J. Noncommut. Geom. (Online first)
DOI 10.4171/JNCG/551

On the growth of Fourier multipliers

Bat-Od Battseren

Abstract. We define a sequence of functions, namely, tame cuts, in the Fourier algebra $A(G)$ of a locally compact group G, which satisfies certain convergence and growth conditions. This new consideration allows us to give a group admitting a Fourier multiplier that is not completely bounded. Furthermore, we show that the induction map $MA(\Gamma) \to MA(G)$ is not always continuous. We also show how Liao's property $(T_{\mathrm{Schur}}, G, K)$ opposes tame cuts. Some examples are provided.

1. Introduction

Approximate identity in the Fourier algebra $A(G)$ of a locally compact group G is a very useful tool for answering analytical, algebraic, or geometrical questions on the locally compact group G. The most well-known applications are from amenability, a-T-menability (also known as Haagerup's property), and weak amenability. It is common that these properties consider approximate identities consisting of Fourier multipliers that are bounded in a related topology. For example, a locally compact group G is said to be *weakly amenable* if there is a net (φ_i) in $A(G)$ such that $\varphi_i \to 1$ uniformly on compact subsets and $\sup \|\varphi_i\|_{M_0A(G)} < \infty$, where $M_0A(G)$ is the algebra of completely bounded Fourier multipliers [5,6,11]. In the present paper, we consider a sequence of compactly supported Fourier multipliers that are generally unbounded in the Fourier multiplier algebra $MA(G)$ but grow at polynomial rate.

Definition 1.1. Let G be a locally compact group and ℓ a proper length function on G. Given $n \in \mathbb{N}$, the *ball of radius* n is the subset $B_n = \{x \in G \mid \ell(x) \leq n\}$. A sequence (φ_n) of compactly supported continuous functions is called *tame cuts* for (G, ℓ) if there exist constants $C, a \geq 0$ such that

$$\varphi_n|_{B_n} \equiv 1 \quad \text{and} \quad \|\varphi_n\|_{MA} \leq Cn^a \quad \text{for all } n \in \mathbb{N}.$$

If in addition we have $\|\varphi_n\|_{M_0A} \leq Cn^a$ for all $n \in \mathbb{N}$, we say that the sequence (φ_n) is *completely bounded tame cuts*.

The main goal is to provide some applications of tame cuts, to show how tame cuts relate to other group properties, and to provide groups with (completely bounded) tame cuts. Our first result is the following theorem in which we provide a group admitting a Fourier multiplier that is not completely bounded.

Mathematics Subject Classification 2020: 43A22 (primary); 22D15 (secondary).
Keywords: locally compact groups, group algebras, Fourier multipliers, weak amenability, rapid decay property.

Theorem A. *Let* Γ *be a uniform lattice in* $G = \mathrm{SL}(3,\mathbb{R})$. *Then,* $M_0A(\Gamma)$ *is a proper subalgebra of* $MA(\Gamma)$.

We recall that a discrete subgroup Γ of a locally compact group G is called a *lattice* in G if there is a finite measure Borel fundamental domain $\Omega \subseteq G$ for the right Γ-action on G. If Ω can be chosen to be relatively compact, we say that Γ is a *uniform* lattice in G.

Theorem A provides many supporting examples for the following conjecture.

Conjecture 1.2. *A discrete countable group* Γ *is amenable if and only if all Fourier multipliers of* Γ *are completely bounded; i.e.,* $M_0A(\Gamma) = MA(\Gamma)$.

The "only if" part is already known even for locally compact groups [18, 23]. On the other hand, in [12, Proposition 4.8], Haagerup–Steenstrup–Szwarc constructed an explicit Fourier multiplier of the free group F_2 of two generators, which is not completely bounded. Also, in [1, Theorem 2], lacunary sets were used to prove the same result. This result implies Theorem A. Indeed, by Tits alternative, any lattice Γ of $\mathrm{SL}(3,\mathbb{R})$ contains a copy of F_2, and any non-completely bounded Fourier multiplier extends to a non-completely bounded Fourier multiplier of Γ. We note, however, that our proof does not rely on the fact that lattices in $\mathrm{SL}(3,\mathbb{R})$ contain a copy of F_2. In fact, our proof relies on the existence of tame cuts for uniform lattices in $\mathrm{SL}(3,\mathbb{R})$.

Our next result is about the *induction map*

$$\Phi : \ell^\infty(\Gamma) \to C_b(G), \quad \varphi \mapsto \widehat{\varphi} = \mathbb{1}_\Omega * (\varphi\mu_\Gamma) * \widetilde{\mathbb{1}}_\Omega.$$

Amenability, a-T-menability, weak amenability, approximation property, and property (T) are known to be inherited by lattices thanks to various continuity results of this induction map. For example, by considering the restriction of Φ, one gets norm decreasing maps $\Phi : A(\Gamma) \to A(G)$ and $\Phi : M_0A(\Gamma) \to M_0A(G)$, and it is used to show that (weak) amenability is inherited by lattices [11, Theorem 2.3]. When G is amenable, the map $\Phi : MA(\Gamma) \to MA(G)$ is also well defined and norm decreasing. One might wonder if this is true for non-amenable groups. The following theorem answers negatively.

Theorem B. *Let* Γ *be a uniform lattice in* $G = \mathrm{SL}(3,\mathbb{R})$, *and let* $\Omega \subseteq G$ *be a relatively compact Borel fundamental domain for the right* Γ*-action on* G. *Then, the restriction of the map*

$$\Phi : \ell^\infty(\Gamma) \to C_b(G), \quad \varphi \mapsto \widehat{\varphi} = \mathbb{1}_\Omega * (\varphi\mu_\Gamma) * \widetilde{\mathbb{1}}_\Omega$$

does not define a bounded map $MA(\Gamma) \to MA(G)$.

The proof of Theorems A and B relies on two results. The first one is that uniform lattices in $\mathrm{SL}(3,\mathbb{R})$ have rapid decay property [15], and the second one is the rigidity inequality of Fourier multiplier norm on $\mathrm{SL}(3,\mathbb{R})$ [16, Lemma 5.2]. It is worth mentioning that the latter rigidity inequality implies that $\mathrm{SL}(3,\mathbb{R})$ does not have tame cuts. If the induction map $\Phi : MA(\Gamma) \to MA(G)$ in Theorem B were continuous, we could say that Γ does not admit tame cuts. In fact, we do not have any example of finitely generated group without tame cuts with respect to the word length function.

Property $(T_{\mathrm{Schur}}, G, K)$ was introduced in [17] to illustrate that all known methods so far to prove the Baum–Connes conjecture for a particular group do not work for a lattice of $G = \mathrm{Sp}(4, \mathbb{F}_q((\pi)))$ because of property $(T_{\mathrm{Schur}}, G, K)$. For our interest, we will show that property $(T_{\mathrm{Schur}}, G, K)$ opposes the existence of tame cuts.

Theorem C. *Let H be an unbounded closed subgroup of a locally compact group G. Suppose that H satisfies property $(T_{\mathrm{Schur}}, G, K, \ell)$ for a compact subgroup K and a proper length function ℓ of G. Then, $(H, \ell|_H)$ does not admit K-bi-invariant tame cuts.*

For discrete groups, we also consider characteristic functions as tame cuts in relation to rapid decay property.

Definition 1.3. Let Γ be a discrete group and ℓ a proper length function on Γ. If (φ_n) is a sequence of characteristic functions that gives (completely bounded) tame cuts for (Γ, ℓ), then we call (φ_n) *(completely bounded) characteristic tame cuts* for (Γ, ℓ).

