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Bilinear control and growth of Sobolev norms for the nonlinear Schrödinger equation

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Abstract. We consider the nonlinear Schrödinger equation (NLS) on a torus of arbitrary dimension. The equation is studied the in presence of an external potential field whose time-dependent amplitude is taken as control. Assuming that the potential satisfies a saturation property, we show that the NLS equation is approximately controllable between any pair of eigenstates in arbitrarily small time. The proof is obtained by developing a multiplicative version of a geometric control approach introduced by Agrachev and Sarychev. We give an application of this result to the study of the large time behaviour of the NLS equation with random potential. More precisely, we assume that the amplitude of the potential is a random process whose law is 1-periodic in time and nondegenerate. Combining the controllability with a stopping time argument and the Markov property, we show that the trajectories of the random equation are almost surely unbounded in regular Sobolev spaces.

Keywords: nonlinear Schrödinger equation, approximate controllability, geometric control theory, growth of Sobolev norms, random perturbation.

0. Introduction

In this paper, we study the controllability and the growth of Sobolev norms for the following nonlinear Schrödinger (NLS) equation on the torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$:

$$i\partial_t\psi = -\Delta\psi + V(x)\psi + \kappa|\psi|^{2p}\psi + \langle u(t), Q(x)\rangle\psi. \tag{0.1}$$

We assume that $V : \mathbb{T}^d \to \mathbb{R}$ is an arbitrary smooth potential, $Q : \mathbb{T}^d \to \mathbb{R}^q$ is a given smooth external field subject to some geometric condition, $d, p \geq 1$ are arbitrary integers, and κ is an arbitrary real number. The role of the control (or the random perturbation) is

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played by the $\mathbb{R}^q$-valued function (or the random process) u which is assumed to depend only on time. Equation (0.1) is equipped with the initial condition

$$\psi(0, x) = \psi_0(x) \tag{0.2}$$

belonging to a Sobolev space $H^s = H^s(\mathbb{T}^d; \mathbb{C})$ of order $s > d/2$, so that the problem is locally well-posed.

The purpose of this paper is to study the NLS equation (0.1) when the driving force u acts multiplicatively through only few low Fourier modes. Referring the reader to the subsequent sections for the general setting, let us formulate in this Introduction some particular cases of our main results. Let $K \subset \mathbb{Z}^d_*$ be the set of d vectors defined by

$$K = \{(1, 0, \dots, 0),\ (0, 1, \dots, 0),\ \dots,\ (0, 0, \dots, 1, 0),\ (1, \dots, 1)\}, \tag{0.3}$$

and assume that the field $Q = (Q_1, \dots, Q_q)$ is such that

$$\{\mathbf{1},\ \sin\langle x, k\rangle,\ \cos\langle x, k\rangle : k \in K\} \subset \operatorname{span}\{Q_j : j = 1, \dots, q\}. \tag{0.4}$$

Let s_d be the least integer strictly greater than $d/2$.

Theorem A. *The problem (0.1), (0.2) is approximately controllable in the following sense: for any integer $s \geq s_d$, $\varepsilon > 0$, $\varkappa > 0$, $\psi_0 \in H^s$, and $\theta \in C^\infty(\mathbb{T}^d; \mathbb{R})$, there is a time $T \in (0, \varkappa)$, a control $u \in C^\infty([0, T]; \mathbb{R}^q)$, and a unique solution $\psi \in C([0, T]; H^s)$ of (0.1), (0.2) such that*

$$\|\psi(T) - e^{i\theta}\psi_0\|_{H^s} < \varepsilon.$$

A more general formulation of this result is given in Theorem 2.2, where the controllability is proved under an abstract saturation condition for the field Q (see Condition (C_1)). Note that the time T may depend on the initial condition ψ_0, the target $e^{i\theta}\psi_0$, and the parameters in the equation. In the second result, we show that, when the initial condition ψ_0 is an eigenstate $\phi_l(x) = (2\pi)^{-d/2} e^{i\langle x, l\rangle}$, $l \in \mathbb{Z}^d$, of the Laplacian and $V = 0$, the system can be controlled in any fixed time $T > 0$ to any L^2-neighbourhood of any target of the form $e^{i\theta}\phi_m$, $m \in \mathbb{Z}^d$.

Theorem B. *For any integer $s \geq s_d$, $\varepsilon > 0$, $l, m \in \mathbb{Z}^d$, $\theta \in C^\infty(\mathbb{T}^d; \mathbb{R})$, and $T > 0$, there is a control $u \in C^\infty([0, T]; \mathbb{R}^q)$ and a unique solution $\psi \in C([0, T]; H^s)$ of (0.1), (0.2) with $V = 0$ and $\psi_0 = \phi_l$ such that*

$$\|\psi(T) - e^{i\theta}\phi_m\|_{L^2} < \varepsilon.$$

The controllability of the Schrödinger equation with time-dependent bilinear (multiplicative) control has attracted a lot of attention during the last fifteen years. In the one-dimensional case, local exact controllability results have been established by Beauchard, Bournissou, Coron, Laurent, Teismann, and the present authors [5, 6, 9, 10, 14, 15, 20]. There is a vast literature on approximate controllability in the multidimensional case. For the first achievements, we refer the reader to the papers by Boscain, Caponigro, Chambrion, Mason, and Sigalotti [11, 17], Mirrahimi [29], and the second author [31].

Except [9, 10, 20], all the papers consider the linear Schrödinger equation, i.e., the one obtained by taking $\kappa = 0$ in (0.1); note that in that case the control problem is still nonlinear in u.

Theorems A and B are the first to deal with the problem of bilinear approximate controllability of the NLS equation. Let us emphasise that controllability between any pair of eigenstates in arbitrarily small time is new even in the linear case $\kappa = 0$. It is interesting to note that Theorem B complements results by Beauchard, Coron, and Teismann [7, 8], who prove that, for some choice of Q, there is a positive minimal time for approximate controllability to some particular states.

The approach used to prove Theorems A and B is quite different from the ones usually applied in the literature to study bilinear control systems. We proceed by developing Agrachev–Sarychev type arguments which were previously employed in the case of additive controls. Let us recall that Agrachev and Sarychev [2, 3] considered the global approximate controllability of the 2D Navier–Stokes and Euler systems. Their approach has been further extended by many authors to different equations. Let us mention, for example, the papers [35, 36] by Shirikyan who considered the approximate controllability of the 3D Navier–Stokes system, Rodrigues [33] who studied the 2D Navier–Stokes system on a rectangle with Lions boundary conditions, and Sarychev [34] who considered the 2D defocusing cubic Schrödinger equation. The configuration we use in the present paper is closer to the one in the recent paper [32], where parabolic PDEs are studied with polynomial nonlinearities. We refer the reader to the reviews [1, 37] and the paper [32] for more references and discussions.

