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Quasi-inner automorphisms of Drinfeld modular groups

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Abstract. Let A be the set of elements in an algebraic function field K over $\mathbb{F}_q$ which are integral outside a fixed place ∞. Let $G = \mathrm{GL}_2(A)$ be a *Drinfeld modular group*. The normalizer of G in $\mathrm{GL}_2(K)$, where K is the quotient field of A, gives rise to automorphisms of G, which we refer to as *quasi-inner*. Modulo the inner automorphisms of G, they form a group $\mathrm{Quinn}(G)$ which is isomorphic to $\mathrm{Cl}(A)_2$, the 2-torsion in the ideal class group $\mathrm{Cl}(A)$. The group $\mathrm{Quinn}(G)$ acts on all kinds of objects associated with G. For example, it acts freely on the cusps and elliptic points of G. If $\mathcal{T}$ is the associated Bruhat–Tits tree, the elements of $\mathrm{Quinn}(G)$ induce non-trivial automorphisms of the quotient graph $G\backslash\mathcal{T}$, generalizing an earlier result of Serre. It is known that the ends of $G\backslash\mathcal{T}$ are in one-to-one correspondence with the cusps of G. Consequently, $\mathrm{Quinn}(G)$ acts freely on the ends. In addition, $\mathrm{Quinn}(G)$ acts transitively on those ends which are in one-to-one correspondence with the vertices of $G\backslash\mathcal{T}$ whose stabilizers are isomorphic to $\mathrm{GL}_2(\mathbb{F}_q)$.

1. Introduction

Let K be an algebraic function field of one variable with constant field $\mathbb{F}_q$, the finite field of order q. Let ∞ be a fixed place of K, and let δ be its degree. The ring A of all those elements of K which are integral outside ∞ is a Dedekind domain. Denote by K_∞ the completion of K with respect to ∞, and let C_∞ be the ∞-completion of an algebraic closure of K_∞. The group $\mathrm{GL}_2(K_\infty)$ (and its subgroup $G = \mathrm{GL}_2(A)$) acts as Möbius transformations on C_∞, K_∞ and hence $\Omega = C_\infty \backslash K_\infty$, the *Drinfeld upper halfplane*. This is part of a far-reaching analogy, initiated by Drinfeld [2], where $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are replaced by K, K_∞, C_∞, respectively. The roles of the classical upper half plane (in $\mathbb{C}$) and the classical modular group $\mathrm{SL}_2(\mathbb{Z})$ are assumed by Ω and G, respectively.

Modular curves, that is quotients of the complex upper half plane by finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$, are an indispensable tool when proving deep theorems about elliptic curves. Of similar importance in the theory of Drinfeld A-modules of rank 2 are *Drinfeld modular curves*, which are (the "compactifications" of) the quotient spaces $H\backslash\Omega$, where H is a finite index subgroup of G. Consequently, we refer to G as a *Drinfeld modular group*.

A complicating factor in this correspondence between $\mathrm{SL}_2(\mathbb{Z})$ and G is that, while the genus of the former is zero, for different choices of K and ∞, the genus of G can take

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many values. The simplest case, where $K = \mathbb{F}_q(t)$ and $A = \mathbb{F}_q[t]$ (equivalently, $g = 0$ and $\delta = 1$), has to date attracted most attention.

An element $\omega \in \Omega$ which is stabilized by a non-scalar matrix in G is called *elliptic*. Let $E(G)$ be the set of all such elements. It is known [3, p. 50] that $E(G) \neq \emptyset$ if and only if δ is odd. Clearly, G acts on $E(G)$ and the elements of the set of G-orbits, $\mathrm{Ell}(G) = G\backslash E(G) = \{G\omega : \omega \in E(G)\}$, are called the *elliptic points* of G. It is known [3, p. 50] that $\mathrm{Ell}(G)$ is finite. See [9] for a detailed treatment of elliptic points.

In addition, G acts on $\mathbb{P}^1(K) = K \cup \{\infty\}$. (Here, of course, ∞ refers to the one point compactification of K.) We refer to the elements of $\mathbb{P}^1(K)$ as *rational points*. For each finite index subgroup H of G, the elements of $\mathrm{Cusp}(H) = H\backslash\mathbb{P}^1(K)$ are called the *cusps* of H. Since A is a Dedekind domain, it is well known that $\mathrm{Cusp}(G)$ can be identified with $\mathrm{Cl}(A)$, the *ideal class group* of A. As Möbius transformations, G acts without inversion on $\mathcal{T}$, the Bruhat–Tits tree associated with $\mathrm{GL}_2(K_\infty)$ and the *ends* of the quotient graph $G\backslash\mathcal{T}$ are determined by $\mathrm{Cusp}(G)$ [11, p. 106, Theorem 9].

Cusps and elliptic points are important for several reasons. If H is a finite index subgroup of G, the quotient space $H\backslash\Omega$ will, after adding $\mathrm{Cusp}(H)$, be the C_∞-analog of a compact Riemann surface, which is called the *Drinfeld modular curve* associated with H. Moreover, in the covering of Drinfeld modular curves induced by the natural map $H\backslash\Omega \to G\backslash\Omega$, ramification can only occur above the cusps and elliptic points of G. Also, for (classical and Drinfeld) modular forms, analyticity at the cusps and elliptic points requires special care.

This paper is a continuation and extension of [9] which is concerned with the elliptic points of G. There the starting point [3, p. 51] is the existence of a bijection between $\mathrm{Ell}(G)$ and $\ker \bar{N}$, where $\bar{N}\colon \mathrm{Cl}(\tilde{A}) \to \mathrm{Cl}(A)$ is the norm map and $\tilde{A} = A.\mathbb{F}_{q^2}$. It can be shown [9] that $\mathrm{Cl}(\tilde{A})_2 \cap \ker \bar{N}$, the 2-torsion subgroup of $\ker \bar{N}$, is in bijection with $\mathrm{Ell}(G)^{=} = \{G\omega : \omega \in E(G), G\omega = G\bar{\omega}\}$, where $\bar{\omega}$, the *conjugate* of ω, is the image of ω under the Galois automorphism of $K.\mathbb{F}_{q^2}/K$. (In [9], $\mathrm{Ell}(G)^{=}$ is denoted by $\mathrm{Ell}(G)_2$.) Here we show that, when δ is odd, $\mathrm{Cl}(A)_2$ and the 2-torsion in $\ker \bar{N}$ are isomorphic. This is the starting point for this paper, where the principal focus of attention is the group $\mathrm{Cl}(A)_2$ and its actions on various objects related to G. *Unless otherwise stated, results hold for all δ.*

Let $g \in N_{\mathrm{GL}_2(K)}(G)$, the normalizer of G in $\mathrm{GL}_2(K)$. Then g, acting by conjugation, induces an automorphism ι_g of G, which we refer to as *quasi-inner*. If $g \in G.Z(K)$, then ι_g reduces to an inner automorphism. If $g \in N_{\mathrm{GL}_2(K)}(G)\backslash G.Z(K)$, we call ι_g *non-trivial*. We denote the quotient group $N_{\mathrm{GL}_2(K)}(G)/G.Z(K)$ by $\mathrm{Quinn}(G)$. It is well known [1] that $\mathrm{Quinn}(G)$ is isomorphic to $\mathrm{Cl}(A)_2$. Hence G has non-trivial quasi-inner automorphisms if and only if $|\mathrm{Cl}(A)|$ is *even*. Now, as an element of $\mathrm{GL}_2(K)$, ι_g acts as a Möbius transformation on the rational points and elliptic elements of G, as well as $\mathcal{T}$. In particular, $\overline{g(\omega)} = g(\bar{\omega})$. Since all of these actions are trivial for scalar matrices, they extend to actions of $\mathrm{Quinn}(G)$ on $\mathrm{Cusp}(G)$, $\mathrm{Ell}(G)$ and the quotient graph, $G\backslash\mathcal{T}$. In this paper, we study the (often surprising) properties of these actions.

Theorem 1.1. *The group* $\mathrm{Quinn}(G)$ *acts freely on*

(i) $\mathrm{Cusp}(G)$*,*

(ii) $\mathrm{Ell}(G)$ *if* δ *is odd.*

From the above, it is clear that $\mathrm{Quinn}(G)$ can be embedded as a subgroup $\mathrm{Ell}(G)^{=}$ (resp. $\mathrm{Cl}(A)_2$) of $\mathrm{Ell}(G)$ (resp. $\mathrm{Cusp}(G)$). We show that the action of $\mathrm{Quinn}(G)$ is equivalent to multiplication by the elements of the subgroup. The "freeness" in this result follows immediately. Restricting to these subsets yields stronger results.

Corollary 1.2. *The group* $\mathrm{Quinn}(G)$ *acts freely and transitively on*

(i) $\mathrm{Cl}(A)_2$*,*

(ii) $\mathrm{Ell}(G)^{=}$ *if* δ *is odd.*

Corollary 1.3. *When* δ *is odd,* $\mathrm{Quinn}(G)$ *acts freely on* $\mathrm{Ell}(G)^{\neq} = \{G\omega : G\omega \neq G\bar{\omega}\}$*. Moreover, if* $\ker \bar{N}$ *has no element of order* 4*, then* $\mathrm{Quinn}(G)$ *acts freely on*

$$\{\{G\omega, G\bar{\omega}\} : G\omega \in \mathrm{Ell}(G)^{\neq}\}.$$

Theorem 1.4. *Every non-trivial element of* $\mathrm{Quinn}(G)$ *determines an automorphism of* $G\backslash\mathcal{T}$ *of order* 2 *which preserves the structure of all its vertex and edge stabilizers.*

Serre [11, p. 117, Exercise 2 (e)] states this result for the special case $K = \mathbb{F}_q(t)$ with δ even. Our result shows that, in general, the quotient graph has symmetries of this type provided $|\mathrm{Cl}(A)|$ is even. (In general, this restriction is necessary.)

We now list more detailed results on the action of $\mathrm{Quinn}(G)$ on $G\backslash\mathcal{T}$. Serre [11, p. 106, Theorem 9] has described the basic structure of $G\backslash\mathcal{T}$. Its *ends* (i.e., the equivalence classes of semi-infinite paths without backtracking) are in one-to-one correspondence with the elements of $\mathrm{Cl}(A)$. To date, the only cases for which the precise structures of $G\backslash\mathcal{T}$ are known are $g = 0$ [4, 6], and $g = \delta = 1$ [14].

Theorem 1.5. *The group* $\mathrm{Quinn}(G)$ *acts freely on the ends of* $G\backslash\mathcal{T}$ *and, in addition, transitively on the ends of* $G\backslash\mathcal{T}$ *corresponding to the elements of* $\mathrm{Cl}(A)_2$*,*

We show that the ends corresponding to $\mathrm{Cl}(A)_2$ are in one-to-one correspondence with those vertices whose stabilizers are isomorphic to $\mathrm{GL}_2(\mathbb{F}_q)$. (Each such vertex is "attached" to the corresponding end.) It is known [8, Corollary 2.12] that if G_v contains a cyclic subgroup of order $q^2 - 1$, then $G_v \cong \mathbb{F}_{q^2}^*$ or $\mathrm{GL}_2(\mathbb{F}_q)$.

The *building map* [3, p. 41] extends to a map $\lambda\colon \mathrm{Ell}(G) \to \mathrm{vert}(G\backslash\mathcal{T})$. This map leads to another action of $\mathrm{Quinn}(G)$ on the quotient graph.

Theorem 1.6. (a) *The group* $\mathrm{Quinn}(G)$ *acts freely and transitively on*

$$\{\tilde{v} \in \mathrm{vert}(G\backslash\mathcal{T}) : G_v \cong \mathrm{GL}_2(\mathbb{F}_q)\}.$$

(b) *Suppose that* δ *is odd and that* $\ker \bar{N}$ *has no element of order* 4*. Then* $\mathrm{Quinn}(G)$ *acts freely on*

$$\{\tilde{v} \in \mathrm{vert}(G\backslash\mathcal{T}) : G_v \cong \mathbb{F}_{q^2}^*\}.$$

As an illustration of our results, especially the existence of reflective symmetries as in Theorem 1.4, we conclude with diagrams of two examples of $G\backslash\mathcal{T}$, for each of which $g = \delta = 1$, the so called "elliptic" case. For these we make use of Takahashi's paper [14]. Special features of these cases include the following. For part (i), see [9, Theorem 5.1].

Corollary 1.7. *Suppose that* $\delta = 1$.

(i) *The isolated (i.e., (graph) valency* 1*) vertices of* $G\backslash\mathcal{T}$ *are precisely those whose stabilizers are isomorphic to* $\mathrm{GL}_2(\mathbb{F}_q)$ *or* $\mathbb{F}_{q^2}^*$.

(ii) *If* $\ker \overline{N}$ *has no element of order* 4*, then* $\mathrm{Quinn}(G)$ *acts freely on the isolated vertices of* $G\backslash\mathcal{T}$.

