J. Noncommut. Geom. (Online first)
DOI 10.4171/JNCG/547

The 3-cyclic quantum Weyl algebras, their prime spectra and a classification of simple modules (q is not a root of unity)

Volodymyr V. Bavula

Abstract. The 3-cyclic quantum Weyl algebra $A = A(\alpha, \beta, \gamma; q^2)$, where $\alpha, \beta, \gamma, q^2 \in \mathbb{K}$ (a ground field), is a quadratic Noetherian domain of Gelfand–Kirillov dimension 3 that is generated by three subalgebras each of them is either a quantum plane or the quantum Weyl algebra. For the algebras A, their prime, completely prime, primitive and maximal spectra are described together with containments of prime ideals (the Zariski–Jacobson topology on the spectrum) and simple A-modules are classified when q^2 is not a root of unity. For each prime ideal, an explicit set of ideal generators is given. The centre $Z(A)$ of A is $\mathbb{K}[\Omega]$ where Ω is a cubic element. A semisimplicity criterion for the category of finite dimensional A-modules is given. Criteria are presented for all ideals of the algebra A to commute and for each ideal of A to be a unique product of primes (up to order).

1. Introduction

In the paper, $\mathbb{K}$ is a field, $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ = \{1, 2, \ldots\}$ and module means a left module.

The 3-cyclic quantum Weyl algebra $A(\alpha, \beta, \gamma)$.

Definition. For $\alpha, \beta, \gamma \in \mathbb{K}$, we define the *3-cyclic quantum Weyl algebra* $A = A(\alpha, \beta, \gamma)$ as an algebra generated by x, y and z subject to the defining relations

$$xy = q^2 yx + \alpha, \tag{1}$$

$$xz = q^{-2} zx + \beta, \tag{2}$$

$$yz = q^2 zy + \gamma. \tag{3}$$

The family of cyclic quantum Weyl algebras appeared naturally when we tried to classify Harish-Chandra modules over the quantized Lorentz algebra [11]. The constant q^2 rather than q is used in the defining relations of the algebra A in order that the results of this paper can be applied without change in [11]. The algebras A belong to the class of bi-quadratic algebras on 3 generators.

Mathematics Subject Classification 2020: 16D25 (primary); 17B10, 16D60, 16D70, 16P50 (secondary).
Keywords: prime ideal, primitive ideal, weight module, the generalized Weyl algebra, simple module, centralizer.

The bi-quadratic algebras on 3 generators, [10]. Let K be a field and $A = K[x_1, x_2, x_3; Q, \mathbb{A}, \mathbb{B}]$ be a bi-quadratic algebra where $Q = (q_1, q_2, q_3) \in K^{\times 3}$,

$$\mathbb{A} = \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{pmatrix}$$

and $\mathbb{B} = (b_1, b_2, b_3) \in K$. So, the algebra A is an algebra that is generated over the field K by the elements x_1, x_2 and x_3 subject to the defining relations

$$\begin{aligned} x_2x_1 - q_1x_1x_2 &= ax_1 + bx_2 + cx_3 + b_1, \\ x_3x_1 - q_2x_1x_3 &= \alpha x_1 + \beta x_2 + \gamma x_3 + b_2, \\ x_3x_2 - q_3x_2x_3 &= \lambda x_1 + \mu x_2 + \nu x_3 + b_3. \end{aligned}$$

An explicit description of all the bi-quadratic algebras on 3 generators is obtained in [10]. There are several dozens of classes.

Examples of bi-quadratic algebras on 3 generators.

(1) The universal enveloping algebra of any 3-dimensional Lie algebra.

(2) The *3-dimensional quantum space* $\mathbb{A}^3_{q_1,q_2,q_3} := K[x_1, x_2, x_3; Q, \mathbb{A} = 0, \mathbb{B} = 0]$.

(3) The algebra $U_q'(so_3)$ is generated over the field K by elements I_1, I_2 and I_3 subject to the defining relations

$$q^{\frac{1}{2}}I_1I_2 - q^{-\frac{1}{2}}I_2I_1 = I_3, \quad q^{\frac{1}{2}}I_2I_3 - q^{-\frac{1}{2}}I_3I_2 = I_1, \quad q^{\frac{1}{2}}I_3I_1 - q^{-\frac{1}{2}}I_1I_3 = I_2,$$

where $q \in K \setminus \{0, \pm 1\}$, [18, 20].

(4) The Askey–Wilson algebras $AW(3)$ introduced by A. Zhedanov, [21]. The algebra $AW(3)$ is generated by three elements K_0, K_1 and K_2 subject to the defining relations

$$\begin{aligned} [K_0, K_1]_w &= K_2, \\ [K_2, K_0]_w &= BK_0 + C_1K_1 + D_1, \\ [K_1, K_2]_w &= BK_1 + C_0K_0 + D_0, \end{aligned}$$

where $B, C_0, C_1, D_0, D_1 \in K$, $[L, M]_w := wLM - w^{-1}ML$ and $w \in K^\times$.

For a particular choice of the parameters α, β and γ, Ito, Terwilliger and Weng [19] showed that the algebra $U_q(\mathfrak{sl}_2)$ is the localization of the 3-cyclic quantum Weyl algebra at the powers of x.

The algebra $A = \mathbb{K}[x][y; \sigma_1, \delta_1][z; \sigma_2, \delta_2]$ is an iterated Ore extension. Therefore, the algebra A is a Noetherian domain of Gelfand–Kirillov dimension 3. The associated graded algebra $\operatorname{gr} A$ with respect to the standard filtration associated with the canonical generators of the algebra A is the *3-dimensional quantum affine space*, i.e.,

$$\operatorname{gr} A(\alpha, \beta, \gamma) \simeq A(0, 0, 0).$$

Cyclic permutation symmetry and the rank of $A(\alpha, \beta, \gamma)$. Notice that cyclicly permuting the canonical generators x, y and z of the algebra A (i.e., $x \to y \to z \to x$) we obtain the 3-cyclic quantum Weyl algebra but for a different choice of the defining parameters. In more detail, let $A(x, y, z; \alpha, \beta, \gamma) = A(\alpha, \beta, \gamma)$; then

$$A(x, y, z; \alpha, \beta, \gamma) = A(z, x, y; q^2\beta, q^{-2}\gamma, \alpha).$$

So, we say that the class of 3-cyclic quantum Weyl algebras admits the *cyclic permutation symmetry*. The *rank* $\mathrm{rk}(A)$ of the algebra $A(\alpha, \beta, \gamma)$ is the number of *nonzero* parameters in the set $\{\alpha, \beta, \gamma\}$. The rank of the algebra is invariant under the cyclic permutation symmetry. The cyclic permutation symmetry does not change the algebra but the way it is parametrized. So, in order to study the algebras $A(\alpha, \beta, \gamma)$ it suffices to consider four cases where $\mathrm{rk}(A) = 0, 1, 2$ and 3. Namely, $A(0, 0, 0)$, $A(\alpha, 0, 0)$, $A(\alpha, \beta, 0)$ and $A(\alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \in \mathbb{K}^*$.

Isomorphism criterion for the algebras $A(\alpha, \beta, \gamma)$. Theorem 1.1 is a criterion for two algebras $A(\alpha, \beta, \gamma)$ and $A(\alpha', \beta', \gamma')$ to be isomorphic.

Theorem 1.1. *Suppose the field $\mathbb{K}$ is not necessarily algebraically closed but is closed under taking square roots ($\sqrt{\mathbb{K}} \subseteq \mathbb{K}$) and q^2 is not a root of unity. Then two 3-cyclic quantum Weyl algebras are isomorphic iff they have the same rank.*

The aim of the paper. The aim of the paper is to give explicit descriptions of prime, completely prime, primitive and maximal ideals, to classify simple modules and ideals of the algebra $A = A(\alpha, \beta, \gamma)$ when q^2 is not a root of unity, and to obtain corollaries of the classification results. In [12], the root of unity case is considered.

The class of algebras $A(\alpha, \beta, \gamma)$ comprises different types of algebras. Properties of the algebras depend on the rank $\mathrm{rk}(A)$ and on the characteristic of $\mathbb{K}$. Each type of algebras requires somewhat different approaches to achieve the aim of the paper. Because of that and for simplicity reason we assume that the field $\mathbb{K}$ is an algebraically closed field. Using the same approach the interested reader may repeat arguments of this paper and obtain similar results with obvious modifications for an arbitrary field (but the paper will be more technical).

Since the opposite algebra A^{op} of the algebra $A(\alpha, \beta, \gamma)$ is an algebra of the type $A(\alpha', \beta', \gamma')$ and $\mathrm{rk}(A^{\mathrm{op}}) = \mathrm{rk}(A)$, a classification of simple *right* A-modules is automatically obtained from a classification of simple *left* $A(\alpha', \beta', \gamma')$-modules, and vice versa. In this paper, we deal with left modules.

In this paper, q^2 *is not a root of unity* (unless it is stated otherwise). The root of unity case is considered in [12].

The centre of the algebra A. The centre $Z(A)$ of the algebra A is a polynomial algebra $\mathbb{K}[\Omega]$ where

$$\Omega = yxz + \frac{q^{-2}\gamma}{q^2 - 1}x - \frac{q^2\beta}{q^2 - 1}y + \frac{\alpha}{q^2 - 1}z,$$

see Theorem 2.3.

Theorem 1.2. *Every nonzero ideal of the algebra $A(\alpha, \beta, \gamma)$ meets the centre.*

Classifications of prime ideals and simple $A(\alpha, \beta, \gamma)$-modules. For a noncommutative infinite dimensional algebra $\mathcal{A}$, classifications of its simple modules (up to isomorphism) $\widehat{\mathcal{A}}$ and prime ideals $\mathrm{Spec}(\mathcal{A})$ are very difficult problems (considered by many as intractable). Another difficult problem is to describe the Zariski–Jacobson topology on $\mathrm{Spec}(\mathcal{A})$, i.e., the complete *containment* information between prime ideals. The first two problems are linked via the map

$$\widehat{\mathcal{A}} \to \mathrm{Spec}(\mathcal{A}), \qquad [M] \mapsto \mathrm{ann}_{\mathcal{A}}(M).$$

The image $\mathrm{Prim}(\mathcal{A})$ of this map, the set of primitive ideals, is far from being $\mathrm{Spec}(\mathcal{A})$. Typically, the set $\widehat{\mathcal{A}}$ is much more massive than $\mathrm{Prim}(\mathcal{A})$, e.g., if $\mathcal{A}$ is a simple algebra (e.g., $\mathcal{A} = A_1 = \mathbb{K}\langle x, \partial \mid \partial x - x\partial = 1\rangle$ is the first Weyl algebra over a field of characteristic zero), then $\mathrm{Prim}(\mathcal{A}) = \mathrm{Spec}(\mathcal{A}) = \{0\}$ but the set $\widehat{\mathcal{A}}$ is huge. A typical situation is that for arbitrary large number of independent parameters one can construct a family of simple infinite dimensional modules that depends on this parameters and different choices of values of the parameters give *non-isomorphic* modules. In the introduction to his book "Enveloping Algebra" [17], Dixmier writes: "But a deeper study reveals the existence of an enormous number of irreducible representations of [Heisenberg Lie algebra]. It seems that these representations defy classification. A similar phenomenon exist for $\mathfrak{g} = \mathfrak{sl}(2)$, and most certainly for all non-commutative Lie algebras."

In 1981, Block classified simple modules over the first Weyl algebra A_1 and $\mathfrak{sl}_2$ over the field of complex numbers $\mathbb{C}$ [16]. In his book "Enveloping Algebra", Dixmier writes: "Even if $\widehat{\mathcal{A}}$ is very large, $\mathrm{Prim}(\mathcal{A})$ can be of reasonable size. N. Jacobson has equipped it with a topology and termed it the structural space of $\mathcal{A}$."

Let D be a (commutative) Dedekind domain, σ be its automorphism and δ be a σ-derivation of D (that is, $\delta(d_1 d_2) = \delta(d_1)d_2 + \sigma(d_1)\delta(d_2)$ for all elements $d_1, d_2 \in D$). Let $D[x; \sigma, \delta]$ be a skew polynomial ring, it is a ring which is generated by the ring D and x subject to the defining relations $xd = \sigma(d)x + \delta(d)$ for all $d \in D$. In [4], the simple modules of the ring $D[x; \sigma, \delta]$ are classified. In [2, 7], simple modules over generalized Weyl algebras $D[X, Y; \sigma, a]$ are classified, see also [3, 6, 15]. In [14], simple modules of generalized cross products with coefficients from D are classified.

One of the key points in obtaining a classification of simple A-modules is to use (not in a straightforward way) a classification of simple modules of some explicit generalized Weyl algebras $\mathcal{A} = D[x, y; \sigma, a]$ and skew polynomial ring $\mathcal{B} = D[x; \sigma, \delta]$ where D is a Dedekind domain (in this paper, $D = \mathbb{K}[z]$ or $D = \mathbb{K}[z, z^{-1}]$), σ is an automorphism of D and δ is a σ-derivation of D.

The paper has the following structure. In Sections 2 and 3, general properties of the algebras $A(\alpha, \beta, \gamma)$ are considered and results are given that are used in classifications of prime ideals and simple A-modules. In Section 5, we first classify the sets of prime, completely prime, primitive and maximal ideals of the algebra $A(\alpha, 0, 0)$ (Theorem 5.2 and Corollary 5.3) where $\alpha \neq 0$. Then using an explicit description of primitive ideals of the algebra $A(\alpha,0,0)$, for each primitive ideal $\mathfrak{p}$ of A, simple $A/\mathfrak{p}$-modules are classified (Theorem 5.4).

For the algebra $A = A(\alpha,\beta,\gamma)$ where $\alpha \neq 0$ and $\beta \neq 0$, the opposite approach is used. First, we classify simple modules and then prime, completely prime, primitive and maximal ideals are classified.

Let us explain our approach in classification of simple A-modules. The set $\widehat{A}$ of isomorphism classes of simple A-modules is a disjoint union

$$\widehat{A} = \bigsqcup_{\omega\in\mathbb{K}} \widehat{\overline{A}(\omega)} \qquad \text{where } \overline{A}(\omega) = A/(\Omega-\omega),$$

see (17), where $\mathbb{K}$ is an algebraically closed field. For each $\omega \in \mathbb{K}$, the set $\widehat{\overline{A}(\omega)}$ is a disjoint union of the three sets (see, (21))

$$\widehat{\overline{A}(\omega)} = \widehat{\overline{A}(\omega)}\,(z\text{-torsion}) \sqcup \widehat{\overline{A}(\omega)}\,(z\text{-torsionfree}, \mathbb{K}[z]\text{-torsion}) \sqcup \widehat{\overline{A}(\omega)}\,(\mathbb{K}[z]\text{-torsionfree}).$$

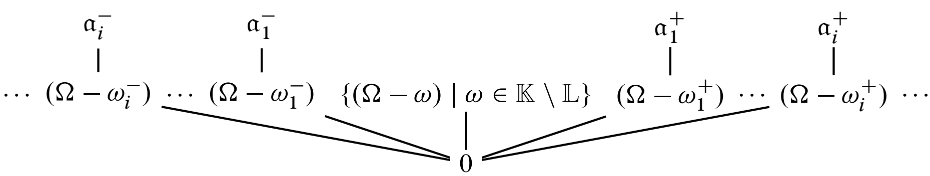
In Section 6, each of the three subsets above are described. The descriptions are too technical to present in the introduction.

An ideal $\mathfrak{p}$ of an algebra R is called a *completely prime ideal* of R if the factor algebra $R/\mathfrak{p}$ is a domain. Each completely prime ideal is a prime ideal but not vice versa, in general. The set of completely prime ideals of R is denoted by $\mathrm{Spec}_c(R)$. An ideal of an algebra R is called a *primitive ideal* of R if it is the annihilator ideal of a simple R-module. Each primitive ideal is a prime ideal but not vice versa, in general. The set of primitive ideals of R is denoted by $\mathrm{Prim}(R)$.

In Section 7, prime, completely prime, primitive and maximal ideals are described for the algebra $A(\alpha,\beta,\gamma)$ in the cases where $\alpha \neq 0$, $\beta \neq 0$, $\gamma = 0$ and $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$, respectively. When $\gamma \neq 0$, there are two cases: $\mathrm{char}(\mathbb{K}) \neq 2$ and $\mathrm{char}(\mathbb{K}) = 2$.

For example, the prime spectrum of the algebra $A = A(\alpha,\beta,\gamma)$ (where $\alpha,\beta,\gamma \in \mathbb{K}^*$ and $\mathrm{char}(\mathbb{K}) \neq 2$) is described by the diagram below (Theorem 7.4),

$$\mathrm{Spec}(A) = \{0, (\Omega-\omega) \mid \omega \in \mathbb{K}\} \sqcup \{\mathfrak{a}_i^{\pm} \mid i \geqslant 1\},$$

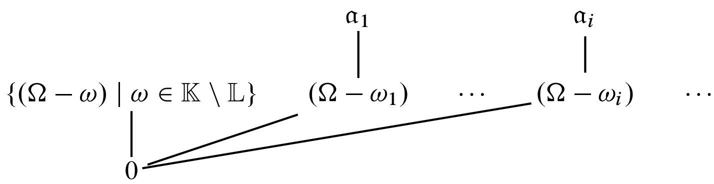


where $i \geqslant 1$, $\mathbb{L} := \{\omega_i^{\pm} \mid i \geqslant 1\}$ and $\omega_i^{\pm} := \pm\frac{1+q^{2i}}{q^2-1}\left(\frac{\alpha\beta\gamma}{(q^2-1)q^{2i}}\right)^{\frac{1}{2}}$ (all numbers in $\mathbb{L}$ are distinct, Lemma 6.7 (3)), and

means $\mathfrak{a} \subsetneq \mathfrak{b}$, if two ideals are not connected by a path of lines then they are *incomparable* ($\mathfrak{a} \not\subseteq \mathfrak{b}$ and $\mathfrak{b} \not\subseteq \mathfrak{a}$).

If $\gamma \neq 0$ and $\mathrm{char}(\mathbb{K}) = 2$, the prime spectrum of the algebra A (where $\alpha, \beta, \gamma \in \mathbb{K}^*$) is described by the diagram below (Theorem 7.6),

$$\mathrm{Spec}(A) = \{0, (\Omega - \omega) \mid \omega \in \mathbb{K}\} \sqcup \{\mathfrak{a}_i \mid i \geqslant 1\},$$



where $i \geqslant 1$, $\mathbb{L} := \{\omega_i \mid i \geqslant 1\}$ and $\omega_i := \frac{1+q^{2i}}{q^2-1}\big(\frac{\alpha\beta\gamma}{(q^2-1)q^{2i}}\big)^{\frac{1}{2}}$ where the numbers ω_i are all distinct (Lemma 6.7 (3)).

Commutativity of ideals and every ideal is a unique product of prime ideals. For an algebra A, we say that ideals *commute* if $IJ = JI$ for all ideals I and J of A. In [9, Section 4], it was shown that ideals of $U(\mathfrak{sl}_2)$ commute and every ideal is a unique product of prime ideals (with multiplicity and up to permutation). For the algebras $A = A(\alpha, \beta, \gamma)$, Theorem 1.3 is a criterion of ideals to commute and for each ideal of A to be a unique product of prime ideals.

Theorem 1.3. *Suppose that $\mathbb{K}$ is an algebraically closed field and q^2 is not a root of unity. Then ideals of the algebra $A = A(\alpha, \beta, \gamma)$ commute iff* $\mathrm{rk}(A) = 2, 3$. *Furthermore, if* $\mathrm{rk}(A) = 2, 3$ *then each ideal of A is a unique product of prime ideals (see Theorem 7.1 and Theorem 1.4, for details).*

Theorem 1.4 shows that every ideal of the algebra $\bar{A}(\alpha, \beta, \gamma)$ where $\alpha, \beta \in \mathbb{K}^*$, is a unique product of primes.