The present paper is the first to deal with Agrachev–Sarychev type arguments in a bilinear setting. To explain the scheme of the proof of Theorem A, let us denote by $R_t(\psi_0, u)$ the solution of the problem (0.1), (0.2) defined up to some maximal time. A central role in the proof is played by the limit

$$e^{-i\delta^{-1/2}\varphi} R_\delta(e^{i\delta^{-1/2}\varphi}\psi_0, \delta^{-1}u) \to e^{-i(\mathbb{B}(\varphi)+\langle u,Q\rangle)}\psi_0 \quad \text{in } H^s \text{ as } \delta \to 0^+, \tag{0.5}$$

which holds for any $\psi_0 \in H^s$, $\varphi \in C^\infty(\mathbb{T}^d; \mathbb{R})$, and constant $u \in \mathbb{R}^q$. Here we denote $\mathbb{B}(\varphi)(x) = \sum_{j=1}^d (\partial_{x_j}\varphi(x))^2$. The limit (0.5) specifies the asymptotic behaviour of the solution of the NLS equation in small time under appropriately scaled large control and rapidly oscillating initial condition. Applying this limit with $\varphi = 0$ and using the assumption (0.4), we see that the equation can be controlled in small time from any initial point ψ_0 to an arbitrary neighbourhood of $e^{i\theta}\psi_0$ for any θ in the vector space

$$H_0 = \operatorname{span}\{\mathbf{1}, \sin\langle x,k\rangle, \cos\langle x,k\rangle : k \in K\}.$$

Applying again the limit (0.5) with functions $\varphi = \theta_j \in H_0$, $j = 0, \dots, n$, we add more directions in θ. That is, we show that the system can be steered from ψ_0 close to $e^{i\theta}\psi_0$, where θ now belongs to a larger vector space H_1 whose elements are of the form

$$\theta_0 - \sum_{j=1}^n \mathbb{B}(\theta_j).$$

We iterate this argument and construct an increasing sequence $\{H_j\}$ of subspaces such that the equation can be approximately controlled to any target $e^{i\theta}\psi_0$ with any $\theta \in H_j$ and $j \geq 1$. Using trigonometric computations, we show that the union $\bigcup_{j=1}^{\infty} H_j$ is dense in $C^k(\mathbb{T}^d;\mathbb{R})$ for any $k \geq 1$ (in other words, H_0 is a saturating space for the NLS equation, see Definition 2.1). This completes the proof of Theorem A.

Theorem B is derived from Theorem A by noticing that the eigenstate ϕ_l can be approximated in L^2 by functions of the form $e^{i\theta}\phi_m$ and that the eigenstates are constant solutions[1] of (0.1) corresponding to some control. This allows to appropriately adjust the controllability time and to choose it the same for any initial condition and target.

As an application of Theorem A, we study the large time behaviour of trajectories of the random NLS equation. We show that if a random process perturbs the same Fourier modes as in the above controllability results, then the energy is almost surely transferred to higher modes resulting in the unboundedness of trajectories in regular Sobolev spaces. More precisely, we replace the control u by an $\mathbb{R}^q$-valued random process η of the form

$$\eta(t) = \sum_{k=1}^{\infty} \mathbb{I}_{[k-1,k)}(t)\eta_k(t-k+1), \tag{0.6}$$

where $\mathbb{I}_{[k-1,k)}$ is the indicator function of the interval $[k-1,k)$ and $\{\eta_k\}$ are independent identically distributed random variables in $L^2([0,1];\mathbb{R}^q)$ with nondegenerate law (see Condition (C_2)). The solution ψ of the problem (0.1), (0.2), (0.6) will be a random process in H^s. We prove the following result.

Theorem C. *For any $s > s_d$ and any nonzero $\psi_0 \in H^s$, the trajectory of* (0.1), (0.2), (0.6) *is almost surely unbounded in H^s.*

The idea of constructing unbounded solutions by using random perturbations is not new. Such results have been obtained by Bourgain [12] and Erdoğan et al. [21] for linear one-dimensional Schrödinger equations. They also provided polynomial lower bounds for the growth. Unboundedness of trajectories for multidimensional linear Schrödinger equations is obtained in [30]. In that paper, the assumptions on the law of the random perturbation are rather general and no estimates for the growth are given; Theorem C is a generalisation of that result to the case of the NLS equations. There are also examples of linear Schrödinger equations with various deterministic time-dependent potentials which admit unbounded trajectories: e.g., see the papers by Bambusi et al. [4], Delort [19], Haus and Maspero [25, 28], and the references therein.

There are only a few results in the case of unperturbed NLS equations. For cubic defocusing Schrödinger equations on bounded domains or manifolds, the existence of unbounded trajectories in regular Sobolev spaces is a challenging open problem (see Bourgain [13]). In different situations, existence of trajectories with arbitrarily large finite growth has been shown by Kuksin [27], Colliander et al. [18], Guardia and Kaloshin [23],

[1] This follows immediately from the assumptions that $V = 0$ and $\mathbf{1} \in H_0$.

and others. Hani et al. [24] prove the existence of unbounded trajectories in the case of the cubic defocusing Schrödinger equation on the infinite cylinder $\mathbb{R} \times \mathbb{T}^d$. In the case of the cubic Szegő equation on the circle, Gérard and Grellier [22] show that the trajectories are generically unbounded in Sobolev spaces. Moreover, they exhibit the existence of a family of solutions with superpolynomial growth.

Theorem C is proved by generalising the ideas of [30]. In this nonlinear case, the main difficulties come from the weaker controllability properties of the equation and the fact that the nonlinear equation is not necessarily well-posed in negative Sobolev spaces. Let us give a brief and not entirely accurate description of the main ideas of the proof of Theorem C. Starting from any initial point $\psi_0 \in H^s$, Theorem A allows appropriately to increase the Sobolev norms of solutions by choosing the control. This, together with a compactness argument and the assumption that the law of the process η is nondegenerate, leads to a uniform estimate of the form

$$c_M = \sup_{\psi_0 \in H^s} \mathbb{P}\left\{ \sup_{t \in [0,1]} \|\psi(t)\|_{H^s} \leq M \right\} < 1$$

for any $M > 0$. Combining the latter with the Markov property, we show that

$$\mathbb{P}\left\{ \sup_{t \in [0,n]} \|\psi(t)\|_{H^s} \leq M \right\} \leq c_M^n$$

for any $\psi_0 \in H^s$. Then, the Borel–Cantelli lemma implies that the norm of any trajectory becomes almost surely larger than M in some random time that is almost surely finite. As M is arbitrary, this proves the required result.

The paper is organised as follows. In Section 1, we discuss the local well-posedness and some stability properties of the NLS equation. In Section 2, we formulate more general versions of Theorems A and B and give their proofs. Section 3 is devoted to the derivation of the limit (0.5). In Section 4, we establish a general criterion for the validity of the saturation property. Finally, in Section 5, we prove Theorem C.

Notation. In what follows, we use the following notation.

- $\langle \cdot, \cdot \rangle$ is the Euclidian scalar product in $\mathbb{R}^q$ and $\| \cdot \|$ is the corresponding norm.
- $H^s = H^s(\mathbb{T}^d; \mathbb{C}),\ s \geq 0$ and $L^p = L^p(\mathbb{T}^d; \mathbb{C}),\ p \geq 1$, are the standard Sobolev and Lebesgue spaces of functions $f : \mathbb{T}^d \to \mathbb{C}$ endowed with the norms $\| \cdot \|_s$ and $\| \cdot \|_{L^p}$. The space L^2 is endowed with the scalar product
$$\langle f, g \rangle_{L^2} = \int_{\mathbb{T}^d} f(x)\overline{g(x)}\, \mathrm{d}x.$$
- $C^s = C^s(\mathbb{T}^d; \mathbb{C})$ for $s \in \mathbb{N} \cup \{\infty\}$ is the space of s-times continuously differentiable functions $f : \mathbb{T}^d \to \mathbb{C}$.
- Let X be a Banach space. We denote by $B_X(a, r)$ the closed ball of radius $r > 0$ centred at $a \in X$.
- We write J_T instead of $[0, T]$ and J instead of $[0, 1]$.

