, \alpha, V)$ is a representation of the unital relative Poisson algebra $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}}, \widetilde{P})$.

Proof. The first part of the conclusion can be proved by checking equations (27) and (28) directly or as follows. Since (μ, ρ, α, V) is a representation of $(A, \cdot, [-,-], P)$, $(A \ltimes_{\mu,\rho} V, P + \alpha)$ is a relative Poisson algebra. Hence there is an extended Jacobi algebra $\widetilde{A \ltimes_{\mu,\rho} V}$ in which the bilinear operations $\cdot_{\widetilde{A \ltimes_{\mu,\rho} V}}$ and $[-,-]_{\widetilde{A \ltimes_{\mu,\rho} V}}$ on $A \oplus V$ are the same as the ones of $(A \ltimes_{\mu,\rho} V, P + \alpha)$ and for all $x \in A, u \in V$,

$$e \cdot_{\widetilde{A \ltimes_{\mu,\rho} V}} x = x, \qquad e \cdot_{\widetilde{A \ltimes_{\mu,\rho} V}} u = u,$$
$$[e, x]_{\widetilde{A \ltimes_{\mu,\rho} V}} = (P + \alpha)x = P(x), \qquad [e, u]_{\widetilde{A \ltimes_{\mu,\rho} V}} = (P + \alpha)u = \alpha(u).$$

On the other hand, the above Jacobi algebra is exactly the Jacobi algebra structure on the direct sum $\widetilde{A} \oplus V = A \oplus V \oplus \mathbb{K}e$ given by the Jacobi algebra $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}})$ and the linear maps $\widetilde{\mu}, \widetilde{\rho} : \widetilde{A} \to \mathrm{End}(V)$ defined by equation (94) through equations (8) and (9). Thus $(\widetilde{\mu}, \widetilde{\rho}, V)$ is a representation of $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}})$. The second part of the conclusion follows from Proposition 2.13. ∎

Corollary 5.7. *Let $T : V \to A$ be an $\mathcal{O}$-operator of a relative Poisson algebra $(A, \cdot, [-,-], P)$ associated to a representation (μ, ρ, α, V). Then $T : V \to A \subset \widetilde{A}$ is also an $\mathcal{O}$-operator of the unital relative Poisson algebra $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}}, \widetilde{P})$ (that is, the extended Jacobi algebra $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}})$) associated to the representation $(\widetilde{\mu}, \widetilde{\rho}, \alpha, V)$.*

Proof. It follows from Lemma 5.5 and Proposition 5.6. ∎

Thus there is the following construction of Frobenius Jacobi algebras from relative pre-Poisson algebras.

Theorem 5.8. *Let $(A, \star, \circ, P)$ be a relative pre-Poisson algebra and $(A, \cdot, [-,-], P)$ be the sub-adjacent relative Poisson algebra. Let $\{e_1, \ldots, e_n\}$ be a basis of A and $\{e_1^*, \ldots, e_n^*\}$ be the dual basis. Then*

$$r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i) \tag{95}$$

is an antisymmetric solution of the $(-\widetilde{P} + P^*)$-RPYBE in the unital relative Poisson algebra $(\widetilde{A} \ltimes A^* := \widetilde{A} \ltimes_{-\widetilde{\mathcal{L}}_\star^*, \widetilde{\mathcal{L}}_\circ^*} A^*, \widetilde{P} - P^*)$. Moreover, there is a relative Poisson bialgebra

$$(\widetilde{A} \ltimes A^*, \cdot_{\widetilde{A} \ltimes A^*}, [-,-]_{\widetilde{A} \ltimes A^*}, \Delta, \delta, \widetilde{P} - P^*, -\widetilde{P} + P^*),$$

where the linear maps $\Delta, \delta : \widetilde{A} \ltimes A^* \to (\widetilde{A} \ltimes A^*) \otimes (\widetilde{A} \ltimes A^*)$ *are defined respectively by equations* (57) *and* (58) *through* r *such that the induced relative Poisson algebra* $(\widetilde{A} \ltimes A^*) \bowtie (\widetilde{A} \ltimes A^*)^*$ *with the bilinear form* $\mathcal{B}_d$ *defined by equation* (46) *is a Frobenius Jacobi algebra.*

Proof. By Proposition 4.21, the identity map id_A is an $\mathcal{O}$-operator of the relative Poisson algebra $(A, \cdot, [-,-], P)$ associated to $(\mathcal{L}_\star, \mathcal{L}_\circ, P, A)$. By Corollary 5.7, id_A is also an $\mathcal{O}$-operator of the unital relative Poisson algebra $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}}, \widetilde{P})$ associated to the representation $(\widetilde{\mathcal{L}_\star}, \widetilde{\mathcal{L}_\circ}, P, A)$. Then by Corollary 4.14, r is an antisymmetric solution of the $(-\widetilde{P} + P^*)$-RPYBE in the relative Poisson algebra $(\widetilde{A} \ltimes A^*, \widetilde{P} - P^*)$. Further, the relative Poisson algebra $(\widetilde{A} \ltimes A^*, \widetilde{P} - P^*)$ is dually represented by $-\widetilde{P} + P^*$. The rest follows from Corollary 5.3. ∎