Theorem 1.4. *Suppose that $\mathbb{K}$ is an algebraically closed field and q^2 is not a root of unity. Let $A = A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$ and $\beta \neq 0$. Then:*

(1) *the ideals of the algebra A commute and each ideal of A is a unique product (up to order) of prime ideals (see statements* (2) *and* (3)*).*

(2) *If $\gamma \neq 0$ and* $\mathrm{char}(\mathbb{K}) \neq 2$ *then every nonzero ideal I of A is a unique product (up to order) of prime ideals,*

$$I = \prod_{\omega \in \mathbb{K}} (\Omega - \omega)^{n(\omega)} \cdot \prod_{i \geqslant 1} (\mathfrak{a}_i^+)^{n_i} \cdot \prod_{j \geqslant 1} (\mathfrak{a}_j^-)^{m_j},$$

where $n(\omega) \in \mathbb{N}$, $n_i, m_j \in \{0, 1\}$ and all but finitely many numbers $n(\omega), n_i$ and m_j are equal to zero.

(3) *If* $\gamma \neq 0$ *and* $\mathrm{char}(\mathbb{K}) = 2$ *then every nonzero ideal* I *of* A *is a unique product (up to order) of prime ideals*

$$I = \prod_{\omega \in \mathbb{K}} (\Omega - \omega)^{n(\omega)} \cdot \prod_{i \geqslant 1} \mathfrak{a}_i^{n_i}$$

where $n(\omega) \in \mathbb{N}$, $n_i \in \{0, 1\}$ *and all but finitely many numbers* $n(\omega)$ *and* n_i *are equal to zero.*

As an application of Theorem 1.4, we show that ideals of $U_q(\mathfrak{sl}_2)$ commute and each ideal is a unique product of primes, Theorem 8.1.

Semisimplicity of the category of finite dimensional modules. We say that a category of finite dimensional A-modules is *semisimple* if every nonzero finite dimensional A-module is a direct sum of simple finite dimensional A-modules. For example, the category of finite dimensional $U(\mathfrak{sl}_2)$-modules and $U_q(\mathfrak{sl}_2)$-modules are semisimple. Theorem 1.5 is a semisimplicity criterion of the category of finite dimensional $A(\alpha, \beta, \gamma)$-modules.

Theorem 1.5. *Suppose that* $\mathbb{K}$ *is an algebraically closed field,* q^2 *is not a root of unity and* $A = A(\alpha, \beta, \gamma)$. *Then the category of finite dimensional* A*-modules is semisimple iff* $\mathrm{rk}(A) = 3$.

Theorem 1.6 is a criterion for the algebra $A(\alpha, \beta, \gamma)$ to have only infinite dimensional modules apart from the zero module.

Theorem 1.6. *Suppose that* $\mathbb{K}$ *is an algebraically closed field,* q^2 *is not a root of unity and* $A = A(\alpha, \beta, \gamma)$. *Then all nonzero* A*-modules are infinite dimensional iff* $\mathrm{rk}(A) = 2$.

Classification of simple finite dimensional $A(\alpha, \beta, \gamma)$-modules. In each of the four cases, $\mathrm{rk}(A) = 0, 1, 2$ and 3, simple finite dimensional A-modules are classified (Corollary 4.4, Corollary 5.5, Corollary 6.6, Corollary 6.9 and Corollary 6.11).

In particular, suppose that $A = A(\alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \in \mathbb{K}^*$. If $\mathrm{char}(\mathbb{K}) \neq 2$ then for each natural number $i = 1, 2, \ldots$ there are only two (non-isomorphic) simple A-modules of dimension i (namely, $L(\omega_i^+)$ and $L(\omega_i^-)$, Theorem 6.8 (2)). If $\mathrm{char}(\mathbb{K}) = 2$, then for each natural number $i = 1, 2, \ldots$ there is only one simple A-module of dimension i (namely, $L(\omega_i)$, Theorem 6.10 (2)).

Criterion for all prime ideals of the algebra $A(\alpha, \beta, \gamma)$ to be completely prime ideals. Theorem 1.7 is such a criterion.

Theorem 1.7. *Suppose that* $\mathbb{K}$ *is an algebraically closed field and* q^2 *is not a root of unity. Then all prime ideals of the algebra* $A = A(\alpha, \beta, \gamma)$ *are completely prime iff* $\mathrm{rk}(A) = 0, 1$ *or* 2.

Theorem 1.8 shows that all prime ideals that are induced from the centre of the algebra $A(\alpha, \beta, \gamma)$ are completely prime.

Theorem 1.8. *Suppose that $\mathbb{K}$ is an algebraically closed field and q^2 is not a root of unity. Then for all $\alpha, \beta, \gamma, \omega \in \mathbb{K}$, the ideal $(\Omega - \omega)$ of the algebra $A(\alpha, \beta, \gamma)$ is a completely prime ideal.*

In Section 8, applications of the classificational results obtained in the previous sections are given. In particular, the above theorems are proved.

2. The 3-cyclic quantum Weyl algebra and its centre

The aim of the section is to prove Theorem 2.3 about the centre of the algebra A. Some algebras that are related to A are introduced and studied. They are examples of generalized Weyl algebras. These algebras are used in finding the prime spectrum of the algebra A and classifying simple A-modules.

The algebra $A_1 = \mathbb{K}\langle x, \partial \mid \partial x - x\partial = 1\rangle$ is called the *Weyl algebra*. The algebra $A_n := A_1^{\otimes n}$ is called the n-th *Weyl algebra*. If $\mathrm{char}(\mathbb{K}) = 0$ then the Weyl algebra is the ring of polynomial differential operators. $A_n = \mathbb{K}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ where $\partial_i := \frac{\partial}{\partial x_i}$.

The central element Ω of A. It is easy to check that the element

$$\Omega = yxz + \frac{q^{-2}\gamma}{q^2-1}x - \frac{q^2\beta}{q^2-1}y + \frac{\alpha}{q^2-1}z \tag{4}$$

belongs to the centre of the algebra A. In fact, if q^2 is not a root of unity then $Z(A) = \mathbb{K}[\Omega]$ (Theorem 2.3). The element Ω can be written in the form

$$\Omega = dz + l \qquad \text{where} \quad d := yx + \frac{\alpha}{q^2-1} \quad \text{and} \quad l := \frac{q^{-2}\gamma}{q^2-1}x - \frac{q^2\beta}{q^2-1}y. \tag{5}$$

The element d satisfies the following relations

$$xd = q^2dx, \qquad yd = q^{-2}dy, \qquad dz = zd + q^{-2}\gamma x + \beta y. \tag{6}$$

For all $n \geqslant 1$,

$$\Omega^n = \xi_n y^n x^n z^n + \cdots \qquad \text{where } \xi_n = (q^2)^{\frac{n(n-1)}{2}} \tag{7}$$

and the three dots mean smaller terms (with respect to the standard filtration on A) of total degree < 3 where $\deg(x) = \deg(y) = \deg(z) = 1$. Indeed,

$$\begin{aligned}\Omega^n &= (dz)^n + \cdots = d^n z^n + \cdots && (\text{since } dz = zd + \cdots)\\ &= (q^2)^{1+2+\cdots+n-1} y^n x^n z^n + \cdots && (\text{by (1)})\\ &= \xi_n y^n x^n z^n + \cdots .\end{aligned}$$

Proposition 2.1. (1) *The multiplicative set $S = \{x^i d^j q^{2k} \mid i, j \in \mathbb{N}, k \in \mathbb{Z}\}$ is an Ore set of A and $A_{x,d} := S^{-1}A = \mathbb{K}[\Omega] \otimes B$ is a tensor product of algebras where $B = \mathbb{K}[d^{\pm 1}][x^{\pm 1}; \sigma]$ is a Noetherian domain of Gelfand–Kirillov dimension 2 and $\sigma(d) = q^2 d$.*

(2) *The algebra B is a central simple algebra.*

Proof. (1) By (5), the multiplicative set S is an Ore set of A. Using (5), the elements y and z can be "replaced" by the elements d and Ω, respectively, and the equality $A_{x,d} = \mathbb{K}[\Omega] \otimes B$ follows from the presentation of the algebra A as an iterated Ore extension. It is obvious that B is a Noetherian domain of Gelfand–Kirillov dimension 2.

(2) Since q^2 is not a root of 1, the algebra B is central and simple. ∎

Definition (Generalized Weyl algebra, [5–7]). Let D be a ring, σ be an automorphism of D and a is an element of the centre of D. *The generalized Weyl algebra* $A := D(\sigma, a) := D[X, Y; \sigma, a]$ is a ring generated by D, X and Y subject to the defining relations

$$X\alpha = \sigma(\alpha)X \quad \text{and} \quad Y\alpha = \sigma^{-1}(\alpha)Y \quad \text{for all } \alpha \in D,\ YX = a \text{ and } XY = \sigma(a).$$

The algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is $\mathbb{Z}$-graded where $A_n = Dv_n$, $v_n = X^n$ for $n > 0$, $v_n = Y^{-n}$ for $n < 0$ and $v_0 = 1$. It follows from the above relations that $v_n v_m = (n, m)v_{n+m} = v_{n+m}\langle n, m\rangle$ for some $(n, m) \in D$. If $n > 0$ and $m > 0$ then

$$\begin{aligned}
n \geq m: \quad & (n, -m) = \sigma^n(a)\cdots\sigma^{n-m+1}(a), \qquad & (-n, m) = \sigma^{-n+1}(a)\cdots\sigma^{-n+m}(a),\\
n \leq m: \quad & (n, -m) = \sigma^n(a)\cdots\sigma(a), \qquad & (-n, m) = \sigma^{-n+1}(a)\cdots a,
\end{aligned}$$

in other cases $(n, m) = 1$. Clearly, $\langle n, m\rangle = \sigma^{-n-m}((n, m))$.

The generalized Weyl algebra $\mathbb{A}$ and its centre. Let $\mathbb{A}$ be the subalgebra of A generated by the elements e, f, z and Ω where

$$e := xz - \frac{q^2\beta}{q^2-1} \qquad \text{and} \qquad f := yz + \frac{\gamma}{q^2-1}. \tag{8}$$

Let σ be the automorphism of the polynomial algebra $\mathbb{D} = \mathbb{K}[z, \Omega]$ given by the rule $\sigma(z) = q^{-2}z$ and $\sigma(\Omega) = \Omega$. The elements e, f, z and Ω satisfy the following relations:

$$\begin{aligned}
&eu = \sigma(u)e, \qquad fu = \sigma^{-1}(u)f \quad \text{for all } u \in \mathbb{D},\\
&fe = a, \quad \text{and} \quad ef = \sigma(a),
\end{aligned} \tag{9}$$

where $a = -\frac{q^2\alpha}{q^2-1}z^2 + q^2\Omega z - \frac{q^2\beta\gamma}{(q^2-1)^2}$. In more detail,

$$\begin{aligned}
fe &= \left(yz + \frac{\gamma}{q^2-1}\right)\left(xz - \frac{q^2\beta}{q^2-1}\right)\\
&= q^2 zyxz + \frac{q^2\gamma}{q^2-1}xz - \frac{q^2\beta}{q^2-1}yz - \frac{q^2\beta\gamma}{(q^2-1)^2}\\
&= q^2 z\left(\Omega - \frac{q^{-2}\gamma}{q^2-1}x + \frac{q^2\beta}{q^2-1}y - \frac{\alpha}{q^2-1}z\right)\\
&\quad + \frac{q^2\gamma}{q^2-1}xz - \frac{q^2\beta}{q^2-1}yz - \frac{q^2\beta\gamma}{(q^2-1)^2}\\
&= -\frac{q^2\alpha}{q^2-1}z^2 + q^2\Omega z - \frac{q^2\beta\gamma}{(q^2-1)^2} = a,
\end{aligned}$$

$$
\begin{aligned}
ef &= \left(xz - \frac{q^2\beta}{q^2-1}\right)\left(yz + \frac{\gamma}{q^2-1}\right)\\
&= zyxz + q^{-2}\alpha z^2 - \frac{\beta}{q^2-1}yz + \frac{\gamma}{q^2-1}xz - \frac{q^2\beta\gamma}{(q^2-1)^2}\\
&= z\left(\Omega - \frac{q^{-2}\gamma}{q^2-1}x + \frac{q^2\beta}{q^2-1}y - \frac{\alpha}{q^2-1}z\right)\\
&\quad + q^{-2}\alpha z^2 - \frac{\beta}{q^2-1}yz + \frac{\gamma}{q^2-1}xz - \frac{q^2\beta\gamma}{(q^2-1)^2}\\
&= -\frac{q^{-2}\alpha}{q^2-1}z^2 + \Omega z - \frac{q^2\beta\gamma}{(q^2-1)^2} = \sigma(a).
\end{aligned}
$$

Let R be a ring and $s \in R$. Suppose that the set $S = \{s^i \mid i \geqslant 0\}$ is a left denominator set in R. Then $R_s := S^{-1}R = \{s^{-i}r \mid r \in R, i \geqslant 0\}$ is the localization of R at S, i.e., the localization of R at the powers of the element s.

The next proposition shows that the algebra $\mathbb{A}$ is a GWA such that $\mathbb{A} \subseteq A \subseteq \mathbb{A}_z = A_z$, it also shows that the centres of the algebras $\mathbb{A}$, A and $\mathbb{A}_z$ are a polynomial algebra $\mathbb{K}[\Omega]$.

Proposition 2.2. (1) *The algebra $\mathbb{A} = \mathbb{D}[e, f; \sigma, a]$ is the GWA where $\mathbb{D} = \mathbb{K}[z, \Omega]$ is a polynomial algebra, $\sigma(z) = q^{-2}z$, $\sigma(\Omega) = \Omega$ and $a = -\frac{q^2\alpha}{q^2-1}z^2 + q^2\Omega z - \frac{q^2\beta\gamma}{(q^2-1)^2}$.*

(2) *$\mathbb{A} \subset A \subset \mathbb{A}_z = A_z$ where $A_z = \mathbb{D}_z[e, f; \sigma, a]$ is a GWA and the subscript "z" means the localization of an algebra at the Ore set $\{z^i \mid i \in \mathbb{N}\}$.*

(3) *$Z(\mathbb{A}) = Z(\mathbb{A}_z) = Z(A_z) = \mathbb{K}[\Omega]$.*

Proof. (1) Let $\mathbb{A}' = \mathbb{D}[e, f; \sigma, a]$ be the GWA in statement (1). By (9), the algebra $\mathbb{A}$ is a factor algebra of the GWA $\mathbb{A}'$. In fact, the canonical epimorphism $\mathbb{A}' \to \mathbb{A}$, $e \mapsto e$, $f \mapsto f$, $z \mapsto z$, $\Omega \mapsto \Omega$ is an isomorphism since the $\mathbb{K}$-basis of the GWA $\mathbb{A}'$, $\{z^i\Omega^j, z^i\Omega^j e^k, z^i\Omega^j f^k \mid i, j \in \mathbb{N}, k \geqslant 1\}$, is mapped by the canonical epimorphism to a set of $\mathbb{K}$-linearly independent elements. This follows from the fact that $\{x^i y^j z^k \mid i, j, k \in \mathbb{N}\}$ is a $\mathbb{K}$-basis of A and explicit expressions for the leading terms of the images of the elements of the basis of the GWA $\mathbb{A}'$.

(2) Statement (2) follows from statement (1) and (8).

(3) Since q^2 is not a root of 1, then $\mathbb{K}[\Omega] \subseteq Z(\mathbb{A}) \subseteq Z(\mathbb{A}_z) = \mathbb{K}[\Omega]$ (the last equality is due to the fact that q^2 is not a root of unity). ■

Suppose that $\alpha \neq 0$. Then the algebra $\mathcal{A} = \mathbb{K}\langle x, y \mid xy = q^2yx + \alpha\rangle$ is isomorphic to the quantum Weyl algebra (by replacing x by $\alpha^{-1}x$). By (6),

$$\mathcal{A} = \mathbb{K}[d][x, y; \sigma, a := d - \alpha_0] \tag{10}$$

is a GWA where $\sigma(d) = q^2 d$ and $\alpha_0 = \frac{\alpha}{q^2-1}$ (see the definition of d in (5)).

Recall that $a = a_2z^2 + a_1z + a_0 \in \mathbb{K}[z]$ (Proposition 2.2) where

$$a_2 = -\frac{q^2\alpha}{q^2-1}, \qquad a_1 = q^2\Omega, \qquad \text{and} \qquad a_0 = -\frac{q^2\beta\gamma}{(q^2-1)^2}. \tag{11}$$

Theorem 2.3 describes the centre $Z(A)$ of the algebra $A(\alpha, \beta, \gamma)$.

Theorem 2.3. *Let $A = A(\alpha, \beta, \gamma)$. Then $Z(A) = \mathbb{K}[\Omega]$.*

Proof. We use notation and results of Proposition 2.1. Since q^2 is not a root of 1, $\mathbb{K}[\Omega] \subseteq Z(A) \subseteq Z(A_{x,d}) = \mathbb{K}[\Omega] \otimes Z(B) = \mathbb{K}[\Omega] \otimes \mathbb{K} = \mathbb{K}[\Omega]$, and so $Z(A) = \mathbb{K}[\Omega]$. ∎

Lemma 2.4. *The algebra A is a free $\mathbb{K}[\Omega]$-module and the set of elements $\{x^i y^j z^k \mid (i,j,k) \in \mathbb{N}^3 \setminus \mathbb{N}_+^3\}$ is a free basis for the $\mathbb{K}[\Omega]$-module A.*

Proof. The statement follows from the fact that xyz is the leading term of the element Ω (see (4)) with respect to the standard filtration associated with the canonical generators x, y and z of the algebra A. ∎

3. Prime ideals of the algebra A

In this section, we explain key ideas and an approach to classification of prime ideals and simple modules for the algebra A. Briefly, using various localizations of the algebra A we split the sets of prime ideals and simple modules into natural subclasses and then we describe elements in each class (using different techniques).

The prime ideals of A and their partition. For an algebra R, let Spec(R) be the set of its prime ideals. The set (Spec$(R), \subseteq$) is a partially ordered set (poset) with respect to inclusion of prime ideals. Each element $r \in R$ determines two maps from R to R, $r\cdot : x \mapsto rx$ and $\cdot r : x \mapsto xr$ where $x \in R$.

A non-empty subset S of R is called a *multiplicative set* if $SS \subseteq S$ and $0 \notin S$. A multiplicative set S is called a *left Ore set* if $Sr \cap Rs \neq \emptyset$ for all $r \in R$ and $s \in S$ (*the left Ore condition*). A left Ore set S is called a *left denominator set* if $rs = 0$ for some $r \in R$ and $s \in S$ then $s'r = 0$ for some $s' \in S$. If S is a left denominator set of R then the ring of the left fraction $S^{-1}R = \{s^{-1}r \mid s \in S, r \in R\}$ is called the *left localization of R at S*. An element $r \in R$ is called a *normal* element of R if $Rr = rR$.

Proposition 3.1 ([13]). *Let R be a Noetherian ring and s be an element of R such that $S_s := \{s^i \mid i \in \mathbb{N}\}$ is a left denominator set of the ring R and $(s^i) = (s)^i$ for all $i \geqslant 1$ (e.g., s is a normal element such that* $\ker(\cdot s_R) \subseteq \ker(s_R\cdot)$*). Then* $\mathrm{Spec}(R) = \mathrm{Spec}(R, s) \sqcup \mathrm{Spec}_s(R)$ *where* $\mathrm{Spec}(R,s) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid s \in \mathfrak{p}\}$, $\mathrm{Spec}_s(R) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid s \notin \mathfrak{q}\}$ *and*

(a) *the map* $\mathrm{Spec}(R, s) \to \mathrm{Spec}(R/(s))$, $\mathfrak{p} \mapsto \mathfrak{p}/(s)$, *is a bijection with the inverse* $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$ *where* $\pi : R \to R/(s)$, $r \mapsto r + (s)$,

(b) *the map* $\mathrm{Spec}_s(R) \to \mathrm{Spec}(R_s)$, $\mathfrak{p} \mapsto \mathcal{S}_s^{-1}\mathfrak{p}$, *is a bijection with the inverse* $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$ *where* $\sigma : R \to R_s := \mathcal{S}_s^{-1}R$, $r \mapsto \frac{r}{1}$.

(c) *For all* $\mathfrak{p} \in \mathrm{Spec}(R, s)$ *and* $\mathfrak{q} \in \mathrm{Spec}_s(R)$, $\mathfrak{p} \not\subseteq \mathfrak{q}$.

In this paper, we *identify* the sets in the statements (a) and (b) via the bijections given there.