By looking at the stabilizers in G of the objects discussed above, we obtain several statements about the action of $\mathrm{Quinn}(G)$ on the conjugacy classes of certain types of subgroups of G. (See Sections 3 and 5.)

For convenience, we begin with a list of notations which will be used throughout this paper.

$\mathbb{F}_q$	the finite field with $q = p^n$ elements;
K	an algebraic function field of one variable with constant field $\mathbb{F}_q$;
g	the genus of K;
∞	a chosen place of K;
δ	the degree of the place ∞;
A	the ring of all elements of K that are integral outside ∞;
K_∞	the completion of K with respect to ∞;
Ω	Drinfeld's halfplane;
$\mathcal{T}$	the Bruhat–Tits tree of $\mathrm{GL}_2(K_\infty)$;
G	the Drinfeld modular group $\mathrm{GL}_2(A)$;
Gx	the orbit of x under the action of G on the object x;
$\widehat{G}$	$\mathrm{GL}_2(K)$;
$Z(K)$	the set of scalar matrices in $\widehat{G}$;
Z	$Z(K) \cap G$;
$\widetilde{K}$	the quadratic constant field extension $K.\mathbb{F}_{q^2}$;
$\widetilde{A}$	$A.\mathbb{F}_{q^2}$, the integral closure of A in $\widetilde{K}$;
$\mathrm{Cl}(R)$	the ideal class group of the Dedekind ring R;
$\mathrm{Cl}^0(F)$	the divisor class group of degree 0 of the function field F;
$\mathrm{Cusp}(G)$	$G\backslash\mathbb{P}^1(K)$, the set of cusps of G;
$E(G)$	the set of elliptic elements of G:
$\mathrm{Ell}(G)$	$G\backslash E(G)$, the set of elliptic points of G;
$\overline{\omega}$	the image of $\omega \in E(G)$ under the Galois automorphism of $\widetilde{K}/K$;
$\mathrm{Ell}(G)^{=}$	$\{G\omega : \omega \in E(G),\ G\omega = G\overline{\omega}\}$;
$\mathrm{Ell}(G)^{\neq}$	$\mathrm{Ell}(G) \setminus \mathrm{Ell}(G)^{=}$;

$S(s)$	the stabilizer in a finite index subgroup S (of G) of $s \in \mathbb{P}^1(K)$;
G^ω	the stabilizer in G of $\omega \in C_\infty \backslash K$;
S_w	the stabilizer in S of $w \in \text{vert}(\mathcal{T}) \cup \text{edge}(\mathcal{T})$;
$\mathcal{H}$	$\{H \leq G : H \cong \text{GL}_2(\mathbb{F}_q)\}$;
$\mathcal{C}$	$\{C \leq G : C \cong \mathbb{F}_{q^2}^*\}$;
$\mathcal{C}_{mf}$	$\{C \in \mathcal{C} : C$ maximally finite in $G\}$;
$\mathcal{C}_{nm}$	$\mathcal{C} \backslash \mathcal{C}_{mf}$;
$\mathcal{V}$	$\{\tilde{v} \in \text{vert}(G\backslash\mathcal{T}) : G_v \in \mathcal{C}\}$

2. Quasi-inner automorphisms

Let F be any field containing A (and hence K), and let $Z(F)$ denote the set of scalar matrices in $\text{GL}_2(F)$. We are interested here in automorphisms of G arising from conjugation by a non-scalar element of $\text{GL}_2(F)$. We first show this problem reduces to $N_{\widehat{G}}(G)$, the normalizer of G in $\widehat{G} = \text{GL}_2(K)$. For each $x \in F$, we use (x) as a shorthand for the fractional ideal Ax.

Lemma 2.1. *Let $M_0 \in \text{GL}_2(F)$ normalize G. Then*

$$M_0 \in Z(F).N_{\widehat{G}}(G).$$

Proof. Let

$$M_0 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Suppose that $\gamma \neq 0$. Replacing M_0 by $\gamma^{-1}M_0$, we may assume that $\gamma = 1$. Now

$$NT(1)N^{-1} \in G,$$

where $N = M_0^{\pm 1}$. It follows that $\det(M_0), \alpha, \delta \in K$ and hence that $\beta = \alpha\delta - \det(M_0) \in K$. The proof for the case where $\gamma = 0$ is similar. ∎

We state a special case ($n = 2$) of a result of Cremona [1] .

Theorem 2.2. *Let*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \hat{G},$$

and define

$$\mathfrak{q}(M) := (a) + (b) + (c) + (d).$$

Then $M \in N_{\widehat{G}}(G)$ if and only if

$$\mathfrak{q}(M)^2 = (\Delta),$$

where $\Delta = \det(M)$.

Corollary 2.3. *Let* $M \in N_{\widehat{G}}(G)$ *with* $\Delta = \det(M)$.

(i) $\Delta^{-1}M^2 \in \mathrm{SL}_2(A)$.

(ii) *If* $\Delta \in A^*$, *then* $M \in G$.

Proof. (i) By Theorem 2.2, every entry of M^2 is in $\mathfrak{q}(M)^2 = (\Delta)$.

For part (ii), let x be any entry of M. Then $x^2 \in A$ by Theorem 2.2 and so $x \in A$, since A is integrally closed. ∎

Another important consequence [1] of Theorem 2.2 is the following.

Theorem 2.4. *The map* $M \mapsto \mathfrak{q}(M)$ *induces an isomorphism*

$$N_{\widehat{G}}(G)/Z(K).G \cong \mathrm{Cl}(A)_2,$$

where $\mathrm{Cl}(A)_2$ *is the subgroup of all involutions in* $\mathrm{Cl}(A)$.

Proof. This is another special case ($n = 2$) of a result in [1]. If $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N_{\widehat{G}}(G)$, it can be shown [1, Remarks 2] that

$$(a) + (b) = (a) + (c) = (d) + (b) = (d) + (c) = \mathfrak{q}(M).$$

Consequently, there is a map from $N_{\widehat{G}}(G)$ to $\mathrm{Cl}(A)_2$, which turns out to be an isomorphism. ∎

Definition 2.5. An automorphism ι_g of G is called *quasi-inner* if

$$\iota_g(x) = gxg^{-1}, \quad x \in G,$$

for some $g \in N_{\widehat{G}}(G)$. We call ι_g *non-trivial* if $g \notin Z(K).G$, i.e., if ι_g does not act like an inner automorphism. We note that

$$\iota_{g_1} = \iota_{g_2} \Leftrightarrow g_1 g_2^{-1} \in Z(K).$$

Finally, we define

$$\mathrm{Quinn}(G) := N_{\widehat{G}}(G)/Z(K).G \cong \mathrm{Cl}(A)_2.$$

So $\mathrm{Quinn}(G)$ is the group of quasi-inner automorphisms modulo the inner ones. We note that, in particular, all quasi-inner automorphisms of G act like inner automorphisms if $|\mathrm{Cl}(A)|$ is odd.

Let $\mathrm{Cl}^0(K)$ be the *group of divisor classes of degree zero* [13, p. 186]. It is known [11, p. 104] that the following exact sequence holds:

$$0 \to \mathrm{Cl}^0(K) \to \mathrm{Cl}(A) \to \mathbb{Z}/\delta\mathbb{Z} \to 0. \tag{1}$$

Our next result is an immediate consequence of Theorem 2.4.

Corollary 2.6. *The group G has non-trivial quasi-inner automorphisms if and only if*

$$|\mathrm{Cl}(A)| = \delta |\mathrm{Cl}^0(K)| \quad \textit{is even.}$$

Example 2.7. We illustrate the results of this section with the simplest case $K = \mathbb{F}_q(t)$, the rational function field over $\mathbb{F}_q$. Then there exists a (monic) polynomial $\pi(t) \in \mathbb{F}_q[t]$, of degree δ, irreducible over $\mathbb{F}_q$, such that

$$A = \left\{ \frac{f}{\pi^m} : f \in k[t],\, m \geq 0,\, \deg f \leq \delta m \right\}.$$

It is known [13, p. 193, Theorem 5.1.15] that here $\mathrm{Cl}^0(K)$ is trivial, so that

$$\mathrm{Cl}(A) \cong \mathbb{Z}/\delta\mathbb{Z}.$$

Hence G has non-trivial quasi-inner automorphisms if and only if δ is even. Hence here either $\mathrm{Quinn}(G)$ is trivial or cyclic of order 2.

For a specific illustration of Theorem 2.4, we restrict further to $\delta = 2$. In this case, $\pi(t) = t^2 + \sigma t + \tau$, where $\sigma \in \mathbb{F}_q$ and $\tau \in \mathbb{F}_q^*$. We begin with the A-ideal generated by π^{-1} and $t\pi^{-1}$ which is *not* principal. Let $\pi(t) = tt' + \tau$ and put

$$g_0 = \begin{pmatrix} \tau & t \\ -t' & 1 \end{pmatrix}.$$

Then by Theorem 2.2, $g_0 \in N_{\widehat{G}}(G)$ and from Theorem 2.4, we see that $g_0 \notin Z(K).G$. Hence g_0 provides a generator of $\mathrm{Cl}(A)_2$.

Remark 2.8. From the theory of Jacobian varieties, we know that the 2-torsion in $\mathrm{Cl}^0(K)$ is bounded by 2^{2g}, and even by 2^g if the characteristic of K is 2 [10, Theorem 11.12]. Hence by the exact sequence (1), it follows that $|\mathrm{Quinn}(G)| = |\mathrm{Cl}(A)_2| \leq 2^{2g+1}$ (and $\leq 2^{g+1}$, when $\mathrm{char}(K) = 2$).

In odd characteristic, we can easily find examples with $|\mathrm{Cl}(A)_2| = 2^{2g}$, provided we are willing to accept a big constant field. Given a function field F of genus g with constant field $\mathbb{F}_{p^r}$, just pick $q = p^{rn}$ such that all 2-torsion points of $\mathrm{Jac}(F)$ are $\mathbb{F}_q$-rational and consider $K = F.\mathbb{F}_q$. Then $\mathrm{Cl}^0(K)_2 \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$. Choosing a place ∞ of K of odd degree δ from the exact sequence (1), we see that $|\mathrm{Cl}(A)_2| = 2^{2g}$.

Similarly, in characteristic 2 examples for which $|\mathrm{Cl}(A)_2| = 2^g$ can be found by choosing F suitably, namely F has to be *ordinary*.

Whether for even δ one can reach the bound 2^{2g+1} (resp. 2^{g+1}) depends on whether or not the induced short exact sequence for the Sylow 2-subgroup of $\mathrm{Cl}(A)$ splits or not.

Definition 2.9. Let R, S be subgroups of a group T. We write

$$R \sim S$$

if and only if $R = S^t = tSt^{-1}$ for some $t \in T$. We put

$$R^T = \{R^t : t \in T\}.$$

Let $\mathcal{S}$ be a set of subgroups of T. We put

$$\mathcal{S}^T = \{S^T : S \in \mathcal{S}\}.$$

This paper is principally concerned with various actions of Quinn(G). It is appropriate at this point to describe in detail the most important of these. Let ι_g be as above.

(i) It is clear that $\mathrm{GL}_2(K_\infty)$ acts on Ω as Möbius transformations and that this action is trivial for all scalar matrices. Then ι_g acts on $E(G)$ since, for all $\omega \in E(G)$,

$$G^{g(\omega)} = (G^\omega)^g \ (\leq G).$$

Recall that $\mathrm{Ell}(G) = \{G\omega : \omega \in E(G)\}$. The map

$$G\omega \mapsto Gg(\omega)$$

extends naturally to a well-defined action of Quinn(G) on Ell(G).

(ii) Clearly, G acts as Möbius transformations on $\mathbb{P}^1(K)$, and it is well known that

$$G\backslash\mathbb{P}^1(K) \leftrightarrow \mathrm{Cl}(A).$$

As we shall see later from the structure of the quotient graph, it follows that, for all $k \in \mathbb{P}^1(K)$, $G(k)$ is infinite, metabelian. Recall that $\mathrm{Cusp}(G) = \{Gk : k \in \mathbb{P}^1(K)\}$. As before, the map

$$Gk \mapsto Gg(k)$$

extends to a well-defined action of Quinn(G) on Cusp(G).

(iii) Serre [11, Chapter II, Section 1.1, p. 67] uses *lattice classes* as a model for the vertices and edges of $\mathcal{T}$. It is clear that $\mathrm{GL}_2(K_\infty)$ acts naturally on these. In particular, the scalar matrices act trivially. The map

$$Gw \mapsto Gg(w),$$

where $w \in \mathrm{vert}(\mathcal{T}) \cup \mathrm{edge}(\mathcal{T})$, extends to a well-defined action of Quinn(G) on the quotient graph $G\backslash\mathcal{T}$. Note that $G_{g(w)} = ((G_w))^g \leq G$. We will use this action to extend a result of Serre.