The behaviors of the operator norm of reduced group C^*-algebra $C^*_\lambda(\Gamma)$ are notoriously difficult to understand. For instance, Valette's conjecture on rapid decay property for uniform lattices in higher rank Lie groups is still open. Different approaches to understand the operator norm are studied for some cases: when G is amenable, we have $\|\lambda(f)\| = \|f\|_1$ for all non-negative functions $f \in C_c(G)$, and when G is a connected simple Lie group, the Harish-Chandra spherical function ϕ_0 satisfies $\|\lambda(f)\| = \int_G \phi_0(x) f(x) dx$ for non-negative $f \in C_c(G)$. On the other hand, the existence of (completely bounded) characteristic tame cuts (φ_n) provides asymptotic information about how the operator norm of a function $f \in C^*_\lambda(\Gamma)$ is changed after truncating (or cutting off) f to $\mathrm{supp}(\varphi_n)$. In many spaces, truncation process defines a norm decreasing operator, for instance, $\ell^p(\Gamma)$ for $1 \le p \le \infty$ and more generally the space $C_c(\Gamma)$ endowed with an unconditional norm N; that is, $N(f) \le N(g)$ whenever $|f| \le |g|$. However, the situation is very different in $C^*_\lambda(\Gamma)$. The simplest illustration can be seen in the case $\Gamma = \mathbb{Z}$. We identify $C^*_\lambda(\mathbb{Z})$ and $C(\mathbb{T})$ via the Gelfand transform. Then, the truncation operator associated with the characteristic function $\varphi_n = \mathbb{1}_{[-n,n]\cap\mathbb{Z}}$ corresponds to Dirichlet's kernel $D_n(t) = \sum_{j=-n}^{n} e^{i2\pi jt} \in L^1(\mathbb{T})$, and it is well known that

$$\left\| M_{\varphi_n} : C^*_\lambda(\mathbb{Z}) \to C^*_\lambda(\mathbb{Z}) \right\| = \|D_n\|_1 = \frac{4}{\pi} \log n + O(1).$$

This shows that the operator norm of $C^*_\lambda(\mathbb{Z})$ is not unconditional. In [21, Theorem 2], it is shown that this logarithmic rate is asymptotically the best while the support of the truncation operator grows at least linearly. This is why we expect the truncation operators on $C^*_\lambda(\Gamma)$ to have bigger norms as their supports grow, and we study characteristic tame cuts to understand their norm growth rate. In the following theorem, we provide some examples with (completely bounded) characteristic tame cuts.

Theorem D. *The following groups have completely bounded characteristic tame cuts.*

(i) $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ *for any* $d \in \mathbb{N}$ *and* $A \in \mathrm{SL}(d, \mathbb{Z})$.

(ii) $\mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{p}{q}} \mathbb{Z}$ *for any coprime* $p, q \in \mathbb{N}$.

(iii) *Lamplighter groups* $\mathbb{Z}_p \wr \mathbb{Z}$ *for any* $p \in \mathbb{N}$.

(iv) *Baumslag–Solitar groups* $\mathrm{BS}(p,q) = \langle a, t \mid ta^p t^{-1} = a^q \rangle$ *for any* $p, q \in \mathbb{N}$.

The paper is organized as follows. In Section 2, we list out preliminary definitions and results. In Section 3, we discuss how the tame cuts are connected to weak amenability and rapid decay property and provide the first examples. In Section 4, we prove Theorems A and B. In Section 5, we prove Theorem C. In Section 6, we prove Theorem D.

2. Preliminaries

2.1. Group algebras

Let G be a locally compact group. We fix a Haar measure dx on G. The left G-translations on itself induces *the (left) regular representation* $\lambda : G \to \mathcal{U}(L^2(G))$. This gives rise to the $*$-representation

$$\lambda : L^1(G) \to \mathcal{B}\big(L^2(G)\big), \quad \lambda(f)\xi = f * \xi.$$

The *(left) reduced C^*-algebra* $C^*_\lambda(G)$ of G is the completion of $\lambda(L^1(G))$ in $\mathcal{B}(L^2(G))$ with respect to the operator norm. The *group von Neumann algebra* $L(G)$ is the completion of $\lambda(L^1(G))$ in $\mathcal{B}(L^2(G))$ with respect to the weak*-topology. Another important algebra is the *Fourier algebra*,

$$A(G) = \Big\{\varphi = \sum_i \xi_i * \eta_i \mid \xi_i, \eta_i \in L^2(G),\ \sum_i \|\xi_i\|_2 \|\eta_i\|_2 < \infty\Big\}.$$

This is a commutative Banach algebra endowed with pointwise multiplication and the norm $\|\varphi\|_A = \inf \sum_i \|\xi_i\|_2 \|\eta_i\|_2$, where ξ_i's and η_i's run in $L^2(G)$ under the condition $\varphi = \sum_i \xi_i * \eta_i$. It turns out that the unique Banach predual of $L(G)$ is the Fourier algebra $A(G)$. The duality is given by $\langle \lambda(f), \varphi \rangle = \int_G f(x)\varphi(x)dx$ for all $f \in L^1(G)$ and $\varphi \in A(G)$. We refer readers to [8] for further properties of these algebras.

Let φ be a function on G, and F any set of functions on G. We say that φ multiplies F if $\varphi F \subseteq F$. A function $\varphi \in C_\mathrm{b}(G)$ is called a *Fourier multiplier* if it multiplies the Fourier algebra $A(G)$. By the closed graph theorem, the multiplication operator $m_\varphi : A(G) \to A(G)$ corresponding to a Fourier multiplier φ is continuous, and the operator norm $\|\varphi\|_{MA} = \sup\{\|\varphi\psi\|_A \mid \psi \in A(G),\ \|\psi\|_A = 1\}$ is well defined. It turns out that the space $MA(G)$ of all Fourier multipliers is a Banach algebra with pointwise multiplication. The duality and restriction arguments show that the following are equivalent [6, Proposition 1.2]:

(i) $\varphi \in MA(G)$.

(ii) m_φ extends to a weak*-continuous map $m^*_\varphi : L(G) \to L(G)$.

(iii) m_φ extends to a norm-continuous map $m'_\varphi : C^*_\lambda(G) \to C^*_\lambda(G)$.

A Fourier multiplier $\varphi \in MA(G)$ is said to be *completely bounded* if m_φ^* (or equivalently m'_φ) is completely bounded; that is, the quantity

$$\|\varphi\|_{M_0A} = \sup_{n\in\mathbb{N}} \left\| m_\varphi^* \otimes I_n : L(G) \otimes M_n(\mathbb{C}) \to L(G) \otimes M_n(\mathbb{C}) \right\|$$

is finite. Denote by $M_0A(G)$ all completely bounded Fourier multipliers. Again, it is a Banach algebra with pointwise multiplication. Compared to the usual Fourier multipliers, the characterizations of completely bounded Fourier multipliers are much more rich. Below, we list out some important results that we will use.

Theorem 2.1 ([2, 6]). *Let φ be a Fourier multiplier of G. Put $K = \mathrm{SU}(2)$. Then, the following conditions are equivalent:*

- (i) $\varphi \in M_0A(G)$ *with* $\|\varphi\|_{M_0A(G)} \le C$.
- (ii) $\varphi \otimes \mathbb{1}_K \in MA(G \times K)$ *with* $\|\varphi \otimes \mathbb{1}_K\|_{MA(G\times K)} \le C$.
- (iii) *For every locally compact group H, we have $\varphi \otimes \mathbb{1}_H \in MA(G \times H)$ with* $\|\varphi \otimes \mathbb{1}_H\|_{MA(G\times H)} \le C$.
- (iv) *There exist bounded continuous maps P, Q from G to a Hilbert space $\mathcal{H}$ with* $\varphi(y^{-1}x) = \langle P(x), Q(y)\rangle$ *for all $x, y \in G$ and* $\sup_{x,y\in G} \|P(x)\|\|Q(y)\| \le C$.

Theorem 2.2 ([18,22]). *When G is amenable, the Banach algebras $M_0A(G)$ and $MA(G)$ coincide, and the inclusion $A(G) \to MA(G)$ is isometric.*

2.2. Weak amenability

Definition 2.3. A locally compact group G is *weakly amenable* if there is a net (φ_i) of compactly supported functions such that

- (i) $\varphi_i \to 1$ uniformly on all compact subsets of G,
- (ii) there is a constant $C > 0$ such that $\|\varphi_i\|_{M_0A} \le C$.

The best upper bound Λ_G is called the *Cowling–Haagerup constant*.

Locally compact amenable groups are trivially weakly amenable. The following are some more weakly amenable groups: the free group F_2, more generally Gromov's hyperbolic groups, groups acting properly on finite-dimensional CAT(0) cubical complexes, and rank one simple Lie groups. The main importance of weak amenability is that the Cowling–Haagerup constant is stable under Measure equivalence and W*-equivalence. We refer to [5, 6, 11] for more information on weak amenability.

In the following, we mention two results that we will use.