Lemma 3.2. *Let* $t \in \{x, y, z\}$. *Then* $(t^i) = (t)^i$ *in* A *for all* $i \geqslant 1$. *Furthermore, if* $(\alpha, \beta) \neq (0, 0)$ *then* $(x^i) = A$ *for all* $i \geqslant 1$; *if* $(\alpha, \gamma) \neq (0, 0)$ *then* $(y^i) = A$ *for all* $i \geqslant 1$; *if* $(\beta, \gamma) \neq (0, 0)$ *then* $(z^i) = A$ *for all* $i \geqslant 1$.

Proof. In view of the cyclic permutation symmetry, we may assume that $t = z$. If $(\beta, \gamma) = (0, 0)$ then the element z is a normal element of A, and so $(z^i) = (z)^i$ for all $i \geqslant 1$. If $(\beta, \gamma) \neq (0, 0)$, say $\beta \neq 0$, then it follows from the equalities

$$xz^i = q^{-2i} z^i x + \frac{1 - q^{-2i}}{1 - q^{-2}} \beta z^{i-1}, \qquad i \geqslant 1, \tag{12}$$

and the induction on i that $(z^i) = A$ for all $i \geqslant 1$. Then clearly $(z)^i = A$ for all $i \geqslant 1$. ■

Lemma 3.3. *Let* $A = A(\alpha, \beta, \gamma)$ *where* $\alpha \neq 0$ *and* $S = \mathbb{K}[\Omega] \setminus \{0\}$. *Then the algebra* $S^{-1}A_z = \mathbb{K}(\Omega)[z^{\pm 1}][e, f; \sigma, a]$ *is a simple GWA where* σ *and* a *are as in Proposition* 2.2.

Proof. By Proposition 2.2, the algebra $S^{-1}A_z$ is the GWA as in the lemma. To prove simplicity of the algebra $S^{-1}A_z$ we check that the conditions for simplicity of a GWA given in [1, Theorem 4.2] are satisfied: the element a and $\sigma(a)$ are regular (since $\alpha \neq 0$), the algebra $D = \mathbb{K}(\Omega)[z^{\pm 1}]$ has no proper σ-invariant ideals (since q^2 is not a root of unity), the automorphisms $\{\sigma^i \mid i \geqslant 1\}$ are not inner automorphisms (since D is a commutative algebra), finally we have to verify that $Da + D\sigma^i(a) = D$ for all $i \geqslant 1$.

Since $\alpha \neq 0$, the ideals Da and $D\sigma^i(a)$ are distinct for each $i \geqslant 1$. To prove the statement it suffices to show that Da is a maximal ideal, i.e., the Laurent polynomial $a \in \mathbb{K}(\Omega)[z^{\pm 1}]$ is irreducible. If $\beta\gamma = 0$ then $a = z(-\frac{q^2\alpha}{q^2-1} z + q^2\Omega)$ is irreducible (since z is a unit of D). If $\beta \neq 0$ and $\gamma \neq 0$ the polynomial (in z) a is irreducible since otherwise it would have a nonzero root $\frac{p}{q} \in \mathbb{K}(\Omega)$ (since $\beta\gamma \neq 0$) where p and q are co-prime polynomials in $\mathbb{K}[\Omega]$. The fact that $\frac{p}{q}$ is a root of a can be written as $\frac{p}{q}(\frac{p}{q} + \nu\Omega) = \mu$ where $\nu, \mu \in \mathbb{K}^*$. Then $p \in \mathbb{K}^*$ since otherwise taking the value of the equality above at a root of p we would have $0 = \mu \neq 0$, a contradiction. The equality above can be written as $p(p + \nu\Omega q) = \mu q^2$. Then $\deg_\Omega(q) = 1$ and taking the equality modulo q we get $p^2 \equiv 0 \pmod{q}$, a contradiction. The proof of the lemma is complete. ■

By Lemma 3.2, $(z^i) = (z)^i$ for all $i \geqslant 1$. Then, by Proposition 3.1,

$$\mathrm{Spec}(A) = \mathrm{Spec}(A/(z)) \sqcup \mathrm{Spec}(A_z) \tag{13}$$

and $\mathfrak{p} \not\subseteq \mathfrak{q}$ for all prime ideals $\mathfrak{p} \in \mathrm{Spec}(A/(z))$ and $\mathfrak{q} \in \mathrm{Spec}(A_z)$.

Since the algebra $S^{-1}A_z$ is simple (Lemma 3.3) and the field $\mathbb{K}$ is an algebraically closed field, every nonzero prime ideal of the algebra A_z contains an element $\Omega - \omega$ for a unique $\omega \in \mathbb{K}$ (since if $\omega \neq \omega'$ then $(\Omega - \omega) + (\Omega - \omega') = (1)$). Therefore,

$$\mathrm{Spec}(A_z) = \{0\} \sqcup \bigsqcup_{\omega \in \mathbb{K}} \mathrm{Spec}(A_z/(\Omega - \omega)) \tag{14}$$

and prime ideals from distinct components in (14) are not comparable (neither $\mathfrak{p} \subseteq \mathfrak{q}$ nor $\mathfrak{p} \supseteq \mathfrak{q}$).

The factor algebras $\bar{A}(\alpha, \beta, \gamma; \omega)$. Given $\omega \in \mathbb{K}$, consider the factor algebra $\bar{A} = \bar{A}(\omega) = \bar{A}(\alpha, \beta, \gamma; \omega) = A/(\Omega - \omega)$.

Lemma 3.4. *The set $\{x^i y^j z^k \mid (i, j, k) \in \mathbb{N}^3 \setminus \mathbb{N}_+^3\}$ is a $\mathbb{K}$-basis for the algebra $\bar{A}(\alpha, \beta, \gamma; \omega)$ where $\mathbb{N}_+ := \{1, 2, \ldots\}$.*

Proof. The statement follows from the fact that xyz is the leading term of Ω with respect to the standard filtration associated with the canonical generators x, y and z of the algebra A. ■

By Proposition 2.2, the factor algebra

$$\bar{A}_z := \bar{A}_z(\omega) := \bar{A}_z(\alpha, \beta, \gamma; \omega) := A_z/(\Omega - \omega) = \mathbb{K}[z^{\pm 1}][e, f; \sigma, a_\omega] \tag{15}$$

is a GWA where $\sigma(z) = q^{-2}z$ and $a_\omega = -\frac{q^2\alpha}{q^2-1}z^2 + q^2\omega z - \frac{q^2\beta\gamma}{(q^2-1)^2}$. Since $\alpha \neq 0$, the algebra $\bar{A}_z$ is a Noetherian domain of Gelfand–Kirillov dimension 2.

Consider the algebra $\Lambda(\lambda, \mu) := \mathbb{K}\langle X, Y \mid XY = \lambda YX + \mu\rangle$ where $\lambda, \mu \in \mathbb{K}$ and $\lambda \neq 0$. Then for all $i \geqslant 1$,

$$XY^i = \lambda^i Y^i X + (1 + \lambda + \cdots + \lambda^{i-1})\mu Y^{i-1} + \cdots \tag{16}$$

where the three dots mean a polynomial in $\mathbb{K}[y]$ of degree $< i - 1$ (use the induction on i). An element r of a ring R is called a *regular* element if the element r is neither left nor right zero divisor. The set of all regular elements of the ring R is denoted by $\mathcal{C}_R$.

Proposition 3.5. *Let $A = A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$ and $\beta \neq 0$. Then:*

(1) *for all $\omega \in \mathbb{K}$, the element z is a regular non-unit in $A/(\Omega - \omega)$;*

(2) *for all $\omega \in \mathbb{K}$, the A-module $V(\omega) := A/A(\Omega - \omega, z) \simeq \bigoplus_{i \geqslant 0} \mathbb{K}x^i\bar{1}$ is simple and z-torsion where $\bar{1} = 1 + A(\Omega - \omega, z)$.*

Proof. (1) Let $\bar{A} = A/(\Omega - \omega)$.

(i) *The map $\cdot z : \bar{A} \to \bar{A}$, $u \mapsto uz$ is an injection but not a bijection.* Let $\mathcal{A} = \mathbb{K}\langle x, y \mid xy = q^2yx + \alpha\rangle$, the quantum Weyl algebra (since q^2 is not a root of unity and $\alpha \neq 0$). Then, by Lemma 3.4,

$$\bar{A} = \mathbb{K}[z] \oplus V_{yz} \oplus V_{xz} \oplus \mathcal{A}d$$

where $V_{yz} = \bigoplus_{i,j\geqslant 0} \mathbb{K} y^{i+1} z^j$ and $V_{xz} = \bigoplus_{i,j\geqslant 0} \mathbb{K} x^{i+1} z^j$. Clearly, the maps $\cdot z : V_{yz} \to V_{yz}, v \mapsto vz$; $\cdot z : V_{xz} \to V_{xz}, v \mapsto vz$ and $\cdot z : \mathcal{A}d \to \mathcal{A}, ud \mapsto udz = u(\omega - l)$ (since $dz = \Omega - l \equiv \omega - l \pmod{(\Omega-\omega)}$) are injections (the last one is an injection since $\beta \neq 0$). Since

$$\overline{A}z = \mathbb{K}[z]z \oplus V_{yz}z \oplus V_{xz}z \oplus \mathcal{A}(\omega - l),$$

the map $\cdot z : \overline{A} \to \overline{A}, u \mapsto uz$ is an injection. The map $\cdot z$ is not a bijection since

$$V(\omega) = \overline{A}/\overline{A}z \simeq \mathcal{A}/\mathcal{A}(\omega - l) \simeq \bigoplus_{i\geqslant 0} \mathbb{K}x^i \bar{1} \simeq_{\mathbb{K}[x]} \mathbb{K}[x]$$

since $l = \frac{q^{-2}\alpha}{q^2-1}x - \frac{q^2\beta}{q^2-1}y \neq 0$ (as $\beta \neq 0$) where $\bar{1} = 1 + \overline{A}z$.

(ii) *The map* $z\cdot : \overline{A} \to \overline{A}$, $u \mapsto zu$ *is an injection but not a bijection*: Since the opposite algebra $A^{\rm op} = A(\alpha', \beta', \gamma')_{q^{-2}}$ to A is the 3-cyclic quantum Weyl algebra with $\alpha' \neq 0$ and $\beta' \neq 0$ (since $\alpha \neq 0$ and $\beta \neq 0$), the map $z\,\cdot$ is an injection, by the statement (i).

(2) The set $S_z = \{z^i \mid i \geqslant 0\}$ is an Ore set of the algebra $\overline{A}(\omega)$ that consists of regular elements of $\overline{A}(\omega)$, by statement (1). Therefore, the set S_z is a denominator set of $\overline{A}(\omega)$ and the $\overline{A}(\omega)$-module $V(\omega)$ is a z-torsion one, i.e., S_z-torsion. In particular, the A-module $V(\omega)$ is a z-torsion one. To finish the proof of statement (2), it suffices to show that the $\overline{A}$-module $V = V(\omega)$ is simple. Notice that ${}_{\mathbb{K}[x]}V \simeq \mathbb{K}[x]$. Let U be a nonzero submodule of V such that $U \neq V$, we seek a contradiction. Then $U = p\mathbb{K}[x]\bar{1}$ for a unique polynomial $p = x^n + \mu x^{n-1} + \cdots \in \mathbb{K}[x]$ of degree $n \geqslant 1$. Recall that $zx = q^2xz - q^2\beta$. By (16),

$$\begin{aligned} U \ni zp\bar{1} &= z(x^n + \mu x^{n-1} + \cdots)\bar{1} \\ &= \big(q^{2n}x^nz - (1 + q^2 + \cdots + q^{2(n-1)})q^2\beta x^{n-1} + \mu q^{2(n-1)}x^{n-1}z + \cdots\big)\bar{1} \\ &= \Big(-\frac{q^{2n}-1}{q^2-1}q^2\beta x^{n-1} + \cdots\Big)\bar{1}. \end{aligned}$$

This contradicts to the choice of p as the degree of the nonzero polynomial in the bracket above is $n - 1 < n = \deg(p)$, since q^2 is not a root of unity. ∎

Lemma 3.6. *Let $A = A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$ and $\beta \neq 0$. Then, for each $\omega \in \mathbb{K}$, the ideal of A, $(\Omega - \omega) = A \cap (\Omega - \omega)A_z$, is a completely prime ideal.*

Proof. Since $\alpha \neq 0$, the algebra $\overline{A}_z \simeq A_z/A_z(\Omega - \omega)$ is a Noetherian domain, see (15). Hence, the ideal $A \cap (\Omega - \omega)A_z$ of A is a completely prime ideal. By Proposition 3.5 (1), $(\Omega - \omega) = A \cap (\Omega - \omega)A_z$. ∎

Simple A-modules and their partition. If $\mathbb{K}$ is an algebraically closed field then

$$\widehat{A} = \bigsqcup_{\omega \in \mathbb{K}} \widehat{\overline{A}(\omega)}. \tag{17}$$

For each $\omega \in \mathbb{K}$,

$$\widehat{\overline{A}(\omega)} = \widehat{\overline{A}(\omega)}\,(z\text{-torsion}) \sqcup \widehat{\overline{A}(\omega)}\,(z\text{-torsionfree}) \tag{18}$$

where a simple A-module M belongs to the first (resp., second) set if $S_z^{-1}M = 0$ (resp., $S_z^{-1}M \neq 0$) and the module M is called *z-torsion* (resp., *z-torsionfree*).

The set $S = \mathbb{K}[z] \setminus \{0\}$ is a (left and right) Ore set of the algebra $\overline{A}(\omega)$ and $\overline{A}(\omega)_z$ which are domains. The localization

$$B = B(\omega) := S^{-1}\overline{A}(\omega) \simeq S^{-1}\overline{A}(\omega)_z = \mathbb{K}(z)[e, e^{-1}; \sigma] = \bigoplus_{i\in\mathbb{Z}} \mathbb{K}(z)e^i \tag{19}$$

is a skew Laurent polynomial algebra and $\overline{A}(\omega) \subseteq B$. The algebra B is a (left and right) principle ideal domain. The set $\widehat{\overline{A}(\omega)}$ of isomorphism classes of simple $\overline{A}(\omega)$-modules is a disjoint union

$$\widehat{\overline{A}(\omega)} = \widehat{\overline{A}(\omega)}\,(\mathbb{K}[z]\text{-torsion}) \sqcup \widehat{\overline{A}(\omega)}\,(\mathbb{K}[z]\text{-torsionfree}) \tag{20}$$

where $\widehat{\overline{A}(\omega)}\,(\mathbb{K}[z]\text{-torsion}) := \{[M] \in \widehat{\overline{A}(\omega)} \mid S^{-1}M = 0\}$ and $\widehat{\overline{A}(\omega)}\,(\mathbb{K}[z]\text{-torsionfree}) := \{[M] \in \widehat{\overline{A}(\omega)} \mid S^{-1}M \neq 0\}$. A simple module from the first (resp., second) set is called *$\mathbb{K}[z]$-torsion* or *S-torsion* (resp., *$\mathbb{K}[z]$-torsionfree* or *S-torsionfree*).

It follows from (18) and (20) that

$$\widehat{\overline{A}(\omega)} = \widehat{\overline{A}(\omega)}(z\text{-torsion}) \sqcup \widehat{\overline{A}(\omega)}(z\text{-torsionfree}, \mathbb{K}[z]\text{-torsion}) \sqcup \widehat{\overline{A}(\omega)}(\mathbb{K}[z]\text{-torsionfree}). \tag{21}$$

4. Classification of prime ideals and simple modules for the algebra $A(0, 0, 0)$

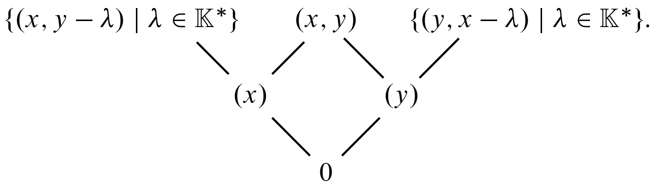
In this section, $\mathbb{K}$ is an algebraically closed field and $A = A(0, 0, 0)$. For the algebra A, the prime, completely prime, primitive and maximal ideals are classified (Theorem 4.1 and Corollary 4.2). Furthermore, the simple A-modules are classified (Theorem 4.3).

Suppose for a moment that the elements α, β and γ are arbitrary. Then by (1), (2) and (3), the factor algebra $\Lambda := \Lambda(\alpha, \beta, \gamma) := A/(z)$ is equal to

$$\Lambda(\alpha, \beta, \gamma) = \begin{cases} \mathbb{K}\langle x, y \mid xy = q^2yx + \alpha\rangle, & \text{if } (\beta, \gamma) = (0, 0), \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha = \beta = \gamma = 0$ then the algebra $\Lambda(0, 0, 0) = \mathbb{K}\langle x, y \mid xy = q^2yx\rangle$ is the, so-called, *quantum plane* and its prime spectrum is well known.

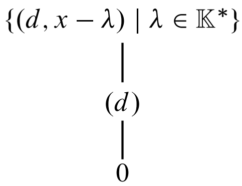
Spec $\Lambda(0,0,0)$:



(22)

If $\alpha \neq 0$ then the algebra $\Lambda(\alpha,0,0) = \mathbb{K}\langle x, y \mid xy = q^2 yx + \alpha\rangle$, is the, so-called, *quantum Weyl algebra*. Its prime spectrum is also well known.

Spec $\Lambda(\alpha,0,0)$:

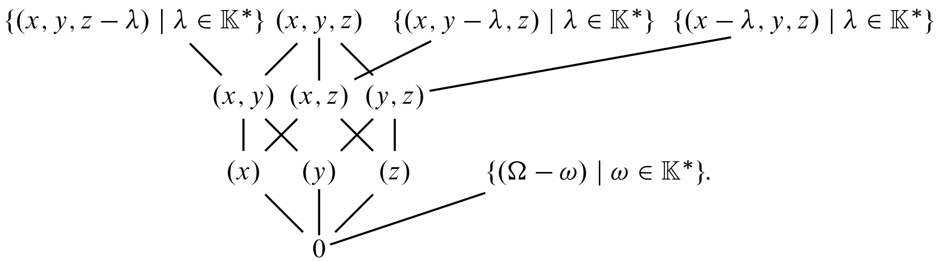


(23)

where $d = yx - \frac{\alpha}{1-q^2}$.

The set Spec $A(0,0,0)$. The next theorem is a description of prime ideals for the algebra $A(0,0,0)$.

Theorem 4.1. *Suppose that the field $\mathbb{K}$ is an algebraically closed field. Then the spectrum of the algebra $A(0,0,0)$ is given below together with all possible containments of prime ideals where $\Omega = yxz$,*



(24)

Proof. Let $A = A(0,0,0)$. Recall that $Z(A) = \mathbb{K}[\Omega]$ where $\Omega = yxz$ (Proposition 2.2 (3), since q^2 is not a root of unity). The elements $x, y, z \in A$ are normal and $d = yx$. Let $S = \{\Omega^i \mid i \geqslant 0\}$. By Proposition 2.1 (1), the algebra $S^{-1}A \simeq S^{-1}A_{x,d} = \mathbb{K}[\Omega^{\pm 1}] \otimes B$ where the algebra B is a central simple Noetherian algebra, and so every nonzero prime ideal of A that does not contain Ω contains an element $\Omega - \omega$ for a unique $\omega \in \mathbb{K}^*$ (since $\mathbb{K}$ is an algebraically closed field). Since $\omega \neq 0$, $yxz = \omega$ in $\overline{A}(\omega)$, and so the normal elements x, y and z are units in $\overline{A}(\omega)$. So, the algebra

$$\overline{A}(\omega) \simeq \frac{S^{-1}A}{S^{-1}A\big(\Omega-\omega\big)} \simeq \frac{\mathbb{K}[\Omega^{\pm 1}]}{(\Omega-\omega)} \otimes B \simeq B$$
$$\simeq \mathbb{K}\langle x^{\pm 1}, y^{\pm 1} \mid xy = q^2 yx\rangle \simeq \mathbb{K}[y^{\pm 1}][x^{\pm 1};\tau] \qquad (25)$$

is a simple algebra since $\tau(y) = q^2 y$ and q is not a root of unity. This means that the ideals $\{(\Omega-\omega) \mid \omega \in \mathbb{K}^*\}$ are maximal ideals of A.