(iv) Suppose that

$$\mathcal{S} = \{H \leq G : H \cong \mathrm{GL}_2(\mathbb{F}_q)\}$$

or $\mathcal{S}$ is a G-conjugacy closed subset of $\mathcal{C} = \{C \leq G : C \cong \mathbb{F}_{q^2}^*\}$.

Then Quinn(G) acts by conjugation on $\mathcal{S}^G$. We use these to define actions of Quinn(G) on significant subsets of vert$(\mathcal{T})$.

3. Action on vertex stabilizers

Almost all the results in this section hold for all δ. We record the important general properties of subgroups of vertex stabilizers.

Lemma 3.1. (i) *G_v is finite for all $v \in \mathrm{vert}(\mathcal{T})$.*

(ii) *Let S be a finite subgroup of G. Then*

$$S \leq G_{v_0}$$

for some $v_0 \in \mathrm{vert}(\mathcal{T})$.

Proof. See [11, p. 76, Proposition 2]. ∎

In this section, we are concerned with subgroups of G_v which contain a cyclic subgroup of order $q^2 - 1$. We record the following result.

Lemma 3.2. *Suppose that G_v contains a cyclic subgroup of order $q^2 - 1$. Then*

$$G_v \cong \mathrm{GL}_2(\mathbb{F}_q) \quad \text{or} \quad G_v \cong \mathbb{F}_{q^2}^*.$$

Proof. See [8, Corollaries 2.2, 2.4 and 2.12]. ∎

In the first part of this section, we look at the action of quasi-inner automorphisms on the following set:

$$\mathcal{H} = \{H \leq G : H \cong \mathrm{GL}_2(\mathbb{F}_q)\}.$$

Lemma 3.3. *Let $H \in \mathcal{H}$. Then there exists $v_0 \in \mathrm{vert}(\mathcal{T})$ for which*

$$H = G_{v_o}.$$

Proof. Follows from Lemmas 3.1 (ii) and 3.2. ∎

Remark 3.4. (i) Every $\mathcal{T}$ contains a particular vertex v_s, usually referred to as *standard* (after Serre), for which

$$G_{v_s} = \mathrm{GL}_2(\mathbb{F}_q).$$

See [11, p. 97, Remark 3].

(ii) On the other hand, for the case $A = \mathbb{F}_q[t]$ (equivalently, $g(K) = 0$, $\delta = 1$), it follows from Nagao's theorem [11, Corollary, p. 87] that here $\mathrm{vert}(\mathcal{T})$ has no stabilizer which is cyclic of order $q^2 - 1$.

Lemma 3.5. *Let $H \in \mathcal{H}$. Then there exists a quasi-inner automorphism $\kappa = \iota_g$ of G such that*

$$H = \kappa(\mathrm{GL}_2(\mathbb{F}_q)).$$

Proof. From the proofs of [8, Theorem 2.6, Corollary 2.8], as well as [8, Corollary 2.12], it is clear that there exists

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\widetilde{K})$$

such that

$$H = g(\mathrm{GL}_2(\mathbb{F}_q))g^{-1}.$$

We denote by $\bar{x}$ the image of $x \in \widetilde{K}$ under the extension of the Galois automorphism of $\mathbb{F}_{q^2}/\mathbb{F}_q$ to $\widetilde{K}$. It is clear that $gE_{ij}g^{-1} \in M_2(A)$, where $1 \leq i, j \leq 2$ and so

$$xy/\Delta \ (= \overline{xy}/\overline{\Delta}) \in A$$

for all $x, y \in \{a, b, c, d\}$, where $\Delta = \det(g)$.

Now we may assume without loss of generality that $c \neq 0$. Let $z \in \{a, b, d\}$. Then

$$c^2/\Delta = \bar{c}^2/\overline{\Delta}, \quad cz/\Delta = \overline{cz}/\overline{\Delta}.$$

It follows that $z/c = \bar{z}/\bar{c}$, so that $z/c \in K$. We now replace $g = M$ by $g_0 = c^{-1}M$. Then by Theorem 2.2, the map $\kappa_0 \colon G \to G$ defined by $\kappa_0(x) = g_0 x g_0^{-1}$ is a quasi-inner automorphism of G. ■

Lemma 3.6. *Let $\kappa_0 = \iota_{g_o}$ be a non-trivial quasi-inner automorphism of G, and let $H \in \mathcal{H}$. Then*

$$\kappa_0(H) \not\sim H.$$

Proof. By definition, $g_0 \in N_{\widehat{G}}(G)\backslash G.Z(K)$. Suppose to the contrary that

$$\kappa_0(H) = gHg^{-1}$$

for some $g \in G$. Replacing g_0 by $g^{-1}g_0$, we may assume that $g = 1$. Now by Lemma 3.5, $H = \kappa_0'(\mathrm{GL}_2(\mathbb{F}_q))$ for some quasi-inner $\kappa_0' = \iota_{g_0'}$, say. It follows that

$$g_1(\mathrm{GL}_2(\mathbb{F}_q))g_1^{-1} = \mathrm{GL}_2(\mathbb{F}_q),$$

where $g_1 = (g_0')^{-1}g_0g_0'$. As $N_{\widehat{G}}(G)/G.Z(K)$ is abelian, this implies that

$$g_1 \equiv g_0 \pmod{Z(K).G},$$

and so we may further assume that $g_1 = g_0$. Let

$$S_p = \{T(a) = E_{12}(a) : a \in \mathbb{F}_q\}.$$

Now S_p is a Sylow p-subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ and so from the above,

$$g_0(S_p)g_0^{-1} = h(S_p)h^{-1}$$

for some $h \in \mathrm{GL}_2(\mathbb{F}_q)$. As above, we may assume then that $h = 1$. It follows that g_0 "fixes" ∞, and so

$$g_0 = \begin{bmatrix} \alpha & * \\ 0 & \beta \end{bmatrix}.$$

By Corollary 2.3 (i), we note that

$$(\det(g_0))^{-1} \operatorname{tr}((g_0)^2) = \gamma + \gamma^{-1} \in A,$$

where $\gamma = \alpha\beta^{-1}$. Since A is integrally closed, it follows that $\gamma \in A^*(= \mathbb{F}_q^*)$. Then we can replace g_0 by $\beta^{-1}g_0$ which belongs to G by Corollary 2.3 (ii). Thus $g_0 \in Z(K).G$. ∎

Lemma 3.7. *Let $e \in \mathrm{edge}(\mathcal{T})$ be incident with v_s. Then*

$$G_e \lneqq \mathrm{GL}_2(\mathbb{F}_q).$$

Proof. The edges attached to v_s are parametrized by $\mathbb{P}^1(\mathbb{F}_{q^\delta})$, and $\mathrm{GL}_2(\mathbb{F}_q)$ acts on these as Möbius transformations. See [11, p. 99, Exercise 6].

If the edge corresponds to $f \in \mathbb{F}_{q^\delta}$, it is not fixed by the translations in $\mathrm{GL}_2(\mathbb{F}_q)$, and if it corresponds to ∞, it is not fixed by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_q)$. ∎

Proposition 3.8. *No edge of $\mathcal{T}$ can have a stabilizer isomorphic to* $\mathrm{GL}_2(\mathbb{F}_q)$.

Proof. For odd δ, this follows from [8, Corollary 2.16]. We provide a proof that holds for all δ. Suppose to the contrary that there is an edge e whose stabilizer is isomorphic to $\mathrm{GL}_2(\mathbb{F}_q)$. Then by Lemma 3.2, the stabilizers of its terminal vertices are both G_e.

By Lemma 3.5 and the action of quasi-inner automorphisms on $\mathcal{T}$, we can assume that

$$G_e = \mathrm{GL}_2(\mathbb{F}_q).$$

It follows that $\mathrm{GL}_2(\mathbb{F}_q)$ stabilizes the geodesic from v_s to one of the terminal vertices of e which includes e, and hence an edge incident with v_s. This contradicts Lemma 3.7. ∎

Corollary 3.9. *Let $H \in \mathcal{H}$. Then there exists a unique vertex $v \in \mathrm{vert}(\mathcal{T})$ such that*

$$G_v = H.$$

Proof. Follows from Lemma 3.3 and Proposition 3.8. ∎

Remark 3.10. Another interesting consequence of Lemma 3.5 and Proposition 3.8 is the following. Suppose that $G_v \in \mathcal{H}$. Then there exists $\kappa = \iota_g$ such that $\kappa(v) = v_s$. Since κ is an automorphism of $\mathcal{T}$, the action of G_v on the $q^\delta + 1$ edges of $\mathcal{T}$ incident with v is identical to the action of $\mathrm{GL}_2(\mathbb{F}_q)$ on the edges of $\mathcal{T}$ incident with v_s, as described in Lemma 3.7.

Definition 3.11. By definition,

$$\mathrm{vert}(G\backslash\mathcal{T}) = \{Gv : v \in \mathrm{vert}(\mathcal{T})\}.$$

We put $\tilde{v} = Gv$ and define its *stabilizer*

$$G_{\tilde{v}} = (G_v)^G.$$

We refer to $G_{\tilde{v}}$ as being *isomorphic to* G_v.

Lemma 3.12. *There exists a bijection*

$$\mathcal{H}^G \leftrightarrow \{\tilde{v} \in \mathrm{vert}(G\backslash\mathcal{T}) : G_v \in \mathcal{H}\}.$$

Proof. Follows from Corollary 3.9 and the above. ∎

It is clear that $\mathrm{Quinn}(G)$ acts on $\mathcal{H}^G$. Since $Z(K)$, represented by scalar matrices, acts trivially on $\mathcal{T}$, it is also clear that $\mathrm{Quinn}(G)$ acts on $G\backslash\mathcal{T}$. We now come to the principal result in this section which follows from Lemmas 3.5, 3.6 and 3.12.

Theorem 3.13. *The group* $\mathrm{Quinn}(G)$ *acts freely and transitively on*

(i) *the G-conjugacy classes of subgroups of G which are isomorphic to* $\mathrm{GL}_2(\mathbb{F}_q)$,

(ii) *the vertices of $G\backslash\mathcal{T}$ whose stabilizers are isomorphic to* $\mathrm{GL}_2(\mathbb{F}_q)$.

A special case of this result is provided by Corollary 2.6.

Corollary 3.14. *Suppose that* $|\mathrm{Cl}(A)|$ *is odd. Then*

(i) *every subgroup H of G isomorphic to* $\mathrm{GL}_2(\mathbb{F}_q)$ *is actually conjugate in G to the natural subgroup* $\mathrm{GL}_2(\mathbb{F}_q)$ *of G obtained from the inclusion $\mathbb{F}_q \subseteq A$,*

(ii) *the only vertex in $G\backslash\mathcal{T}$ whose stabilizer is isomorphic to* $\mathrm{GL}_2(\mathbb{F}_q)$ *is $\tilde{v}_s$, the image of the standard vertex v_s.*

4. Action on elliptic points

Throughout this section, we assume that δ is *odd*. Recall that

$$\mathrm{Ell}(G) = \{G\omega : \omega \in E(G)\}$$

denotes the elliptic points of the Drinfeld modular curve $G\backslash\Omega$.

Definition 4.1. We define

$$\mathrm{Ell}(G)^{=} = \{G\omega : G\omega = G\bar{\omega}\} \quad \text{and} \quad \mathrm{Ell}(G)^{\neq} = \{G\omega : G\omega \neq G\bar{\omega}\}.$$

(In [9, Section 3] $\mathrm{Ell}(G)^{=}$ is denoted by $\mathrm{Ell}(G)_2$.)

The action of an element of $\mathrm{GL}_2(K_\infty)$ on an element of Ω will always refer to its action as a Möbius transformation. We record the following.

Lemma 4.2. *Let $g \in N_{\widehat{G}}(G)$ and $\omega \in E(G)$. Then*

(i) $g(\omega) \in E(G)$,

(ii) $\overline{g(\omega)} = g(\bar{\omega})$.

It is clear then that Quinn(G) acts on both Ell$(G)^{=}$ and Ell$(G)^{\neq}$.

In this section, our approach is based on [9, Sections 3 and 4]. We recall some details.

Definition 4.3. Let I be an A-ideal (resp. $\tilde{A}$-ideal). Then $[I]$ denotes the image of I in Cl(A) (resp. Cl$(\tilde{A})$).

Fix $\varepsilon \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. By [9, Theorem 2.5], any elliptic point ω of G can be written as $\omega = \frac{\varepsilon+s}{t}$, where $s, t \in A$ and t divides $(\bar{\varepsilon}+s)(\varepsilon+s)$ in A. Now let

$$J_\omega = A(\varepsilon+s) + At.$$

It is known [9, Lemmas 3.1 and 3.2] that

(i) J_ω is an $\tilde{A}$-ideal.

(ii) J_ω is independent of the choice of $\varepsilon \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.