- $C(J_T; X)$ is the space of continuous functions $f : J_T \to X$ with the norm

$$\|f\|_{C(J_T;X)} = \max_{t \in J_T} \|f(t)\|_X.$$

- $L^p(J_T; X)$, $1 \leq p < \infty$, is the space of Borel-measurable functions $f : J_T \to X$ with

$$\|f\|_{L^p(J_T;X)} = \left(\int_0^T \|f(t)\|_X^p \,\mathrm{d}t\right)^{1/p} < \infty.$$

- s_d is the least integer strictly greater than $d/2$.
- $\mathbf{1}$ is the function identically equal to 1 on $\mathbb{T}^d$.

1. Preliminaries

In this section, we consider the NLS equation (0.1), where u is a deterministic $\mathbb{R}^q$-valued function and $V : \mathbb{T}^d \to \mathbb{R}$ and $Q : \mathbb{T}^d \to \mathbb{R}^q$ are arbitrary smooth functions. In what follows, we shall always assume that the parameters $d \geq 1$, $p \geq 1$, and $\kappa \in \mathbb{R}$ are arbitrary. Here we formulate two propositions that will be used in the proofs of our main results. The first one gathers some well-known facts about the local well-posedness and stability of the NLS equation in regular Sobolev spaces.

Proposition 1.1. *For any $s > d/2$, $\hat{\psi}_0 \in H^s$, and $\hat{u} \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^q)$, there is a maximal time $T = T(\hat{\psi}_0, \hat{u}) > 0$ and a unique solution $\hat{\psi}$ of the problem* (0.1), (0.2) *with $(\psi_0, u) = (\hat{\psi}_0, \hat{u})$ whose restriction to the interval J_T belongs to $C(J_T; H^s)$ for any $T < T$. If $T < \infty$, then $\|\hat{\psi}(t)\|_s \to \infty$ as $t \to T^-$. Furthermore, for any $T < T$, there are constants $\delta = \delta(T, \Lambda) > 0$ and $C = C(T, \Lambda) > 0$, where*

$$\Lambda = \|\hat{\psi}\|_{C(J_T;H^s)} + \|\hat{u}\|_{L^2(J_T;\mathbb{R}^q)},$$

such that the following properties hold:

(i) *For any $\psi_0 \in H^s$ and $u \in L^2(J_T; \mathbb{R}^q)$ satisfying*

$$\|\psi_0 - \hat{\psi}_0\|_s + \|u - \hat{u}\|_{L^2(J_T;\mathbb{R}^q)} < \delta, \tag{1.1}$$

the problem (0.1), (0.2) *has a unique solution $\psi \in C(J_T; H^s)$.*

(ii) *Let R be the resolving operator for* (0.1), *i.e., the mapping taking a couple (ψ_0, u) satisfying* (1.1) *to the solution ψ. Then*

$$\|R(\psi_0, u) - R(\hat{\psi}_0, \hat{u})\|_{C(J_T;H^s)} \leq C(\|\psi_0 - \hat{\psi}_0\|_s + \|u - \hat{u}\|_{L^2(J_T;\mathbb{R}^q)}).$$

The proof of this proposition is rather standard, so we omit it (e.g., see [38, Section 3.3] or [16, Section 4.10] for similar results). Let S be the unit sphere in L^2. As the functions V, Q, and u are real-valued, the solution ψ belongs to S throughout its lifespan provided that $\psi_0 \in S \cap H^s$.

Before formulating the second proposition, let us introduce some notation. For any $\psi_0 \in H^s$ and $T > 0$, let $\Theta(\psi_0, T)$ be the set of functions $u \in L^2(J_T; \mathbb{R}^q)$ such that the problem (0.1), (0.2) has a solution in $C(J_T; H^s)$. By the previous proposition, the set $\Theta(\psi_0, T)$ is open in $L^2(J_T; \mathbb{R}^q)$. For any $\varphi \in C^1(\mathbb{T}^d; \mathbb{R})$, let

$$\mathbb{B}(\varphi)(x) = \sum_{j=1}^{d} (\partial_{x_j} \varphi(x))^2. \tag{1.2}$$

The following asymptotic property plays a key role in this paper.

Proposition 1.2. *For any integer $s \geq s_d$, $\psi_0 \in H^s$, $u \in \mathbb{R}^q$, and $\varphi \in C^r(\mathbb{T}^d; \mathbb{R})$, where $r = s + 2$, there is a constant $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$, we have*[2] *$\delta^{-1} u \in \Theta(e^{i\delta^{-1/2}\varphi}\psi_0, \delta)$ and*

$$e^{-i\delta^{-1/2}\varphi} R_\delta(e^{i\delta^{-1/2}\varphi}\psi_0, \delta^{-1}u) \to e^{-i(\mathbb{B}(\varphi)+\langle u, Q\rangle)}\psi_0 \quad \text{in } H^s \text{ as } \delta \to 0^+. \tag{1.3}$$

Here R_δ is the restriction of the solution at time $t = \delta$.

The proof of this proposition is postponed to Section 3. The limit (1.3) is a multiplicative version of a limit established in [32, Proposition 2] in the case of parabolic PDEs with additive controls.

2. Approximate controllability

In what follows, we assume that $s \geq s_d$ is an integer and denote $r = s + 2$ as in Proposition 1.2. We start this section with a definition of a saturation property inspired by [3,35]. Let H be a finite-dimensional subspace of $C^r(\mathbb{T}^d; \mathbb{R})$, and let $F(H)$ be the largest subspace of $C^r(\mathbb{T}^d; \mathbb{R})$ whose elements can be represented in the form

$$\theta_0 - \sum_{j=1}^{n} \mathbb{B}(\theta_j)$$

for some integer $n \geq 1$ and functions $\theta_j \in H$, $j = 0, \dots, n$, where $\mathbb{B}$ is given by (1.2). As $\mathbb{B}$ is quadratic, $F(H)$ is well-defined and finite-dimensional. Let us define a nondecreasing sequence $\{H_j\}$ of finite-dimensional subspaces by $H_0 = H$ and $H_j = F(H_{j-1})$, $j \geq 1$, and denote

$$H_\infty = \bigcup_{j=1}^{\infty} H_j. \tag{2.1}$$

Definition 2.1. A finite-dimensional subspace $H \subset C^r(\mathbb{T}^d; \mathbb{R})$ is said to be *saturating* if H_∞ is dense in $C^r(\mathbb{T}^d; \mathbb{R})$.

[2] For any vector $u \in \mathbb{R}^q$, with a slight abuse of notation, we denote by the same letter the constant function equal to u.

We assume that the following condition is satisfied:

(C_1) The field $Q = (Q_1, \dots, Q_q)$ is saturating, i.e., the subspace

$$H = \operatorname{span}\{Q_j : j = 1, \dots, q\}$$

is saturating in the sense of Definition 2.1.

In this section, we prove the following result. As we will see below, it implies Theorems A and B formulated in the Introduction.

Theorem 2.2. *Assume that Condition* (C_1) *is satisfied. Then for any* $\varepsilon > 0$, $\varkappa > 0$, $\psi_0 \in H^s$, *and* $\theta \in C^r(\mathbb{T}^d; \mathbb{R})$, *there is a time* $T \in (0, \varkappa)$ *and a control* $u \in \Theta(\psi_0, T) \cap C^\infty(J_T; \mathbb{R}^q)$ *such that*

$$\|R_T(\psi_0, u) - e^{i\theta}\psi_0\|_s < \varepsilon. \tag{2.2}$$

Proof. Using an induction argument in N, we show that the approximate controllability property in this theorem is true for any $\theta \in H_N$ and $N \geq 0$. More precisely, we prove the following property:

(P_N) For any $\theta \in H_N$ and $\psi_0 \in H^s$, there is a family $\{u_\tau\}_{\tau>0} \subset L^2(J_1; \mathbb{R}^q)$ such that $u_\tau \in \Theta(\psi_0, \tau)$ for sufficiently small $\tau > 0$ and

$$R_\tau(\psi_0, u_\tau) \to e^{i\theta}\psi_0 \quad \text{in } H^s \text{ as } \tau \to 0^+. \tag{2.3}$$

Combined with the saturation hypothesis, this leads to approximate controllability for any $\theta \in C^r(\mathbb{T}^d; \mathbb{R})$.