Example 5.9. Let $(A, \star, \circ, P)$ be the 3-dimensional relative pre-Poisson algebra given by Example 4.20. Then the sub-adjacent relative Poisson algebra $(A, \cdot, [-,-], P)$ is given by the following non-zero products:

$$e_1 \cdot e_1 = 2e_3, \qquad e_1 \cdot e_2 = e_3, \qquad [e_1, e_2] = e_3.$$

By Lemma 5.5, we have the extended Jacobi algebra $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}})$ whose non-zero products are given by the above products and the following products:

$$e \cdot_{\widetilde{A}} e_1 = e_1, \qquad e \cdot_{\widetilde{A}} e_2 = e_2, \qquad e \cdot_{\widetilde{A}} e_3 = e_3, \qquad e \cdot_{\widetilde{A}} e = e,$$
$$[e, e_1]_{\widetilde{A}} = P(e_1) = e_1 + e_2, \qquad [e, e_2]_{\widetilde{A}} = P(e_2) = 2e_2, \qquad [e, e_3]_{\widetilde{A}} = P(e_3) = 3e_3.$$

Let $\{e_1^*, e_2^*, e_3^*\}$ be the basis of A^* which is dual to $\{e_1, e_2, e_3\}$. By Proposition 5.6, there is a Jacobi algebra $(\widetilde{A} \ltimes A^* := \widetilde{A} \ltimes_{-\widetilde{\mathcal{L}_\star}^*, \widetilde{\mathcal{L}_\circ}^*} A^*, \cdot_{\widetilde{A} \ltimes A^*}, [-,-]_{\widetilde{A} \ltimes A^*})$, where the non-zero products are given by the above non-zero products of $(\widetilde{A}, \cdot_{\widetilde{A}}, [-,-]_{\widetilde{A}})$ and the following products:

$$\begin{aligned}
e \cdot_{\widetilde{A} \ltimes A^*} e_1^* &= -\widetilde{\mathcal{L}_\star}^*(e) e_1^* = e_1^*, \\
e \cdot_{\widetilde{A} \ltimes A^*} e_2^* &= -\widetilde{\mathcal{L}_\star}^*(e) e_2^* = e_2^*, \\
e \cdot_{\widetilde{A} \ltimes A^*} e_3^* &= -\widetilde{\mathcal{L}_\star}^*(e) e_3^* = e_3^*, \\
e_1 \cdot_{\widetilde{A} \ltimes A^*} e_3^* &= -\mathcal{L}_\star^*(e_1) e_3^* = e_1^* + e_2^*, \\
[e, e_1^*]_{\widetilde{A} \ltimes A^*} &= \widetilde{\mathcal{L}_\circ}^*(e) e_1^* = -P^*(e_1^*) = -e_1^*, \\
[e, e_2^*]_{\widetilde{A} \ltimes A^*} &= \widetilde{\mathcal{L}_\circ}^*(e) e_2^* = -P^*(e_2^*) = -e_1^* - 2e_2^*, \\
[e, e_3^*]_{\widetilde{A} \ltimes A^*} &= \widetilde{\mathcal{L}_\circ}^*(e) e_3^* = -P^*(e_3^*) = -3e_3^*, \\
[e_1, e_3^*]_{\widetilde{A} \ltimes A^*} &= \mathcal{L}_\circ^*(e_1) e_3^* = -e_1^* - e_2^*.
\end{aligned}$$

In order to simplify the notations, we replace $\widetilde{A} \ltimes A^*$ by J, $\cdot_{\widetilde{A} \ltimes A^*}$ by $\cdot$, $[-,-]_{\widetilde{A} \ltimes A^*}$ by $[-,-]$, and $\{e, e_1, e_2, e_3, e_1^*, e_2^*, e_3^*\}$ by $\{E, E_1, E_2, E_3, E_4, E_5, E_6\}$. Then $(J, \cdot, [-,-])$ is a Jacobi algebra with a basis $\{E, E_1, E_2, E_3, E_4, E_5, E_6\}$, in which E is the unit and the other non-zero products are given by

$$\begin{array}{lll} E_1\cdot E_1 = 2E_3, & E_1\cdot E_2 = E_3, & E_1\cdot E_6 = E_4+E_5,\\ [E,E_1] = E_1+E_2, & [E,E_2] = 2E_2, & [E,E_3] = 3E_3,\\ [E,E_4] = -E_4, & [E,E_5] = -E_4-2E_5, & [E,E_6] = -3E_6,\\ [E_1,E_2] = E_3, & [E_1,E_6] = -E_4-E_5. & \end{array}$$