(iii) Let $\omega, \omega' \in E(G)$. Then

$$G\omega = G\omega' \Leftrightarrow [J_\omega] = [J_{\omega'}] \quad \text{in Cl}(\tilde{A}).$$

Let α be the Galois automorphism of $\tilde{K}/K$ (which extends that of $\mathbb{F}_{q^2}/\mathbb{F}_q$). Let $k \in \tilde{K}$. Then the *norm* of k is $k\bar{k}$, where $\bar{k} = \alpha(k)$. Now α restricts to $\tilde{A}$ and so acts on its ideals and hence its ideal class group. For each $\tilde{A}$-ideal, J, the *norm* of J, $N(J) = A \cap (J\bar{J})$, which is an A-ideal. We now come to the *norm map*

$$\bar{N}\colon \text{Cl}(\tilde{A}) \to \text{Cl}(A),$$

where $\bar{N}([I]) = [(I\bar{I}) \cap A]$. Then

$$[I] \in \ker \bar{N} \Leftrightarrow (I\bar{I}) \cap A \text{ is a principal } A\text{-ideal.}$$

We restate [9, Theorem 3.4].

Theorem 4.4. *The map $\omega \mapsto [J_\omega]$ induces a one-to-one correspondence*

$$\text{Ell}(G) \leftrightarrow \ker \bar{N}.$$

For each ω, it is known that

(i) $\overline{J_\omega} = J_{\bar{\omega}}$,

(ii) $J_\omega J_{\bar{\omega}}$ is a principal A-ideal.

It follows that

$$\ker \bar{N} = \{[J_\omega] : [J_{\bar{\omega}}] = [J_\omega]^{-1}\}.$$

We recall from Theorem 2.4 that Quinn(G) can be identified with Cl$(A)_2$. From this and Theorem 4.4, we are able to study the action of Quinn(G) on Ell(G). For this purpose, we require two further lemmas.

Lemma 4.5. *Let* $\iota\colon \mathrm{Cl}(A) \to \mathrm{Cl}(\widetilde{A})$ *be the canonical map, where* $\iota([I]) = [I\widetilde{A}]$ $(I \trianglelefteq A)$*. Then*

(i) ι *is injective,*

(ii) $\{[I] \in \mathrm{Cl}(\widetilde{A}); [I] = [\overline{I}]\} = \iota(\mathrm{Cl}(A))$.

Proof. The analogous statements are known to hold for the canonical map

$$\mathrm{Cl}^0(K) \to \mathrm{Cl}^0(\widetilde{K}).$$

See [10, Corollary to Proposition 11.10]. The results follow from the exact sequence in Section 2, since δ is odd and the infinite place is inert in $\widetilde{K}$. ■

Lemma 4.6. *With the above notation, the* 2*-torsion in* $\mathrm{Cl}(A)$*,*

$$\mathrm{Cl}(A)_2 \cong \iota(\mathrm{Cl}(A)_2) = (\ker \overline{N})_2,$$

the 2*-torsion in* $\ker \overline{N}$.

Proof. Let $[I] \in \mathrm{Cl}(A)_2$. Then $\iota([I])$ has order 2 in $\mathrm{Cl}(\widetilde{A})$ by Lemma 4.5. Now

$$\overline{N}(\iota([I])) = \iota([I])\overline{\iota([I])} = (\iota([I]))^2 = 1$$

by Lemma 4.5 (ii). Hence $\iota([I]) \in \ker \overline{N}$. Conversely, let $[J] \in Cl(\widetilde{A})$ have order 2 and lie in $\ker \overline{N}$. Then $[J]^2 = 1$ and $[J][\overline{J}] = \overline{N}([J]) = 1$. Hence $[J] = [\overline{J}]$, and so $[J] \in \iota(\mathrm{Cl}(A)_2)$ again by Lemma 4.5 (ii). ■

Any element of $N_{\widehat{G}}(G)$ can be represented by a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widehat{G}.$$

By multiplying M by a suitable scalar matrix, we may assume that $a, b, c, d \in A$. As before, let

$$\mathfrak{q}(M) := (a) + (b) + (c) + (d).$$

Then

(i) $\mathfrak{q}(M)^2 = (\Delta)$.

(ii) $(a) + (b) = (a) + (c) = (d) + (b) = (d) + (c) = \mathfrak{q}(M)$.

See Theorem 2.2 and [1, Remarks 2]. Thus, $\mathfrak{q}$ induces an isomorphism from $\mathrm{Quinn}(G)$ onto $\mathrm{Cl}(A)_2$, and so $\iota \circ \mathfrak{q}$ provides an embedding of $\mathrm{Quinn}(G)$ into $\mathrm{Cl}(\widetilde{A})$.

As before, each $\omega \in E(G)$ can be represented as $\omega = \frac{\varepsilon+s}{t}$, where $s, t \in A$ and t divides $(\varepsilon^q + s)(\varepsilon + s)$ in A. The element M acts as a Möbius transformation on ω by multiplying the column vector $\binom{\varepsilon+s}{t}$ on the left by the matrix M. It follows that $J_{M(\omega)}$ is the $\widetilde{A}$-ideal generated by $a(\varepsilon + s) + bt$ and $c(\varepsilon + s) + dt$. Our next result, the most important in this section, shows that the action of $\mathrm{Quinn}(G)$ on $\mathrm{Ell}(G)$ is equivalent to group multiplication in $\ker \overline{N}$.

Theorem 4.7. *With the above notation,*

$$[J_{M(\omega)}] = [\iota(\mathfrak{q}(M))J_\omega] = [\iota(\mathfrak{q}(M))][J_\omega] \quad \textit{in } \ker \overline{N}.$$

Proof. From the above, it is clear that $J_{M(\omega)} \le \mathfrak{q}(M)J_\omega$. Since $\widetilde{A}$ is a Dedekind domain, there is an integral ideal I_1 of $\widetilde{A}$ such that

$$J_{M(\omega)} = \mathfrak{q}(M)J_\omega I_1.$$

By the same argument, there exists an integral ideal I_2 of $\widetilde{A}$ with

$$J_{M^2(\omega)} = \mathfrak{q}(M)J_{M(\omega)}I_2 = \mathfrak{q}(M)^2 J_\omega I_1 I_2 = \Delta J_\omega I_1 I_2.$$

On the other hand, from part (i) of Corollary 2.3, we see that $J_{M^2(\omega)} = \Delta J_\omega$. Hence $I_1 = I_2 = \widetilde{A}$, and the result follows. ■

An immediate consequence is the following.

Corollary 4.8. *The group* Quinn(G) *acts freely on* Ell(G)*. More precisely, a quasi-inner automorphism that fixes an elliptic point in $G \setminus \Omega$ must necessarily be inner.*

Since

$$G\omega = G\overline{\omega} \;\Leftrightarrow\; [J_{\overline{\omega}}] = [J_\omega] = [J_\omega]^{-1},$$

we can identify Ell$(G)^=$ with $\iota(\mathrm{Cl}(A)_2) \cong$ Quinn(G). Combining Lemma 4.6 and Corollary 4.8, we obtain the following result.

Theorem 4.9. *The group* Quinn(G) *acts freely and transitively on* Ell$(G)^=$.

Theorem 3.13 (ii), which holds for all δ, provides an alternative proof of Theorem 4.9. Applying the former for the case of odd δ, the latter then follows from the existence of a Quinn(G)-invariant one-to-one correspondence between Ell$(G)^=$ and $\{\widetilde{v} \in \mathrm{vert}(G\backslash\mathcal{T}) : G_v \cong \mathrm{GL}_2(\mathbb{F}_q)\}$.

From the above, it is clear that $|\mathrm{Ell}(G)| = n_E|\mathrm{Ell}(G)^=|$, where

$$n_E = |\ker \overline{N} : \iota(\mathrm{Cl}(A)_2)|.$$

It follows that $|\mathrm{Ell}(G)^{\neq}| = (n_E - 1)|\mathrm{Ell}(G)^=|$.

We recall that the *building map* [3, p. 41] restricts to a map

$$\lambda\colon\ E(G) \to \mathrm{vert}(\mathcal{T}),$$

for which $G^\omega \le G_{\lambda(\omega)}$. Let κ be a quasi-inner automorphism. Then by [3, p. 44, item (iii)],

$$\lambda(\kappa(\omega)) = \kappa(\lambda(\omega)).$$

Then λ induces a map

$$\mathrm{Ell}(G) \mapsto \mathrm{vert}\,\mathcal{T}.$$

By Lemma 4.2 (ii), Theorem 3.13 and [9, Proposition 3.4], this leads to two Quinn(G)-invariant one-to-one correspondences,

$$\mathrm{Ell}(G)^{=} \leftrightarrow \{\tilde{v} \in \mathrm{vert}(G\backslash\mathcal{T}) : G_v \cong \mathrm{GL}_2(\mathbb{F}_q)\},$$
$$\mathcal{G} = \{\{G\omega, G\bar{\omega}\} : G\omega \neq G\bar{\omega}\} \leftrightarrow \mathcal{V} = \{\tilde{v} \in \mathrm{vert}(G\backslash\mathcal{T}) : G_v \cong \mathbb{F}_{q^2}^*\}.$$

Note that $|\mathcal{G}| = \frac{1}{2}|\mathrm{Ell}(G)^{\neq}|$.

Lemma 4.10. *Let $G\omega \in \mathrm{Ell}(G)^{\neq}$, and let κ be a quasi-inner automorphism represented by $M \in N_{\hat{G}}(G)$. Then $\kappa(G\omega) = G\bar{\omega}$ if and only if $[\iota(\mathfrak{q}(M))] = [J_\omega]^2$ and $[J_\omega]$ has order 4 in $\ker \bar{N}$.*

Proof. Let $n > 2$ be the order of $[J_\omega]$ in $\ker \bar{N}$. If $\kappa(G\omega) = G\bar{\omega}$, then by Theorem 4.7,

$$[\iota(\mathfrak{q}(M))][J_\omega] = [J_\omega]^{n-1}.$$

Hence $[\iota(\mathfrak{q}(M))] = [J_\omega]^{n-2} = [J_\omega]^{-2}$, and so $n = 4$. The converse is straightforward. ■

The following is an immediate consequence.

Lemma 4.11. *Let $\tilde{v} \in \mathcal{V}$, and let $\{G\omega, G\bar{\omega}\}$ be the corresponding elliptic element of $\tilde{v}$. Then the length of the orbit of $\tilde{v}$ under the action of Quinn(G) is $\frac{1}{2}|\mathrm{Quinn}(G)|$ if $[J_\omega]$ has order 4 in $\ker \bar{N}$ and $|\mathrm{Quinn}(G)|$ otherwise.*

Proposition 4.12. *Suppose that $|\mathrm{Ell}(G)^{=}| < |\mathrm{Ell}(G)|$. Then*

(a) Quinn(G) *acts transitively on* $\mathrm{Ell}(G)^{\neq}$ *if and only if* $n_E = 2$.

(b) Quinn(G) *acts transitively on* $\mathcal{V}$ *if and only if* $n_E \in \{2, 3\}$.

(c) Quinn(G) *acts freely on* $\mathcal{V}$ *if and only if* n_E *is odd.*

(d) Quinn(G) *acts freely and transitively on* $\mathcal{V}$ *if and only if* $n_E = 3$.

Proof. (a) Since Quinn(G) acts freely on $\mathrm{Ell}(G)^{\neq}$, the action is transitive if and only if $|\mathrm{Quinn}(G)| = |\mathrm{Ell}(G)^{=}| = |\mathrm{Ell}(G)^{\neq}|$ that is if $n_E = 2$.

(b) If Quinn(G) acts transitively on $\mathcal{V}$, then $|\mathcal{G}| \le |\mathrm{Ell}(G)^{=}|$ and so $n_E \in \{2, 3\}$. When $n_E = 2$, (a) applies. When $n_E = 3$, the two Quinn(G)-orbits represented by $G\omega$ and $G\bar{\omega}$ are identified in $\mathcal{G}$.

(c) By Lemma 4.10, the action of Quinn(G) on $\mathcal{G}$ is *not* free if and only if there exists $[J_\omega]$ of order 4, and such an element exists if and only if n_E is even.

(d) follows from (b) and (c). ■

Remark 4.13. Suppose that $g(K) = g > 0$. The 2-torsion rank of an abelian variety of dimension g is bounded by $2g$. Applying this to $\mathrm{Cl}^0(\tilde{K})$ or $\mathrm{Cl}(\tilde{A})$ (and using the fact that δ is odd), it follows that

$$|\mathrm{Ell}(G)^{=}| \le 2^{2g}.$$

See [10, Chapter 11]. On the other hand by the Riemann hypothesis for function fields [13, Theorems 5.1.15 (e) and 5.2.1],

$$|\mathrm{Ell}(G)| = L_K(-1) \geq (\sqrt{q}-1)^{2g}.$$

If $n_E = 2$, then

$$2^{2g+1} \geq (\sqrt{q}-1)^{2g}.$$

(a) If $q \geq 16$ (and $g > 0$), then $\mathrm{Quinn}(G)$ cannot act transitively on $\mathrm{Ell}(G)^{\neq}$.

Another consequence follows using an identical argument.