The ideal $(\Omega) = (yxz) = (x)(y)(z)$ is a product of three completely prime ideals (x), (y), (z). In particular, the ideal (Ω) is not a prime ideal and every prime ideal of the algebra A that contains the element Ω necessarily contains one of the prime ideals (x), (y) or (z). Using the fact that each of the factor algebras $A/(x)$, $A/(y)$ and $A/(z)$ is a quantum plane, and (22), we see that the set of prime ideals of A containing one of the prime ideals (x), (y) or (z) is as in the diagram (24). The proof of the theorem is complete. ∎

The next corollary describes the sets of maximal, primitive and completely prime ideals of the algebra $A(0,0,0)$.

Corollary 4.2. *Suppose that the field $\mathbb{K}$ is an algebraically closed field and $A = A(0,0,0)$. Then:*

(1) $\mathrm{Max}(A) = \{(\Omega-\omega) \mid \omega \in \mathbb{K}^*\} \sqcup \mathcal{M}$ *where* $\mathcal{M} := \{(x,y,z), (x-\lambda, y, z), (x, y-\lambda, z), (x, y, z-\lambda) \mid \lambda \in \mathbb{K}^*\}$. *For every $\omega \in \mathbb{K}^*$, the factor algebra $A/(\Omega-\omega) \simeq \mathbb{K}[y^{\pm 1}][x^{\pm 1};\tau]$ is a simple Noetherian domain of Gelfand–Kirillov dimension 2 where $\tau(y) = q^2 y$ and, for all maximal ideals $\mathfrak{m} \in \mathcal{M}$, $A/\mathfrak{m} \simeq \mathbb{K}$.*

(2) $\mathrm{Prim}(A) = \mathrm{Max}(A) \sqcup \{(x), (y), (z)\}$.

(3) $\mathrm{Spec}(A) = \mathrm{Spec}_c(A)$, *all prime ideals of A are completely prime ideals.*

(4) *Every nonzero prime ideal of A meets the centre.*

Proof. The statements follow from (24) and the proof of Theorem 4.1. ∎

Let $\mathcal{A}$ be an algebra. For each primitive ideal $\mathfrak{p} \in \mathrm{Prim}(\mathcal{A})$, let $\widehat{\mathcal{A}}(\mathfrak{p}) := \{[M] \in \widehat{\mathcal{A}} \mid \mathrm{ann}_{\mathcal{A}}(M) = \mathfrak{p}\}$. Then

$$\widehat{\mathcal{A}} = \bigsqcup_{\mathfrak{p}\in \mathrm{Prim}(\mathcal{A})} \widehat{\mathcal{A}}(\mathfrak{p}).$$

Recall that $\mathrm{Prim}(A) = \mathrm{Max}(A) \sqcup \{(x), (y), (z)\}$ and $\mathrm{Max}(A) = \{(\Omega-\omega) \mid \omega \in \mathbb{K}^*\} \sqcup \mathcal{M}$ (Proposition 4.2) provided q^2 is not a root of unity.

Classification of simple $A(0,0,0)$-modules. A classification of simple modules over the quantum plane is given in [3]. The next theorem is a classification of simple A-modules where q^2 is not a root of unity.

Theorem 4.3. *Suppose that the field $\mathbb{K}$ is an algebraically closed field. Then:*

(1) *for every $\omega \in \mathbb{K}^*$, $\widehat{A}((\Omega-\omega)) = \widehat{\overline{A}(\omega)}$ and the classification of simple $\overline{A}(\omega)$-modules of the algebra $\overline{A}(\omega) := A/(\Omega-\omega) \simeq \mathbb{K}[y^{\pm 1}][x^{\pm 1};\sigma]$ (where $\sigma(y) = q^2 y$, see (25)) is given in [6,7].*

(2) *For each maximal ideal $\mathfrak{m} \in \mathcal{M} = \{(x,y,z), (x-\lambda,y,z), (x,y-\lambda,z), (x,y,z-\lambda) \mid \lambda \in \mathbb{K}^*\}$, $\widehat{A}(\mathfrak{m}) = \{A/\mathfrak{m}\}$ and $A/\mathfrak{m} \simeq \mathbb{K}$.*

(3) *For each $\mathfrak{p} \in \{(x),(y),(z)\}$, the factor algebra $A/\mathfrak{p}$ is a quantum plane (e.g., $A/(X) = \mathbb{K}\langle y,z \mid yz = q^2 zy\rangle$) and*

$$\widehat{A}(\mathfrak{p}) = \widehat{A/\mathfrak{p}}(\textit{weight, linear}) \sqcup \widehat{A/\mathfrak{p}}(\mathbb{K}[H]\textit{-torsionfree})$$

where the sets $\widehat{A/\mathfrak{p}}(\textit{weight, linear})$ and $\widehat{A/\mathfrak{p}}(\mathbb{K}[H]\textit{-torsionfree})$ are described in [3].

The next corollary is a classification of simple finite dimensional A-modules where q^2 is not a root of unity.

Corollary 4.4. *Suppose that $\mathbb{K}$ is an algebraically closed field. Then*

$$\widehat{A}(\textit{fin. dim.}) = \{A/\mathfrak{m} \mid \mathfrak{m} \in \mathcal{M}\}$$

is the set of isomorphism classes of simple finite dimensional A-modules. Furthermore, $\dim(A/\mathfrak{m}) = 1$ *for all $\mathfrak{m} \in \mathcal{M}$.*

Proof. The corollary follows from Theorem 4.3. ∎

It remains to consider the remaining cases where $\mathrm{rk}(A) = 1,2,3$, i.e., $(\alpha,0,0)$, $(\alpha,\beta,0)$ and (α,β,γ), respectively, where $\alpha,\beta,\gamma \in \mathbb{K}^*$.

- *So, we assume till the end of the paper that $\alpha \neq 0$.*

5. Classification of prime ideals and simple modules for the algebra $A(\alpha,0,0)$ where $\alpha \neq 0$

In this section, $\mathbb{K}$ is an algebraically closed field and $A = A(\alpha,0,0)$ where $\alpha \neq 0$. For the algebra A, its prime, completely prime, primitive and maximal ideals and the simple A-modules are classified (Theorem 5.2, Corollary 5.3 and Theorem 5.4).

The prime spectrum of the algebra $A(\alpha,0,0)$.

Lemma 5.1. *Suppose that $A = A(\alpha,0,0)$ where $\alpha \neq 0$. Then:*

(1) *the algebra $A = \mathbb{K}[z,d][x,y;\sigma,d-\frac{\alpha}{q^2-1}]$ is a GWA where $\sigma(z) = q^{-2}z$ and $\sigma(d) = q^2 d$. In particular, the elements z and d of A are normal regular elements.*

(2) *For all $\omega \in \mathbb{K}^*$, $(\Omega-\omega) = A \cap A_z(\Omega-\omega)$ is a completely prime ideal of A, and the algebra $A/(\Omega-\omega)$ is a simple Noetherian domain.*

Proof. (1) Statement (1) follows at once from (1), (2) and (3).

(2) By statement (1) and the equality $\Omega = dz$, the factor algebra

$$A/(\Omega - \omega) \simeq \mathbb{K}[d^{\pm 1}]\Big[x, y; \sigma, d - \frac{\alpha}{q^2 - 1}\Big]$$

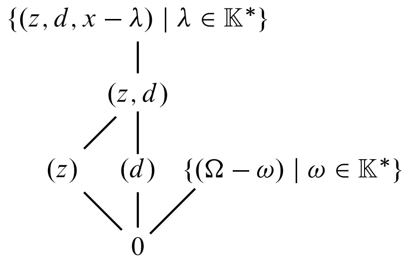
is a GWA which is a simple, Noetherian domain, by [1, Theorem 4.2]. In particular, the ideal $(\Omega - \omega)$ of A is a completely prime ideal. Similarly,

$$A_z/A_z(\Omega - \omega) \simeq \mathbb{K}[d^{\pm 1}]\Big[x, y; \sigma, d - \frac{\alpha}{q^2 - 1}\Big] = A/(\Omega - \omega).$$

Therefore, $(\Omega - \omega) = A \cap A_z(\Omega - \omega)$. ■

The next theorem is a description of prime ideals of the algebra $A(\alpha, 0, 0)$ where $\alpha \neq 0$.

Theorem 5.2. *Suppose that $\mathbb{K}$ is an algebraically closed field. Let $A = A(\alpha, 0, 0)$ where $\alpha \neq 0$. Then the spectrum of the algebra A is given by the diagram below where $\Omega = dz$ and $d = yx + \frac{\alpha}{q^2-1}$:*



(26)

(1) *For each $\omega \in \mathbb{K}^*$, $A/(\Omega - \omega) \simeq \mathbb{K}[d^{\pm 1}][x, y; \sigma, d - \frac{\alpha}{q^2-1}]$ is a GWA which is a simple Noetherian domain of Gelfand–Kirillov dimension 2.*

(2) *$A/(z) \simeq \mathbb{K}\langle x, y \mid xy = q^2 yx + \alpha \rangle$ is a quantum Weyl algebra.*

(3) *$A/(d) \simeq \mathbb{K}[z][x^{\pm 1}; \sigma]$ is a skew Laurent polynomial ring where $\sigma(z) = q^{-2} z$.*

(4) *$A/(z, d) \simeq \mathbb{K}[x^{\pm 1}]$ is a Laurent polynomial ring.*

(5) *$A/(z, d, x - \lambda) \simeq \mathbb{K}$ for all $\lambda \in \mathbb{K}^*$.*

Proof. By Lemma 5.1, statements (1)–(5) hold. So, all the ideals in the diagram (26) are completely prime ideals since the algebras in statements (1)–(5) are domains. The inclusions in the diagram (26) are obvious. Since $\Omega = dz$, we have that $(\Omega) \subseteq (d)$ and $(\Omega) \subseteq (z)$. So, there are no other inclusions in the diagram (26) (since $(\Omega) + (\Omega - \omega) = (1)$ for all $\omega \in \mathbb{K}^*$). In view of (13), (14), and the fact that the algebras $A_z/A_z(\Omega - \omega) \simeq A/(\Omega - \omega)$ are simple (Lemma 5.1 (2)) there are no new prime ideals in $\mathrm{Spec}(A)$ apart from the ones given in the diagram (26). ■

The next corollary describes the sets of maximal, prime and completely prime ideals of A.

Corollary 5.3. *Suppose that $\mathbb{K}$ is an algebraically closed field. Let $A = A(\alpha, 0, 0)$ where $\alpha \neq 0$. Then*

(1) $\mathrm{Max}(A) = \{(z, d, x - \lambda) \mid \lambda \in \mathbb{K}^*\} \sqcup \{(\Omega - \omega) \mid \omega \in \mathbb{K}^*\}$.

(2) $\mathrm{Prim}(A) = \mathrm{Max}(A) \sqcup \{(z), (d)\}$.

(3) $\mathrm{Spec}_c(A) = \mathrm{Spec}(A)$.

(4) *Every nonzero prime ideal of A meets the centre of A.*

Proof. The theorem follows from the explicit description of prime factor algebras given in Theorem 5.2. ∎

Classification of simple $A(\alpha, 0, 0)$-modules.

Theorem 5.4. *Suppose that $\mathbb{K}$ is an algebraically closed field. Let $A = A(\alpha, 0, 0)$ where $\alpha \neq 0$. Then*

$$\widehat{A} = \widehat{A}(z) \sqcup \widehat{A}(d) \sqcup \bigsqcup_{\mathfrak{m} \in \mathrm{Max}(A)} \widehat{A}(\mathfrak{m})$$

where

(1) *If $\mathfrak{m} = \mathfrak{m}_\lambda := (z, d, x - \lambda)$ where $\lambda \in \mathbb{K}^*$ then $\widehat{A}(\mathfrak{m}_\lambda) = \{A/\mathfrak{m}_\lambda \simeq \mathbb{K}\}$.*

(2) *If $\mathfrak{m} = (\Omega - \omega)$ where $\omega \in \mathbb{K}^*$ then $\widehat{A}(\Omega - \omega) = \widehat{A/(\Omega - \omega)}$ and the simple $A/(\Omega - \omega)$-modules for the GWA $A/(\Omega - \omega) = \mathbb{K}[d^{\pm 1}][x, y; \sigma, d - \frac{\alpha}{q^2 - 1}]$ (where $\sigma(d) = q^2 d$) are classified in [7] (see also [4, 14]).*

(3) *$\widehat{A}(d) = \widehat{A/(d)} \setminus \{A/\mathfrak{m}_\lambda \mid \lambda \in \mathbb{K}^*\}$ and the simple modules of the algebra $A/(d) \simeq \mathbb{K}[z][x^{\pm 1}; \sigma]$ (where $\sigma(z) = q^{-2} z$) are classified in [7] (see also [4, 14]).*

(4) *$\widehat{A}(z) = \widehat{A/(z)} \setminus \{A/\mathfrak{m}_\lambda \mid \lambda \in \mathbb{K}^*\}$ and the simple modules of the algebra $A/(z) \simeq \mathbb{K}\langle x, y \mid xy = q^2 yx + \alpha\rangle$ are classified in [3].*

The set $\widehat{A}$(fin. dim.). Corollary 5.5 is a classification of simple finite dimensional A-modules where q^2 is not a root of unity.

Corollary 5.5. *Suppose that $\mathbb{K}$ is an algebraically closed field. Let $A = A(\alpha, 0, 0)$ where $\alpha \neq 0$. Then*

$$\widehat{A}(\textit{fin. dim.}) = \{A/\mathfrak{m}_\lambda \mid \lambda \in \mathbb{K}^*\}$$

where $\mathfrak{m}_\lambda = (z, d, x - \lambda)$ is a maximal ideal of A and $\dim_{\mathbb{K}}(A/\mathfrak{m}_\lambda) = 1$.

Proof. The corollary follows from Theorem 5.4 since for the algebras in statements (2)–(4) simple modules are infinite dimensional. ∎

6. Classification of simple $A(\alpha, \beta, \gamma)$-modules where $\alpha \neq 0$ and $\beta \neq 0$

In this section, $\mathbb{K}$ is an algebraically closed field and $A = A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$ and $\beta \neq 0$. A classification of simple A-modules is given.

Classification of simple $A(\alpha, \beta, \gamma)$-modules. In view of (17) and (21), in order to classify simple $A(\alpha, \beta, \gamma)$-modules we have to classify simple modules in each of the three subsets in (21). This is done below. In each of the three cases a different approach is used.

The set $\widehat{A(\omega)}(z$-torsion). Theorem 6.1 describes the set $\widehat{A(\omega)}(z\text{-torsion})$.

Theorem 6.1. *Suppose that $\mathbb{K}$ is an algebraically closed field. Let $A = A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$ and $\beta \neq 0$. Then $\widehat{\bar{A}(\omega)}(z\text{-torsion}) = \{V(\omega)\}$ where $V(\omega) = \bar{A}(\omega)/\bar{A}(\omega)z$.*

Proof. Recall that each z-torsion simple $\bar{A}(\omega)$-module is an epimorphic image of the $\bar{A}(\omega)$-module $V(\omega)$. The $\bar{A}(\omega)$-module $V(\omega)$ is simple (Proposition 3.5 (2)), and the theorem follows. ∎

In the $\bar{A}(\omega)$-module $V(\omega) = \bigoplus_{i \geqslant 0} \mathbb{K} x^i \bar{1}$ (see Proposition 3.5), for all $i \geqslant 0$,

$$ex^i\bar{1} = \left(\frac{q^{2(i+1)}}{1-q^2}\beta x^i + \cdots\right)\bar{1}. \tag{27}$$

In more detail, recall that $e = xz - \beta_0$ where $\beta_0 = \frac{q^2\beta}{q^2-1}$. Then

$$\begin{aligned} ex^i\bar{1} &= (xzx^i - \beta_0 x^i)\bar{1} \\ &\overset{(16)}{=} \Big(x\big(q^{2i}x^i z - (1 + q^2 + \cdots + q^{2(i-1)})q^2\beta x^{i-1}\big) - \beta_0 x^i + \cdots\Big)\bar{1} \\ &= -\left(\frac{q^{2i}-1}{q^2-1} + \frac{1}{q^2-1}\right)q^2\beta x^i\bar{1} + \cdots \\ &= \frac{q^{2(i+1)}}{1-q^2}\beta x^i\bar{1} + \cdots . \end{aligned}$$

- So, $\mathcal{E}_\beta := \{\frac{q^{2(i+1)}}{1-q^2}\beta \mid i \geqslant 0\}$ *is the set of eigenvalues of the linear map $e\cdot : V(\omega) \to V(\omega)$, $v \mapsto ev$.*

Lemma 6.2. *Given a nonzero element $b = b(z, e) \in \mathbb{K}[z][e;\sigma]$ where $\sigma(z) = q^{-2}z$. If the map $b\cdot : V(\omega) \to V(\omega)$, $v \mapsto bv$ has nonzero kernel then the polynomial $b(0, e) \in \mathbb{K}[e] \simeq \mathbb{K}[z][e;\sigma]/(z)$ has a root in the set $\mathcal{E}_\beta = \{\frac{q^{2(i+1)}}{1-q^2}\beta \mid i \geqslant 0\}$.*

Proof. Recall that $V(\omega) \simeq \mathbb{K}[x]\bar{1} \simeq_{\mathbb{K}[x]} \mathbb{K}[x]$. So, the $\bar{A}(\omega)$-module $V(\omega)$ admits a filtration by the degree of x, $V(\omega) = \bigcup_{i \geqslant 0} V_i$ where $V_i = \{p\bar{1} \mid \deg(p) \leqslant i\}$. For all $i \geqslant 0$, $zV_i \subseteq V_{i-1}$, by (16). The element $b = b(z, e) = b(0, e) + b'z$ is a unique sum where $b' \in \mathbb{K}[z][e;\sigma]$. The field $\mathbb{K}$ is an algebraically closed field. So, $b(0, e) = \mu \prod_{i=1}^s (e - \lambda_i)$ where $\mu \in \mathbb{K}^*$ and $\lambda_1, \ldots, \lambda_s$ are the roots of the polynomial $b(0, e) \in \mathbb{K}[e]$. By (27), for all $i \geqslant 0$, $eV_i \subseteq V_i$ and

$$bx^i\bar{1} = b(0, e)x^i\bar{1} + \cdots = \mu \prod_{j=1}^s \left(\frac{q^{2(i+1)}\beta}{1-q^2} - \lambda_j\right)x^i\bar{1} + \cdots .$$

So, if an element $v = x^i\bar{1} + \cdots$ belongs to the kernel of the map $b\cdot : V(\omega) \to V(\omega)$ then necessarily $\lambda_j = \frac{q^{2(i+1)}}{1-q^2}\beta \in \mathcal{E}_\beta$ for some j, as required. ∎

The set $\widehat{\overline{A}(\omega)}(\mathbb{K}[z]$-torsionfree). Theorem 6.3 is an explicit description of the set $\widehat{\overline{A}(\omega)}$ ($\mathbb{K}[z]$-torsionfree). Recall that $\overline{A}(\omega) \subset \overline{A}(\omega)_z \subset B = B(\omega)$, see (19). The GWA $\overline{A}(\omega)_z = \mathbb{K}[z^{\pm 1}][e, f; \sigma, a]$ is also the GWA $\overline{A}(\omega)_z = \mathbb{K}[z^{\pm 1}][f, e; \sigma^{-1}, \sigma(a_\omega)]$.

For nonzero elements $p, q \in \mathbb{K}[z^{\pm 1}]$, we write $p <_{\sigma^{-1}} q$ if there is no maximal ideal $\mathfrak{m}$ of $\mathbb{K}[z^{\pm 1}]$ such that $q \in \mathfrak{m}$ and $p \in (\sigma^{-1})^i(\mathfrak{m})$ for some $i \geqslant 0$.

Definition. An element $b = \sum_{i=0}^m e^i b_i \in \mathbb{K}[z][e;\sigma] \in \overline{A}(\omega)_z$ (where $b_i \in \mathbb{K}[z]$, $b_0 \neq 0$, $b_m \neq 0$ and $m \geqslant 1$) is called r-*normal* if it is l-*normal* as an element of the GWA $\overline{A}(\omega)_z = \mathbb{K}[z^{\pm 1}][f, e; \sigma^{-1}, \sigma(a_\omega)]$, i.e., $b_0 <_{\sigma^{-1}} b_m$ and $b_0 <_{\sigma^{-1}} \sigma(a_\omega)$.