Obviously it is isomorphic to the Jacobi algebra $(\widetilde{A}\ltimes A^*, \cdot_{\widetilde{A}\ltimes A^*}, [-,-]_{\widetilde{A}\ltimes A^*})$. By equation (95), we set

$$r = E_1\otimes E_4 - E_4\otimes E_1 + E_2\otimes E_5 - E_5\otimes E_2 + E_3\otimes E_6 - E_6\otimes E_3.$$

Let $\Delta, \delta : J \to J\otimes J$ be linear maps given by equations (57) and (58) respectively. Then

$$\begin{aligned} &\Delta(E_1) = \Delta(E_2) = -E_3\otimes E_4 - E_4\otimes E_3,\\ &\Delta(E_6) = -2E_4\otimes E_4 - E_4\otimes E_5 - E_5\otimes E_4,\\ &\delta(E_1) = \delta(E_2) = -E_3\otimes E_4 + E_4\otimes E_3,\\ &\delta(E_6) = -E_4\otimes E_5 + E_5\otimes E_4,\\ &\Delta(E) = \Delta(E_3) = \Delta(E_4) = \Delta(E_5) = \delta(E) = \delta(E_3) = \delta(E_4) = \delta(E_5) = 0. \end{aligned}$$

Hence $(J, \cdot, [-,-], \Delta, \delta, \mathrm{ad}_J(E), -\mathrm{ad}_J(E))$ is a relative Poisson bialgebra. Moreover, the non-zero products on the relative Poisson algebra J^* are given by

$$\begin{array}{lll} E_3^*\cdot E_4^* = -E_1^*-E_2^*, & E_4^*\cdot E_4^* = -2E_6^*, & E_4^*\cdot E_5^* = -E_6^*,\\ [E_3^*,E_4^*] = -E_1^*-E_2^*, & [E_4^*,E_5^*] = -E_6^*. & \end{array}$$

With the above relative Poisson algebra structures on J and J^* together, $J\bowtie J^*$ is a relative Poisson algebra with the unit E, in which the other non-zero products are given by equations (30) and (31) with respect to the matched pair

$$((J, \mathrm{ad}_J(E)), (J^*, -\mathrm{ad}_J(E)^*), -\mathcal{L}_J^*, -\mathcal{L}_{J^*}^*, \mathrm{ad}_J^*, \mathrm{ad}_{J^*}^*)$$

as follows:

$$\begin{array}{lll} E_1\cdot E_1^* = E^*, & E_1\cdot E_3^* = -E_4+2E_1^*+E_2^*, & E_1\cdot E_4^* = -E_3+E_6^*,\\ E_1\cdot E_5^* = E_6^*, & E_2\cdot E_2^* = E^*, & E_2\cdot E_3^* = -E_4+E_1^*,\\ E_2\cdot E_4^* = -E_3, & E_3\cdot E_3^* = E^*, & E_4\cdot E_4^* = E^*,\\ E_5\cdot E_5^* = E^*, & E_6\cdot E_4^* = -2E_4-E_5+E_1^*, & E_6\cdot E_5^* = -E_4+E_1^*,\\ E_6\cdot E_6^* = E^*, & [E,E_1^*] = -E_1^*, & [E,E_2^*] = -E_1^*-2E_2^*,\\ [E,E_3^*] = -3E_3^*, & [E,E_4^*] = E_4^*+E_5^*, & [E,E_5^*] = 2E_5^*, \end{array}$$

$$
\begin{aligned}
&[E, E_6^*] = 3E_6^*, && [E_1, E_1^*] = E^*, && [E_1, E_2^*] = E^*,\\
&[E_1, E_3^*] = -E_4 - E_2^*, && [E_1, E_4^*] = E_3 + E_6^*, && [E_1, E_5^*] = E_6^*,\\
&[E_2, E_2^*] = 2E^*, && [E_2, E_3^*] = -E_4 + E_1^*, && [E_2, E_4^*] = E_3,\\
&[E_3, E_3^*] = 3E^*, && [E_4, E_4^*] = -E^*, && [E_5, E_4^*] = -E^*,\\
&[E_5, E_5^*] = -2E^*, && [E_6, E_4^*] = -E_5 - E_1^*, && [E_6, E_5^*] = E_4 - E_1^*,\\
&[E_6, E_6^*] = -3E^*.
\end{aligned}
$$

Therefore, by Theorem 5.8, $(J \bowtie J^*, \mathcal{B}_d)$ is a 14-dimensional Frobenius Jacobi algebra, where $\mathcal{B}_d$ is given by equation (46).

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