(b) If $q \geq 23$ (and $g > 0$), then $\mathrm{Quinn}(G)$ cannot act transitively on $\mathcal{V}$.

Remark 4.14. It is known [8, Corollary 2.12, Theorem 5.1] that a vertex $\widetilde{v}$ of $G\backslash\mathcal{T}$ is *isolated* if and only if $\delta = 1$ and $G_v \cong \mathrm{GL}_2(\mathbb{F}_q)$ or $\mathbb{F}_{q^2}^*$. Hence when $\delta = 1$, therefore Theorem 4.9, Proposition 4.12 and Remark 4.13 can be interpreted as statements about the action of $\mathrm{Quinn}(G)$ on the isolated vertices of $G\backslash\mathcal{T}$.

5. Action on cyclic subgroups

Our focus of attention in this section are the subgroups of G which are cyclic of order q^2-1. As distinct from Section 3, some of the results require δ to be *odd*.

Definition 5.1. A finite subgroup S of G is *maximally finite* if every subgroup of G which properly contains it is infinite.

Lemma 5.2. *Let C be a cyclic subgroup of G of order q^2-1 which is not maximally finite. Then there exists $H \in \mathcal{H}$ which contains C. Moreover, H is unique if δ is odd.*

Proof. By Lemma 3.1 (ii), there exists G_v which properly contains C. Hence $G_v \in \mathcal{H}$ by Lemma 3.2.

Suppose now that δ is odd. If H is not unique, then

$$C \leq G_{v_1} \cap G_{v_2},$$

where $v_1 \neq v_2$. It follows that C fixes the geodesic in $\mathcal{T}$ joining v_1 and v_2, including all its edges. This contradicts [8, Corollary 2.16]. ∎

Lemma 5.3. *Let C, C_0 be cyclic subgroups of order q^2-1 contained in some $H \in \mathcal{H}$. Then C, C_0 are conjugate in H.*

Proof. By Lemma 3.5, we may assume that $H = \mathrm{GL}_2(\mathbb{F}_q)$. This then becomes a well-known result. In the absence of a suitable reference, we sketch a proof which lies within the context of this paper.

By the proof of [8, Theorem 2.6] (based on [8, Lemma 1.4]), it follows that

$$C = F^{\mu} = \{g \in \mathrm{GL}_2(\mathbb{F}_q) : g(\mu) = \mu\}$$

for some $\mu \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q$. Let $C_0 = F^{\mu_0}$.

Now $\mu_0 = \alpha\mu + \beta$ for some $\alpha, \beta \in \mathbb{F}_q$, where $\alpha \neq 0$. Then $C_0 = g_0 C g_0^{-1}$, where

$$g_0 = \begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}. \qquad \blacksquare$$

Definition 5.4. Let

$$\begin{aligned} \mathcal{C} &= \{C \le G : C, \text{ cyclic of order } q^2 - 1\}, \\ \mathcal{C}_{mf} &= \{C \in \mathcal{C} : C, \text{ maximally finite}\}, \\ \mathcal{C}_{nm} &= \mathcal{C}\backslash\mathcal{C}_{mf}. \end{aligned}$$

Clearly, every automorphism of G acts on both $\mathcal{C}_{mf}$ and $\mathcal{C}_{nm}$.

Proposition 5.5. *The quasi-inner automorphisms act transitively on all cyclic subgroups of G of order $q^2 - 1$ that are not maximally finite.*

Proof. Let $C \in \mathcal{C}_{nm}$. Then by Lemmas 3.5 and 5.2, there exists $g_0 \in N_{\hat{G}}(G)$ such that

$$C^{g_0} \in \mathrm{GL}_2(\mathbb{F}_q).$$

The rest follows from Lemma 5.3. $\blacksquare$

The next result follows from Proposition 5.5 and Theorem 3.13.

Proposition 5.6. *If δ is odd,* Quinn(G) *acts freely and transitively on the conjugacy classes (in G) of cyclic subgroups of G of order $q^2 - 1$ that are not maximally finite.*

The restrictions on δ in Lemma 5.2 and Proposition 5.6 are necessary.

Example 5.7. Consider the case where $g(K) = 0$, $\delta = 2$. This case is studied in detail in [7, Section 3]. By the exact sequence in Section 2, it is known that here

$$\mathrm{Cl}(A) = \mathrm{Cl}(A)_2 \cong \mathrm{Quinn}(G) \cong \mathbb{Z}/2\mathbb{Z}.$$

There exists a vertex v_0 adjacent to the standard vertex v_s and $g_0 \in N_{\hat{G}}(G)\backslash G$ such that

$$G_{v_0} = \mathrm{GL}_2(\mathbb{F}_q)^{g_0} \quad \text{and} \quad G_{v_s} \cap G_{v_0} \in \mathcal{C}_{nm}.$$

Hence the restriction on δ in part of Lemma 5.2 is necessary.

It is known [7, Theorem 3.3] that in this case,

$$G = \mathrm{GL}_2(\mathbb{F}_q) *_C \mathrm{GL}_2(\mathbb{F}_q)^{g_0},$$

where $C\ (= \mathrm{GL}_2(\mathbb{F}_q) \cap \mathrm{GL}_2(\mathbb{F}_q)^{g_0}) \in \mathcal{C}_{nm}$. It follows by Lemma 5.3 that there exists $g \in \mathrm{GL}_2(\mathbb{F}_q)$ for which

$$C^g = C^{g_0}.$$

In this case, therefore $\mathrm{Quinn}(G)$, which is non-trivial, fixes C^G. The restriction on δ in Proposition 5.6 is therefore necessary.

We conclude this section with some remarks about $\mathcal{C}_{mf}$.

Lemma 5.8. *Suppose that δ is odd. Then*

$$C \in \mathcal{C}_{mf} \ \Leftrightarrow\ C = G_v \cong \mathbb{F}_{q^2}^*.$$

Proof. Suppose that $C = G_v \cong \mathbb{F}_{q^2}^*$ and that $C \in \mathcal{C}_{nm}$. Then by Lemmas 3.1 and 3.3, it follows that $C \leq G_v \cap G_{v_0}$ for some $v_0 \neq v$, which contradicts [8, Corollary 2.16]. The rest follows from Lemma 3.1. ■

When δ is odd, there is therefore a one-to-one correspondence

$$(\mathcal{C}_{mf})^G \leftrightarrow \mathcal{V}.$$

For the case where δ is odd, this shows that the results in Proposition 4.12 apply to the action of $\mathrm{Quinn}(G)$ on $(\mathcal{C}_{mf})^G$.

Remark 5.9. As a Möbius transformation, every member of G fixes an element of C_∞. Suppose now that δ is even and that C is a cyclic subgroup of order $q^2 - 1$ (maximally finite or not). Then from the proof of [9, Proposition 2.3], it follows that C fixes $\mu \in K.\mathbb{F}_{q^2} \backslash K$. In this case, however $\mu \in K_\infty$ as δ is even. So μ, which is not in Ω and not in K, can neither be an inner point nor a cusp of the Drinfeld modular curve $G\backslash\Omega$. We refer to μ as *pseudo-elliptic*.

On the other hand, suppose that δ is odd. Let g be any element of infinite order in G, and let g fix λ. Then $\lambda \in K_\infty \backslash K$.

6. Action on cusps

As distinct from Section 4, the results here hold *for all* δ. Any element of $\widehat{G}$ acts on $\mathbb{P}^1(K) = K \cup \{\infty\}$ as a Möbius transformation. In this way, $\mathrm{Quinn}(G)$ acts on $G\backslash\mathbb{P}^1(K) = \mathrm{Cusp}(G)$. Every element of $\mathrm{Cusp}(G)$ can be represented in the form $(a : b)$, where $a, b \in A$. Since A is a Dedekind ring, this gives rise to a one-to-one correspondence

$$\mathrm{Cusp}(G) \leftrightarrow \mathrm{Cl}(A).$$

Hence the action of $\mathrm{Quinn}(G)$ on $\mathrm{Cusp}(G)$ translates to an action of $\mathrm{Cl}(A)_2$ on $\mathrm{Cl}(A)$. The principal result in this section is similar to but simpler than Theorem 4.7. It translates this action into multiplication in the group $\mathrm{Cl}(A)$. We sketch a proof.

We can represent any cusp, c, by an element $(x : y) \in \mathbb{P}^1(K)$, where $x, y \in A$. Let

$$J_c = xA + yA,$$

and let $[J_c]$ be its image in $\mathrm{Cl}(A)$.

Now let κ be a non-trivial element of $\mathrm{Quinn}(G)$. Then as before by Theorem 2.2, κ can be represented by a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widehat{G},$$

where we may assume that $a, b, c, d \in A$. Let $\mathfrak{q}(M)$ be the A-ideal generated by a, b, c, d.

The action of κ on c is given by the action of M multiplying the column vector $\binom{x}{y}$ on the left by M. In this way,

$$J_{\kappa(c)} = J_{M(c)} = (ax + by)A + (cx + dy)A.$$

Theorem 6.1. *Under the identification of* $\mathrm{Cusp}(G)$ *with* $\mathrm{Cl}(A)$ *and* $\mathrm{Quinn}(G)$ *with* $\mathrm{Cl}(A)_2$*, the action of* $\mathrm{Quinn}(G)$ *on the cusps translates into multiplication in the group* $\mathrm{Cl}(A)$. *More precisely,*

$$[J_{\kappa(c)}] = [\mathfrak{q}(M)J_c] = [\mathfrak{q}(M)][J_c] \quad \textit{in } \mathrm{Cl}(A).$$

Proof. Since A is a Dedekind domain, there exists an A-ideal I_1 such that

$$J_{M(c)} = \mathfrak{q}(M)J_cI_1.$$

By Corollary 2.3 (i), there exists an A-ideal I_2 with

$$\Delta J_c = J_{M^2(c)} = \mathfrak{q}(M)J_{M(c)}I_2 = \mathfrak{q}(M)^2J_cI_1I_2 = \Delta J_cI_1I_2,$$

where $\Delta = \det(M)$. Hence $I_1 = I_2 = A$, and the result follows. ∎

As in the previous section, we have the following immediate consequence.

Corollary 6.2. *If a non-trivial quasi-inner automorphism κ fixes any cusp, then κ reduces to an inner automorphism. In particular,* $\mathrm{Quinn}(G)$ *acts freely on* $\mathrm{Cusp}(G)$.

Remark 6.3. The group $\mathrm{Quinn}(G)$ acts transitively on $\mathrm{Cusp}(G)$ if and only if $\mathrm{Cl}(A)_2 = \mathrm{Cl}(A)$.

From the exact sequence in Section 2, a necessary condition for this is $\delta \in \{1, 2\}$. If $g(K) = 0$, this condition is also sufficient, as then $\mathrm{Cl}(A) \cong \mathbb{Z}/\delta\mathbb{Z}$.

But if $g(K) = g > 0$, the action cannot be transitive for $q > 9$ by an argument very similar to that used in Remark 4.13. The inequality

$$\frac{|\mathrm{Cl}^0(K)|}{|\mathrm{Cl}^0(K)_2|} \geq \frac{(\sqrt{q}-1)^{2g}}{2^{2g}}$$

shows that for fixed $q > 9$, the number of orbits of $\mathrm{Quinn}(G)$ on $\mathrm{Cusp}(G)$ tends to ∞ with $g(K)$.

The cusp ∞ $(= \binom{1}{0})$ corresponds to the principal A-ideals. Its orbit under $\mathrm{Quinn}(G)$ corresponds to the 2-torsion in $\mathrm{Cl}(A)$ and in the sense of Theorem 6.1, the action of $\mathrm{Quinn}(G)$ on it translates into $\mathrm{Cl}(A)_2$ acting on itself by multiplication.

For every cusp c, represented by the ideal class $[J_c]$ in $\mathrm{Cl}(A)$, there corresponds its (group) inverse $[J_c]^{-1}$ in $\mathrm{Cl}(A)$. We can partition $\mathrm{Cl}(A)$ thus

$$\mathrm{Quinn}(G) \leftrightarrow \mathrm{Cl}(A)_2 = \{[J_c] : [J_c] = [J_c]^{-1}\},$$
$$\mathrm{Cl}(A) \setminus \mathrm{Cl}(A)_2 = \{[J_c] : [J_c] \neq [J_c]^{-1}\}.$$

Our next result follows from Theorem 6.1 in the same way as Lemma 4.10 follows from Theorem 4.7.

Lemma 6.4. *A quasi-inner automorphism κ, represented by $M \in N_{\hat{G}}(G)$, maps the cusp c corresponding to $[J_c]$ in $\mathrm{Cl}(A) \setminus \mathrm{Cl}(A)_2$, to the cusp corresponding to $[J_c]^{-1}$ if and only if $[J_c]$ has order 4 and $[J_c]^2 = \mathfrak{q}(M)$.*

In the next section, we will use the results of Sections 5 and 6, together with Theorem 3.13 (ii), to examine in detail the action of $\mathrm{Quinn}(G)$ on $G \backslash \mathcal{T}$.