Theorem 6.3. *Let $\mathbb{K}$ be an algebraically closed field. Let $A = A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$, $\beta \neq 0$ and $\omega \in \mathbb{K}$. Then*

$$\widehat{\overline{A}(\omega)}(\mathbb{K}[z]\text{-torsionfree}) = \Big\{[M_b := \overline{A}(\omega)/\overline{A}(\omega) \cap Bb] \mid b = b(z,e) \in \mathbb{K}[z][e;\sigma] \text{ is an } r\text{-normal, irreducible element of } B \text{ such that the polynomial } b(0,e) \in \mathbb{K}[e] \text{ has no root in the set } \mathcal{E}_\beta\Big\}$$

where $\mathcal{E}_\beta = \{\frac{q^{2(i+1)}}{1-q^2}\beta \mid i \geqslant 0\}$ and the algebra $B = B(\omega)$ is defined in (19); and $M_b \simeq M_{b'}$ iff the elements b and b' are similar (i.e., $B/Bb \simeq B/Bb'$ as B-modules). All modules M_b are infinite dimensional.

Proof. Let L and R be the left-hand side and right-hand side of the equality in the theorem and $\overline{A} = \overline{A}(\omega)$. We have to show that $R = L$.

(i) $L \subseteq R$: Let M be a $\mathbb{K}[z]$-torsionfree, simple $\overline{A}$-module, i.e., $M \in L$. Since M_z is a simple $\overline{A}_z$-module, $M_z \simeq \overline{A}_z/\overline{A}_z \cap Bb$ for some element $b = b(z,e) \in \mathbb{K}[z][e;\sigma]$ which is an r-normal, irreducible element of B (by [7, Theorem 5]). Now, for all $n \geqslant 0$,

$$M_z = \left(\frac{\overline{A}z^n + \overline{A} \cap Bb}{\overline{A} \cap Bb}\right)_z \simeq \left(\frac{\overline{A}z^n}{\overline{A}z^n \cap \overline{A} \cap Bb}\right)_z = \left(\frac{\overline{A}z^n}{\overline{A}z^n \cap Bb}\right)_z$$
$$= \left(\frac{\overline{A}}{\overline{A} \cap Bbz^{-n}}\right)_z = \left(\frac{\overline{A}}{\overline{A} \cap B\omega_{z^n}(b)}\right)_z$$

where $\omega_{z^n}(b) = z^n b z^{-n}$. The element $b = b(z,e)$ is a unique sum $b = b_0 + b'z$ where $b_0 = b(0,e) \in \mathbb{K}[e]$ and $b' \in \mathbb{K}[z][e;\sigma]$. Notice that $\omega_{z^n}(b) = \omega_{z^n}(b_0) + \omega_{z^n}(b')z$ and so $\omega_{z^n}(b)|_{z=0} = \omega_{z^n}(b_0) \in \mathbb{K}[e]$. The field $\mathbb{K}$ is an algebraically closed field. In particular, all the elements $\omega_{z^n}(b') \in \mathbb{K}[z][e;\sigma]$ are r-normal, irreducible elements of B. Let $b_0 = \mu \prod_{i=1}^s (e - \lambda_i)$ where $\lambda_1, \ldots, \lambda_s$ are roots of the polynomial $b_0 \in \mathbb{K}[e]$ and $\mu \in \mathbb{K}^*$. Since $\omega_{z^n}(e) = q^{2n}e$,

$$\omega_{z^n}(e) = \mu' \prod_{i=1}^s (e - q^{-2n}\lambda_i) \qquad \text{for some } \mu' \in \mathbb{K}^*.$$

So, for all sufficiently large n, the polynomial $\omega_{z^n}(b_0)$ has no roots in the set $\mathcal{E}_\beta$ (since q^2 is not a root of unity). Fix one such n and let $M' = \overline{A}/\overline{A} \cap B\omega_{z^n}(b)$. Then $M = \mathrm{soc}_{\overline{A}}(M_z) = M'$, by simplicity of M and M'. Therefore, $M \in R$.

(ii) $L \supseteq R$: We have to show that each $\overline{A}$-module $M' = \overline{A}/\overline{A} \cap Bb$ in R is simple, i.e., the left ideal $I = \overline{A} \cap Bb$ is a maximal left ideal of $\overline{A}$. Let J be a left ideal of $\overline{A}$ such that $I \subsetneq J \subsetneq \overline{A}$, we seek a contradiction. By the very definition, the $\overline{A}$-module M' is $\mathbb{K}[z]$-torsionfree.

Localizing the short exact sequence of the $\overline{A}$-modules

$$0 \to J/I \to M' = A/I \to \bar{M} := A/J \to 0$$

at the powers of the element z we obtain a short exact sequence of $\overline{A}$-modules

$$0 \to (J/I)_z \to M'_z \to \bar{M}_z \to 0.$$

The $\overline{A}_z$-module M'_z is simple and $\mathbb{K}[z]$-torsionfree (since b is r-normal) and the $\overline{A}_z$-module $(J/I)_z$ is a nonzero $\overline{A}_z$-module (since the $\overline{A}$-module M' is z-torsionfree). Therefore, $(J/I)_z = M'_z$, and so $\bar{M}_z = 0$. So, $J \supseteq I + Az^i$ for some $i \geqslant 1$. The $\overline{A}$-module $N := A/J$ is z-torsion. By Theorem 6.1, the unique z-torsion simple $\overline{A}$-module V is an epimorphic image of the $\overline{A}$-module N. In particular, there is a nonzero vector $0 \neq v \in V$ such that $Jv = 0$. In particular, $bv = 0$. By Lemma 6.2, the polynomial $b(0, e) \in \mathbb{K}[e]$ has root in the set $\mathcal{E}_\beta$, a contradiction. ■

The set $\widehat{\overline{A}(\omega)}$ (z-torsionfree, $\mathbb{K}[z]$-torsion). The classification of simple z-torsionfree, $\mathbb{K}[z]$-torsion $\overline{A}(\omega)$-modules is done in two steps. First, we show that this class coincides with the set of simple, $\mathbb{K}[z]$-torsion $\overline{A}(\omega)_z$-modules (Theorem 6.4). Second, since the algebra $\overline{A}(\omega)$ is a GWA with Dedekind base ring that belongs to the class of GWAs considered in [7], we apply the classification results of [7] to our algebra (Theorem 6.5, Theorem 6.8 and Theorem 6.10).

Theorem 6.4. *Suppose that $\mathbb{K}$ is an algebraically closed field. Let $A = A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$, $\beta \neq 0$ and $\omega \in \mathbb{K}$. Then*

$$\widehat{\overline{A}(\omega)}(z\text{-torsionfree}, \mathbb{K}[z]\text{-torsion}) = \widehat{\overline{A}(\omega)_z}(\mathbb{K}[z]\text{-torsion}),$$

i.e., every simple, $\mathbb{K}[z]$-torsion $A(\omega)_z$-module is a simple, z-torsionfree, $\mathbb{K}[z]$-torsion $A(\omega)$-module (by restriction to the algebra $\overline{A}(\omega)$), and vice versa.

Proof. Let L and R be the left-hand side and right-hand side of the equality. Then the map $L \to R$, $[M] \mapsto [M_z]$ is an injection. Since every element N of R is a *semisimple* $\mathbb{K}[z^{\pm 1}]$-module (by [7, Theorem 1]) and $\mathbb{K}$ is an algebraically closed field, it is also a simple $A(\omega)$-module since $z\cdot : N \to N, n \mapsto zn$ is a bijection. Therefore, the map $L \to R$ is a bijection. ■

The case $\gamma = 0$. In Theorem 6.5, we consider the case when $\gamma = 0$. We denote by $\mathbb{K}^*/\langle q^2 \rangle$ the factor group of the multiplicative group $\mathbb{K}^*$ by the subgroup $\langle q^2 \rangle = \{q^{2i} \mid i \in \mathbb{Z}\}$ generated by the element $\langle q^2 \rangle$. Elements of the group $\mathbb{K}^*/\langle q^2 \rangle$ are cosets $\mathcal{O} = \lambda \langle q^2 \rangle$ where $\lambda \in \mathbb{K}^*$. For each coset $\mathcal{O}$, we fix an element, say $\lambda_{\mathcal{O}}$. So, $\mathcal{O} = \lambda_{\mathcal{O}} \langle q^2 \rangle$.

Theorem 6.5. *Suppose that $\mathbb{K}$ is an algebraically closed field. Let $A = A(\alpha, \beta, 0)$ where $\alpha \neq 0$, $\beta \neq 0$ and $\omega \in \mathbb{K}$. Then:*

(1) $\widehat{\overline{A}(0)}$*(z-torsionfree,* $\mathbb{K}[z]$*-torsion)*$= \{M_{\mathcal{O}} = \overline{A}(0)_z / \overline{A}(0)_z(z - \lambda_{\mathcal{O}}) \mid \mathcal{O} \in \mathbb{K}^*/\langle q^2\rangle\}$ *where $\lambda_{\mathcal{O}}$ is an arbitrary but fixed element of the coset $\mathcal{O} = \lambda_{\mathcal{O}}\langle q^2\rangle$.*

(2) *For $\omega \neq 0$,*

$$\begin{aligned}\widehat{\overline{A}(\omega)}&(z\text{-torsionfree}, \mathbb{K}[z]\text{-torsion})\\ &= \big\{L = \overline{A}(\omega)_z / \overline{A}(\omega)_z(z - (q^2-1)\alpha^{-1}\omega, e),\\ &\quad L' = \overline{A}(\omega)_z / \overline{A}(\omega)_z(z - q^2(q^2-1)\alpha^{-1}\omega, f),\\ &\quad M_{\mathcal{O}} = \overline{A}(\omega)_z / \overline{A}(\omega)_z(z - \lambda_{\mathcal{O}})\\ &\quad \mid \mathcal{O} \in \mathbb{K}^*/\langle q^2\rangle, \mathcal{O} \neq (q^2-1)\alpha^{-1}\omega\langle q^2\rangle\big\}\end{aligned}$$

where $\lambda_{\mathcal{O}}$ is an arbitrary but fixed element of the coset $\mathcal{O} = \lambda_{\mathcal{O}}\langle q^2\rangle$.

All modules in statements (1) *and* (2) *are infinite dimensional.*

Proof. Since $\gamma = 0$, $a_\omega = -q^2(q^2-1)^{-1}\alpha z(z - (q^2-1)\alpha^{-1}\omega)$. Now, the theorem follows from Theorem 6.4 and [7, Theorem 1]. ∎

Corollary 6.6. *Suppose that $\mathbb{K}$ is an algebraically closed field. Let $A = A(\alpha, \beta, 0)$ where $\alpha \neq 0$ and $\beta \neq 0$. Then the zero module is the only finite dimensional A-module.*

Proof. The corollary follows from Theorem 6.1, Theorem 6.3 and Theorem 6.5. ∎

The case $\gamma \neq 0$. It remains to consider the case when $\gamma \neq 0$.

Lemma 6.7. *Suppose that $\mathbb{K}$ is an algebraically closed field, $\alpha, \beta, \gamma \in \mathbb{K}^*$ and $\omega \in \mathbb{K}$. Let $\lambda_0 := (\frac{\beta\gamma}{(q^2-1)\alpha})^{\frac{1}{2}}$. Then:*

(1) *both (necessarily nonzero) roots of the polynomial $a_\omega = -\frac{q^2\alpha}{q^2-1}z^2 + q^2\omega z - \frac{q^2\beta\gamma}{(q^2-1)^2}$ belong to a single coset in $\mathbb{K}^*/\langle q^2\rangle$ iff either*

(a) *$\omega \neq 0$, and in this case either $\omega = \omega_i^\pm := \pm\frac{1+q^{2i}}{q^2-1}(\frac{\alpha\beta\gamma}{(q^2-1)q^{2i}})^{\frac{1}{2}}$, $i \geqslant 1$, ($\{\lambda_i^\pm, q^{2i}\lambda_i^\pm\}$ are two distinct roots of the polynomial $a_{\omega_i^\pm}$ where $\lambda_i^\pm := \frac{(q^2-1)\omega_i^\pm}{(1+q^{2i})\alpha}$) or $i = 0$,* $\mathrm{char}(\mathbb{K}) \neq 2$*, $\omega = \omega_0^\pm := \pm\frac{2}{q^2-1}(\frac{\alpha\beta\gamma}{q^2-1})^{\frac{1}{2}}$, ($\lambda_0^\pm = \frac{q^2-1}{2\alpha}\omega_0^\pm = \pm\lambda_0$ is a double root of the polynomial $a_{\omega_0^\pm}$), or*

(b) *$\omega = 0$, and in this case* $\mathrm{char}(\mathbb{K}) = 2$ *and λ_0 is a double root of the polynomial a_ω.*

(2) *The polynomial a_ω has a double root iff either $\omega = \omega_0^\pm = \pm\frac{2}{q^2-1}(\frac{\alpha\beta\gamma}{q^2-1})^{\frac{1}{2}}$,* $\mathrm{char}(\mathbb{K}) \neq 2$ *and in this case $\lambda_0^\pm = \frac{(q^2-1)\omega_0^\pm}{2\alpha} = \pm\lambda_0$ is the double root of the polynomial $a_{\omega_0^\pm}$ or $\omega = 0$,* $\mathrm{char}(\mathbb{K}) = 2$ *and λ_0 is the double root of a_0.*

(3) *If* $\mathrm{char}(\mathbb{K}) \neq 2$ *then the elements $\{\omega_i^+, \omega_i^- \mid i \geqslant 0\}$ are distinct. If* $\mathrm{char}(\mathbb{K}) = 2$ *then the elements $\{\omega_i \mid i \geqslant 1\}$ are distinct and nonzero where $\omega_i = \omega_i^+ = \omega_i^-$.*

Proof. (1) The polynomial $a_\omega = -\frac{q^2\alpha}{q^2-1}(z^2-(q^2-1)\alpha^{-1}\omega z+\frac{\beta\gamma}{(q^2-1)\alpha})$ has two necessarily *nonzero* roots (since $\beta,\gamma\in\mathbb{K}^*$) that may coincide. Both roots belong to the same coset in $\mathbb{K}^*/\langle q^2\rangle$ iff they belong to the set $\{\lambda, q^{2i}\lambda\}$ for some $\lambda\in\mathbb{K}^*$ and $i\geqslant 0$ iff $(1+q^{2i})\lambda=(q^2-1)\alpha^{-1}\omega$ and $q^{2i}\lambda^2=\frac{\beta\gamma}{(q^2-1)\alpha}$ iff either $\omega\neq 0$ and in this case either $\lambda=\frac{(q^2-1)\omega}{(1+q^{2i})\alpha}$ and $\omega=\pm\frac{1+q^{2i}}{q^2-1}(\frac{\alpha\beta\gamma}{(q^2-1)q^{2i}})^{\frac{1}{2}}$ for some natural number $i\geqslant 1$ or $i=0$ and $\mathrm{char}(\mathbb{K})\neq 2$ (since otherwise $0=2\lambda=(q^2-1)\alpha^{-1}\omega\neq 0$, a contradiction); or $\omega=0$ and in this case $\mathrm{char}(\mathbb{K})=2$ and $\lambda=(\frac{\beta\gamma}{(q^2-1)\alpha})^{\frac{1}{2}}=\lambda_0$ is a double root of a_0.

(2) Statement (2) follows from statement (1).

(3) Statement (3) follows from the following fact: $\omega_i^\pm=\omega_j^*$ where $*\in\{+,-\}$ for some $i\neq j$ iff $q^i+q^{-i}=\pm(q^j+q^{-j})$ where $*=\pm$ iff $(q^i\mp q^j)(1\mp q^{-(i+j)})=0$ iff q^2 is a root of unity, a contradiction. ∎

By Lemma 6.7, we have two cases: $\mathrm{char}(\mathbb{K})\neq 2$ and $\mathrm{char}(\mathbb{K})=2$. In each case, let us summarize some facts that will be used in Theorem 6.8 and Theorem 6.10 and their proofs, respectively. We keep the notation of Lemma 6.7.

Suppose that $\mathrm{char}(\mathbb{K})\neq 2$. Then the following holds.

- The polynomial a_ω has double root iff $\omega\in\{\omega_0^+,\omega_0^-\}$. In this case, $\omega_0^+\neq\omega_0^-$ and for $\omega=\omega_0^\pm$, $\lambda_0^\pm=\pm\lambda_0$ is the double root of $a_{\omega_0^\pm}$ and $\lambda_0^+\langle q^2\rangle\neq\lambda_0^-\langle q^2\rangle$.
- The polynomial a_ω has two distinct roots that belong to the same coset in $\mathbb{K}^*/\langle q^2\rangle$ iff $\omega\in\{\omega_i^+,\omega_i^-\mid i\geqslant 1\}$. In this case, the elements of the set $\{\omega_i^+,\omega_i^-\mid i\geqslant 1\}$ are distinct and for each $\omega=\omega_i^\pm$, $\{\lambda_i^\pm, q^{2i}\lambda_i^\pm\}$ are the two distinct roots of the polynomial $a_{\omega_i^\pm}$.
- The polynomial a_ω has two (necessarily distinct) roots that belong to distinct cosets in $\mathbb{K}^*/\langle q^2\rangle$ iff $\omega\in\mathbb{K}^*\setminus\{\omega_i^+,\omega_i^-\mid i\geqslant 0\}$. In this case, for each ω let $\lambda_{\omega,1}$ and $\lambda_{\omega,2}$ be the two (distinct) roots of the polynomial a_ω.
- The elements $\{\omega_i^+,\omega_i^-\mid i\geqslant 0\}$ are distinct.

Suppose that $\mathrm{char}(\mathbb{K})=2$. Then the following holds.

- The polynomial a_ω has double root iff $\omega=0$. In this case, λ_0 is a double root of the polynomial a_0.
- The polynomial a_ω has two distinct roots that belong to the same coset in $\mathbb{K}^*/\langle q^2\rangle$ iff $\omega\in\{\omega_i\mid i\geqslant 1\}$. In this case, the elements of the set $\{\omega_i\mid i\geqslant 1\}$ are distinct nonzero elements and for each $\omega=\omega_i$, $\{\lambda_i, q^{2i}\lambda_i\}$ are the two distinct roots of the polynomial a_{ω_i}.
- The polynomial a_ω has two (necessarily distinct) roots that belong to distinct cosets in $\mathbb{K}^*/\langle q^2\rangle$ iff $\omega\in\mathbb{K}^*\setminus\{0,\omega_i\mid i\geqslant 1\}$. In this case, for each ω let $\lambda_{\omega,1}$ and $\lambda_{\omega,2}$ be the two (distinct) roots of the polynomial a_ω.
- The elements $\{0,\omega_i\mid i\geqslant 1\}$ are distinct.

For each coset $\mathcal{O}\in\mathbb{K}^*/\langle q^2\rangle$, we fixed a representative $\lambda_\mathcal{O}\in\mathcal{O}$. So, $\mathcal{O}=\lambda_\mathcal{O}\langle q^2\rangle$.

The set $\widehat{A(\omega)}$ (z-torsionfree, $\mathbb{K}[z]$-torsion) where $\alpha, \beta, \gamma \in \mathbb{K}^*$ and char($\mathbb{K}$) $\neq 2$. Theorem 6.8 gives an explicit description of the set $\widehat{A(\omega)}$(z-torsionfree, $\mathbb{K}[z]$-torsion) in the case where $\alpha, \beta, \gamma \in \mathbb{K}^*$ and char($\mathbb{K}$) $\neq 2$.

Theorem 6.8. *We keep the notation of Lemma 6.7. Suppose that $\mathbb{K}$ is an algebraically closed field with* char($\mathbb{K}$) $\neq 2$. *Let $A = A(\alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \in \mathbb{K}^*$, and*

$$\mathcal{M}(\omega) := \widehat{\overline{A}(\omega)}\,(z\text{-torsionfree}, \mathbb{K}[z]\text{-torsion}) = \widehat{\overline{A}(\omega)_z}\,(\mathbb{K}[z]\text{-torsion}) \qquad (\text{Theorem 6.4}).$$

Then:

(1) *suppose that $\omega \in \{\omega_0^+, \omega_0^-\}$, i.e., the polynomial $a_{\omega_0^\pm}$ has the double root $\lambda_0^\pm = \pm\lambda_0$ where $\lambda_0 = (\frac{\beta\gamma}{(q^2-1)\alpha})^{\frac{1}{2}}$. Then*

$$\mathcal{M}_{\omega_0^\pm} = \left\{L_+(\omega_0^\pm),\ L_-(\omega_0^\pm),\ M_{\mathcal{O}}(\omega_0^\pm) \mid \mathcal{O} \in \mathbb{K}^*/\langle q^2\rangle,\ \mathcal{O} \neq \lambda_0^\pm\langle q^2\rangle\right\}$$

where

$$\begin{aligned} L_-(\omega_0^\pm) &= \overline{A}(\omega_0^\pm)_z/\overline{A}(\omega_0^\pm)_z(e, z-\lambda_0^\pm),\\ L_+(\omega_0^\pm) &= \overline{A}(\omega_0^\pm)_z/\overline{A}(\omega_0^\pm)_z(f, z-q^2\lambda_0^\pm),\\ M_{\mathcal{O}}(\omega_0^\pm) &= \overline{A}(\omega_0^\pm)_z/\overline{A}(\omega_0^\pm)_z(z-\lambda_{\mathcal{O}}). \end{aligned}$$

All modules in $\mathcal{M}_{\omega_i^\pm}$ are infinite dimensional.