Lemma 2.4 ([5, Proposition 1.6]). *Suppose that K is a compact subgroup of a locally compact group G. Let $\varphi \in M_0A(G)$ (resp. $\varphi \in MA(G)$), and $\dot\varphi$ denotes the function obtained by averaging φ over the double cosets KxK ($x \in G$); that is,*

$$\dot\varphi(x) = \int_K \int_K \varphi(k_1 x k_2) dk_1 dk_2 \quad \textit{for all } x \in G,$$

where dk_1 and dk_2 are normalized Haar measures on K. Then, $\dot{\varphi} \in M_0A(G)$ and $\|\dot{\varphi}\|_{M_0A} \le \|\varphi\|_{M_0A}$ (resp. $\dot{\varphi} \in MA(G)$ and $\|\dot{\varphi}\|_{MA} \le \|\varphi\|_{MA}$). Furthermore, if G admits a closed amenable subgroup P such that $G = PK$, then

$$\|\dot{\varphi}\|_{M_0A(G)} = \|\dot{\varphi}\|_{MA(G)} = \|\dot{\varphi}|_P\|_{MA(P)}.$$

Theorem 2.5 ([16, Lemma 5.2, case $p = \infty$ and $\varepsilon = 1/3$]). *Let $G = \mathrm{SL}(3,\mathbb{R})$, and $K = \mathrm{SO}(3,\mathbb{R})$. There is a constant $C > 0$ such that for any K-bi-invariant function $\varphi \in C_0(G)$ and any $t > 0$, we have*

$$Ce^{t/3}\left|\varphi\begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t}\end{pmatrix}\right| \le \|\varphi\|_{MA}. \tag{2.1}$$

In the original version [16, Lemma 5.2], the right-hand side of the above inequality is given by the multiplier norm $\|\varphi\|_{MS^p(L^2(G))}$ on p-Schatten class. In the case of $p = \infty$, the latter norm coincides with the completely bounded Fourier multiplier norm $\|\varphi\|_{M_0A}$ as shown in [16, Theorem 1.7]. Recall that the group P of upper triangle matrices of $G = \mathrm{SL}(3,\mathbb{R})$ is amenable, and set theoretically, we have $G = PK$. Moreover, since we assume that φ is K-bi-invariant, we have $\dot{\varphi} = \varphi$ and $\|\varphi\|_{M_0A} = \|\varphi\|_{MA}$ by Lemma 2.4. Therefore, MA-norm is valid in (2.1).

The same result was proven for $\mathrm{Sp}(2,\mathbb{R})$ in [7, Proposition 3.2]. Combined together, these two inequalities show that all higher rank simple Lie groups with finite center are not weakly amenable.

2.3. Rapid Decay property (RD)

First observed by Uffe Haagerup in [10] for free groups, and later developed by Paul Jolissaint in [13], Rapid Decay property (RD) took significant attention of mathematicians for being a part of tools to establish important examples of groups satisfying Baum–Connes conjecture. The class of groups with RD is quite large and includes Gromov's hyperbolic groups, co-compact cubical CAT(0) groups, Coxeter groups, Mapping class groups, Braid groups, large type Artin groups, 3-manifold groups not containing Sol, Wise non-Hopfian group, some small cancelation groups, and uniform lattices of $\mathrm{SL}(3,\mathbb{R})$. We refer the readers to [4] for a comprehensive survey.

Recall that a *length function* $\ell : G \to \mathbb{R}_+$ on a locally compact group G is a measurable function such that $\ell(e) = 0$, $\ell(x) = \ell(x^{-1})$, and $\ell(xy) \le \ell(x) + \ell(y)$ for all $x, y \in G$, where $e \in G$ is the neutral element. The ball of radius $n \ge 0$ with respect to ℓ is the set $B_n = \{x \in G \mid \ell(x) \le n\}$. We say that a length function ℓ_1 *dominates* another length function ℓ_2 if there is a constant $c > 0$ such that $\ell_2(x) \le c\ell_1(x) + c$ for all $x \in G$. If G is generated by a compact subset $K \subseteq G$, then the *word length function*

$$\ell_K(x) = \min\left\{n \in \mathbb{Z}_+ \mid x \in (K \cup K^{-1})^n\right\}$$

dominates any other length functions. If two length functions dominate each other, we say that they are *equivalent*. Let us provide an interesting example of equivalent length function. Consider the group $G = \mathrm{SL}(3, \mathbb{R})$. Take any *word length function* ℓ_C associated with a compact generating subset $C \subseteq G$. Another length function on G can be defined as $L(x) = \log \|x\| + \log \|x^{-1}\|$, where $\|x\|$ is the operator norm of x acting on $\ell^2(\{1, 2, 3\})$. It turns out that ℓ_C and L are equivalent. Let Γ be any uniform lattice in G. Since Γ satisfies Kazhdan's property (T), there is a finite generating set $S \subseteq \Gamma$. The length functions ℓ_S, $\ell_C|_S$, and $L|_S$ on Γ are all equivalent on Γ [19, 20].

Definition 2.6. A locally compact group G has RD with respect to a length function ℓ if there exist non-negative constants C and a such that for all $n \geq 0$ and $f \in C_c(G)$ with $\mathrm{supp}(f) \subseteq B_n$, we have

$$\|\lambda(f)\| \leq C(1 + n)^a \|f\|_2.$$

We highlight one of Lafforgue's results, which was the last step to prove that uniform lattices in $\mathrm{SL}(3, \mathbb{R})$ satisfy Baum–Connes' conjecture.

Theorem 2.7 ([15]). *Uniform lattices in* $\mathrm{SL}(3, \mathbb{R})$ *have RD.*

This theorem is the main ingredient to prove Theorem A.

2.4. Property $(T_{\mathrm{Schur}}, G, K)$

Property $(T_{\mathrm{Schur}}, G, K)$ was introduced in [17] as an analogue of Lafforgue's property (T_{Schur}). Suppose that G is a reductive group over a local field, K is its maximal compact subgroup, and Γ is a lattice of G satisfying property $(T_{\mathrm{Schur}}, G, K)$. Then, Liao showed that all known methods to prove the Baum–Connes conjecture fail for Γ.

Definition 2.8. Let G be a locally compact group, K a compact subgroup, H a closed subgroup of G, and ℓ a proper length function of G. For $n \in \mathbb{N}$ and $f \in C(G)$, define the quantity

$$\|f\|_{MA(H,G,\ell,n)} = \sup\left\{\left\|\lambda_H(f|_H\varphi)\right\| \mid \varphi \in C_c(H),\ \mathrm{supp}(\varphi) \subseteq B_n,\ \left\|\lambda_H(\varphi)\right\| \leq 1\right\}.$$

When G and ℓ are already fixed, we also write $\|f\|_{MA(H,G,\ell,n)} = \|f\|_{MA(H,n)}$. We say that H has *property* $(T_{\mathrm{Schur}}, G, K, \ell)$ if there exist a positive constant $s > 0$ and a function $\phi \in C_0(G)$ vanishing at infinity such that for any $D > 0$ and K-bi-invariant function $\varphi \in C(G)$ with

$$\|\varphi\|_{MA(H,n)} \leq De^{sn} \quad \text{for all } n \in \mathbb{N},$$

there exists a limit $\varphi_\infty \in \mathbb{C}$ to which φ tends uniformly rapidly

$$\left|\varphi(x) - \varphi_\infty\right| \leq D\phi(x) \quad \text{for all } x \in G.$$

The only known example given in [17] is as follows. Let $\mathbb{F}_q$ be a finite field of characteristic different from 2 with cardinality q. Let G be the symplectic group $\mathrm{Sp}(4, \mathbb{F}_q((\pi)))$ over the local field $\mathbb{F}_q((\pi))$ and $K = \mathrm{Sp}(4, \mathbb{F}_q[[\pi]])$ the maximal compact subgroup of G.

Let Γ be the non-uniform lattice $\mathrm{Sp}(4, \mathbb{F}_q[\pi^{-1}])$ in G. Let $H < \Gamma$ be the subgroup consisting of the elements of the form

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \in \Gamma.$$

For $i, j \in \mathbb{N}_0$, denote $D(i, j) = \mathrm{diag}(\pi^{-i}, \pi^{-j}, \pi^{j}, \pi^{i})$. By Cartan's decomposition theorem, every element $x \in G$ can be written as $x = kD(i, j)k'$ for some $k, k' \in K$ and a unique $(i, j) \in \mathbb{N}_0^2$ with $i \geq j$. Moreover, the length function $\ell : kD(i, j)k' \mapsto i + j$ is equivalent to the word length function of G, and even its restriction to the lattice Γ is equivalent to the word length function of Γ. Then, H and Γ have property $(T_{\mathrm{Schur}}, G, K, \ell)$.