7. Action on the quotient graph

The model used by Serre for $\mathcal{T}$ [11, Chapter II, Section 1.1] is based on two-dimensional so called *lattice classes*. Since every quasi-inner automorphism, ι_g, can be represented by a matrix in $\hat{G}$, it acts on $\mathcal{T}$, and hence $\mathrm{Quinn}(G)$ acts on $G \backslash \mathcal{T}$.

In this section, we investigate the action of a quasi-inner automorphism on the quotient graph $H \backslash \mathcal{T}$, where H is a finite index subgroup of G. In the process, we extend a result of Serre [11, p. 117, Exercise 2 (e)] which motivated our interest in this question. We begin with a detailed account of Serre's classical description of $G \backslash \mathcal{T}$. Serre's original proof [11, p. 106, Theorem 9] is based on the theory of vector bundles. For a more detailed version which refers explicitly to matrices, see [5]. In addition, we use the results of the previous sections to shed new light on the structure of $G \backslash \mathcal{T}$.

Definition 7.1. A *ray* $\mathcal{R}$ in a graph $\mathcal{G}$ is an infinite half-line, without backtracking. In accordance with Serre's terminology [11, p. 104], we call $\mathcal{R}$ *cuspidal* if all its non-terminal vertices have valency 2 (in $\mathcal{G}$).

Let $\{g_1, \dots, g_s\} \subseteq \hat{G}$, where $s \geq 1$, be a complete system of representatives for $\mathrm{Cl}(A)_2$ $(\cong N_{\hat{G}}(G)/G.Z(K))$. Let $c_i = g_i(\infty)$, $1 \leq i \leq s$. We will assume that $c_1 = \infty$. If $\mathrm{Cl}(A) = \mathrm{Cl}(A)_2$, then $\{c_1, \dots, c_s\}$ is a complete system of representatives for $\mathrm{Cl}(A)$. If $\mathrm{Cl}(A) \neq \mathrm{Cl}(A)_2$, we can find further elements $h_1, \dots, h_t \in \hat{G}$, where $t \geq 1$ so that

$$\mathcal{S} = \{c_1, \dots, c_s,\ d_1, \dots, d_t\}$$

is a complete set of representatives for $\mathrm{Cl}(A)$, where $d_j = h_j(\infty)$, $1 \leq j \leq t$.

Theorem 7.2. *There exists a complete system of representatives* $\mathcal{C}$ *(*$\subseteq \mathbb{P}^1(K)$*) for* $\mathrm{Cusp}(G)$ *(equivalently,* $\mathrm{Cl}(A)$*) of the above type such that*

$$G\backslash\mathcal{T} = X \cup \Big(\bigcup_{1\le i\le s} \mathcal{R}(c_i)\Big)\Big(\bigcup_{1\le j\le t} \mathcal{R}(d_j)\Big),$$

where

(i) X *is finite,*

(ii) *each* $\mathcal{R}(c_i)$, $\mathcal{R}(d_j)$ *is a cuspidal ray (in* $G\backslash\mathcal{T}$*), whose only intersection with* X *consists of a single vertex,*

(iii) *the* $|\mathrm{Cl}(A)|$ *cuspidal rays are pairwise disjoint.*

Moreover, if $\mathcal{R}(e)$ *is any of these cuspidal rays, then it has a lift,* $\overline{\mathcal{R}(e)}$*, to* $\mathcal{T}$ *with the following properties. Let* $\mathrm{vert}(\overline{\mathcal{R}(c)}) = \{v_1, v_2, \dots\}$*. Then*

(i) $G_{v_i} \le G_{v_{i+1}}$, $i \ge 1$,

(ii) $\bigcup_{i\ge 1} G_{v_i} = G(c)$*, where* $G(c)$ *is the stabilizer (in* G*) of the cusp* c*.*

For each j, let $\tilde{d}_j$ be the element of $\{d_1, \dots, d_t\}$ corresponding to $h_j^{-1}(\infty)$. We may relabel the latter set as $\{d_1, \tilde{d}_1, \dots, d_{t'}, \tilde{d}_{t'}\}$, where $t' = \frac{t}{2}$. We can use the results of Section 3 to elaborate on the structure of the above cuspidal rays. We recall that

$$\mathcal{H} = \{H \le G : H \cong \mathrm{GL}_2(\mathbb{F}_q)\}.$$

Corollary 7.3. *For the above set of* $|\mathrm{Cl}(A)|$ *cuspidal rays,*

(i) $\mathcal{R}_1 = \{\mathcal{R}(c_1), \dots, \mathcal{R}(c_s)\} \leftrightarrow \{\tilde{v} \in \mathrm{vert}(G\backslash\mathcal{T}) : G_v \in \mathcal{H}\} \leftrightarrow \mathrm{Cl}(A)_2$.

(ii) $\mathcal{R}_2 = \{\mathcal{R}(d_j), \mathcal{R}(\tilde{d}_j) : 1 \le j \le t'\} \leftrightarrow \mathrm{Cl}(A)\backslash \mathrm{Cl}(A)_2$.

Proof. Let $\tilde{v} \in \mathrm{vert}(G\backslash\mathcal{T})$, where $G_v \in \mathcal{H}$, and let $H \in \mathcal{H}$ be any representative of its stabilizer. Then, for some *unique* i,

$$H = gg_i(\mathrm{GL}_2(\mathbb{F}_q))(gg_i)^{-1},$$

where $g \in G$, by Lemmas 3.5 and 3.6. Now let u be any unipotent element of H. Then u fixes $gg_ih(\infty)$ for some $h \in \mathrm{GL}_2(\mathbb{F}_q)$. It follows that

$$u \in G(c) \Leftrightarrow c = g'c_i,$$

where $g' \in G$. The rest follows from Corollary 3.9 together with Theorem 3.13. ∎

Remark 7.4. Let $\tilde{v} \in \mathrm{vert}(G\backslash\mathcal{T})$, where $G_v \in \mathcal{H}$. Then it is shown in Corollary 7.3 that $\tilde{v}$ is adjacent in $G\backslash\mathcal{T}$ to a vertex whose stabilizer (up to conjugacy in G) is contained in $G(c_i)$, for some *unique* i. In this way, $\tilde{v}$ can be thought of as *closer* in $G\backslash\mathcal{T}$ to $\mathcal{R}(c_i)$ than to any other cuspidal ray. For the case $\delta = 1$ (and only for this case), $\tilde{v}$ is *isolated* in $G\backslash\mathcal{T}$ by [8, Theorem 5.1]. As in Takahashi's example [14], such a $\tilde{v}$ then appears as a "spike" next to its associated cuspidal ray.

For each subgroup H of G, we recall that the elements of $H\backslash\mathcal{T}$ are

$$\text{vert}(H\backslash\mathcal{T}) = \{Hv : v \in \text{vert}(\mathcal{T})\} \quad \text{and} \quad \text{edge}(H\backslash\mathcal{T}) = \{He : e \in \text{edge}(\mathcal{T})\}.$$

Definition 7.5. Let H, H^* be isomorphic subgroups of G. An isomorphism of graphs

$$\phi\colon H\backslash\mathcal{T} \to H^*\backslash\mathcal{T}$$

is said to be *stabilizer invariant* if the following condition holds.

For any $w \in \text{vert}(\mathcal{T}) \cup \text{edge}(\mathcal{T})$, let

$$\phi(Hw) = H^*w^*$$

(where $w^* \in \text{vert}(\mathcal{T})$ if and only if $w \in \text{vert}(\mathcal{T})$). Then, for all $u \in H_w$ and $u^* \in H^*w^*$,

$$H_u \cong H^*_{u^*}.$$

As we shall see, it is easy to find examples of isomorphisms of quotient graphs which are not stabilizer invariant.

Theorem 7.6. *Let $\kappa = \iota_g$, where $g \in N_{\widehat{G}}(G)$, and let H be a subgroup of G. Then the map*

$$\overline{\kappa}_H\colon H\backslash\mathcal{T} \to \kappa(H)\backslash\mathcal{T},$$

defined by

$$\overline{\kappa}_H(Hw) = H'w',$$

where $H' = H^g = gHg^{-1}$, $w' = g(w)$ and $w \in \text{vert}(\mathcal{T}) \cup \text{edge}(\mathcal{T})$, defines a stabilizer invariant isomorphism of the quotient graphs

$$\kappa(H)\backslash\mathcal{T} \cong H\backslash\mathcal{T}.$$

Proof. Note that $\overline{\kappa}_H$ is well defined since if $\kappa(x) = g_1xg_1^{-1}$, where $g_1 \in N_{\widehat{G}}(G)$, then $gg_1^{-1} \in Z(K)$ and Z_∞, the set of scalar matrices in $\text{GL}_2(K_\infty)$, stabilizes every w. The rest is obvious (since g acts on $\mathcal{T}$). ■

Let H be any finite index subgroup of G, and let M be the largest normal subgroup of G contained in H. Then $N = M \cap M^g$ is the largest (finite index) subgroup of G, contained in H, which is normalized by G, $Z(K)$ and g. (See Section 2.)

Corollary 7.7. *Suppose that κ is non-trivial (i.e., $g \notin G.Z(K)$). Let N be a finite index normal subgroup of G normalized by κ. Then the map*

$$\overline{\kappa}_N\colon N\backslash\mathcal{T} \to N\backslash\mathcal{T},$$

defined as above, is a non-trivial stabilizer invariant automorphism whose order n is even. Moreover, if $Z \leq N$, then $n = 2m$, where m divides $|G : N|$.

Proof. To prove that $\overline{\kappa}_N$ is non-trivial, it suffices to prove that $\overline{\kappa}_G$ is not the identity map. There exists $v_0 \in \mathrm{vert}(\mathcal{T})$ for which (non-central) $G_{v_0} \leq G(\infty)$ [5, Lemma 3.2]. Suppose to the contrary that $\overline{\kappa}_G$ fixes Gv_0. Then there exists $g_0 \in G$ such that $g' = gg_0 \in G(\infty)$ which implies that

$$g' = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

We may assume that $a, b, c \in A$. By Theorem 2.2, together with an argument used in the proof of Theorem 2.4, it follows that

$$a^2 A = c^2 A = acA.$$

Hence $a, c \in \mathbb{F}_q$. Thus $g' \in G$ and so $g \in G.Z(K)$.

For the second part, n is the smallest n (> 0) such that $g^n \in N.Z(K)$. Now $g^2 \in G.Z(K)$ by Corollary 2.3 (i). If n is odd, then $g \in G.Z(K)$. Hence $n = 2m$ is even. In addition, when $Z \leq N$, m divides $|G.Z(K) : N.Z(K)| = |G : N|$. ∎

A special case of Corollary 7.7, combined with Corollary 2.6, is the following.

Corollary 7.8. *Suppose that* $|\mathrm{Cl}(A)| = |\mathrm{Cusp}(G)|$ *is even. Then there exists a stabilizer invariant automorphism of* $G\backslash\mathcal{T}$ *of order* 2.

Serre [11, p. 117, Exercise 2 (e)] states this result for the case where $g(K) = 0$ (i.e., $K = \mathbb{F}_q(t)$) and δ even. The restriction here is necessary. For the case $g(K) = 0$, $\delta = 1$, in which case $A = \mathbb{F}_q[t]$ and $|\mathrm{Cl}(A)| = 1$, it is known by Nagao's theorem [11, p. 87, Corollary] that $G\backslash\mathcal{T}$ is a cuspidal ray whose terminal vertex is isolated. Here then the only (graph) automorphism is trivial.

Corollary 7.8 shows that Quinn(G) acts non-trivially on $G\backslash\mathcal{T}$. This extends to an action on its cuspidal rays which we now describe. We use the notation of Theorem 7.2.

Definition 7.9. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be rays in a graph $\mathcal{G}$. We write

$$\mathcal{R}_1 \sim \mathcal{R}_2$$

if and only if $|\mathcal{R}_i \backslash \mathcal{R}_1 \cap \mathcal{R}_2| < \infty$, $i = 1, 2$. This a well-known equivalence relation. The equivalence class containing the ray $\mathcal{R}$ is called the *end* (of $\mathcal{G}$) determined by $\mathcal{R}$. In the notation of Theorem 7.2, we denote by $\mathcal{E}(e)$ the end (in $G\backslash\mathcal{T}$) determined by $\mathcal{R}(e)$, where $e = c_i, d_j$.

Now let $\kappa = \iota_g$, where $g \in N_{\widehat{G}}(G)\backslash G.Z(K)$, be a non-trivial quasi-inner automorphism, and let $\widehat{\kappa}$ be the corresponding (non-trivial) element of Quinn(G). Now fix $e \in \mathcal{S}$. Let $e^* = \kappa(e)$. Then by Corollary 6.2, $e \neq e^*$, and we may assume that $e^* \in \mathcal{S}$.