(2) *Suppose that $\omega \in \{\omega_i^+, \omega_i^- \mid i \geqslant 1\}$, i.e., the polynomial $a_{\omega_i^\pm}$ has two distinct roots $\{\lambda_i^\pm, q^{2i}\lambda_i^\pm\}$ that belong to the same coset in $\mathbb{K}^*/\langle q^2\rangle$. Then for $\omega = \omega_i^\pm$,*

$$\mathcal{M}_{\omega_i^\pm} = \left\{L_+(\omega_i^\pm), L_-(\omega_i^\pm), L(\omega_i^\pm), M_{\mathcal{O}}(\omega_i^\pm) \mid \mathcal{O} \in \mathbb{K}^*/\langle q^2\rangle \text{ and } \mathcal{O} \neq \lambda_i^\pm\langle q^2\rangle\right\}$$

where

$$\begin{aligned} L_-(\omega_i^\pm) &= \overline{A}(\omega_i^\pm)_z/\overline{A}(\omega_i^\pm)_z(e, z-\lambda_i^\pm),\\ L_+(\omega_i^\pm) &= \overline{A}(\omega_i^\pm)_z/\overline{A}(\omega_i^\pm)_z(f, z-q^{2(i+1)}\lambda_i^\pm),\\ L(\omega_i^\pm) &= \overline{A}(\omega_i^\pm)_z/\overline{A}(\omega_i^\pm)_z(e, z-q^{2i}\lambda_i^\pm, f^i),\\ M_{\mathcal{O}}(\omega_i^\pm) &= \overline{A}(\omega_i^\pm)_z/\overline{A}(\omega_i^\pm)_z(z-\lambda_{\mathcal{O}}). \end{aligned}$$

For all $i \geqslant 1$, $\dim_{\mathbb{K}}(L(\omega_i^\pm)) = i$ *and the module $L(\omega_i^\pm)$ is the only finite dimensional module in the set $\mathcal{M}_{\omega_i^\pm}$.*

(3) *Suppose that $\omega \in \mathbb{K}^* \setminus \{\omega_i^\pm \mid i \geqslant 0\}$, i.e., the polynomial a_ω has two (distinct) roots, say $\lambda_{\omega,1}$ and $\lambda_{\omega,2}$, that belong to distinct cosets ($\lambda_{\omega,1}\langle q^2\rangle \neq \lambda_{\omega,2}\langle q^2\rangle$). Then*

$$\mathcal{M}_\omega = \big\{L_-(\omega, \lambda_{\omega,i}),\ L_+(\omega, \lambda_{\omega,i}),\ M_{\mathcal{O}}(\omega) \mid i = 1, 2 \text{ and } \mathcal{O} \in \mathbb{K}^*/\langle q^2\rangle \setminus \{\lambda_{\omega,1}\langle q^2\rangle, \lambda_{\omega,2}\langle q^2\rangle\}\big\}$$

where

$$L_-(\omega,\lambda_{\omega,i}) = \bar{A}(\omega)_z/\bar{A}(\omega)_z(e, z-\lambda_{\omega,i}),$$
$$L_+(\omega,\lambda_{\omega,i}) = \bar{A}(\omega)_z/\bar{A}(\omega)_z(f, z-q^2\lambda_{\omega,i}),$$
$$M_{\mathcal{O}}(\omega) = \bar{A}(\omega)_z/\bar{A}(\omega)_z(z-\lambda_{\mathcal{O}}).$$

All modules in $\mathcal{M}_\omega$ are infinite dimensional.

The set $\widehat{A}$ (fin. dim.) where $\alpha,\beta,\gamma\in\mathbb{K}^*$ and char($\mathbb{K}$) $\neq 2$. We denote by $\widehat{A}$(fin. dim.) the set of isomorphism classes of simple finite dimensional A-modules. Corollary 6.9 classifies all simple finite dimensional $A(\alpha,\beta,\gamma)$-modules where $\alpha,\beta,\gamma\in\mathbb{K}^*$ and char($\mathbb{K}$) $\neq 2$. It shows that for each natural number $i\geqslant 1$ there are only 2 simple non-isomorphic A-modules of dimension i.

Corollary 6.9. *We keep the notation of Theorem 6.8. Suppose that $\mathbb{K}$ is an algebraically closed field with* char($\mathbb{K}$) $\neq 2$. *Let $A = A(\alpha,\beta,\gamma)$ where $\alpha,\beta,\gamma\in\mathbb{K}^*$. Then*

$$\widehat{A}(\textit{fin. dim.}) = \{L(\omega_i^+),\ L(\omega_i^-)\mid i=1,2,\dots\}$$

and $\dim_{\mathbb{K}}(L(\omega_i^{\pm})) = i$ *for all* $i\geqslant 1$.

Proof. The corollary follows from Theorem 6.8 and [7, Theorem 1]. ■

The set $\widehat{A}(\omega)$ (z-torsionfree, $\mathbb{K}[z]$-torsion) where $\alpha,\beta,\gamma\in\mathbb{K}^*$ and char($\mathbb{K}$) $=2$. Theorem 6.10 gives an explicit description of the set $\widehat{A}(\omega)$ (z-torsionfree, $\mathbb{K}[z]$-torsion) in the case where $\alpha,\beta,\gamma\in\mathbb{K}^*$ and char($\mathbb{K}$) $=2$.

Theorem 6.10. *We keep the notation of Lemma 6.7. Suppose that $\mathbb{K}$ is an algebraically closed field of* char($\mathbb{K}$) $=2$. *Let $A=A(\alpha,\beta,\gamma)$ where $\alpha,\beta,\gamma\in\mathbb{K}^*$ and*

$$\mathcal{M}_\omega := \widehat{A(\omega)}(z\textit{-torsionfree}, \mathbb{K}[z]\textit{-torsion}) = \widehat{A(\omega)_z}(\mathbb{K}[z]\textit{-torsion}) \qquad \textit{(Theorem 6.4)}.$$

Then:

(1) *suppose that $\omega=0$, i.e., the polynomial a_0 has double root $\lambda_0 = (\frac{\beta\gamma}{(q^2-1)\alpha})^{\frac{1}{2}}$. Then*

$$\mathcal{M}_0 = \left\{L_+(0),\ L_-(0),\ M_{\mathcal{O}}(0)\mid \mathcal{O}\in\mathbb{K}^*/\langle q^2\rangle,\ \mathcal{O}\neq\lambda_0\langle q^2\rangle\right\}$$

where

$$L_-(0) = \bar{A}(0)_z/\bar{A}(0)_z(e, z-\lambda_0),$$
$$L_+(0) = \bar{A}(0)_z/\bar{A}(0)_z(f, z-q^2\lambda_0),$$
$$M_{\mathcal{O}}(0) = \bar{A}(0)_z/\bar{A}(0)_z(z-\lambda_{\mathcal{O}}).$$

All modules in $\mathcal{M}_0$ are infinite dimensional.

(2) *Suppose that* $\omega \in \{\omega_i \mid i \geqslant 1\}$, *i.e., the polynomial* a_{ω_i} *has two distinct roots* $\{\lambda_i, q^{2i}\lambda_i\}$ *that belong to the same coset in* $\mathbb{K}^*/\langle q^2\rangle$. *Then, for* $\omega = \omega_i$,

$$\mathcal{M}_{\omega_i} = \{L_+(\omega_i),\ L_-(\omega_i),\ L(\omega_i),\ M_{\mathcal{O}}(\omega_i) \mid \mathcal{O} \in \mathbb{K}^*/\langle q^2\rangle \text{ and } \mathcal{O} \neq \lambda_i\langle q^2\rangle\}$$

where

$$\begin{aligned} L_-(\omega_i) &= \bar{A}(\omega_i)_z/\bar{A}(\omega_i)_z(e, z-\lambda_i),\\ L_+(\omega_i) &= \bar{A}(\omega_i)_z/\bar{A}(\omega_i)_z(f, z-q^{2(i+1)}\lambda_i),\\ L(\omega_i) &= \bar{A}(\omega_i)_z/\bar{A}(\omega_i)_z(e, z-q^{2i}\lambda_i, f^i),\\ M_{\mathcal{O}}(\omega_i) &= \bar{A}(\omega_i)_z/\bar{A}(\omega_i)_z(z-\lambda_{\mathcal{O}}). \end{aligned}$$

For all $i \geqslant 1$, $\dim_{\mathbb{K}} L(\omega_i) = i$ *and the module* $L(\omega_i)$ *is the only finite dimensional module in the set* $\mathcal{M}_{\omega_i}$.

(3) *Suppose that* $\omega \in \mathbb{K}^* \setminus \{\omega_i \mid i \geqslant 1\}$, *i.e., the polynomial* a_ω *has two (distinct) roots, say* $\lambda_{\omega,1}$ *and* $\lambda_{\omega,2}$, *that belong to distinct cosets in* $\mathbb{K}^*/\langle q^2\rangle$, $(\lambda_{\omega,1}\langle q^2\rangle \neq \lambda_{\omega,2}\langle q^2\rangle)$. *Then*

$$\begin{aligned} \mathcal{M}_\omega = \{&L_-(\omega, \lambda_{\omega,i}),\ L_+(\omega, \lambda_{\omega,i}),\ M_{\mathcal{O}}(\omega)\\ &\mid i = 1,2 \text{ and } \mathcal{O} \in \mathbb{K}^*/\langle q^2\rangle \setminus \{\lambda_{\omega,1}\langle q^2\rangle, \lambda_{\omega,2}\langle q^2\rangle\}\} \end{aligned}$$

where

$$\begin{aligned} L_-(\omega, \lambda_{\omega,i}) &= \bar{A}(\omega)_z/\bar{A}(\omega)_z(e, z-\lambda_{\omega,i}),\\ L_+(\omega, \lambda_{\omega,i}) &= \bar{A}(\omega)_z/\bar{A}(\omega)_z(f, z-q^2\lambda_{\omega,i}),\\ M_{\mathcal{O}}(\omega) &= \bar{A}(\omega)_z/\bar{A}(\omega)_z(z-\lambda_{\mathcal{O}}). \end{aligned}$$

All modules in $\mathcal{M}_\omega$ *are infinite dimensional.*

Proof. The theorem follows from [7, Theorem 1] and Lemma 6.7. ∎

The set $\widehat{A}$(fin. dim.) where $\alpha, \beta, \gamma \in \mathbb{K}^*$ and char($\mathbb{K}$) = 2. Corollary 6.11 classifies all simple finite dimensional $A(\alpha, \beta, \gamma)$-modules where $\alpha, \beta, \gamma \in \mathbb{K}^*$ and char($\mathbb{K}$) = 2. It shows that for each natural number $i \geqslant 1$ there is only one simple A-module of dimension i (up to isomorphism).

Corollary 6.11. *We keep the notation of Theorem* 6.10. *Suppose that* $\mathbb{K}$ *is an algebraically closed field with* char($\mathbb{K}$) = 2. *Let* $A = A(\alpha, \beta, \gamma)$ *where* $\alpha, \beta, \gamma \in \mathbb{K}^*$. *Then*

$$\widehat{A}(\text{fin. dim.}) = \{L(\omega_i) \mid i = 1, 2, \ldots\}$$

and $\dim_{\mathbb{K}}(L(\omega_i)) = i$ *for all* $i \geqslant 1$.

Proof. The corollary follows from Theorem 6.10 and [7, Theorem 1]. ∎

7. Classification of prime ideals for the algebra $A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$ and $\beta \neq 0$

In this section, $\mathbb{K}$ is an algebraically closed field and $A = A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$ and $\beta \neq 0$. For the algebra A, the prime, completely prime, primitive and maximal ideals are classified. The results and their proofs are different in the following cases: $\gamma = 0$; $\gamma \neq 0$ and $\mathrm{char}(\mathbb{K}) \neq 2$; and $\gamma \neq 0$ and $\mathrm{char}(\mathbb{K}) = 2$, (see Theorem 7.1, Theorem 7.4 and Theorem 7.6).

Since $\beta \neq 0$, $(z^i) = A$ for all $i \geqslant 1$, by Lemma 3.2. Now, by (13) and (14),

$$\mathrm{Spec}(A) = \{0\} \bigsqcup_{\omega \in \mathbb{K}} \mathrm{Spec}(\overline{A}(\omega)_z). \tag{28}$$

Since $\beta \neq 0$, $(z^i) = A$ for all $i \geqslant 1$, by Lemma 3.2. Then $(z^i) = (1)$ for all $i \geqslant 1$ in $\overline{A}(\omega) = A/(\Omega - \omega)$. Now, by Proposition 3.1, the map

$$\mathrm{Spec}(\overline{A}(\omega)) \to \mathrm{Spec}(\overline{A}(\omega)_z), \qquad \mathfrak{p} \mapsto \mathfrak{p}_z \tag{29}$$

is a bijection with the inverse $\mathfrak{q} \mapsto \overline{A}(\omega) \cap \mathfrak{q}$ where $\mathfrak{p}_z$ is the localization of the prime ideal $\mathfrak{p}$ at the powers of z. By Proposition 3.5 (1), the element z is a regular non-unit of the algebra $\overline{A}(\omega)$, and so $\overline{A}(\omega) \subseteq \overline{A}(\omega)_z$. By (15), $\overline{A}(\omega)_z = \overline{A}_z(\omega) = \mathbb{K}[z^{\pm 1}][e, f; \sigma, a_\omega]$ and $a_\omega \neq 0$ for all ω (since $\alpha \neq 0$). Therefore, the algebra $\overline{A}(\omega)_z$ is a domain, hence so is the algebra $\overline{A}(\omega)$ since $\overline{A}(\omega) \subseteq \overline{A}(\omega)_z$. So, the ideal $(\Omega - \omega)$ of the algebra A is a completely prime ideal, and so,

$$\{0, (\Omega - \omega) \mid \omega \in \mathbb{K}\} \subseteq \mathrm{Spec}_c(A). \tag{30}$$

The set $T := \mathbb{K}[\Omega] \setminus \{0\}$ is a denominator set in A_z that consists of central regular elements. Then

$$A \subset \mathbb{A}_z = A_z \subset T^{-1}A_z = T^{-1}\mathbb{A}_z = \mathbb{K}(\Omega)[z^{\pm 1}][e, f; \sigma, a] \tag{31}$$

where $\mathbb{K}(\Omega) = T^{-1}\mathbb{K}[\Omega]$ is the field of rational functions in the variable Ω over $\mathbb{K}$ and the algebra $T^{-1}A_z = T^{-1}\mathbb{A}_z$ is the GWA $\mathbb{K}(\Omega)[z^{\pm 1}][e, f; \sigma, a]$ with coefficients in the Laurent polynomial ring $\mathbb{K}(\Omega)[z, z^{-1}]$ over the field $\mathbb{K}(\Omega)$.

The set $\mathrm{Spec}(A(\alpha, \beta, 0))$. The next theorem is a description of prime, maximal, primitive and completely prime ideals of the algebra $A(\alpha, \beta, 0)$. For an algebra R, we denote by $\mathcal{I}(R)$ the set of ideals of R.

Theorem 7.1. *Suppose that $\mathbb{K}$ is an algebraically closed field. Let $A = A(\alpha, \beta, 0)$ where $\alpha \neq 0$ and $\beta \neq 0$. Then:*

(1) $\mathrm{Spec}(A) = \{0, (\Omega - \omega) \mid \omega \in \mathbb{K}\}$. *In particular, every nonzero prime ideal meets the centre.*

(2) $\mathrm{Max}(A) = \mathrm{Prim}(A) = \{(\Omega - \omega) \mid \omega \in \mathbb{K}\}$.

(3) $\mathrm{Spec}_c(A) = \mathrm{Spec}(A)$.

(4) *The algebra* $A/(\Omega - \omega)$ *is a simple Noetherian domain for all* $\omega \in \mathbb{K}$.

(5) *The map* $\mathcal{I}(A) \to \mathcal{I}(\mathbb{K}[\Omega])$, $I \mapsto I \cap \mathbb{K}[\Omega]$ *is a bijection with the inverse map* $\mathfrak{a} \mapsto A\mathfrak{a}$. *In particular, all ideals of the algebra* A *commute* ($IJ = JI$) *and every ideal* I *is a unique product of maximal ideals* ($I = \prod_{i=1}^{s}(\Omega - \mu_i)^{n_i}$ *where* $\mu_i \in \mathbb{K}$ *and* $n_i \geqslant 1$).

Proof. (4) Recall that, $\overline{A}(\omega)_z = \overline{A_z}(\omega) = \mathbb{K}[z^{\pm 1}][e, f; \sigma, a_\omega]$ is a GWA (by (15)) where $a_\omega = z(-\frac{q^2\alpha}{q^2-1}z + q^2\omega)$. Using the simplicity criterion for generalized Weyl algebras [1, Theorem 4.2], we see that the algebra $\overline{A}(\omega)_z$ is simple for all $\omega \in \mathbb{K}$. Then, statement (4) follows from (29) and Lemma 3.6.

(1) Recall that $A_z \subseteq T^{-1}A_z$. By the simplicity criterion for GWAs [1, Theorem 4.2], the GWA $T^{-1}A_z$ is a simple algebra. Hence, any nonzero prime ideal $\mathfrak{p}$ of the algebra A_z contains a non-scalar polynomial in the variable Ω, say $t = \prod_{i=1}^{n}(\Omega - \omega_i)^{n_i}$ where $\omega_1, \ldots, \omega_n$ are distinct roots of t. Hence, $(\Omega - \omega_i) \subseteq \mathfrak{p}$ for some i, by statement (4). The ideal $(\Omega - \omega_i)$ is a maximal ideal, by statement (4). Therefore, $\mathfrak{p} = (\Omega - \omega_i)$, as required.

(2) Statement (2) follows from statement (1).

(3) Statement (3) follows from statement (4).

(5) Statement (5) follows from statement (1). ∎

The set $\mathrm{Spec}(A(\alpha, \beta, \gamma))$ where $\gamma \neq 0$. Recall that, for each $\omega \in \mathbb{K}$, $\overline{\mathbb{A}}(\omega) = \mathbb{A}/(\Omega - \omega)$. The algebra $\overline{\mathbb{A}}(\omega) = \mathbb{K}[z][e, f; \sigma, a_\omega]$ is a GWA where $\sigma(z) = q^{-2}z$. Further, by Proposition 2.2 (1), (2), $\overline{\mathbb{A}}(\omega) \subseteq \overline{\mathbb{A}}(\omega)_z \simeq \mathbb{A}_z/(\Omega - \omega)_z$. We have seen already that $\overline{A}(\omega) \subseteq \overline{A}(\omega)_z$. By Proposition 2.2 (2), $A_z = \mathbb{A}_z$, hence $\overline{A}(\omega)_z = \overline{\mathbb{A}}(\omega)_z$ and so

$$\overline{\mathbb{A}}(\omega) \subseteq \overline{A}(\omega) \subseteq \overline{A}(\omega)_z = \overline{\mathbb{A}}(\omega)_z. \tag{32}$$

Lemma 7.2. (1) *Suppose that* $\mathrm{char}(\mathbb{K}) \neq 2$. *We keep the notation of Theorem 6.8. Then*

$$\mathrm{Ext}^1_{\mathbb{A}}(L(\omega_i^{\pm}), L(\omega_i^{\pm})) = 0 \qquad \textit{for all } i \geqslant 1.$$

(2) *Suppose that* $\mathrm{char}(\mathbb{K}) = 2$. *We keep the notation of Theorem 6.10. Then*

$$\mathrm{Ext}^1_{\mathbb{A}}(L(\omega_i), L(\omega_i)) = 0 \qquad \textit{for all } i \geqslant 1.$$

Proof. We give a proof of both statements simultaneously. Let $L = L(\omega_i^{\pm})$ if $\mathrm{char}(\mathbb{K}) \neq 2$, (respectively, $L = L(\omega_i)$ if $\mathrm{char}(\mathbb{K}) = 2$). Suppose that $\mathrm{Ext}^1_{\mathbb{A}}(L, L) \neq 0$, i.e., there exists a non-split sequence of $\mathbb{A}$-modules

$$0 \to L \to M \to L \to 0,$$

we seek a contradiction. Notice that the $\mathbb{A}$-module M is an epimorphic image of the $\mathbb{A}$-module $M' := \mathbb{A}/\mathbb{A}(e, (z - q^{2i}\lambda_i^{\pm})^2, f^i)$, (respectively, $M' := \mathbb{A}/\mathbb{A}(e, (z - q^{2i}\lambda)^2, f^i)$).