Step 1. Case $N = 0$. Applying Proposition 1.2 with $\varphi = 0$ and $u \in \mathbb{R}^q$ such that $\theta = -\langle u, Q\rangle$, we obtain

$$R_\delta(\psi_0, \delta^{-1}u) \to e^{i\theta}\psi_0 \quad \text{in } H^s \text{ as } \delta \to 0^+.$$

This implies (2.3) with $\tau = \delta$ and $u_\tau = \delta^{-1}u$.

Step 2. Case $N \geq 1$. We assume that Property (P_{N-1}) is true. Let $\tilde\theta \in H_N$ be of the form

$$\tilde\theta = \theta_0 - \sum_{j=1}^n \mathbb{B}(\theta_j),$$

where $n \geq 1$ and $\theta_j \in H_{N-1}$, $j = 0, \dots, n$. Applying Proposition 1.2 with $\varphi = \theta_1$ and $u = 0$, we get

$$e^{-i\delta^{-1/2}\theta_1} R_\delta(e^{i\delta^{-1/2}\theta_1}\psi_0, 0) \to e^{-i\mathbb{B}(\theta_1)}\psi_0 \quad \text{in } H^s \text{ as } \delta \to 0^+. \tag{2.4}$$

The fact that $\theta_1 \in H_{N-1}$ and the induction hypothesis imply that, for any $\delta > 0$, there are families of controls $\{u^1_{\tau,\delta}\} \subset L^2(J_1; \mathbb{R}^q)$ and $\{u^2_{\tau,\delta}\} \subset L^2(J_1; \mathbb{R}^q)$ such that $u^1_{\tau,\delta} \in \Theta(\psi_0, \tau)$ and $u^2_{\tau,\delta} \in \Theta(R_\delta(e^{i\delta^{-1/2}\theta_1}\psi_0, 0), \tau)$ for sufficiently small $\tau > 0$ and

$$R_\tau(\psi_0, u^1_{\tau,\delta}) \to e^{i\delta^{-1/2}\theta_1}\psi_0, \tag{2.5}$$

$$R_\tau(R_\delta(e^{i\delta^{-1/2}\theta_1}\psi_0,0),u^2_{\tau,\delta}) \to e^{-i\delta^{-1/2}\theta_1}R_\delta(e^{i\delta^{-1/2}\theta_1}\psi_0,0) \tag{2.6}$$

in H^s as $\tau \to 0^+$. Combining these controls and using Proposition 1.1, we construct a new family $\{u^1_\tau\} \subset L^2(J_1;\mathbb{R}^q)$ such that $u^1_\tau \in \Theta(\psi_0,\tau)$ for sufficiently small $\tau > 0$ and

$$R_\tau(\psi_0,u^1_\tau) \to e^{-i\mathbb{B}(\theta_1)}\psi_0 \quad \text{in } H^s \text{ as } \tau \to 0^+.$$

Iterating this argument with $\theta_j \in H_{N-1}$, $j = 0,\dots,n$, we obtain a family $\{u^n_\tau\} \subset L^2(J_1;\mathbb{R}^q)$ such that $u^n_\tau \in \Theta(\psi_0,\tau)$ for small $\tau > 0$ and

$$R_\tau(\psi_0,u^n_\tau) \to e^{i(\theta_0-\sum_{j=1}^n \mathbb{B}(\theta_j))}\psi_0 = e^{i\tilde\theta}\psi_0 \quad \text{in } H^s \text{ as } \tau \to 0^+.$$

As $\tilde\theta \in H_N$ is arbitrary, this proves the required Property (P_N) for N.

Step 3. Conclusion. Finally, let $\theta \in C^r(\mathbb{T}^d;\mathbb{R})$ be arbitrary. By the saturation hypothesis, H_∞ is dense in $C^r(\mathbb{T}^d;\mathbb{R})$. This implies that we can find $N \ge 1$ and $\tilde\theta \in H_N$ such that

$$\|e^{i\theta}\psi_0 - e^{i\tilde\theta}\psi_0\|_s < \varepsilon.$$

Applying Property (P_N) for $\tilde\theta \in H_N$, we find $T \in (0,\varkappa)$ and $u \in \Theta(\psi_0,T)$ such that (2.2) holds. Proposition 1.1 and a density argument show that we can take $u \in \Theta(\psi_0,T) \cap C^\infty(J_T;\mathbb{R}^q)$. ■

As a consequence of this result, we have the following two theorems.

Theorem 2.3. *Under the conditions of Theorem 2.2, for any $M > 0$, $\varkappa > 0$, and nonzero $\psi_0 \in H^s$, there is a time $T \in (0,\varkappa)$ and a control $u \in \Theta(\psi_0,T)$ such that*

$$\|R_T(\psi_0,u)\|_s > M.$$

Proof. It suffices to apply Theorem 2.2 by choosing $\theta \in C^r(\mathbb{T}^d;\mathbb{R})$ such that

$$\|e^{i\theta}\psi_0\|_s > M.$$

To find such θ, we take any $\theta_1 \in C^r(\mathbb{T}^d;\mathbb{R})$ satisfying $\|e^{i\theta_1}\psi_0\|_1 \neq 0$, put $\theta = \lambda\theta_1$ with sufficiently large $\lambda > 0$, and use the inequality $\|\cdot\|_1 \le \|\cdot\|_s$. ■

Theorem 2.4. *Assume that the conditions of Theorem 2.2 are satisfied and*

$$\mathbf{1} \in \operatorname{span}\{Q_j : j = 1,\dots,q\} \quad \textit{and} \quad V = 0. \tag{2.7}$$

Then, for any $\varepsilon > 0$, $l,m \in \mathbb{Z}^d$, $\theta \in C^r(\mathbb{T}^d;\mathbb{R})$, and $T > 0$, there is a control $u \in \Theta(\phi_l,T) \cap C^\infty(J_T;\mathbb{R}^q)$ such that

$$\|R_T(\phi_l,u) - e^{i\theta}\phi_m\|_{L^2} < \varepsilon. \tag{2.8}$$

Proof. Let us take any $\theta \in C^r(\mathbb{T}^d;\mathbb{R})$ and let $\theta_1 \in C^r(\mathbb{T}^d;\mathbb{R})$ be such that

$$\|e^{i\theta_1}\phi_l - e^{i\theta}\phi_m\|_{L^2} < \varepsilon/2. \tag{2.9}$$

Applying Theorem 2.2, we find a time $T_1 \in (0, T)$ and a control $u_1 \in \Theta(\phi_l, T_1)$ such that

$$\|R_{T_1}(\phi_l, u) - e^{i\theta_1}\phi_l\|_s < \varepsilon/2. \tag{2.10}$$

Combining this with (2.9), we arrive at

$$\|R_{T_1}(\phi_l, u) - e^{i\theta}\phi_m\|_{L^2} < \varepsilon.$$

Now, notice that ϕ_l is a stationary solution of (0.1) corresponding to a control $u_0 \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^q)$ satisfying the relation

$$\langle u_0(t), Q(x)\rangle = -|l|^2 - \kappa(2\pi)^{-dp} \quad \text{for any } t \geq 0 \text{ and } x \in \mathbb{T}^d.$$

Such a choice of u_0 is possible in view of assumption (2.7). Thus, $u_0 \in \Theta(\phi_l, t)$ and $\phi_l = R_t(\phi_l, u_0)$ for any $t \geq 0$. Setting

$$u(t) = \begin{cases} u_0(t) & \text{for } t \in [0, T - T_1], \\ u_1(t - T + T_1) & \text{for } t \in (T - T_1, T], \end{cases}$$

we get (2.8). Perturbing u, we obtain a control in $\Theta(\phi_l, T) \cap C^\infty(J_T; \mathbb{R}^q)$ that still satisfies (2.8). ∎

Remark 2.5. It is not clear how to generalise Theorem 2.4 to the case of the H^s Sobolev norm. It is tempting to try to choose $\theta_1(x) = \theta(x) + \langle x, m - l\rangle$ in (2.10); however, the function θ_1 is not periodic, so Theorem 2.2 cannot be applied.