It seems that Kazhdan's property (T) has less to do with property $(T_{\mathrm{Schur}}, G, K, \ell)$. For example, compact groups obviously satisfy property (T), while it is not the case for property $(T_{\mathrm{Schur}}, G, K, \ell)$ as the following lemma shows.

Lemma 2.9. *Let G be an unbounded locally compact group endowed with a proper length function ℓ. Suppose that K and H are compact subgroups of G. Then, H does not have property $(T_{\mathrm{Schur}}, G, K, \ell)$.*

Proof. Assume, by contradiction, that H has property $(T_{\mathrm{Schur}}, G, K, \ell)$. Let $s > 0$ and $\phi \in C_0(G)$ be as in Definition 2.8. Define

$$L_1(x) = \begin{cases} 0, & \text{if } \ell(x) = 0, \\ \lfloor \ell(x) \rfloor + 1, & \text{otherwise,} \end{cases}$$

$$L_2(x) = \int_K L_1(kxk^{-1})dk,$$

$$L_3(x) = \min\left\{L_2(k_1xk_2) \mid k_1, k_2 \in K\right\},$$

$$L_4(x) = \begin{cases} 0, & \text{if } x \in K, \\ 1, & \text{otherwise,} \end{cases}$$

for all $x \in G$. This construction is from [13, Lemma 2.1.3]. Note that $L_3 + L_4$ is a length function equivalent to ℓ. If necessary, replacing ℓ by $L_3 + L_4$, we can assume that ℓ takes 0 on K so that the balls are K-bi-invariant. Choose a large enough $r \in \mathbb{N}$ such that B_r has a non-empty interior and contains H. For each $m \geq r$, construct a non-negative function $f_m \in A(G)$ such that f_m takes 1 on B_m and 0 outside B_{2m}^0 using Eymard's trick [8, Lemma 3.2]. Then, $\psi_m = f_{4m} - f_m$ is a non-negative compactly supported function in $A(G)$ such that $\psi_m|_H = 0$ and $\psi_m(x) \neq 0$ for some $x \in G$ with $\ell(x) \geq m$. We use $K \times K$ double averaging and normalization on ψ_m in order to have a K-bi-invariant function $\varphi_m \in C_c(G)$ such that $\varphi_m|_H = 0$ and $\varphi_m(x) = 1$ for some $x \in G$ with $\ell(x) \geq m$. Now, we have $\|\varphi_m\|_{MA(H,n)} = 0 \leq e^{sn}$ for all $n \in \mathbb{N}$ and $m \geq r$. It follows that $|\varphi_m(x)| \leq \phi(x)$

for all $x \in G$ and $m \geq r$. Taking $\lim_{x\to\infty} \sup_{m\geq r}$ on the left- and right-hand side of the inequality, we get a desired contradiction. ∎

In [17, Proposition 2.3], it was proven that if a discrete subgroup Γ of G has property $(T_{\mathrm{Schur}}, G, K, \ell)$, then (Γ, ℓ) does not have RD. As its analogue, we have Theorem C.

3. Groups with tame cuts

In this section, we will give the first examples of groups with tame cuts by investigating its connection to weak amenability and RD. If there exists a proper length function ℓ for which (G, ℓ) has (completely bounded) [characteristic] tame cuts, we simply say that G has (completely bounded) [characteristic] tame cuts. It is also worth noting that when G is amenable, G has (characteristic) tame cuts if and only if G has completely bounded (characteristic) tame cuts since the Banach algebras $M_0A(G)$ and $MA(G)$ coincide in that case. Here are the first examples.

Proposition 3.1. *The infinite cyclic group* $\Gamma = \mathbb{Z}$ *has characteristic tame cuts with respect to the logarithmic length function* $\log(1 + |\cdot|)$ *but does not with respect to the double logarithmic length function* $\log(1 + \log(1 + |\cdot|))$.

Proof. We use the Fourier transform and bounds for $L^1(\mathbb{T})$-norm of trigonometric polynomials.

(i) [14, p. 71, Exercise 1.1] For $n \in \mathbb{N}$, we have

$$\begin{aligned}\|\mathbb{1}_{\mathbb{Z}\cap[-n,n]}\|_{M_0A} = \|\mathbb{1}_{\mathbb{Z}\cap[-n,n]}\|_A &= \left\|\mathcal{F}(\mathbb{1}_{\mathbb{Z}\cap[-n,n]})\right\|_{L^1(\mathbb{T})} \\ &= \left\|\sum_{k=-n}^{n} e^{i2\pi kt}\right\|_{L^1(\mathbb{T})} = \frac{4}{\pi}\log(n) + O(1). \qquad (3.1)\end{aligned}$$

This shows that the sequence $(\mathbb{1}_{3^n})_{n\in\mathbb{N}}$ forms characteristic tame cuts for $(\mathbb{Z}, \log(1 + |\cdot|))$.

(ii) [21, Theorem 2] There is a constant $C > 0$ such that for any finite subset $B \subseteq \mathbb{Z}$, we have

$$\|\mathbb{1}_B\|_{M_0A} = \left\|\sum_{k\in B} e^{i2\pi kt}\right\|_{L^1(\mathbb{T})} \geq C \log|B|. \qquad (3.2)$$

If φ_n is a characteristic function on $\mathbb{Z}$ such that its support is finite and contains the ball B_n of radius n, then the set $\mathrm{supp}(\varphi_n)$ would contain at least $e^{e^n} \approx |B_n|$ elements. Inequality (3.2) shows that $\|\varphi_n\|_{M_0A} \geq Ce^n$. Thus, there are no characteristic tame cuts for $(\mathbb{Z}, \log(1 + \log(1 + |\cdot|)))$. ∎

Remark 3.2. The sequence $(\mathbb{1}_{\mathbb{Z}\cap[-n,n]})$ gives characteristic tame cuts for $(\mathbb{Z}, \log(1+|\cdot|))$, and the multiplier norms grow linearly. Inequality (3.2) shows that this linear rate is actually the best.

Proposition 3.3. *Let G be a locally compact group, and let ℓ be a proper length function on it. Then, (Γ, ℓ) has completely bounded tame cuts with $a = 0$ (cf. Definition 1.1) if and only if G is weakly amenable.*

Proof. See [5, Proposition 1.1]. ■

Remark 3.4. Similar results can be attained for other approximation properties such as multiplier approximation property and n-positive approximation property.

Remark 3.5. By Propositions 3.1 and 3.3, the group $(\mathbb{Z}, \log(1 + \log(1 + |\cdot|)))$ has tame cuts but no characteristic tame cuts.

Proposition 3.6. *Let Γ be a discrete group satisfying RD with respect to a proper length function ℓ. Then, (Γ, ℓ) has characteristic tame cuts.*

Proof. For any non-zero function $f \in C_c(\Gamma)$, we have

$$\left\|\lambda(f)\right\| \geq \frac{\langle \lambda(f)\delta_e | f \rangle}{\|f\|_2} = \|f\|_2.$$

Put $\varphi_n = \mathbb{1}_{B_n}$. Then, we have

$$\left\|\lambda(\varphi_n f)\right\| \leq C(1+n)^a \|\varphi_n f\|_2 \leq C(1+n)^a \|f\|_2 \leq C(1+n)^a \left\|\lambda(f)\right\|$$

for all $f \in C_c(\Gamma)$. It follows that $\|\varphi_n\|_{MA} \leq C(1+n)^a$, completing the proof. ■

4. Proof of Theorems A and B

As a motivation to investigate tame cuts, we provide two applications. The first one is about the continuity of the induction map. The following lemma shows the importance of induction map and is used to prove that amenability, a-T-menability, weak-amenability, and Kazhdan's property (T) are inherited by lattices.

Lemma 4.1 ([11, Lemma 2.1]). *Let Γ be a lattice in a locally compact group G. Let $\sigma : G/\Gamma \to G$ be a Borel cross-section, and $\Omega \subseteq G$ its image. Denote by μ_Γ the counting measure on G corresponding to Γ. Then,*

$$\Phi : \ell^\infty(\Gamma) \to C_b(G), \quad \varphi \mapsto \widehat{\varphi} = \mathbb{1}_\Omega * (\varphi \mu_\Gamma) * \widetilde{\mathbb{1}}_\Omega$$

is well defined. Moreover, the restrictions of this map give norm decreasing linear maps

$$\Phi : A(\Gamma) \to A(G) \quad \text{and} \quad \Phi : M_0A(\Gamma) \to M_0A(G).$$

Moreover, if φ is a positive definite function on Γ, so is $\widehat{\varphi}$ on G.