Since in both cases $\dim_{\mathbb{K}}(M') = 2i$, we must have $M = M'$, as $\dim_{\mathbb{K}}(M) = 2\dim_{\mathbb{K}}(L) = 2i$. Notice that

$$M' = \bigoplus_{j=0}^{i-1} f^j \mathbb{K}[z]/(z - q^{2i}\lambda_i^{\pm})^2\bar{1} \qquad \text{where } \bar{1} = 1 + \mathbb{A}(e, (z - q^{2i}\lambda_i^{\pm})^2, f^i),$$

respectively,

$$M' = \bigoplus_{j=0}^{i-1} f^j \mathbb{K}[z]/(z - q^{2i}\lambda_i)^2\bar{1} \qquad \text{where } \bar{1} = 1 + \mathbb{A}(e, (z - q^{2i}\lambda_i)^2, f^i)).$$

In both cases, $0 = e\bar{1}$ and so $0 = fe\bar{1} = a(z, \Omega)\bar{1} = a(q^{2i}\lambda_i^{\pm}, \Omega)\bar{1}$, (respectively, $0 = fe\bar{1} = a(z, \Omega)\bar{1} = a(q^{2i}\lambda_i, \Omega)\bar{1}$). Hence, $\Omega = \omega_i^{\pm}$ (respectively, $\Omega = \omega_i$). Therefore, $(\Omega - \omega_i^{\pm})M' = 0$ (respectively, $(\Omega - \omega_i)M' = 0$). This means that the $\mathbb{A}$-module M' is also an $\overline{\mathbb{A}}(\omega_i^{\pm})$-module (respectively, $\overline{\mathbb{A}}(\omega_i)$-module). By Proposition 7.3 (1), $(\bar{\mathfrak{b}}_i^{\pm})^2 = \bar{\mathfrak{b}}_i^{\pm}$ where $\bar{\mathfrak{b}}_i^{\pm} = \mathrm{ann}_{\overline{\mathbb{A}}(\omega_i^{\pm})}(L(\omega_i^{\pm}))$ (respectively, by Proposition 7.5 (1), $(\bar{\mathfrak{b}}_i)^2 = \bar{\mathfrak{b}}_i$ where $\bar{\mathfrak{b}}_i = \mathrm{ann}_{\overline{\mathbb{A}}(\omega_i)}(L(\omega_i))$). So, in both cases $M \simeq L \oplus L$, a contradiction. ∎

For a natural number $n \geqslant 1$, we denote by $M_n(\mathbb{K})$ the algebra of $n \times n$ matrices over the field $\mathbb{K}$. There are two cases to consider: $\mathrm{char}(\mathbb{K}) \neq 2$ and $\mathrm{char}(\mathbb{K}) = 2$.

The case char($\mathbb{K}$) ≠ 2.

Proposition 7.3. *We keep the notation of Theorem 6.8. Suppose that $\mathbb{K}$ is an algebraically closed field,* $\mathrm{char}(\mathbb{K}) \neq 2$ *and* $i \in \mathbb{N}_+ = \{1, 2, \dots\}$. *Then:*

(1) *the simple i-dimensional $\overline{A}(\omega_i^{\pm})$-module $L(\omega_i^{\pm})$ is also a simple $\overline{\mathbb{A}}(\omega_i^{\pm})$-module (via the restriction of scalars (32)),*

$$(\bar{\mathfrak{b}}_i^{\pm})^2 = \bar{\mathfrak{b}}_i^{\pm} \qquad \text{and} \qquad \overline{\mathbb{A}}(\omega_i^{\pm})/\bar{\mathfrak{b}}_i^{\pm} \simeq M_i(\mathbb{K})$$

where $\bar{\mathfrak{b}}_i^{\pm} = \mathrm{ann}_{\overline{\mathbb{A}}(\omega_i^{\pm})}(L(\omega_i^{\pm}))$.

(2) *Let $\mathfrak{b}_i^{\pm}$ be the pre-image of the ideal $\bar{\mathfrak{b}}_i^{\pm}$ under the epimorphism $\mathbb{A} \to \overline{\mathbb{A}}(\omega_i^{\pm})$, $\alpha \mapsto \alpha + (\Omega - \omega_i^{\pm})$. Then $(\mathfrak{b}_i^{\pm})^2 = \mathfrak{b}_i^{\pm}$ and $\mathbb{A}/\mathfrak{b}_i^{\pm} \simeq \overline{\mathbb{A}}(\omega_i^{\pm})/\bar{\mathfrak{b}}_i^{\pm} \simeq M_i(\mathbb{K})$.*

Proof. (1) The algebra $\overline{\mathbb{A}}(\omega) = \mathbb{K}[z][e, f; \sigma, a_\omega]$ is a GWA. By [7, Theorem 1], the simple i-dimensional $A(\omega_i^{\pm})$-module $L(\omega_i^{\pm})$ is also a simple $\overline{\mathbb{A}}(\omega_i^{\pm})$-module (by restriction since $\overline{\mathbb{A}}(\omega_i) \subseteq \overline{A}(\omega_i)$). Then $\overline{\mathbb{A}}(\omega_i^{\pm})/\bar{\mathfrak{b}}_i^{\pm} \simeq M_i(\mathbb{K})$. By [8, Corollary 4], $(\bar{\mathfrak{b}}_{i,z}^{\pm})^2 = \bar{\mathfrak{b}}_{i,z}^{\pm}$ (since the polynomial $a_{\omega_i^{\pm}} \in \mathbb{K}[z]$ has degree 2), where $\bar{\mathfrak{b}}_{i,z}^{\pm}$ is the localization of the ideal $\bar{\mathfrak{b}}_i^{\pm}$ of the algebra $\overline{\mathbb{A}}(\omega_i^{\pm})$ at the powers of z. So,

$$0 = \bar{\mathfrak{b}}_{i,z}^{\pm}/(\bar{\mathfrak{b}}_{i,z}^{\pm})^2 = \big(\bar{\mathfrak{b}}_i^{\pm}/(\bar{\mathfrak{b}}_i^{\pm})^2\big)_z.$$

We must have $(\bar{\mathfrak{b}}_i^{\pm})^2 = \bar{\mathfrak{b}}_i^{\pm}$. Suppose that this is not the case, i.e., $N := \bar{\mathfrak{b}}_i^{\pm}/(\bar{\mathfrak{b}}_i^{\pm})^2 \neq 0$, we seek a contradiction. Then the $\mathbb{A}(\omega_i^{\pm})$-module N is annihilated by the ideal $\bar{\mathfrak{b}}_i^{\pm}$. Since,

$\mathbb{A}(\omega_i^{\pm})/\bar{\mathfrak{b}}_i^{\pm} \simeq M_i(\mathbb{K})$, the $\mathbb{A}(\omega_i^{\pm})$-module N is isomorphic to $L(\omega_i^{\pm})^n$, a direct sum of n copies of the $\mathbb{A}(\omega_i^{\pm})$-module $L(\omega_i^{\pm})$ for some $n \geqslant 1$. Since the map $z\cdot : L(\omega_i^{\pm}) \to L(\omega_i^{\pm})$, $v \mapsto zv$ is a bijection, so is the map $z\cdot : N \to N$, $u \mapsto zu$. Therefore, $0 \neq N = N_z = 0$, a contradiction. Ideals of the GWAs like the GWA $\bar{\mathbb{A}}(\omega_i^{\pm})_z$ were classified in [8].

(2) By the very definition of the ideal $\mathfrak{b}_i^{\pm}$,

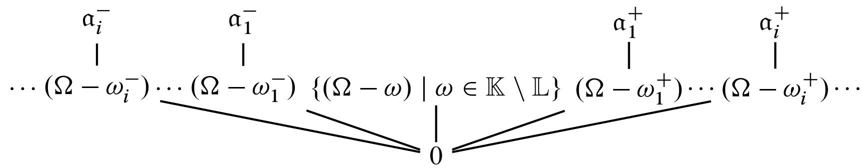
$$\mathbb{A}/\mathfrak{b}_i^{\pm} \simeq \bar{\mathbb{A}}(\omega_i^{\pm})/\bar{\mathfrak{b}}_i^{\pm} \simeq M_i(\mathbb{K}).$$

By Lemma 7.2 (1), $(\mathfrak{b}_i^{\pm})^2 = \mathfrak{b}_i^{\pm}$. ∎

The next theorem is a description of prime, maximal, primitive and completely prime ideals of the algebra $A(\alpha, \beta, \gamma)$.

Theorem 7.4. *Suppose that $\mathbb{K}$ is an algebraically closed field and* $\mathrm{char}(\mathbb{K}) \neq 2$. *Let $A = A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$. Then:*

(1) $\mathrm{Spec}(A) = \{0, (\Omega - \omega) \mid \omega \in \mathbb{K}\} \sqcup \{\mathfrak{a}_i^{\pm} \mid i \geqslant 1\}$, *where $\mathfrak{a}_i^{\pm} = \mathrm{ann}_A(L(\omega_i^{\pm}))$, see Theorem 6.8, $(\mathfrak{a}_i^{\pm})^2 = \mathfrak{a}_i^{\pm}$ and $A/\mathfrak{a}_i^{\pm} \simeq M_i(\mathbb{K})$, the algebra of $i \times i$ matrices over $\mathbb{K}$. The containments of prime ideals of the algebra A are given by the diagram below:*



where $i \geqslant 1$, $\mathbb{L} = \{\omega_i^{\pm} \mid i \geqslant 1\}$ and $\omega_i^{\pm} = \pm\frac{1+q^{2i}}{q^2-1}\left(\frac{\alpha\beta\gamma}{(q^2-1)q^{2i}}\right)^{\frac{1}{2}}$ (all numbers in $\mathbb{L}$ are distinct, Lemma 6.7 (3)).

(2) $\mathrm{Max}(A) = \mathrm{Prim}(A) = \{(\Omega - \omega) \mid \omega \in \mathbb{K} \setminus \mathbb{L}\} \sqcup \{\mathfrak{a}_i^{\pm} \mid i \geqslant 1\}$.

(3) $\mathrm{Spec}_c(A) = \{(\Omega - \omega) \mid \omega \in \mathbb{K}\} \sqcup \{\mathfrak{a}_1^+, \mathfrak{a}_1^-\}$.

(4) *The algebra $\bar{A}(\omega)$ is a Noetherian domain for all $\omega \in \mathbb{K}$.*

(5) *The algebra $\bar{A}(\omega)$ is simple iff $\omega \notin \mathbb{L} = \{\omega_i^{\pm} \mid i \geqslant 1\}$.*

(6) *For each $\omega = \omega_i^{\pm}$ where $i \geqslant 1$, the ideal $\bar{\mathfrak{a}}_i^{\pm} := \mathfrak{a}_i^{\pm}/(\Omega - \omega_i^{\pm}) = \mathrm{ann}_{\bar{A}(\omega_i^{\pm})}(L(\omega_i^{\pm}))$ is a unique proper ideal of the algebra $\bar{A}(\omega_i^{\pm})$, $(\bar{\mathfrak{a}}_i^{\pm})^2 = \bar{\mathfrak{a}}_i^{\pm}$, $\bar{A}(\omega_i^{\pm})/\bar{\mathfrak{a}}_i^{\pm} \simeq M_i(\mathbb{K})$, $\bar{\mathfrak{a}}_i^{\pm} = (\bar{\mathfrak{b}}_i^{\pm}) = \bar{A}(\omega_i^{\pm})\bar{\mathfrak{b}}_i^{\pm}\bar{A}(\omega_i^{\pm})$ and $\bar{\mathfrak{b}}_i^{\pm} = \bar{\mathbb{A}}(\omega_i^{\pm}) \cap \bar{\mathfrak{a}}_i^{\pm}$.*

(7) *For all $i \geqslant 1$, $(\mathfrak{a}_i^{\pm})^2 = \mathfrak{a}_i^{\pm}$ and $A/\mathfrak{a}_i^{\pm} \simeq M_i(\mathbb{K})$.*

(8) *Every nonzero prime ideal meets the centre of A.*

Proof. (4) Statement (4) is obvious.

(5) Since $\beta \neq 0$, $(z^i) = (1)$ for all $i \geqslant 1$ in $\bar{A}(\omega)$ (Lemma 3.2). By (29), $\mathrm{Spec}(\bar{A}(\omega)) = \mathrm{Spec}(\bar{A}(\omega)_z)$. Now, statement (5) follows from the simplicity criterion [1, Theorem 4.2] for the GWA $\bar{A}(\omega)_z = \bar{\mathbb{A}}(\omega)_z$.

(6) Let $\overline{\mathbb{A}} = \overline{\mathbb{A}}(\omega_i^{\pm})$ and $\overline{A} = \overline{A}(\omega_i^{\pm})$. The inclusion of algebras $\overline{\mathbb{A}} \subseteq \overline{A}$ yields the inclusions

$$\overline{\mathfrak{b}}_i^{\pm} \subseteq \overline{A} \cap \overline{\mathfrak{a}}_i^{\pm} \subseteq (\overline{\mathfrak{b}}_i^{\pm}) \subseteq \overline{\mathfrak{a}}_i^{\pm}$$

and the algebra homomorphisms (Lemma 7.3)

$$M_i(\mathbb{K}) \simeq \overline{\mathbb{A}}/\overline{\mathfrak{b}}_i^{\pm} \to \overline{\mathbb{A}}/A \cap \overline{\mathfrak{a}}_i^{\pm} \to \overline{A}/(\overline{\mathfrak{b}}_i^{\pm}) \to \overline{A}/(\overline{\mathfrak{a}}_i^{\pm}) \simeq M_i(\mathbb{K}).$$

Clearly, all homomorphisms are isomorphisms, and so the inclusions above are equalities. By Proposition 7.3 (1), $(\overline{\mathfrak{b}}_i^{\pm})^2 = \overline{\mathfrak{b}}_i^{\pm}$. Now,

$$(\overline{\mathfrak{a}}_i^{\pm})^2 = \left((\overline{\mathfrak{b}}_i^{\pm})\right)^2 \supseteq \left((\overline{\mathfrak{b}}_i^{\pm 2})\right) = (\overline{\mathfrak{b}}_i^{\pm}) = \overline{\mathfrak{a}}_i^{\pm},$$

and so $(\overline{\mathfrak{a}}_i^{\pm})^2 = \overline{\mathfrak{a}}_i^{\pm}$.

(7) By Proposition 7.3 (2), $(\mathfrak{b}_i^{\pm})^2 = \mathfrak{b}_i^{\pm}$. By statement (6), $\overline{\mathfrak{a}}_i^{\pm} = (\overline{\mathfrak{b}}_i^{\pm})$, and so $\mathfrak{a}_i^{\pm} = (\mathfrak{b}_i^{\pm})$. Now,

$$(\mathfrak{a}_i^{\pm})^2 = \left((\mathfrak{b}_i^{\pm})\right)^2 \supseteq \left((\mathfrak{b}_i^{\pm})^2\right) = (\mathfrak{b}_i^{\pm}) = \mathfrak{a}_i^{\pm},$$

and so $(\mathfrak{a}_i^{\pm})^2 = \mathfrak{a}_i^{\pm}$.

(1) Statement (1) follows from statements (5)–(7) and Lemma 3.6.

(2) Statement (2) follows from statement (1).

(3) Statement (3) follows from statement (1), (30) and statement (7).

(8) Statement (8) follows from statement (1). ∎

The case char($\mathbb{K}$) = 2.

Proposition 7.5. *We keep the notation of Theorem 6.10. Suppose that $\mathbb{K}$ is an algebraically closed field,* char$(\mathbb{K}) = 2$ *and $i \in \mathbb{N}_+ = \{1, 2, \ldots\}$. Then:*

(1) *the simple i-dimensional $\overline{A}(\omega_i)$-module $L(\omega_i)$ is also a simple $\overline{\mathbb{A}}(\omega_i)$-module (via the restriction of scalars (32)),*

$$(\overline{\mathfrak{b}}_i)^2 = \overline{\mathfrak{b}}_i \qquad \textit{and} \qquad \overline{\mathbb{A}}(\omega_i)/\overline{\mathfrak{b}}_i \simeq M_i(\mathbb{K})$$

where $\overline{\mathfrak{b}}_i := \mathrm{ann}_{\overline{\mathbb{A}}(\omega_i)}(L(\omega_i))$.

(2) *Let $\mathfrak{b}_i$ be the pre-image of the ideal $\overline{\mathfrak{b}}_i$ under the epimorphism $\mathbb{A} \to \overline{A}(\omega_i)$, $\alpha \mapsto \alpha + (\Omega - \omega_i)$. Then $\mathfrak{b}_i^2 = \mathfrak{b}_i$ and $\mathbb{A}/\mathfrak{b}_i \simeq \overline{\mathbb{A}}(\omega_i)/\overline{\mathfrak{b}}_i \simeq M_i(\mathbb{K})$.*

Proof. (1) The algebra $\overline{\mathbb{A}}(\omega) = \mathbb{K}[z][e, f; \sigma, a_\omega]$ is a GWA. By [7, Theorem 1], the simple i-dimensional $A(\omega_i)$-module $L(\omega_i)$ is also a simple $\overline{\mathbb{A}}(\omega_i)$-module. Then $\overline{\mathbb{A}}(\omega_i)/\overline{\mathfrak{b}}_i \simeq M_i(\mathbb{K})$. By [8, Corollary 4], $(\overline{\mathfrak{b}}_{i,z})^2 = \overline{\mathfrak{b}}_{i,z}$ (since the polynomial $a_{\omega_i} \in \mathbb{K}[z]$ has degree 2), where $\overline{\mathfrak{b}}_{i,z}$ is the localization of the ideal $\overline{\mathfrak{b}}_i$ of the algebra $\overline{\mathbb{A}}(\omega_i)$ at the powers of z. So,

$$0 = \overline{\mathfrak{b}}_{i,z}/(\overline{\mathfrak{b}}_{i,z})^2 = \left(\overline{\mathfrak{b}}_i/(\overline{\mathfrak{b}}_i)^2\right)_z.$$

We must have $(\overline{\mathfrak{b}}_i)^2 = \overline{\mathfrak{b}}_i$. Suppose that this is not the case, i.e., $N := \overline{\mathfrak{b}}_i/(\overline{\mathfrak{b}}_i)^2 \neq 0$, we seek a contradiction. Then the $\mathbb{A}(\omega_i)$-module N is annihilated by the ideal $\overline{\mathfrak{b}}_i$. Since

$\mathbb{A}(\omega_i)/\overline{\mathfrak{b}}_i \simeq M_i(\mathbb{K})$, the $\mathbb{A}(\omega_i)$-module N is isomorphic to $L(\omega_i)^n$, a direct sum of n copies of the $\mathbb{A}(\omega_i)$-module $L(\omega_i)$ for some $n \geqslant 1$. Since the map $z\cdot : L(\omega_i) \to L(\omega_i)$, $v \mapsto zv$ is a bijection, so is the map $z\cdot : N \to N$, $u \mapsto zu$. Therefore, $0 \neq N = N_z = 0$, a contradiction. Ideals of the GWAs like the GWA $\overline{\mathbb{A}}(\omega_i)_z$ were classified in [8].