Let us close this section with an example of a saturating subspace. Let $I \subset \mathbb{Z}^d_*$ be a finite set and let

$$H = H(I) = \mathrm{span}\,\{\mathbf{1}, \sin\langle x, k\rangle, \cos\langle x, k\rangle : k \in I\}. \tag{2.11}$$

Recall that I is a *generator* if any vector of $\mathbb{Z}^d$ is a linear combination of vectors of I with integer coefficients. We write $m \perp l$ when the vectors $m, l \in \mathbb{R}^d$ are orthogonal and $m \not\perp l$ when they are not. The following proposition is proved in Section 4.

Proposition 2.6. *The subspace $H(I)$ is saturating in the sense of Definition 2.1 if and only if I is a generator and for any $l, m \in I$, there are vectors $\{n_j\}_{j=1}^{\sigma} \subset I$ such that $l \not\perp n_1$, $n_j \not\perp n_{j+1}$ for $j = 1, \ldots, \sigma - 1$, and $n_\sigma \not\perp m$.*

Clearly, the set $K \subset \mathbb{Z}^d_*$ defined by (0.3) satisfies the condition in this proposition. Therefore, the subspace $H(K)$ is saturating, and Theorems A and B follow from Theorems 2.2 and 2.4, respectively.

3. Proof of Proposition 1.2

Let us fix any $R > 0$ and assume that $\psi_0 \in H^s$, $\varphi \in C^r(\mathbb{T}^d; \mathbb{R})$, and $u \in \mathbb{R}^q$ are such that

$$\|\psi_0\|_s + \|\varphi\|_{C^r} + \|u\|_{\mathbb{R}^q} \leq R. \tag{3.1}$$

For any $\delta > 0$, we denote $\phi(t) = e^{-i\delta^{-1/2}\varphi} R_t(e^{i\delta^{-1/2}\varphi}\psi_0, \delta^{-1}u)$. According to Proposition 1.1, $\phi(t)$ exists up to some maximal time $T^\delta = T(e^{i\delta^{-1/2}\varphi}\psi_0, \delta^{-1}u)$, and

$$\|e^{i\delta^{-1/2}\varphi}\phi(t)\|_s \to \infty \quad \text{as } t \to T^{\delta-} \text{ if } T^\delta < \infty.$$

We need to show that

(a) there is a constant $\delta_0 > 0$ such that $T^\delta > \delta$ for any $\delta < \delta_0$;

(b) $\phi(\delta) \to e^{-i(\mathbb{B}(\varphi)+\langle u,Q\rangle)}\psi_0$ in H^s as $\delta \to 0^+$.

To prove these properties, we introduce the functions

$$w(t) = e^{-i(\mathbb{B}(\varphi)+\langle u,Q\rangle)t}\psi_0^\delta, \quad v(t) = \phi(\delta t) - w(t), \tag{3.2}$$

where $\psi_0^\delta \in H^r$ is such that[3]

$$\|\psi_0^\delta\|_s \le C \qquad \text{for } \delta \le 1, \tag{3.3}$$

$$\|\psi_0^\delta\|_r \le C\delta^{-1/4} \quad \text{for } \delta \le 1, \tag{3.4}$$

$$\|\psi_0 - \psi_0^\delta\|_s \to 0 \quad \text{as } \delta \to 0^+.$$

For example, we can define ψ_0^δ by using the heat semigroup: $\psi_0^\delta = e^{\delta^{1/4}\Delta}\psi_0$. In view of (3.1)–(3.4), we have

$$\|w(t)\|_s \le C, \qquad t \ge 0, \tag{3.5}$$

$$\|w(t)\|_r \le C\delta^{-1/4}, \quad t \ge 0. \tag{3.6}$$

Furthermore, $v(t)$ is well-defined for $t < \delta^{-1}T^\delta$ and satisfies the equation

$$\begin{aligned} i\partial_t v = {} & -\delta\Delta(v+w) + \delta V(v+w) + \delta\kappa|v+w|^{2p}(v+w) \\ & - i\delta^{1/2}\mathbb{D}(v+w,\varphi) + \mathbb{B}(\varphi)v + \langle u, Q\rangle v, \end{aligned} \tag{3.7}$$

and the initial condition

$$v(0) = \psi_0 - \psi_0^\delta, \tag{3.8}$$

where

$$\mathbb{D}(v+w,\varphi) = (v+w)\Delta\varphi + 2\sum_{j=1}^d \partial_{x_j}(v+w)\partial_{x_j}\varphi.$$

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ be such that $|\alpha| = |\alpha_1| + \cdots + |\alpha_d| \le s$. We take the scalar product of (3.7) with $\partial^{2\alpha}v$ in L^2 and integrating by parts, we obtain

$$\begin{aligned} \partial_t\|\partial^\alpha v\|_{L^2}^2 \le C\Big(& \delta|\langle\Delta w, \partial^{2\alpha}v\rangle_{L^2}| + \delta|\langle V(v+w), \partial^{2\alpha}v\rangle_{L^2}| \\ & + \delta|\langle|v+w|^{2p}(v+w), \partial^{2\alpha}v\rangle_{L^2}| + \delta^{1/2}|\langle\mathbb{D}(v+w,\varphi), \partial^{2\alpha}v\rangle_{L^2}| \\ & + |\langle\mathbb{B}(\varphi)v + \langle u,Q\rangle v, \partial^{2\alpha}v\rangle_{L^2}|\Big) = \sum_{j=1}^5 I_j. \end{aligned} \tag{3.9}$$

[3]In what follows, C denotes positive constants which may change from line to line. These constants depend on the parameters R, V, Q, κ, p, d, s, but not on δ.