As we have a disjoint union

$$G = \bigsqcup_{\gamma \in \Gamma} \Omega\gamma,$$

the map $\gamma : G \to \Gamma$ such that $x \in \Omega\gamma(x)$ is well defined, and we have

$$\widehat{\varphi}(x) = \int_\Omega \varphi\big(\gamma(x\omega)\big) d\omega \quad \text{for all } x \in G,$$

where $d\omega$ is the normalized Haar measure on G such that $\int_\Omega d\omega = 1$.

Proof of Theorem B. Recall that the length functions $\ell_S, \ell_C|_\Gamma$, and $L|_\Gamma$ are all equivalent on Γ (cf. Section 2.3). We choose L as the main length function on both Γ and G.

By contradiction, suppose that the map $\Phi : MA(\Gamma) \to MA(G)$ is well defined and bounded. By Theorem 2.7, Γ has RD and a fortiori characteristic tame cuts, so there exist finitely supported characteristic functions φ_n such that

$$\|\varphi_n\|_{MA(\Gamma)} \leq Cn^a \quad \text{and} \quad \varphi_n|_{B_n} = 1 \quad \text{for all } n \in \mathbb{N}.$$

By continuity of Φ, there is a constant $C' > 0$ such that

$$\|\widehat{\varphi}_n\|_{MA(G)} \leq C'n^a \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 2.4, the K-K-averaging $\dot{\widehat{\varphi}}_n$ of $\widehat{\varphi}_n$ is a K-bi-invariant Fourier multiplier of G with

$$\|\dot{\widehat{\varphi}}_n\|_{MA(G)} \leq \|\widehat{\varphi}_n\|_{MA(G)} \leq C'n^a \quad \text{for all } n \in \mathbb{N}.$$

Applying the rigidity inequality (2.1) on $\dot{\widehat{\varphi}}_n$, we get

$$C''e^{t/3} \left| \dot{\widehat{\varphi}}_n \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \right| \leq \|\dot{\widehat{\varphi}}_n\|_{MA} \leq C'n^a \quad \text{for all } n \in \mathbb{N},\ t \in \mathbb{R}_+. \tag{4.1}$$

Put $c = \max\{L(\omega) \mid \omega \in \Omega\}$ and choose $t = n/4 - c$ so that

$$L\big(\operatorname{diag}(e^t, 1, e^{-t})\big) = 2t < n - 2c.$$

Recall that we have

$$\dot{\widehat{\varphi}}_n(x) = \int_K \int_K \int_\Omega \varphi_n\big(\gamma(k_1xk_2\omega)\big) d\omega dk_1 dk_2.$$

By construction, note that

$$L\big(\gamma(k_1xk_2\omega)\big) = L(\omega'k_1xk_2\omega) \leq L(k_1xk_2) + L(\omega') + L(\omega) \leq 2c + L(x)$$

for some $\omega' \in \Omega^{-1}$. Thus, if $L(x) \leq n - 2c$, we have $\dot{\widehat{\varphi}}_n(x) = 1$. Therefore, the left-hand side of (4.1) grows exponentially while the right-hand side is polynomial, which gives the desired contradiction. ∎

Remark 4.2. We do not know if the map $\Phi : MA(\Gamma) \to MA(G)$ is well defined in general.

Proof of Theorem A. Suppose $M_0A(\Gamma) = MA(\Gamma)$. Since the inclusion

$$M_0A(\Gamma) \to MA(\Gamma)$$

is a contraction between two Banach spaces, two norms $\|\cdot\|_{MA}$ and $\|\cdot\|_{M_0A}$ are equivalent by the closed graph theorem applied on the inverse map.

Let us use the functions φ_n, $\widehat{\varphi}_n$, and $\dot{\widehat{\varphi}}_n$ from Theorem B. Since two norms are equivalent and (φ_n) forms tame cuts, this sequence is also completely bounded tame cuts. It follows that there are constants $C, a \geq 0$ such that

$$\|\varphi_n\|_{M_0A} \leq Cn^a \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 4.1, we have $\|\dot{\widehat{\varphi}}_n\|_{M_0A} \leq \|\widehat{\varphi}_n\|_{M_0A} \leq Cn^a$. Again, the rigidity inequality on $\dot{\widehat{\varphi}}_n$ gives a desired contradiction for the same choice of t as in Theorem B. ■

5. Proof of Theorem C

Definition 5.1. Let G be a locally compact group, H a closed subgroup of G, and K a compact subgroup of G. A function $\varphi \in C(H)$ is said to be *K-bi-invariant* if there is a K-bi-invariant continuous function on G whose restriction on H is exactly φ.

Proof of Theorem C. We prove by contradiction. Assume that there exist K-bi-invariant tame cuts $(\varphi_m)_{m\in\mathbb{N}}$ for $(H, \ell|_H)$. There are constants $C, a \geq 0$ such that

$$\|\varphi_m\|_{MA(H)} \leq Cm^a, \tag{5.1}$$

$$\varphi_m|_{B_m} \equiv 1 \tag{5.2}$$

for all $m \in \mathbb{N}$. Take any function $f \in C_c(H)$ with $\operatorname{supp}(f) \subseteq B_n$. If $m \geq n$, the Fourier multiplier φ_m acts trivially on f, so $\|\lambda_H(\varphi_m f)\| = \|\lambda_H(f)\|$. If $m < n$, then we have

$$\left\|\lambda_H(\varphi_m f)\right\| \leq Cm^a \left\|\lambda_H(f)\right\| \leq Cn^a \left\|\lambda_H(f)\right\|$$

by (5.1). Unifying these two cases, if we denote by $\varphi'_m \in C(G)$ a K-bi-invariant extension of φ_m, we get the inequality $\|\varphi'_m\|_{MA(H,n)} \leq Cn^a$ for all $n, m \in \mathbb{N}$. Let $s > 0$ and $\phi \in C_0(G)$ be from property $(T_{\text{Schur}}, G, K, \ell)$. Put $D = \sup_{n\in\mathbb{N}} Cn^a e^{-sn}$ so that we get

$$\|\varphi'_m\|_{MA(H,n)} \leq De^{sn} \quad \text{for all } m, n \in \mathbb{N}.$$

By property $(T_{\text{Schur}}, G, K, \ell)$, we get

$$\left|\varphi_m(x)\right| = \left|\varphi'_m(x)\right| \leq D\phi(x) \quad \text{for all } x \in H,\ m \in \mathbb{N}. \tag{5.3}$$

Now, if we take the sequential limits $\lim_{x\to\infty} \lim_{m\to\infty}$ on (5.3), the left-hand side goes to 1 whereas the right-hand side goes to 0. This gives a desired contradiction. ■

Corollary 5.2. *If G is unbounded and has property $(T_{\mathrm{Schur}}, G, K, \ell)$, then (G, ℓ) does not have tame cuts.*

Proof. By Lemma 2.4, the K-bi-averaging process does not increase the multiplier norm. Thus, we can assume that tame cuts, if they exist, are K-bi-invariant. Now, the rest follows from the theorem by choosing $H = G$. ∎

Suppose that G is a finitely generated infinite group and H is a finitely generated subgroup of G. Recall that H is *polynomially distorted* in G if there exists $k \geq 0$ such that $\ell_H(x) \leq k\ell_G(x)^k + k$ for all $x \in H$, where ℓ_G and ℓ_H are the word length functions of G and H, respectively.

Corollary 5.3. *Suppose that G is a finitely generated infinite group and H is a finitely generated subgroup of G that is at most polynomially distorted in G. If H has property $(T_{\mathrm{Schur}}, G, \{e\}, \ell_G)$, then (H, ℓ_H) does not have tame cuts.*

Even though these two corollaries suggest ways of constructing an example of a finitely generated group without tame cuts with respect to the word length function, we do not have any example fitting to the corollaries above.

6. Proof of Theorem D

In this section, we prove Theorem D case by case. We need the following three lemmas.