(2) By the very definition of the ideal $\mathfrak{b}_i$,

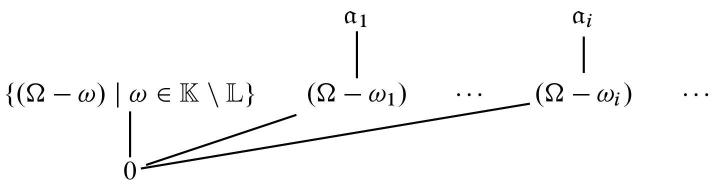
$$\mathbb{A}/\mathfrak{b}_i \simeq \overline{\mathbb{A}}(\omega_i)/\overline{\mathfrak{b}}_i \simeq M_i(\mathbb{K}).$$

By Lemma 7.2 (1), $\mathfrak{b}_i^2 = \mathfrak{b}_i$. ■

The next theorem is a description of prime, maximal, primitive and completely prime ideals of the algebra $A(\alpha, \beta, \gamma)$ in the case where $\alpha, \beta, \gamma \in \mathbb{K}^*$ and $\mathrm{char}(\mathbb{K}) = 2$.

Theorem 7.6. *Suppose that $\mathbb{K}$ is an algebraically closed field and* $\mathrm{char}(\mathbb{K}) = 2$. *Let $A = A(\alpha, \beta, \gamma)$ where $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$. Then:*

(1) $\mathrm{Spec}(A) = \{0, (\Omega - \omega) \mid \omega \in \mathbb{K}\} \sqcup \{\mathfrak{a}_i \mid i \geqslant 1\}$, *where* $\mathfrak{a}_i = \mathrm{ann}_A(L(\omega_i))$, *see Theorem 6.10, $\mathfrak{a}_i^2 = \mathfrak{a}_i$ and $A/\mathfrak{a}_i \simeq M_i(\mathbb{K})$, the algebra of $i \times i$ matrices over $\mathbb{K}$. The containments of prime ideals of the algebra A are given by the diagram below:*



where $i \geqslant 1$, $\mathbb{L} = \{\omega_i \mid i \geqslant 1\}$ and $\omega_i = \frac{1+q^{2i}}{q^2-1}\left(\frac{\alpha\beta\gamma}{(q^2-1)q^{2i}}\right)^{\frac{1}{2}}$ (all numbers in $\mathbb{L}$ are distinct, Lemma 6.7 (3)*).*

(2) $\mathrm{Max}(A) = \mathrm{Prim}(A) = \{(\Omega - \omega) \mid \omega \in \mathbb{K} \setminus \mathbb{L}\} \sqcup \{\mathfrak{a}_i \mid i \geqslant 1\}$.

(3) $\mathrm{Spec}_c(A) = \{(\Omega - \omega) \mid \omega \in \mathbb{K}\} \sqcup \{\mathfrak{a}_1\}$.

(4) *The algebra $\overline{A}(\omega)$ is a Noetherian domain for all $\omega \in \mathbb{K}$.*

(5) *The algebra $\overline{A}(\omega)$ is simple iff $\omega \notin \mathbb{L} = \{\omega_i \mid i \geqslant 1\}$.*

(6) *For each $\omega = \omega_i$ where $i \geqslant 1$, the ideal $\overline{\mathfrak{a}}_i := \mathfrak{a}_i/(\Omega - \omega_i) = \mathrm{ann}_{\overline{A}(\omega_i)}(L(\omega_i))$ is a unique proper ideal of the algebra $\overline{A}(\omega_i)$, $(\overline{\mathfrak{a}}_i)^2 = \overline{\mathfrak{a}}_i$, $\overline{A}(\omega_i)/\overline{\mathfrak{a}}_i \simeq M_i(\mathbb{K})$, $\overline{\mathfrak{a}}_i = (\overline{\mathfrak{b}}_i) = \overline{A}(\omega_i)\overline{\mathfrak{b}}_i\overline{A}(\omega_i)$ and $\overline{\mathfrak{b}}_i = \overline{\mathbb{A}}(\omega_i) \cap \overline{\mathfrak{a}}_i$.*

(7) *For all $i \geqslant 1$, $\mathfrak{a}_i^2 = \mathfrak{a}_i$ and $A/\mathfrak{a}_i \simeq M_i(\mathbb{K})$.*

(8) *Every nonzero prime ideal meets the centre of A.*

Proof. (4) Statement (4) is obvious.

(5) Since $\beta \neq 0$, $(z^i) = (1)$ for all $i \geqslant 1$ in $\overline{A}(\omega)$ (Lemma 3.2). By (29), $\mathrm{Spec}(\overline{A}(\omega)) = \mathrm{Spec}(\overline{A}(\omega)_z)$. Now, statement (5) follows from the simplicity criterion [1, Theorem 4.2] for the GWA $\overline{A}(\omega)_z = \overline{\mathbb{A}}(\omega)_z$.

(6) Let $\overline{\mathbb{A}} = \overline{\mathbb{A}}(\omega_i)$ and $\overline{A} = \overline{A}(\omega_i)$. The inclusion of algebras $\overline{\mathbb{A}} \subseteq \overline{A}$ yields the inclusions

$$\overline{\mathfrak{b}}_i \subseteq \overline{A} \cap \overline{\mathfrak{a}}_i \subseteq (\overline{\mathfrak{b}}_i) \subseteq \overline{\mathfrak{a}}_i$$

and the algebra homomorphisms (Lemma 7.3)

$$M_i(\mathbb{K}) \simeq \overline{\mathbb{A}}/\overline{\mathfrak{b}}_i \to \overline{\mathbb{A}}/A \cap \overline{\mathfrak{a}}_i \to \overline{A}/(\overline{\mathfrak{b}}_i) \to \overline{A}/(\overline{\mathfrak{a}}_i) \simeq M_i(\mathbb{K}).$$

Clearly, all homomorphisms are isomorphisms, and so the inclusions above are equalities. By Proposition 7.3 (1), $\overline{\mathfrak{b}}_i^2 = \overline{\mathfrak{b}}_i$. Now,

$$\overline{\mathfrak{a}}_i^2 = \left((\overline{\mathfrak{b}}_i)\right)^2 \supseteq \left((\overline{\mathfrak{b}}_i^2)\right) = (\overline{\mathfrak{b}}_i) = \overline{\mathfrak{a}}_i,$$

and so $\overline{\mathfrak{a}}_i^2 = \overline{\mathfrak{a}}_i$.

(7) By Proposition 7.3 (2), $\mathfrak{b}_i^2 = \mathfrak{b}_i$. By statement (6), $\overline{\mathfrak{a}}_i = (\overline{\mathfrak{b}}_i)$, and so $\mathfrak{a}_i = (\mathfrak{b}_i)$. Now,

$$\mathfrak{a}_i^2 = \left((\mathfrak{b}_i)\right)^2 \supseteq \left((\mathfrak{b}_i)^2\right) = (\mathfrak{b}_i) = \mathfrak{a}_i,$$

and so $\mathfrak{a}_i^2 = \mathfrak{a}_i$.

(1) Statement (1) follows from statements (5)–(7) and Lemma 3.6.

(2) Statement (2) follows from statement (1).

(3) Statement (3) follows from statement (1), (30) and statement (7).

(8) Statement (8) follows from statement (1). ■

8. Applications and corollaries

In this section, applications and corollaries of the classifications are given. In particular, proofs of Theorems 1.1, 1.2, 1.6, 1.5 and 1.3 are given.

Proof of Theorem 1.2. (i) *If I is a prime ideal then $I \cap Z(A) \neq 0$:* By Theorem 1.1, there are only 4 cases to consider. Now, statement (i) follows from Corollary 4.2 (4), Corollary 5.3 (4), Theorem 7.1 (1), Theorem 7.4 (8) and Theorem 7.6 (8).

(ii) *If I is a nonzero ideal then $I \cap Z(A) \neq 0$:* Let $\mathfrak{n} = \mathfrak{n}(I)$ be the prime radical of I, i.e., $\mathfrak{n} = \bigcap_{P \in \min(I)} P$ is the intersection of minimal prime ideals over I. The algebra A is Noetherian, hence there is a natural number $s \geqslant 1$ such that

$$I \supseteq \mathfrak{n}^s \supseteq \prod_{P \in \min(I)} P^s =: \mathcal{P},$$

the order of multiples in the product is arbitrary. The algebra A is a prime algebra, hence $\mathcal{P} \neq 0$. Then

$$\mathcal{P} \supseteq \mathcal{Q} := \prod_{P \in \min(I)} (P \cap Z(A))^s \neq 0,$$

since the centre $Z(A) = \mathbb{K}[\Omega]$ is a domain and all $P \cap Z(A) \neq 0$. Then $I \cap Z(A) \neq 0$, since $I \cap Z(A) \supseteq Q \neq 0$. ■

Proof of Theorem 1.1. By the cyclic permutation symmetry, we have 4 cases to consider for (α, β, γ): $(0, 0, 0)$, $(\alpha, 0, 0)$, $(\alpha, \beta, 0)$ and (α, β, γ) where $\alpha, \beta, \gamma \in \mathbb{K}^*$. Using the change of generators of the algebra $A = A(\alpha, \beta, \gamma)$ from (x, y, z) to $(\lambda x, \mu y, \nu z)$ where $\lambda, \mu, \nu \in \mathbb{K}^*$ and the fact that $\sqrt{\mathbb{K}} \subseteq \mathbb{K}$ we may assume that each of the nonzero scalars α, β and γ is 1. So, we may assume that (α, β, γ) is one of the four options (according to $\mathrm{rk}(A) = 0, 1, 2$ and 3): $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$ and $(1, 1, 1)$. Suppose that $\mathbb{K}$ is an algebraically closed field. In these cases, the posets $(\mathrm{Spec}(A), \subseteq)$ are not isomorphic (use the description of $(\mathrm{Spec}(A), \subseteq)$). Therefore, the algebras are not isomorphic.

Suppose that $\mathbb{K}$ is not necessarily an algebraically closed field and $\overline{\mathbb{K}}$ be its algebraic closure. If two algebras in the list of four above are isomorphic over $\mathbb{K}$ then they are automatically isomorphic over $\overline{\mathbb{K}}$ and so they must coincide. ■

Proof of Theorem 1.6. The theorem follows from the classification of simple A-modules (by Theorem 7.1 (4), if $\mathrm{rk}(A) = 2$ then all nonzero A-modules are infinite dimensional). ■

Proof of Theorem 1.8. The statement of Theorem 1.8 holds for all ranks $\mathrm{rk}(A) = 0, 1, 2$ and 3, see the classifications of prime ideals for each rank above. ■

Proof of Theorem 1.7. Let $A = A(\alpha, \beta, \gamma)$. If $\mathrm{rk}(A) = 0, 1$ or 2 then every prime ideal of A is completely prime (Corollary 4.2 (3), Corollary 5.3 and Theorem 7.1 (3)). If $\mathrm{rk}(A) = 3$ then the result follows from Theorem 7.4 (3) and Theorem 7.6 (3). ■

Proof of Theorem 1.5. Let $A = A(\alpha, \beta, \gamma)$. If $\mathrm{rk}(A) = 0, 1$, then the category of finite dimensional A-modules is not semisimple (use the fact that the category of finite dimensional $\mathbb{K}[t]$-modules is not semisimple). If $\mathrm{rk}(A) = 2$, the category of finite dimensional A-modules contains only zero module (Theorem 1.6). Hence, it is not semisimple. Suppose that $\mathrm{rk}(A) = 3$. To finish the proof of the theorem it suffices to show that, for each $\omega \in \mathbb{K}$, the category $\mathcal{F}_\omega$ of finite dimensional A-modules is semisimple provided $\mathcal{F}_\omega \neq 0$ where $M \in \mathcal{F}_\omega$ iff $\dim_{\mathbb{K}}(M) < \infty$ and $(\Omega - \omega)^n M = 0$ for some $n = n(M) \geqslant 1$. If $\mathcal{F}_\omega \neq 0$ then the category $\mathcal{F}_\omega$ contains a unique simple module, say $U = U_\omega$ (Theorem 6.8 (2) and Theorem 6.10 (2)). Since its annihilator $\mathfrak{a} = \mathrm{ann}_A(U_\omega)$ is an idempotent ideal ($\mathfrak{a}^2 = \mathfrak{a}$) (Theorem 7.4 (1) and Theorem 7.6 (1)) such that $A/\mathfrak{a} \simeq M_n(\mathbb{K})$ is the $n \times n$ matrix algebra over $\mathbb{K}$ where $n = \dim_{\mathbb{K}}(U)$, the category $\mathcal{F}_\omega$ is a semisimple category. ■

The algebra $U = U_q(\mathfrak{sl}_2)$. If $q^2 \neq 1$ then the algebra

$$U_q(\mathfrak{sl}_2) = \mathbb{K}\Big\langle K^{\pm 1}, E, F \mid KE = q^2 EK,\ KF = q^{-2} FK,\ [E, F] = \frac{K - K^{-1}}{q - q^{-1}}\Big\rangle$$

is isomorphic to the algebra $A = A(\alpha, \beta, \gamma; q^2)_z$ where $\alpha = 1 - q^2$, $\beta = 1 - q^{-2}$ and $\gamma = 1 - q^2$ via the isomorphism

$$U_q(\mathfrak{sl}_2) \to A, \qquad K \mapsto z, \qquad E \mapsto \frac{1 - zx}{1 - q^{-2}}, \qquad F \mapsto \frac{y - z^{-1}}{q - q^{-1}}$$

and $x \mapsto K^{-1} + (1 - q^2) K^{-1} E$, $y \mapsto K^{-1} + (q - q^{-1}) F$ and $z \mapsto K$ is its inverse, [19].

The algebra $U_q(\mathfrak{sl}_2)$ is a localization A_z of the algebra $A = A(\alpha, \beta, \gamma; q^2)$ at the powers of the element z with $\mathrm{rk}(A) = 3$. It follows from Theorem 1.5 that the category of finite dimensional $U_q(\mathfrak{sl}_2)$-modules are semisimple (this result is known). It follows from Theorem 1.3, that the ideals of the algebra $U_q(\mathfrak{sl}_2)$ commute and each ideal of $U_q(\mathfrak{sl}_2)$ is a unique product of primes (Theorem 8.1).

Theorem 8.1. *Suppose that $\mathbb{K}$ is an algebraically closed field, q^2 is not a root of unity and $U = U_q(\mathfrak{sl}_2)$. We keep the notation of Theorem 6.8 and Theorem 6.10. Then:*

(1) *the algebra U is isomorphic to the localization A_z of the algebra $A = A(1,1,1)$ at the powers of the element z.*

(2) $$\mathrm{Spec}(A_z) = \begin{cases} \{(0), (\Omega-\omega)_z \mid \omega \in \mathbb{K}\} \sqcup \{(\mathfrak{a}_i^{\pm})_z \mid i \geqslant 1\}, & \text{if } \mathrm{char}(\mathbb{K}) \neq 2, \\ \{(0), (\Omega-\omega)_z \mid \omega \in \mathbb{K}\} \sqcup \{(\mathfrak{a}_i)_z \mid i \geqslant 1\}, & \text{if } \mathrm{char}(\mathbb{K}) = 2. \end{cases}$$

(3) *Suppose that* $\mathrm{char}(\mathbb{K}) \neq 2$. *Then the ideals of U commute and every nonzero ideal I of U is a unique product (up to permutation) of prime ideals,*

$$I = \prod_{\omega \in \mathbb{K}} (\Omega-\omega)_z^{n(\omega)} \cdot \prod_{i \geqslant 1} (\mathfrak{a}_i^+)_z^{n_i} \cdot \prod_{j \geqslant 1} (\mathfrak{a}_j^-)_z^{m_j},$$

where $n(\omega) \in \mathbb{N}$, $n_i, m_j \in \{0,1\}$ and all but finitely many numbers $n(\omega), n_i$ and m_j are equal to zero.

(4) *Suppose that* $\mathrm{char}(\mathbb{K}) = 2$. *Then the ideals of U commute and every nonzero ideal I of U is a unique product (up to permutation) of prime ideals,*

$$I = \prod_{\omega \in \mathbb{K}} (\Omega-\omega)^{n(\omega)} \cdot \prod_{i \geqslant 1} (\mathfrak{a}_i)_z^{n_i}.$$

Proof. The algebra U is a particular case of GWAs that are considered in [9] and the theorem is a particular case of [9, Theorem 1]. ∎

For an algebra R, we denote by $\mathcal{I}(R)$ the set of its ideals.

Proof of Theorem 1.4. (1) Statement (1) follows from Theorem 7.1 (5) (when $\gamma = 0$), statement (2) (when $\gamma \neq 0$, $\mathrm{char}(\mathbb{K}) \neq 2$) and statement (3) (when $\gamma \neq 0$, $\mathrm{char}(\mathbb{K}) = 2$).

(2) and (3) We prove statements (2) and (3) simultaneously. Recall that $A_z = U = U_q(\mathfrak{sl}_2)$ for which analogues of statements (2) and (3) hold (Theorem 8.1). Let I be a nonzero ideal of the algebra A. By Theorem 8.1 (3), (4),

$$I_z = g \cdot \prod_{i \geqslant 1} (\mathfrak{a}_i^+)_z^{n_i} \cdot \prod_{j \geqslant 1} (\mathfrak{a}_j^-)_z^{m_j} \qquad \text{if } \mathrm{char}(\mathbb{K}) \neq 2$$

(respectively, $I_z = g \cdot \prod_{i \geqslant 1} (\mathfrak{a}_i)_z^{n_i}$ if $\mathrm{char}(\mathbb{K}) = 2$) for some monomial polynomial $g \in \mathbb{K}[\Omega]$. Since $A \subset A_z$ and $I \subseteq I_z \cap A$, we see that $g \in I$. Notice that

$$A = \bigoplus_{\substack{i,j,k \geqslant 0, \\ ijk=0}} \mathbb{K}[\Omega] x^i y^j z^k \subset A_z = \bigoplus_{\substack{i,j \geqslant 0, k \in \mathbb{Z}, \\ ijk=0}} \mathbb{K}[\Omega] x^i y^j z^k.$$

Hence, Ig^{-1} is an ideal of A such that $(Ig^{-1}) = I_z g^{-1}$. So, replacing the ideal I by the ideal Ig^{-1} we may assume that $g = 1$, in both cases.

Let $\min(I)$ and $\min(I_z)$ be the sets of minimal primes of the ideals $I \lhd A$ and $I_z \lhd A_z$ over the ring A and A_z, respectively. By (29), the map

$$\min(I) \to \min(I_z), \qquad \mathfrak{p} \mapsto \mathfrak{p}_z$$

is a bijection with the inverse $\mathfrak{q} \mapsto A \cap \mathfrak{q}$. Clearly, $\min(I_z) = \{(\mathfrak{a}_i^+)_z, (\mathfrak{a}_j^-)_z \mid n_i \neq 0, m_j \neq 0\}$ (respectively, $\min(I_z) = \{(\mathfrak{a}_i)_z \mid n_i \neq 0\}$). Hence, $\min(I) = \{\mathfrak{a}_i^+, \mathfrak{a}_j^- \mid n_i \neq 0, m_j \neq 0\}$ (respectively, $\min(I) = \{\mathfrak{a}_i, \mid n_i \neq 0\}$) since $(\mathfrak{a}_i^+)_z \cap A = \mathfrak{a}_i^+$ and $(\mathfrak{a}_j^-)_z \cap A = \mathfrak{a}_j^-$, by the maximality of the ideals $\mathfrak{a}_i^+$ and $\mathfrak{a}_j^-$ (respectively, since $(\mathfrak{a}_i)_z \cap A = \mathfrak{a}_i$, by the maximality of the ideal $\mathfrak{a}_i$). The algebra A is a Noetherian algebra. Hence, the ideal I contains a product of powers of its minimal primes. The minimal primes of I are idempotent ideals and they commute. Hence,

$$I = \prod_{i \geqslant 1} (\mathfrak{a}_i^+)^{n_i} \cdot \prod_{j \geqslant 1} (\mathfrak{a}_j^-)^{m_j}$$

(respectively, $I = \prod_{i \geqslant 1} \mathfrak{a}_i^{n_i}$), as required. ■

Proof of Theorem 1.3. If the rank r of the algebra $A(\alpha, \beta, \gamma)$ is 0 or 1, then it follows at once from the descriptions of prime ideals that not all of them commute. If $r = 2, 3$ then the ideals commute and each ideal is a unique product of prime ideals, by Theorem 1.4. ■

Funding. This work was supported by the Royal Society.

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Received 18 August 2022.

Volodymyr V. Bavula
Department of Pure Mathematics, University of Sheffield, Sheffield S3 7RH, UK;
v.bavula@sheffield.ac.uk