As in Theorem 7.2,

$$\mathrm{vert}(\mathcal{R}(e)) = \{\tilde{v}_1, \tilde{v}_2, \ldots\} \quad \text{and} \quad \mathrm{vert}(\mathcal{R}(e^*)) = \{\tilde{v}_1^*, \tilde{v}_2^*, \ldots\}.$$

Recall that

$$\bigcup_{i \geq 1} G_{v_i} = G(e),$$

and that $G_{v_i} \leq G_{v_{i+1}}$, $i \geq 1$. In addition, it is known [8, Theorem 2.1 (a)] that there exists a normal subgroup N_i of G_{v_i} such that

$$G_{v_i}/N_i \cong \mathbb{F}_q^* \times \mathbb{F}_q^*,$$

where $N_i \cong V_i^+$, the additive group of an $\mathbb{F}_q$-vector space of dimension n_i. It is also known that $n_i < n_{i+1}$. Corresponding results hold for $\mathcal{R}(e^*)$.

Now let

$$m_X = \max\{|G_v| : v \in \operatorname{vert}(X)\}.$$

(Recall that X is *finite*.) Now choose any $m > m_X$. By the definition of graph automorphism $\overline{\kappa_G}$ determined by the non-trivial element $\hat{\kappa}$ of $\operatorname{Quinn}(G)$, together with Theorem 7.2, there exists $n > m_X$ such that

$$\overline{\kappa}_G\colon \tilde{v}_{m+i} \mapsto \tilde{v}^*_{n+i}$$

for all $i \geq 0$. This gives rise to a map

$$\hat{\kappa}\colon \mathcal{E}(e) \mapsto \mathcal{E}(e^*),$$

which in turn defines a $\operatorname{Quinn}(G)$-action on the ends defined by the cuspidal rays in $G\backslash\mathcal{T}$ (Theorem 7.2). Since this action coincides precisely with the action of $\operatorname{Quinn}(G)$ on $\operatorname{Cusp}(G)$, the following result is an immediate consequence of Theorem 3.13 (ii), Corollary 6.2 and Lemma 6.4.

Corollary 7.10. *With the notation of Theorem 7.2,*

(i) $\operatorname{Quinn}(G)$ *acts (simultaneously) freely and transitively on*

$$\{\mathcal{E}(c_1), \dots, \mathcal{E}(c_s)\} \quad \text{and} \quad \{\tilde{v} \in \operatorname{vert}(G\backslash\mathcal{T}) : G_v \cong \operatorname{GL}_2(\mathbb{F}_q)\}.$$

(ii) $\operatorname{Quinn}(G)$ *acts freely on*

$$\{\mathcal{E}(d_j), \mathcal{E}(\tilde{d}_j) : 1 \leq j \leq t'\}.$$

(iii) $\operatorname{Quinn}(G)$ *acts on*

$$\{\{\mathcal{E}(d_j), \mathcal{E}(\tilde{d}_j)\} : 1 \leq j \leq t'\}.$$

(iv) *Under the action of* $\operatorname{Quinn}(G)$*, some* $\mathcal{E}(d_j)$ *is mapped to* $\mathcal{E}(\tilde{d}_j)$ *if and only if* d_j *has order* 4 *in* $\operatorname{Cl}(A)$.

We recall from Proposition 4.12 that when δ is odd, $\operatorname{Quinn}(G)$ also acts on $\{\tilde{v} \in \operatorname{vert}(G\backslash\mathcal{T}) : G_v \cong \mathbb{F}^*_{q^2}\}$.

Our final result in this section concerns the action of $N_{\hat{G}}(G)$ on $\mathcal{T}$. It is known [11, p. 75, Corollary] that G acts *without inversion* (on the edges) of $\mathcal{T}$.

Proposition 7.11. *Suppose that δ is odd. Then every ι_g acts without inversion on $\mathcal{T}$, and hence on every quotient graph $H\backslash\mathcal{T}$.*

Proof. As in Theorem 2.2, we can represent ι_g with a matrix M in $\widehat{G}$, and we can assume that all its entries lie in A. Let $\Delta = \det(M)$. Then the A-ideal generated by Δ is the *square* of an ideal in A, again by Theorem 2.2. It follows that, for all places $v \neq v_\infty$, $v(\Delta)$ is even.

By the product formula, then $\delta v_\infty(\Delta)$ and hence $v_\infty(\Delta)$ is even. The result follows from [11, p. 75, Corollary]. ∎

Example 7.12. To conclude this section, we consider the case where $g(K) = 0$ and $\delta = 2$. We recall that there exists a quadratic polynomial $\pi \in \mathbb{F}_q[t]$, irreducible over $\mathbb{F}_q$, such that

$$A = \left\{\frac{f}{\pi^m} : f \in \mathbb{F}_q[t],\ m \geq 0,\ \deg f \leq 2m\right\}.$$

In this case, it is known that $\mathrm{Cl}(A)_2 = \mathrm{Cl}(A) \cong \mathrm{Quinn}(G) \cong \mathbb{Z}/2\mathbb{Z}$. It is well known that $G\backslash\mathcal{T}$ is a doubly infinite line, without backtracking. See [11, p. 113, §2.4.2 (a)] and, for a more detailed description, [7, Section 3]. It is known that $G\backslash\mathcal{T}$ lifts to a doubly infinite line $\mathcal{D}$ in $\mathcal{T}$, which we now describe in detail. For some $g_0 \in N_{\widehat{G}}(G)\backslash G.Z(K)$, $\mathrm{vert}(\mathcal{D}) = \{v_0, v_0^*, v_1, v_1^*, \dots\}$, where

(i) $v_i^* = g_0(v_i)$, $i \geq 0$,

(ii) $G_{v_i^*} = (G_{v_i})^{g_0}$, $i \geq 0$,

(iii) $G_{v_0} = \mathrm{GL}_2(\mathbb{F}_q)$,

(iv) for each $i \geq 1$,

$$G_{v_i} = \left\{\begin{bmatrix} \alpha & c\pi^{-i} \\ 0 & \beta \end{bmatrix} : \alpha, \beta \in \mathbb{F}_q^*,\ \deg c \leq 2i\right\}.$$

Then $\mathcal{D}$ maps onto (and is isomorphic to) $G\backslash\mathcal{T}$ which has the following structure:

The action of the (essentially only) non-trivial quasi-inner automorphism of $G\backslash\mathcal{T}$ (represented by g_0) is given by

$$\overline{v_i} \leftrightarrow \overline{v_i^*}, \quad i \geq 0.$$

We note two features of $\mathcal{D}$ which are of interest relevant to this section.

(i) From the structure of $\mathcal{D}$, it is clear that the non-trivial quasi-inner automorphism determined by g_0 *inverts* the edge joining v_0 and v_0^*, which shows that the restriction on δ in Proposition 7.11 is necessary.

(ii) For this case, there is only one stabilizer invariant involution. However, the *graph* $G\backslash\mathcal{T}$ has many automorphisms. Infinitely many examples include translations (which have infinite order) and reflections in any vertex (which are involutions).

8. Two instructive examples

We conclude with two examples which demonstrate how our results apply to the structure of the quotient graph $G\backslash\mathcal{T}$. Both are *elliptic* function fields $K/\mathbb{F}_q$. We record some of their basic properties.

Definition 8.1. A function field $K/\mathbb{F}_q$ is *elliptic* [13, p. 217] if $g(K) = 1$ and K has a place ∞ of degree 1.

Theorem 8.2. *Suppose that $K/\mathbb{F}_q$ is elliptic. Then*

(i) *We have*

$$K = \mathbb{F}_q(x, y),$$

where x, y satisfy a (smooth) Weierstrass equation $F(x, y) = 0$ with

$$F(x, y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 \in \mathbb{F}_q[x, y].$$

(ii) $\mathrm{Cl}^0(K)$ *(*$\cong \mathrm{Cl}(A)$*) is isomorphic to $E(\mathbb{F}_q)$, the group of $\mathbb{F}_q$-rational points,* $\{(\alpha, \beta) \in \mathbb{F}_q \times \mathbb{F}_q : F(\alpha, \beta) = 0\} \cup \{(\infty, \infty)\}$. *Here the group operation is point addition $\oplus$ according to the chord-tangent law.*

Proof. For item (i), see [13, Proposition 6.1.2]. For item (ii), see [13, Propositions 6.1.6 and 6.1.7]. ∎

Here a rational point $(a, b) \in E(\mathbb{F}_q)$ corresponds to the ideal class of $A(x - a) + A(y - b)$.

We also require some "elliptic" properties of $\widetilde{K} = K.\mathbb{F}_{q^2}$ (which is a *constant field extension* of K).

Corollary 8.3. *Suppose that $K/\mathbb{F}_q$ is elliptic. Then $\widetilde{K}/\mathbb{F}_{q^2}$ is also elliptic and defined by the same Weierstrass equation.*

Proof. From the above, $\widetilde{K} = \mathbb{F}_{q^2}(x, y)$, where $F(x, y) = 0$. The rest follows from [13, Proposition 6.1.3]. ∎

With our choice of infinite place, we have

$$A = \mathbb{F}_q[x, y] \quad \text{and} \quad \widetilde{A} = \mathbb{F}_{q^2}[x, y],$$

where x and y satisfy the Weierstrass equation $F(x, y) = 0$. In an analogous way,

$$\mathrm{Cl}(\widetilde{A}) \cong \mathrm{Cl}^0(\widetilde{K}) \cong E(\mathbb{F}_{q^2}).$$

We recall that the image of any $\alpha \in \mathbb{F}_{q^2}$ under the Galois automorphism of $\mathbb{F}_{q^2}/\mathbb{F}_q$ is denoted by $\overline{\alpha}$. For each rational point $P = (\alpha, \beta) \in E(\mathbb{F}_{q^2})$, we put $\overline{P} = (\overline{\alpha}, \overline{\beta})$.

Corollary 8.4. *Suppose that $K/\mathbb{F}_q$ is elliptic. Under the identifications of* $\mathrm{Cl}^0(\tilde{K})$ *(resp.* $\mathrm{Cl}^0(K)$*) with $E(\mathbb{F}_{q^2})$ (resp. $E(\mathbb{F}_q)$), the norm map* $N\colon \mathrm{Cl}^0(\tilde{K}) \to \mathrm{Cl}^0(K)$ *translates to a map $N_E\colon E(\mathbb{F}_{q^2}) \to E(\mathbb{F}_q)$ defined by*

$$N_E(P) = P \oplus \bar{P},$$

so that

$$P \in \ker N_E \;\Leftrightarrow\; \bar{P} = -P.$$

Takahashi [14] has described in detail the quotient graph for an elliptic function field over *any* field of constants. In all cases, $G\backslash\mathcal{T}$ is a *tree*. Since $\delta = 1$, for the case of a finite field of constants, the isolated vertices of $G\backslash\mathcal{T}$ are precisely those whose stabilizer is isomorphic to $\mathrm{GL}_2(\mathbb{F}_q)$ or $\mathbb{F}_{q^2}^*$ by [8, Theorem 5.1]. For each cusp $c \in \mathrm{Cl}(A)_2$, the cuspidal ray $\mathcal{R}(c)$ in $G\backslash\mathcal{T}$ has attached to its terminal vertex (appearing as a "spike") an isolated vertex with stabilizer isomorphic to $\mathrm{GL}_2(\mathbb{F}_q)$. The remaining cuspidal rays consist of $\frac{1}{2}|\mathrm{Cl}(A)\setminus \mathrm{Cl}(A)_2|$ inverse pairs $\{\mathcal{R}(c), \mathcal{R}(c^{-1})\}$ which share a terminal vertex (appearing in $G\backslash\mathcal{T}$ as the "prongs" of a "fork").

In both our examples $q = 7$ in which case the Weierstrass equation can be assumed to take the short form

$$y^2 = f(x) = x^3 + ax + b,$$

where $a, b \in \mathbb{F}_q$ and $f(x)$ has no repeated roots.

Example 8.5. Let $K = \mathbb{F}_7(x, y)$, $A = \mathbb{F}_7[x, y]$ with $y^2 = x^3 - 3x$.