We estimate the terms I_1, I_2, I_3, and I_5 by integrating by parts and by using (3.1), (3.5), and (3.6):

$$
\begin{aligned}
|I_1| &\leq C\delta \|w\|_r \|v\|_s \leq C\delta^{3/4}\|v\|_s, \\
|I_2| &\leq C\delta \|v+w\|_s \|v\|_s \leq C\delta\|v\|_s^2 + C\delta\|v\|_s, \\
|I_3| &\leq C\delta \|v+w\|_s^{2p+1} \|v\|_s \leq C\delta\|v\|_s^{2(p+1)} + C\delta\|v\|_s, \\
|I_5| &\leq C\|v\|_s^2.
\end{aligned}
$$

We estimate I_4 as follows:

$$
|I_4| \leq C\delta^{1/2}\|v\|_s^2 + C\delta^{1/2}\|w\|_{s+1}\|v\|_s \leq C\delta^{1/2}\|v\|_s^2 + C\delta^{1/4}\|v\|_s.
$$

In the last relation, we have used again integration by parts, the identities (3.1), (3.5) and (3.6), and the equality

$$
\langle \partial_{x_j}\varphi \partial_{x_j}\partial^\alpha v, \partial^\alpha v\rangle_{L^2} = \tfrac{1}{2}\langle \partial_{x_j}\varphi, \partial_{x_j}|\partial^\alpha v|^2\rangle_{L^2} = -\langle \partial_{x_j}^2\varphi, |\partial^\alpha v|^2\rangle_{L^2}.
$$

Summing up inequalities (3.9) for all $\alpha \in \mathbb{N}^d$, $|\alpha| \leq s$, combining the resulting inequality with the estimates for I_j and the Young inequality, and recalling that $\delta \leq 1$, we obtain

$$
\partial_t \|v\|_s^2 \leq C\delta^{1/2} + C(1+\delta^{1/2})\|v\|_s^2 + C\delta\|v\|_s^{2(p+1)}, \quad t \leq \delta^{-1}T^\delta.
$$

This inequality, together with (3.8) and the Gronwall inequality, implies that

$$
\|v(t)\|_s^2 \leq e^{C(1+\delta^{1/2})t}\left(C\delta^{1/2}t + \|\psi_0 - \psi_0^\delta\|_s^2 + C\delta\int_0^t \|v(y)\|_s^{2(p+1)}\,\mathrm{d}y\right) \tag{3.10}
$$

for $t \leq \delta^{-1}T^\delta$. Let us take $\delta_0 \in (0,1)$ so small that, for $\delta < \delta_0$,

$$
\|\psi_0 - \psi_0^\delta\|_s^2 < 1, \tag{3.11}
$$

$$
e^{C(1+\delta^{1/2})}(C\delta^{1/2} + \|\psi_0 - \psi_0^\delta\|_s^2) < 1/2, \tag{3.12}
$$

and denote

$$
\tau^\delta = \sup\{t < \delta^{-1}T^\delta : \|v(t)\|_s < 1\}.
$$

From (3.8) and (3.11) it follows that $\tau^\delta > 0$ for $\delta < \delta_0$. Let us show that $\tau^\delta > 1$ provided that

$$
\delta_0 < (2Ce^{2C})^{-1}. \tag{3.13}
$$

Assume, to reach a contradiction, that $\tau^\delta \leq 1$. Let $t = \tau^\delta$ in (3.10). By using (3.12) and (3.13), we obtain

$$
1 = \|v(\tau^\delta)\|_s^2 < \frac{1}{2} + \frac{1}{2}\int_0^{\tau^\delta} \|v(y)\|_s^{2(p+1)}\,\mathrm{d}y \leq 1.
$$

This contradiction shows that $\tau^\delta > 1$ for $\delta < \delta_0$, hence also $1 < \delta^{-1}T^\delta$. Thus, property (a) is proved. Taking $t = 1$ in (3.10), we arrive at

$$
\|v(1)\|_s^2 \leq e^{C(1+\delta^{1/2})}(C\delta^{1/2} + \|\psi_0 - \psi_0^\delta\|_s^2 + C\delta) \to 0 \quad \text{as } \delta \to 0^+.
$$

This implies (b) and completes the proof.

4. Saturating subspaces

Proof of Proposition 2.6. The proof is divided into four steps.

Step 1. First, let us assume that $I \subset \mathbb{Z}_*^d$ is an arbitrary finite set, $H_0(I) = H(I)$ is the subspace defied by (2.11), $H_j(I) = F(H_{j-1}(I))$ for $j \geq 1$, and $H_\infty(I)$ is defined by (2.1).

Step 1.1. Let us show that, if

$$\cos\langle x, m\rangle,\ \sin\langle x, m\rangle \in H_\infty(I) \quad \text{for some } m \in \mathbb{Z}_*^d,$$

then

$$\mathbb{B}(\cos\langle x, m\rangle),\ \mathbb{B}(\sin\langle x, m\rangle) \in H_\infty(I).$$

Indeed, assume that

$$\cos\langle x, m\rangle,\ \sin\langle x, m\rangle \in H_N(I) \quad \text{for some } N \geq 0. \tag{4.1}$$

The equalities

$$\cos\langle x, 2m\rangle = 1 - \frac{2}{|m|^2}\mathbb{B}(\cos\langle x, m\rangle) = \frac{2}{|m|^2}\mathbb{B}(\sin\langle x, m\rangle) - 1, \tag{4.2}$$

the assumptions $\mathbf{1} \in H(I)$ and (4.1), and the definition of F imply that

$$\cos\langle x, 2m\rangle \in H_{N+1}(I). \tag{4.3}$$

As a consequence of (4.2) and (4.3), we have

$$\begin{aligned}
\mathbb{B}(\cos\langle x, m\rangle) &= \frac{|m|^2}{2}(1 - \cos\langle x, 2m\rangle) \in H_{N+1}(I),\\
\mathbb{B}(\sin\langle x, m\rangle) &= \frac{|m|^2}{2}(1 + \cos\langle x, 2m\rangle) \in H_{N+1}(I),
\end{aligned}$$

which implies the required result.

Step 1.2. Let us show that, if

$$\cos\langle x, m\rangle, \sin\langle x, m\rangle, \cos\langle x, l\rangle, \sin\langle x, l\rangle \in H_\infty(I)$$

for some $m, l \in \mathbb{Z}_*^d$ such that $m \not\parallel l$, then

$$\cos\langle x, m+l\rangle,\ \sin\langle x, m+l\rangle \in H_\infty(I).$$

Indeed, this follows immediately from the equalities

$$\begin{aligned}
\cos\langle x, m+l\rangle = {} & \pm\frac{1}{\langle m, l\rangle}\big(\mathbb{B}(\sin\langle x, m\rangle \pm \sin\langle x, l\rangle) + \mathbb{B}(\cos\langle x, m\rangle \mp \cos\langle x, l\rangle)\\
& - \mathbb{B}(\sin\langle x, m\rangle) - \mathbb{B}(\sin\langle x, l\rangle) - \mathbb{B}(\cos\langle x, m\rangle) - \mathbb{B}(\cos\langle x, l\rangle)\big),\\
\sin\langle x, m+l\rangle = {} & \pm\frac{1}{\langle m, l\rangle}\big(\mathbb{B}(\sin\langle x, m\rangle \mp \cos\langle x, l\rangle) + \mathbb{B}(\cos\langle x, m\rangle \mp \sin\langle x, l\rangle)\\
& - \mathbb{B}(\sin\langle x, m\rangle) - \mathbb{B}(\sin\langle x, l\rangle) - \mathbb{B}(\cos\langle x, m\rangle) - \mathbb{B}(\cos\langle x, l\rangle)\big)
\end{aligned}$$

and the result of Step 1.1.

Step 2. Now, let us suppose that $I \subset \mathbb{Z}_*^d$ is a finite set such that, for any $l, m \in I$, there are vectors $\{n_j\}_{j=1}^{\sigma} \subset I$ satisfying $l \not\perp n_1, n_j \not\perp n_{j+1}$ for $j = 1, \dots, \sigma - 1$, and $n_\sigma \not\perp m$. Let $N = \text{card}(I)$ and $I = \{m_1, \dots, m_N\}$. Arguing by induction on N, we show in this step that

$$\cos\langle x, a_1 m_1 + \dots + a_N m_N\rangle, \sin\langle x, a_1 m_1 + \dots + a_N m_N\rangle \in H_\infty(I) \tag{4.4}$$

for any $a_1, \dots, a_N \in \mathbb{Z}$.