Lemma 6.1. *Restrictions of (completely bounded) [characteristic] tame cuts on a closed subgroup are again (completely bounded) [characteristic] tame cuts with respect to the restricted length function.*

Proof. It directly follows from [6, Proposition 1.12] which states that if G is a locally compact group and H is its closed subgroup, then the restriction maps

$$MA(G) \to MA(H) \quad \text{and} \quad M_0A(G) \to M_0A(H)$$

are norm decreasing. ∎

Lemma 6.2. *Let G be a locally compact group and H an open subgroup. The extensions by identity on H and zero on $G \setminus H$*

$$A(H) \to A(G), \quad MA(H) \to MA(G), \quad \textit{and} \quad M_0A(H) \to M_0A(G)$$

are isometric.

Proof. The case $A(H) \to A(G)$ is proven in [8, Lemma 3.21]. Denote by $\iota : MA(H) \to MA(G)$ the trivial extension. The calculation

$$\left\|\iota(\varphi)\right\|_{MA(G)} = \sup\left\{\left\|\iota(\varphi)\psi\right\|_{A(G)} \mid \psi \in A(G),\ \|\psi\|_{A(G)} \leq 1\right\}$$

$$= \sup\left\{\|\varphi\psi|_H\|_{A(H)} \mid \psi \in A(G),\ \|\psi\|_{A(G)} \leq 1\right\}$$
$$= \sup\left\{\|\varphi\psi'\|_{A(H)} \mid \psi' \in A(H),\ \|\psi'\|_{A(H)} \leq 1\right\}$$
$$= \|\varphi\|_{MA(H)}$$

shows that ι is isometric. The case $M_0A(H) \to M_0A(G)$ is done similarly by considering $H \times \mathrm{SU}(2)$ and $G \times \mathrm{SU}(2)$. ∎

Lemma 6.3. *Let Γ be a discrete group, $\ell : \Gamma \to \mathbb{Z}_+$ a proper length function, and H a subgroup of Γ. Assume that $(H, \ell|_H)$ has (completely bounded) [characteristic] tame cuts and that H has polynomial co-growth. Then, Γ has (completely bounded) [characteristic] tame cuts.*

Here, we say that H has polynomial co-growth with respect to ℓ if there exist constants $C', b \geq 0$ such that

$$\#\left\{xH \mid x \in \Gamma,\ \ell(x) \leq n\right\} \leq C'n^b \quad \text{for all } n \in \mathbb{N}.$$

When H is normal in G, it is equivalent to say that the quotient group G/H has polynomial growth with respect to the length function given by $\ell^H(xH) = \min\{\ell(xh) \mid h \in H\}$.

Proof of Lemma 6.3. By hypothesis, there exist constants $C, a \geq 0$ and a sequence of finitely supported functions (ψ_n) on H with

$$\psi_n|_{B_n \cap H} \equiv 1 \quad \text{and} \quad \|\psi_n\|_{MA(H)} \leq Cn^a \quad \text{for all } n \in \mathbb{N}. \tag{6.1}$$

We identify ψ_n with the function on Γ extending ψ_n by 0 on $G \setminus H$. By Lemma 6.2,

$$\|\psi_n\|_{MA(H)} = \|\psi_n\|_{MA(\Gamma)}.$$

Choose the cross-section $\sigma : \Gamma/H \to \Gamma$ defined by $\ell(\sigma(xH)) = \min_{h \in H} \ell(xh)$. We can choose such σ since the length function ℓ takes value in $\mathbb{Z}_+$. Then, by polynomial co-growth, the sets

$$S_n = \left\{y \in \sigma(\Gamma/H) \mid \ell(y) \leq n\right\}$$

grow at polynomial rate, say $C'n^b$. Also, note that

$$B_n \subseteq S_n(B_{2n} \cap H) \quad \text{for all } n \in \mathbb{N}. \tag{6.2}$$

To see that, take any $x \in B_n$. By the choice of the cross-section, we have $\sigma(xH) \in S_n \subseteq B_n$, and since x and $\sigma(xH)$ represent the same class, we have $\sigma(xH)^{-1}x \in B_{2n} \cap H$. Now, $x \in \sigma(xH)(B_{2n} \cap H) \subseteq S_n(B_{2n} \cap H)$.

For every $n \in \mathbb{N}$, define a finitely supported function by $\varphi_n = \sum_{y \in S_n} \delta_y * \psi_{2n}$ on Γ. The conditions (6.2) and (6.1) show that $\varphi_n|_{B_n} \equiv 1$. Since Γ acts on $MA(\Gamma)$ by isometries, we have

$$\|\varphi_n\|_{MA(\Gamma)} \leq \sum_{y \in S_n} \|\delta_y * \psi_{2n}\|_{MA(\Gamma)} \leq C''n^b \|\psi_{2n}\|_{MA(\Gamma)} \leq CC''2^a n^{a+b}.$$

This completes the proof for tame cuts. The same proof works for the other cases. ∎

Proposition 6.4. *Let F be a finite group, and let P be a finitely generated group with polynomial growth. Then, the wreath product $F \wr P$ has characteristic tame cuts. In particular, the Lamplighter group $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$ has characteristic tame cuts.*

Proof. Recall that $F \wr P$ is the semidirect product group $(\bigoplus_{n \in P} F) \rtimes_\rho P$ where ρ acts by shifts. We choose a word length function ℓ on Γ associated with any finite generating set. By Lemma 6.3, it is enough to show that $(H, \ell|_H)$ has characteristic tame cuts, where $H = \bigoplus_{n \in P} F$.

For $n \in \mathbb{N}$, define H_n, the subgroup of H generated by $B_n \cap H$. Since H is locally finite, H_n is a finite group. Define $\varphi_n = \mathbb{1}_{H_n}$. By Lemma 6.2, we have $\|\varphi_n\|_{M_0 A(H)} = \|\varphi_n\|_{M_0 A(H_n)} = 1$ because the multiplier $\varphi_n = \mathbb{1}_{H_n}$ acts on $A(H_n)$ trivially. The support of φ_n is clearly finite and contains the relative ball $\{x \in H \mid \ell(x) \leq n\}$ of radius n. ∎

Proposition 6.5. *For any coprime $p, q \in \mathbb{N}$, the group $\Gamma = \mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{p}{q}} \mathbb{Z}$ has characteristic tame cuts.*

Proof. Recall that the group $\Gamma = \mathbb{Z}[\frac{1}{pq}] \rtimes_{\frac{p}{q}} \mathbb{Z}$ is isomorphic to the subgroup

$$\Gamma \cong \left\{ \begin{pmatrix} (\frac{p}{q})^k & P \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z},\ P \in \mathbb{Z}\left[\frac{1}{pq}\right] \right\}$$

of $\mathrm{GL}_2(\mathbb{R})$ generated by

$$S = \left\{ s = \begin{pmatrix} \frac{p}{q} & 0 \\ 0 & 1 \end{pmatrix},\ t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let ℓ be the associated word length function on Γ. We only need to prove that the subgroup

$$H \cong \left\{ \begin{pmatrix} 1 & P \\ 0 & 1 \end{pmatrix} \mid P \in \mathbb{Z}\left[\frac{1}{pq}\right] \right\} \cong \mathbb{Z}\left[\frac{1}{pq}\right]$$

has characteristic tame cuts with respect to the restricted length function $\ell|_H$.