It can be easily shown that

$$E(\mathbb{F}_7) = \{(\infty,\infty), (0,0), (2,\pm 3), (3,\pm 2), (6,\pm 3)\}.$$

Since E is in the short Weierstrass form, the 8 points are listed as (additive) inverse pairs. In particular, $(0,0)$ is the only such 2-torsion point. It follows that $\mathrm{Quinn}(G) \cong \mathrm{Cl}(A)_2 \cong \mathbb{Z}/2\mathbb{Z}$ and hence that $\mathrm{Cl}(A) \cong \mathbb{Z}/8\mathbb{Z}$. Let κ be a non-trivial quasi-inner automorphism of G representing the non-trivial element of $\mathrm{Quinn}(G)$. In $E(\mathbb{F}_7)$, κ is represented by $(0,0)$ and, by Theorem 6.1, its action on $\mathrm{Cusp}(G)$ is determined by its action (via point addition $\oplus$) in $E(\mathbb{F}_7)$. In a diagram of $G\backslash\mathcal{T}$, as described in [14], we wish to ensure that its involution provided by κ, Corollary 7.8, is given by the reflection in the vertical axis (see Figure 1 below). We begin by labeling appropriately its 8 cuspidal rays (corresponding to $E(\mathbb{F}_7)$). By Corollary 6.2, κ acts freely on these. By Corollary 7.3 (i), it is clear that κ interchanges the cusps (∞,∞) and $(0,0)$. Attached to each of these is a "spike" consisting of an isolated vertex whose stabilizer is isomorphic to $\mathrm{GL}_2(\mathbb{F}_7)$. Since κ is a graph automorphism, it interchanges these vertices, namely, g_1 and g_2. By means of the duplication formula [12, p. 53], it is easily checked that the rational 4-torsion points are $(2,\pm 3)$. Then κ interchanges $(2,3)$ and $(2,-3)$ by Lemma 6.4. For the remaining cusps, κ interchanges $(3,\pm 2)$ and $(6,\pm 3)$. To make this more precise, we use the addition formulae [12, p. 53] which show that $(0,0) \oplus (3,2) = (6,3)$. Hence κ interchanges $(3,2)$ and $(6,3)$ by Theorem 6.1.

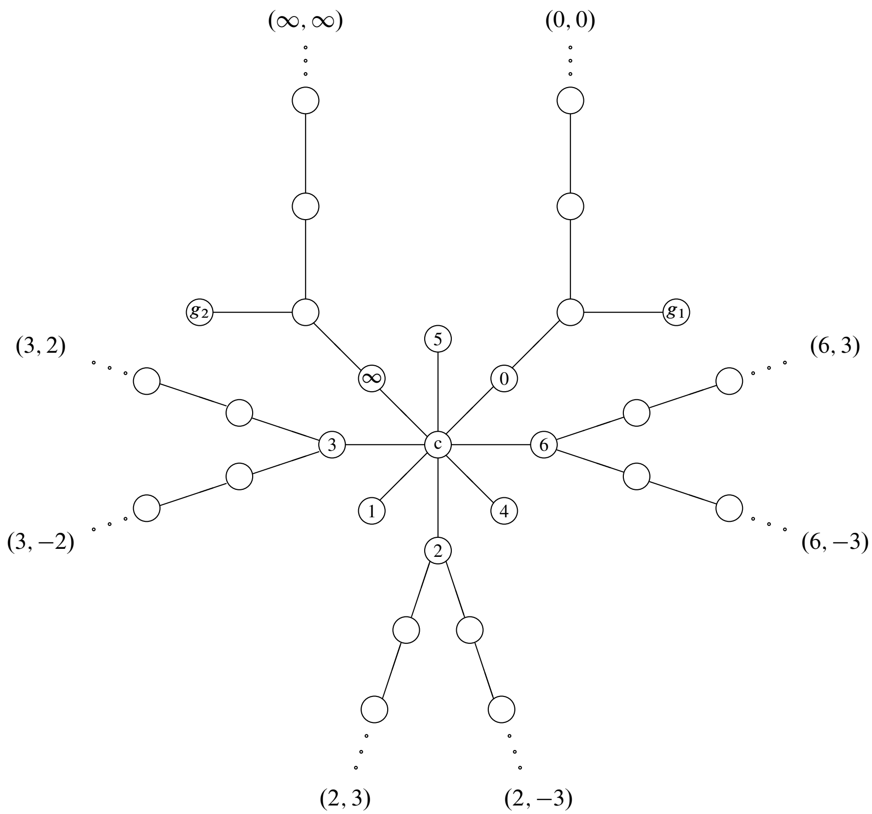

Figure 1. Quotient graph for Example 8.5.

There remain the isolated vertices 1, 4 and 5, each of whose stabilizer is isomorphic to $\mathbb{F}_{49}^*$. We deal with these via their connection with elliptic points. We recall from Theorem 4.4 and the above that there exists a one-to-one correspondence

$$\mathrm{Ell}(G) \leftrightarrow \ker N_E = \{(\alpha, \beta) \in E(\mathbb{F}_{49}) : (\overline{\alpha}, \overline{\beta}) = (\alpha, -\beta)\}$$

since the Weierstrass equation is in the short form.

Now let i denote one of the two square roots of -1 in $\mathbb{F}_{q^2}^*$. Then

$$N_E = \{(\rho, \varepsilon i) \in E(\mathbb{F}_{49}) : \rho, \varepsilon \in \mathbb{F}_q\}.$$

We conclude then that $\mathrm{Ell}(G) \leftrightarrow \{(\infty, \infty), (0, 0), (1, \pm 3i), (4, \pm 2i), (5, \pm 3i)\}$. Here $\mathrm{Ell}(G)$ is identified with a subgroup of $E(\mathbb{F}_{49})$ listed as (additive) inverse pairs. Since there is only one 2-torsion point, $\mathrm{Ell}(G) \cong \mathbb{Z}/8\mathbb{Z}$. (In this case, $|\mathrm{Cl}(A)| = |\mathrm{Ell}(G)|$. However, this not a general feature. For this particular K, its L-polynomial is $L_K(t) = 1 + 7t^2$, so that $L_K(1) = L_K(-1)$.)

As with $\mathrm{Cusp}(G)$, the *free* action (Corollary 4.8) of $\mathrm{Quinn}(G)$ on $\mathrm{Ell}(G)$ is represented by the action of $(0, 0)$ in N_E (by point addition).

By identifications in Section 4, the pairs $(1, \pm 3i)$, $(4, \pm 2i)$, $(5, \pm 3i)$ correspond to the vertices 1, 4 and 5, respectively. By means of the duplication formula, it is readily verified that the two points of order 4 in $\mathrm{Ell}(G)$ are $(5, \pm 3i)$. By Lemma 4.10, it follows that κ fixes vertex 5 and that κ interchanges vertices 1, 4. For a more precise version of the latter statement, we note that $(0,0) \oplus (1, 3i) = (4, 2i)$, and so $(0,0) \oplus (1, -3i) = (4, -2i)$.

It is of interest to use Theorem 2.2 to construct a matrix M which represents κ. We begin with the A-ideal, $Ax + Ay$ whose square is Ax. In determining a possible M, we recall from the proof of Theorem 2.4 the observation of Cremona [1] that every row and column of M generates $\mathfrak{q}(M)$. Two possibilities which arise are

$$M = \begin{bmatrix} y & x^2 \\ x & y \end{bmatrix} \quad \text{or} \quad M = \begin{bmatrix} y & -x^2 \\ x & -y \end{bmatrix}.$$

The latter is simpler since its square is a scalar matrix.

Example 8.6. Let $K = \mathbb{F}_7(x, y)$, $A = \mathbb{F}_7[x, y]$ with $y^2 = x^3 - x$.

It is easily verified that

$$E(\mathbb{F}_7) = \{(\infty, \infty), (0,0), (1,0), (6,0), (4, \pm 2), (5, \pm 1)\},$$

listed as (additive) inverse pairs. The 2-torsion points are $(0,0)$, $(1,0)$, $(6,0)$, and so

$$\mathrm{Quinn}(G) \cong \mathrm{Cl}(A)_2 \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$
$$\mathrm{Cusp}(G) \cong \mathrm{Cl}(A) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

Let the non-trivial quasi-inner automorphisms $\kappa_0, \kappa_1, \kappa_6$ represent $(0,0)$, $(1,0)$, $(6,0)$, respectively, where $\kappa_0 = \kappa_1 \kappa_6$. In the diagram representing $G \backslash \mathcal{T}$ (see Figure 2), we label the 8 cusps with the above rational points in such a way that (i) the action of κ_6 is the reflection about the vertical axis, (ii) the action of κ_1 is the reflection about the horizontal axis, and (iii) (consequently) the action of κ_0 is a rotation of 180 degrees about the "central" vertex c.

There are 4 vertices whose stabilizers are isomorphic to $\mathrm{GL}_2(\mathbb{F}_7)$ which appear as "spikes" attached to the 4 cusps given by the 2-torsion points in $E(\mathbb{F}_7)$, and so κ_6, κ_1 and κ_0 interchange the vertex pairs $\{g_1, g_2\}$, $\{g_1, g_4\}$ and $\{g_1, g_3\}$, respectively.

In $\mathrm{Cl}(A)$, there are 4 points of order 4, namely $(4, \pm 2)$ and $(5, \pm 1)$, and it is easily verified that the square of each is $(1,0)$. By Lemma 6.4, it follows that κ_1 interchanges the cusps $(4,2)$, $(4,-2)$ as well as $(5,1)$, $(5,-1)$. On the other hand, κ_6 interchanges the pairs $(4, \pm 2)$ and $(5, \pm 1)$. In more detail, κ_6 maps $(5,1)$ to $(4,-2)$, since $(6,0) \oplus (5,1) = (4,-2)$.

There remain two vertices 2 and 3 whose stabilizers are cyclic order $q^2 - 1$. As in the previous example, we consider the elliptic function field $\widetilde{K} = K.\mathbb{F}_{49} = \mathbb{F}_{49}(x, y)$: $y^2 = x^3 - x$. As before, let i denote one of the square roots of -1 in $\mathbb{F}_{49}$. It can be verified that $\mathrm{Ell}(G) \leftrightarrow N_E = \{(\infty, \infty), (0,0), (1,0), (6,0), (2, \pm i), (3, \pm 2i)\}$, listed as additive inverse pairs in $E(\mathbb{F}_{49})$.

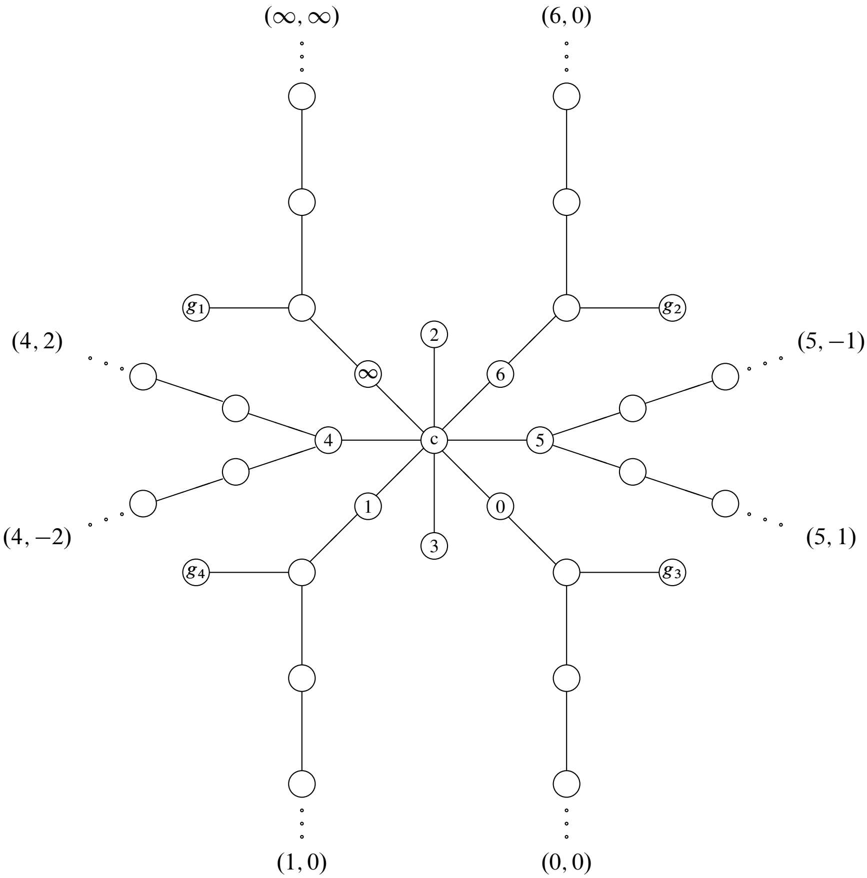

Figure 2. Quotient graph for Example 8.6.

As before, $|\mathrm{Cl}(A)| = |\mathrm{Ell}(G)| = 8$. (Again this is purely coincidental because $L_K(t) = 1 + 7t^2$.) By correspondences discussed in Section 4, the 2 vertices of interest here correspond to the pairs $(2, \pm i)$ and $(3, \pm 2i)$. It is easily verified that the squares of all 4 of these points are $(6, 0)$. It follows from Lemma 4.10 that κ_6 fixes 2 and 3. On the other hand, $(1, 0) \oplus (2, i) = (3, -2i)$ and so κ_1 interchanges 2 and 3.

Finally, using Theorem 2.2 the following matrices M_0, M_1, $M_6 = M_0 M_1$ represent $\kappa_0, \kappa_1, \kappa_6$, respectively,

$$M_0 = \begin{bmatrix} y & -x^2 \\ x & -y \end{bmatrix} \quad \text{and} \quad M_1 = \begin{bmatrix} y & -(x-1)(x+2) \\ x-1 & -y \end{bmatrix}.$$

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