Step 2.1. Let $I = \{m_1, m_2\} \subset \mathbb{Z}_*^d$ with $m_1 \not\perp m_2$. By the result of Step 1.2, we have

$$\cos\langle x, a_1 m_1\rangle, \sin\langle x, a_1 m_1\rangle, \cos\langle x, a_2 m_2\rangle, \sin\langle x, a_2 m_2\rangle \in H_\infty(I)$$

for any $a_1, a_2 \in \mathbb{Z}$. Again, in view of Step 1.2, this implies that

$$\cos\langle x, a_1 m_1 + a_2 m_2\rangle, \sin\langle x, a_1 m_1 + a_2 m_2\rangle \in H_\infty(I)$$

for any $a_1, a_2 \in \mathbb{Z}$.

Step 2.2. Assume that the required property is true if the cardinality of the set I is less than or equal to $N - 1$. Let $I \subset \mathbb{Z}_*^d$ be such that $N = \text{card}(I)$ and $I = \{m_1, \dots, m_N\}$. Without loss of generality, we can assume $m_{N-1} \not\perp m_N$ and the set $\{m_1, \dots, m_{N-1}\}$ satisfies the condition formulated at the beginning of Step 2. Let us take any $a_1, \dots, a_N \in \mathbb{Z}$ and $k \geq 1$ and write

$$\begin{aligned} a_1 m_1 + \dots + a_N m_N = (&a_1 m_1 + \dots + a_{N-2} m_{N-2} + (a_{N-1} - k) m_{N-1}) \\ &+ (k m_{N-1} + a_N m_N). \end{aligned} \tag{4.5}$$

Then

$$\begin{aligned} \langle a_1 m_1 + \dots + (a_{N-1} - k) m_{N-1}, k m_{N-1} + a_N m_N\rangle = (a_{N-1} - k) k \|m_{N-1}\|^2 \\ + O(k) \quad \text{as } k \to \infty. \end{aligned}$$

As $m_{N-1} \neq 0$, for sufficiently large $k \geq 1$ we have

$$a_1 m_1 + \dots + a_{N-2} m_{N-2} + (a_{N-1} - k) m_{N-1} \not\perp k m_{N-1} + a_N m_N. \tag{4.6}$$

Relation (4.4) is proved by combining (4.5) and (4.6), the induction hypothesis, and the assumption that $m_{N-1} \not\perp m_N$.

Step 3. We conclude from Step 2 that, if $I \subset \mathbb{Z}_*^d$ is a set satisfying the conditions of Proposition 2.6, then

$$\cos\langle x, m\rangle, \sin\langle x, m\rangle \in H_\infty(I) \quad \text{for any } m \in \mathbb{Z}_*^d.$$

This implies that $H_\infty(I)$ is dense in $C^r(\mathbb{T}^d; \mathbb{R})$ for any $r \geq 0$, hence $H(I)$ is saturating.

Step 4. Finally, let us assume that the conditions of the proposition are not satisfied for $I \subset \mathbb{Z}_*^d$. We distinguish between two cases.

Step 4.1. If I is not a generator, we can find a vector $n \in \mathbb{Z}_*^d$ which does not belong to the set $\tilde{I}$ of linear combinations of vectors of I with integer coefficients. It is easy to see that

$$H_\infty(I) \subset \operatorname{span}\{\sin\langle x, m\rangle, \cos\langle x, m\rangle : m \in \tilde{I}\}.$$

Thus, the functions $\sin\langle x, n\rangle$ and $\cos\langle x, n\rangle$ are orthogonal to the vector space $H_\infty(I)$ in the Sobolev spaces $H^j(\mathbb{T}^d; \mathbb{R})$ for any $j \geq 0$. We conclude that $H_\infty(I)$ is not dense in $C^r(\mathbb{T}^d; \mathbb{R})$, thus the subspace $H(I)$ is not saturating.

Step 4.2. If I does not satisfy the second condition in the theorem, then it is of the form

$$I = \bigcup_{j=1}^{k} \{m_1^j, \ldots, m_{n_j}^j\},$$

where $k \geq 2$ and $m_{i_1}^{j_1} \perp m_{i_2}^{j_2}$ for any integers $1 \leq j_1 < j_2 \leq k$, $1 \leq i_1 \leq n_{j_1}$, and $1 \leq i_2 \leq n_{j_2}$. Using the arguments of Steps 1 and 2, it is easy to verify that the function $\cos\langle x, m_1^{j_1} + m_2^{j_2}\rangle$ is orthogonal to $H_\infty(I)$ in $H^j(\mathbb{T}^d; \mathbb{R})$ for any $j \geq 0$. Thus, the space $H_\infty(I)$ is not dense in $C^r(\mathbb{T}^d; \mathbb{R})$. ■

5. Growth of Sobolev norms

Let us consider the NLS equation

$$i\partial_t \psi = -\Delta\psi + V(x)\psi + \kappa|\psi|^{2p}\psi + \langle \eta(t), Q(x)\rangle\psi, \tag{5.1}$$

$$\psi(0) = \psi_0 \tag{5.2}$$

with potential V and parameters d, p, κ as in the previous sections. We assume that the field Q satisfies Condition (C_1) and η is a random process of the form (0.6) with the following condition satisfied for the random variables $\{\eta_k\}$. We denote $J = [0, 1]$ and $E = L^2(J; \mathbb{R}^q)$.

(C_2) $\{\eta_k\}$ are independent random variables in E with common law ℓ such that

$$\int_E \|y\|_E^2 \, \ell(\mathrm{d}y) < \infty \quad \text{and} \quad \operatorname{supp} \ell = E.$$

For example, this condition is satisfied if the random variables $\{\eta_k\}$ are of the form

$$\eta_k(t) = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(t), \quad t \in J,$$

where $\{b_j\}$ are nonzero real numbers satisfying $\sum_{j=1}^{\infty} b_j^2 < \infty$, $\{e_j\}$ is an orthonormal basis in E, and $\{\xi_{jk}\}$ are independent real-valued random variables. Furthermore, the law of ξ_{jk} should be assumed to have a continuous density ρ_j with respect to the Lebesgue measure such that

$$\int_{-\infty}^{\infty} x^2 \rho_j(x) \, \mathrm{d}x = 1, \quad \rho_j(x) > 0 \quad \text{for all } x \in \mathbb{R} \text{ and } j \geq 1.$$

By Proposition 1.1, the problem (5.1), (5.2) is locally well-posed in H^s for any $s > d/2$ up to some (random) maximal time $T = T(\psi_0, \eta) > 0$. Let $\mathbb{P}_{\psi_0}$ be the probability measure corresponding to the trajectories issued from ψ_0 (e.g., see [26, Section 1.3.1]). Recall that S is the unit sphere in L^2.

Theorem 5.1. *Under the Conditions* (C_1) *and* (C_2), *for any $s > s_d$ and any $\psi_0 \in H^s \cap S$, we have*

$$\mathbb{P}_{\psi_0}\Big\{\limsup_{t\to T^-} \|\psi(t)\|_s = \infty\Big\} = 1. \tag{5.3}$$

By the blow-up alternative, equality (5.3) gives new information in the case $T(\psi_0, \eta) = \infty$.