Suppose that $x = \left(\begin{smallmatrix} 1 & P \\ 0 & 1 \end{smallmatrix}\right) \in H$ and $\ell(x) \leq n$. Then, there is a decomposition

$$x = s_1, \dots, s_n$$

with

$$s_i = \begin{pmatrix} (\frac{p}{q})^{\varepsilon_i} & \delta_i \\ 0 & 1 \end{pmatrix} \in S^{\pm 1},$$

$\varepsilon_i, \delta_i \in \{-1, 0, 1\}$, $1 \leq i \leq n$ and $\sum \varepsilon_i = 0$. Moreover, we have

$$\begin{aligned} P &= \delta_1 + \delta_2 \left(\frac{p}{q}\right)^{\varepsilon_1} + \delta_3 \left(\frac{p}{q}\right)^{\varepsilon_1 + \varepsilon_2} + \dots + \delta_n \left(\frac{p}{q}\right)^{\varepsilon_1 + \dots + \varepsilon_{n-1}} \\ &= \frac{\sum_{i=1}^n \delta_i\, p^{n + \sum_{j=1}^{i-1} \varepsilon_j} q^{n - \sum_{j=1}^{i-1} \varepsilon_j}}{q^n p^n}. \end{aligned}$$

Evidently, the cyclic subgroup H_n generated by

$$x_n = \begin{pmatrix} 1 & \frac{1}{q^n p^n} \\ 0 & 1 \end{pmatrix} \in H$$

contains the relative ball $B_n \cap H = \{x \in H \mid \ell(x) \leq n\}$. Moreover, for any element $x \in B_n \cap H$, its power in $H_n = \langle x_n \rangle$ has an upper bound

$$|x| = \left| \sum_{i=1}^{n} \delta_i \, p^{n+\sum_{j=1}^{i-1} \varepsilon_j} q^{n-\sum_{j=1}^{i-1} \varepsilon_j} \right| \leq n q^{2n} p^{2n}.$$

Denote by A_n the subset of H_n containing all elements with absolute power less than $nq^{2n}p^{2n}$. Note that A_n is a finite set containing $B_n \cap H$. Considering that H is amenable and by Lemma 6.2 and (3.1), we get

$$\begin{aligned} \|\mathbb{1}_{A_n}\|_{M_0A(H)} &= \|\mathbb{1}_{A_n}\|_{A(H)} = \|\mathbb{1}_{A_n}\|_{A(H_n)} = \left\|\mathcal{F}(\mathbb{1}_{A_n})\right\|_{L^1(\mathbb{T})} \\ &= \frac{4}{\pi}\log(nq^{2n}p^{2n}) + O(1). \end{aligned} \tag{6.3}$$

This completes the proof since (6.3) is at most polynomial. ■

Proposition 6.6. *For any $A \in \mathrm{SL}(d, \mathbb{Z})$, the group $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ has characteristic tame cuts.*

Proof. The proof is essentially similar to Proposition 6.5. Recall that the group

$$\Gamma = \mathbb{Z}^d \rtimes_A \mathbb{Z}$$

can be identified as a subgroup of $\mathrm{SL}(d+1, \mathbb{Z})$ generated by the finite subset

$$S = \left\{ e_1 = I_{d+1} + E_{1,d+1}, \ldots, e_d = I_{d+1} + E_{d,d+1},\ t = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

where $E_{1,d+1} \in M_{d+1}(\mathbb{R})$ is the matrix with 1 on the cell $(i, d+1)$ and 0 elsewhere, and $I_{d+1} \in M_{d+1}(\mathbb{R})$ is the identity matrix. It is practical to use the following unique canonical form for every element $x \in \Gamma$:

$$x = (v, k), \quad v \in \mathbb{Z}^d,\ k \in \mathbb{Z}.$$

In this form, the group law is given by

$$(v, k)(w, l) = (v + A^k w, k + l).$$

We endow Γ with the length function ℓ associated with S. Denote by H the subgroup of Γ generated by $\{e_1, \ldots, e_d\}$. Since $\Gamma = H \rtimes_A \mathbb{Z}$, by Lemma 6.3, it is enough to show that $(H, \ell|_H)$ has characteristic tame cuts. Suppose that $(v, 0) \in H$ and $\ell(v, 0) \leq n$. Then, we can write $(v, 0) = x_1, \ldots, x_n$ for some $x_i = (\varepsilon_i, \delta_i) \in S^{\pm 1}$, $1 \leq i \leq n$, with $\sum \delta_i = 0$

and $v = \varepsilon_1 + A^{\delta_1}\varepsilon_2 + \cdots + A^{\delta_1+\cdots+\delta_{n-1}}\varepsilon_n$. Notice that

$$\|v\|_\infty \le \|v\|_2 \le 1 + \|A^{\delta_1}\| + \cdots + \|A^{\delta_1+\cdots+\delta_{n-1}}\| \le n \max\left(\|A\|, \|A^{-1}\|\right)^n \le nC^n.$$

Thus, the set $A_n = \{(v,0) \in H \mid v \in \mathbb{Z}^d,\ \|v\|_\infty \le nC^n\}$ is finite and contains the relative ball $H \cap B_n$ of radius n. Moreover, by (3.1), we have

$$\begin{aligned} \|\mathbb{1}_{A_n}\|_{M_0A(G)} &= \left\|\mathcal{F}(\mathbb{1}_{A_n})\right\|_{L^1(\mathbb{T}^d)} = \|D_{nC^n} \otimes \cdots \otimes D_{nC^n}\|_{L^1(\mathbb{T}^d)} \\ &= \|D_{nC^n}\|^d_{L^1(\mathbb{T})} = \left(\frac{4}{\pi}\log(nC^n) + O(1)\right)^d, \end{aligned} \tag{6.4}$$

where $D_N(t) = \sum_{k=-N}^{N} e^{i2\pi kt}$ is the Dirichlet kernel. Since (6.4) is at most polynomial, we conclude. ∎

Proposition 6.7. *For any $p, q \in \mathbb{N}$, the Baumslag–Solitar group*

$$\mathrm{BS}(p,q) = \langle a, t \mid ta^pt^{-1} = a^q\rangle$$

has completely bounded characteristic tame cuts.

The proof idea is essentially the same as [9] where it is proven that Baumslag–Solitar groups are weakly amenable and a-T-menable using the following theorem.

Theorem 6.8 ([9, a particular case of Theorem 1]). *Let $p, q \in \mathbb{N}$, $p' = p/\gcd(p,q)$, and $q' = q/\gcd(p,q)$. Let T be the Bass–Serre tree of HNN-extension*

$$\mathrm{BS}(p,q) = \mathrm{HNN}(\mathbb{Z}, p\mathbb{Z} \sim q\mathbb{Z}).$$

Denote by $j_1 : \mathrm{BS}(p,q) \to \mathrm{Aut}(T)$ the obvious group morphism. Denote $j_2 : \mathrm{BS}(p,q) \to \mathbb{Z}[\frac{1}{p'q'}] \rtimes_{\frac{p'}{q'}} \mathbb{Z}$ the group morphism defined by

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \frac{q'}{p'} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, the diagonal morphism

$$j = (j_1, j_2) : \mathrm{BS}(p,q) \to \mathrm{Aut}(T) \times \left(\mathbb{Z}\left[\frac{1}{p'q'}\right] \rtimes_{\frac{p'}{q'}} \mathbb{Z}\right)$$

is a closed embedding. Here, $\mathrm{Aut}(T)$ is endowed with the compact-open topology.

We also need the following lemma.

Lemma 6.9 ([3, Proposition 2.1]). *Suppose that a discrete group Γ acts on a tree $T = (V, E)$ by isometries. Then, for any base point $v_0 \in V$ and $n \in \mathbb{N}$, the characteristic function ϕ_n of the set $B_n = \{x \in \Gamma \mid d(v_0, xv_0) \le n\}$ defines a completely bounded Fourier multiplier on Γ with $\|\phi_n\|_{M_0A} \le 2n + 1$.*

Proof of Proposition 6.7. Theorem 6.8 allows us to identify $\mathrm{BS}(p,q)$ as a discrete subgroup of the locally compact group

$$\mathrm{Aut}(T) \times \left(\mathbb{Z}\left[\frac{1}{p'q'}\right] \rtimes_{\frac{p'}{q'}} \mathbb{Z}\right).$$

For all $n \in \mathbb{N}$, take $\phi_n \in M_0A(\mathrm{Aut}(T)_{\mathrm{d}})$ as in Lemma 6.9 for $\mathrm{Aut}(T)_{\mathrm{d}}$-action on the tree T, where $\mathrm{Aut}(T)_{\mathrm{d}}$ is the discrete realization of $\mathrm{Aut}(T)$. Let $(\psi_n)_{n\in\mathbb{N}}$ be tame cuts of $\mathbb{Z}[\frac{1}{p'q'}] \rtimes_{\frac{p'}{q'}} \mathbb{Z}$. For $n \in \mathbb{N}$, define a function φ_n as $\varphi_n(x,y) = \phi_n(x)\psi_n(y)$ for all $x \in \mathrm{Aut}(T)$ and $y \in \mathbb{Z}[\frac{1}{p'q'}] \rtimes_{\frac{p'}{q'}} \mathbb{Z}$. Then, by Lemma 6.1 and [5, Lemma 1.4], we have

$$\|\varphi_n\|_{M_0A(\mathrm{BS}(p,q))} \le \|\phi_n\|_{M_0A(\mathrm{Aut}(T)_{\mathrm{d}})} \|\psi_n\|_{M_0A(\mathbb{Z}[\frac{1}{p'q'}] \rtimes_{\frac{p'}{q'}} \mathbb{Z})}.$$