Proof. Step 1. Reduction. Together with (5.1), let us consider the following truncated NLS equation:

$$i\partial_t\psi = -\Delta\psi + V(x)\psi + \kappa\chi_R(\|\psi\|_s)|\psi|^{2p}\psi + \langle \eta(t), Q(x)\rangle\psi, \tag{5.4}$$

where $R > 0$ and $\chi_R \in C_0^\infty(\mathbb{R})$ is such that $0 \le \chi_R(x) \le 1$ for $x \in \mathbb{R}$ and $\chi_R(x) = 1$ for $|x| \le R$. Let F_k, $k \ge 1$, be the σ-algebra generated by the family $\{\eta_j\}_{j=1}^k$. The problem (5.4), (5.2) is globally well-posed. The following proposition is proved at the end of this section.

Proposition 5.2. *For any $\psi_0 \in H^s$ and $R > 0$, the problem* (5.4), (5.2) *has a unique solution $\psi^R \in C(\mathbb{R}_+; H^s)$. Moreover, for any $\boldsymbol{\psi}_0 \in C(J; H^s)$, define a $C(J; H^s)$-valued process $\boldsymbol{\psi}_k^R$ by*

$$\boldsymbol{\psi}_0^R = \boldsymbol{\psi}_0, \quad \boldsymbol{\psi}_k^R = \psi^R(k-1+\cdot)|_J, \quad k \ge 1, \tag{5.5}$$

where $\psi^R(\cdot)$ is the solution of (5.4), (5.2) *with $\psi_0 = \boldsymbol{\psi}_0^R(1)$. Then the process $\boldsymbol{\psi}_k^R$ is Markov with respect to the filtration F_k.*

Let us fix any $0 < M < R$ and consider the stopping time

$$\tau_{M,R} = \min\{k \ge 0 : \|\boldsymbol{\psi}_k^R\|_{C(J;H^s)} > M\}, \quad \boldsymbol{\psi}_0 \in C(J; H^s),$$

where the minimum over an empty set is ∞. Assume we have shown that

$$\mathbb{P}_{\boldsymbol{\psi}_0}\{\tau_{M,R} < \infty\} = 1, \quad \boldsymbol{\psi}_0 \in C(J; H^s \cap S). \tag{5.6}$$

Since $R > M$, this implies that

$$\mathbb{P}_{\psi_0}\{\tau_M < \infty\} = 1, \quad \psi_0 \in H^s \cap S, \tag{5.7}$$

where

$$\tau_M = \min\Big\{k \ge 0 : \sup_{t\in J,\, k+t<T} \|\psi(k+t)\|_s > M\Big\}$$

and again the minimum over an empty set is ∞. As $M > 0$ is arbitrary, we conclude that (5.3) holds.

Step 2. Proof of (5.6). Assume that

$$c = c(M, R) = \sup_{\boldsymbol{\psi}_0 \in C(J; H^s \cap S)} \mathbb{P}_{\boldsymbol{\psi}_0}\{\tau_{M,R} \geq 1\} < 1. \tag{5.8}$$

Combining this with the Markov property, we obtain

$$\begin{aligned} \mathbb{P}_{\boldsymbol{\psi}_0}\{\tau_{M,R} \geq n\} &= \mathbb{E}_{\boldsymbol{\psi}_0}(\mathbb{I}_{\{\tau_{M,R} \geq n-1\}} \mathbb{P}_{\boldsymbol{\phi}}\{\tau_{M,R} \geq 1\}|_{\boldsymbol{\phi}=\psi_{n-1}^R}) \\ &\leq c\, \mathbb{P}_{\boldsymbol{\psi}_0}\{\tau_{M,R} \geq n-1\}, \end{aligned}$$

where $\mathbb{E}_{\boldsymbol{\psi}_0}$ is the expectation corresponding to $\mathbb{P}_{\boldsymbol{\psi}_0}$. Iterating this inequality, we get

$$\mathbb{P}_{\boldsymbol{\psi}_0}\{\tau_{M,R} \geq n\} \leq c^n.$$

This, together with the Borel–Cantelli lemma, implies (5.6).

Step 3. Proof of (5.8). By Theorem 2.3, for any $\psi_0 \in H^{s_d} \cap S$, there is a control $u \in E$ such that

$$\sup_{t \in J, t<T} \|\psi(t)\|_{s_d} > M. \tag{5.9}$$

On the other hand, Condition (C_2) implies that

$$\mathbb{P}\{\|u - \eta\|_E < \delta\} > 0$$

for any $\delta > 0$. Combining this with Proposition 1.1 and inequality (5.9), we see that there is a number $\delta > 0$ such that

$$\inf_{\psi_0' \in B_{H^{s_d}}(\psi_0, \delta) \cap S} \mathbb{P}_{\psi_0'}\left\{\sup_{t \in J, t<T'} \|\psi(t)\|_{s_d} > M\right\} > 0,$$

where $T' = T(\psi_0', \eta)$. As $R > M$, we also have

$$\inf_{\psi_0' \in B_{H^{s_d}}(\psi_0, \delta) \cap S} \mathbb{P}_{\psi_0'}\left\{\sup_{t \in J} \|\psi^R(t)\|_{s_d} > M\right\} > 0.$$

Since the ball $B_{H^s}(0, M)$ is compact in H^{s_d} and $\|\cdot\|_{s_d} \leq \|\cdot\|_s$, we derive

$$\inf_{\psi_0 \in B_{H^s}(0, M) \cap S} \mathbb{P}_{\psi_0}\left\{\sup_{t \in J} \|\psi^R(t)\|_s > M\right\} > 0.$$

The latter and the fact that

$$\mathbb{P}_{\boldsymbol{\psi}_0}\{\tau_{M,R} = 0\} = 1 \quad \text{if } \|\boldsymbol{\psi}_0\|_{C(J; H^s \cap S)} > M$$

imply (5.8). This completes the proof of the theorem. ■

Proof of Proposition 5.2. The local well-posedness of (5.4), (5.2) is proved by standard arguments. As the H^s-norm of the solution remains bounded on any bounded time interval, the solution can be extended to any $t > 0$. For any $k \geq 1$ and $\boldsymbol{\psi}_0 \in C(J; H^s)$, let $\boldsymbol{\psi}_k^R(\boldsymbol{\psi}_0, \eta_1, \ldots, \eta_k)$ be the $C(J; H^s)$-valued process defined by (5.5). Let us show that it is Markov. Indeed, we have

$$\psi^R_{k+n}(\psi_0, \eta_1, \dots, \eta_{k+n}) = \psi^R_n(\psi^R_k(\psi_0, \eta_1, \dots, \eta_k), \eta_{k+1}, \dots, \eta_{k+n}).$$

As $\{\eta_j\}_{j \geq k+1}$ is independent of F_k and ψ^R_k is F_k-measurable, the following equality holds:

$$\begin{aligned} &\mathbb{E}\big(f(\psi^R_{k+n}(\psi_0, \eta_1, \dots, \eta_{k+n})) \mid F_k\big) \\ &\qquad = (\mathbb{E} f(\psi^R_n(\psi, \eta_{k+1}, \dots, \eta_{k+n})))|_{\psi = \psi^R_k(\psi_0, \eta_1, \dots, \eta_k)} \end{aligned} \tag{5.10}$$

for any bounded measurable function $f : C(J; H^s) \to \mathbb{R}$. The vectors $(\eta_1, \dots, \eta_n)$ and $(\eta_{k+1}, \dots, \eta_{k+n})$ have the same law, so

$$\mathbb{E} f(\psi^R_n(\psi, \eta_{k+1}, \dots, \eta_{k+n})) = \mathbb{E} f(\psi^R_n(\psi, \eta_1, \dots, \eta_n)).$$

Combining this and (5.10), we arrive at the required result. ∎

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