Note that the right-hand side grows polynomially. Also note that φ_n is a characteristic function since so are ϕ_n and ψ_n. It is left to prove that the support of φ_n is finite and contains the relative ball of radius n. For that, first we explain which length function we want to choose. As $\mathrm{BS}(p,q)$ is finitely generated, we can choose any proper length function on it to prove the statement. On $\mathrm{Aut}(T)_{\mathrm{d}}$, we choose the length function defined by $\ell_1(x) = d(v_0, xv_0)$. Since the tree T is locally finite, the $\mathrm{Aut}(T)$-action is proper, implying that the length function ℓ_1 is proper (when $\mathrm{Aut}(T)$ is endowed with the compact-open topology). Thus, the support of ϕ_n is compact. On $\mathbb{Z}[\frac{1}{p'q'}] \rtimes_{\frac{p'}{q'}} \mathbb{Z}$, we chose any word length function ℓ_2 associated with a finite generating set. One can easily check that the function $\ell(x,y) = \max(\ell_1(x), \ell_2(y))$ on the direct product is also a proper length function. Now, it is easily seen that the support

$$\mathrm{supp}(\varphi_n) = \big(\mathrm{supp}(\phi_n) \times \mathrm{supp}(\psi_n)\big) \cap \mathrm{BS}(p,q)$$

is finite and contains the relative ball $\{z \in \mathrm{BS}(p,q) \mid \ell(z) \le n\}$. ∎

Remark 6.10. Among all the examples in this section, only the following ones satisfy RD:

(i) Some $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ with $\mathrm{sp}(A) \subseteq S^1$.

(ii) $\mathbb{Z}[\frac{1}{1}] \rtimes_{\frac{1}{1}} \mathbb{Z} \cong \mathbb{Z}^2$.

(iii) $\mathrm{BS}(p,p)$ for $p \in \mathbb{N}$.

Concerning non-examples, the rigidity inequality of Theorem 2.5 shows that higher rank simple Lie groups with finite center do not have tame cuts. It follows that uniform lattices in a higher rank simple Lie group with finite center do not have completely bounded tame cuts. We do not have an example of a finitely generated group without (characteristic) tame cuts with respect to the word length function. Our best guess is the group given by the presentation

$$G = \langle a, b, c \mid aba^{-1} = b^2,\ cac^{-1} = a^2\rangle$$

because this group has a cyclic subgroup that is doubly exponentially distorted. The lattices $\mathrm{SL}(3,\mathbb{Z})$ and $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ are also good candidates. We also conjecture that cocompact lattices in higher rank semisimple Lie groups have characteristic tame cuts, which follows from Valette's conjecture [24, Conjecture 7].

Acknowledgments. This work is a part of my doctoral dissertation at the Université Côte d'Azur. I am sincerely grateful to my supervisor, Indira Chatterji, for her guidance and valuable comments. I would like to acknowledge that the notion of tame cuts was inspired by Vincent Lafforgue.

Funding. This work was supported by the French government doctoral degree training scholarship (927352B).

References

[1] M. Bożejko, Remark on Herz-Schur multipliers on free groups. *Math. Ann.* **258** (1981/82), no. 1, 11–15 Zbl 0483.43004 MR 641665

[2] M. Bożejko and G. Fendler, Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. *Boll. Un. Mat. Ital. A (6)* **3** (1984), no. 2, 297–302 Zbl 0564.43004 MR 753889

[3] M. Bożejko and M. A. Picardello, Weakly amenable groups and amalgamated products. *Proc. Amer. Math. Soc.* **117** (1993), no. 4, 1039–1046 Zbl 0780.43002 MR 1119263

[4] I. Chatterji, Introduction to the rapid decay property. In *Around Langlands correspondences*, pp. 53–72, Contemp. Math. 691, American Mathematical Society, Providence, RI, 2017 Zbl 1404.20032 MR 3666050

[5] M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.* **96** (1989), no. 3, 507–549 Zbl 0681.43012 MR 996553

[6] J. De Cannière and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Amer. J. Math.* **107** (1985), no. 2, 455–500 Zbl 0577.43002 MR 784292

[7] T. de Laat, Approximation properties for noncommutative L^p-spaces associated with lattices in Lie groups. *J. Funct. Anal.* **264** (2013), no. 10, 2300–2322 Zbl 1291.46054 MR 3035056

[8] P. Eymard, L'algèbre de Fourier d'un groupe localement compact. *Bull. Soc. Math. France* **92** (1964), 181–236 Zbl 0169.46403 MR 228628

[9] Ś. R. Gal and T. Januszkiewicz, New a-T-menable HNN-extensions. *J. Lie Theory* **13** (2003), no. 2, 383–385 Zbl 1041.20028 MR 2003149

[10] U. Haagerup, An example of a nonnuclear C^*-algebra, which has the metric approximation property. *Invent. Math.* **50** (1978/79), no. 3, 279–293 Zbl 0408.46046 MR 520930

[11] U. Haagerup, Group C^*-algebras without the completely bounded approximation property. *J. Lie Theory* **26** (2016), no. 3, 861–887 Zbl 1353.22004 MR 3476201

[12] U. Haagerup, T. Steenstrup, and R. Szwarc, Schur multipliers and spherical functions on homogeneous trees. *Internat. J. Math.* **21** (2010), no. 10, 1337–1382 Zbl 1272.43003 MR 2748193

[13] P. Jolissaint, Rapidly decreasing functions in reduced C^*-algebras of groups. *Trans. Amer. Math. Soc.* **317** (1990), no. 1, 167–196 Zbl 0711.46054 MR 943303

[14] Y. Katznelson, *An introduction to harmonic analysis*. 3rd edn., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004 Zbl 1055.43001 MR 2039503

[15] V. Lafforgue, A proof of property (RD) for cocompact lattices of $SL(3, \mathbf{R})$ and $SL(3, \mathbf{C})$. *J. Lie Theory* **10** (2000), no. 2, 255–267 Zbl 0981.46046 MR 1774859

[16] V. Lafforgue and M. De la Salle, Noncommutative L^p-spaces without the completely bounded approximation property. *Duke Math. J.* **160** (2011), no. 1, 71–116 Zbl 1267.46072 MR 2838352

[17] B. Liao, About the obstacle to proving the Baum-Connes conjecture without coefficient for a non-cocompact lattice in Sp_4 in a local field. *J. Noncommut. Geom.* **10** (2016), no. 4, 1243–1268 Zbl 1366.46058 MR 3597144

[18] V. Losert, Properties of the Fourier algebra that are equivalent to amenability. *Proc. Amer. Math. Soc.* **92** (1984), no. 3, 347–354 Zbl 0573.43004 MR 759651

[19] A. Lubotzky, S. Mozes, and M. S. Raghunathan, Cyclic subgroups of exponential growth and metrics on discrete groups. *C. R. Acad. Sci. Paris Sér. I Math.* **317** (1993), no. 8, 735–740 Zbl 0786.22016 MR 1244421

[20] A. Lubotzky, S. Mozes, and M. S. Raghunathan, The word and Riemannian metrics on lattices of semisimple groups. *Publ. Math. Inst. Hautes Études Sci.* (2000), no. 91, 5–53 (2001) Zbl 0988.22007 MR 1828742

[21] O. C. McGehee, L. Pigno, and B. Smith, Hardy's inequality and the L^1 norm of exponential sums. *Ann. of Math. (2)* **113** (1981), no. 3, 613–618 Zbl 0473.42001 MR 621019

[22] C. Nebbia, Multipliers and asymptotic behaviour of the Fourier algebra of nonamenable groups. *Proc. Amer. Math. Soc.* **84** (1982), no. 4, 549–554 Zbl 0488.43008 MR 643747

[23] P. F. Renaud, Centralizers of Fourier algebra of an amenable group. *Proc. Amer. Math. Soc.* **32** (1972), 539–542 Zbl 0258.43010 MR 293019

[24] A. Valette, *Introduction to the Baum-Connes conjecture. From notes taken by Indira Chatterji. With an appendix by Guido Mislin*. Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 2002 Zbl 1136.58013 MR 1907596

Received 4 October 2022.

Bat-Od Battseren
Laboratoire Jean Alexandre Dieudonné (LJAD), Université Côte d'Azur, 06000 Nice, France; current address: Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, USA; batoddd@gmail.com