J. Noncommut. Geom. (Online first)
DOI 10.4171/JNCG/553

Published by EMS Press

C^*-algebraic approach to the principal symbol. III

Yuri Kordyukov, Fedor Sukochev, and Dmitriy Zanin

Abstract. We treat the notion of principal symbol mapping on a compact smooth manifold as a $*$-homomorphism of C^*-algebras. Principal symbol mapping is built from the ground, without referring to the pseudodifferential calculus on the manifold. Our concrete approach allows us to extend Connes trace theorem for compact Riemannian manifolds.

1. Introduction

This paper is motivated by the theory of pseudodifferential operators. A central notion of that theory is that of a principal symbol, which is roughly a homomorphism from the algebra of pseudo-differential operators into an algebra of functions, [14, Lemma 5.1], [13, Theorem 5.5], [31, pp. 54–55]. Usually, it is defined in a manner inhospitable for operator theorists. However, in [29], a new approach to a principal symbol mapping on a certain C^*-subalgebra Π in $B(L_2(\mathbb{R}^d))$ is proposed; this mapping turns out to be a $*$-homomorphism from Π into a commutative C^*-algebra. The C^*-algebra Π contains all classical compactly based pseudodifferential operators. This provides a very simple and algebraic approach to the theory.

Whereas our approach is more elementary than the classical approach, the C^*-algebra Π introduced in [29] (see also [19]) is much wider than the class of classical compactly based pseudo-differential operators of order 0 on $\mathbb{R}^d$. The aim of this paper is to extend this C^*-algebraic approach to the setting of smooth compact manifolds.

The C^*-algebra Π in Definition 1.1 below is the closure (in the uniform norm) of the $*$-algebra of all compactly supported *classical* pseudodifferential operators of order 0. However, we use an elementary definition of Π which does not involve pseudodifferential operators. The idea to consider this closure may be discerned yet in [3] (see Proposition 5.2, p. 512). For the recent development of this idea, we refer to [19, 29].

Let $D_k = \frac{\partial}{i\partial t_k}$ be the k-th partial derivative operator on $\mathbb{R}^d$ (these are unbounded self-adjoint operators on $L_2(\mathbb{R}^d)$). In what follows,

$$\nabla = (D_1, \dots, D_d) \quad \text{and} \quad \Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial^2 t_k} = -\sum_{k=1}^{d} D_k^2.$$

Mathematics Subject Classification 2020: 58J42 (primary); 47L20, 47L80 (secondary).
Keywords: principal symbol, smooth manifold, singular traces.

Let the d-dimensional vector $\frac{\nabla}{(-\Delta)^{\frac{1}{2}}}$ be defined by the functional calculus. Let M_f be the multiplication operator by the function f.

Definition 1.1. Let $\pi_1 : L_\infty(\mathbb{R}^d) \to B(L_2(\mathbb{R}^d))$, $\pi_2 : L_\infty(\mathbb{S}^{d-1}) \to B(L_2(\mathbb{R}^d))$ be defined by setting

$$\pi_1(f) = M_f, \quad \pi_2(g) = g\Big(\frac{\nabla}{\sqrt{-\Delta}}\Big), \quad f \in L_\infty(\mathbb{R}^d),\ g \in L_\infty(\mathbb{S}^{d-1}).$$

Let $\mathcal{A}_1 = \mathbb{C} + C_0(\mathbb{R}^d)$ and $\mathcal{A}_2 = C(\mathbb{S}^{d-1})$. Let Π be the C^*-subalgebra in $B(L_2(\mathbb{R}^d))$ generated by the algebras $\pi_1(\mathcal{A}_1)$ and $\pi_2(\mathcal{A}_2)$.

According to [29], there exists a $*$-homomorphism

$$\mathrm{sym} : \Pi \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \simeq C\big(\mathbb{S}^{d-1}, \mathbb{C} + C_0(\mathbb{R}^d)\big) \tag{1.1}$$

such that

$$\mathrm{sym}\,\big(\pi_1(f)\big) = f \otimes 1, \quad \mathrm{sym}\,\big(\pi_2(g)\big) = 1 \otimes g.$$

Here, $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ is the minimal tensor product of the C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ (see [24, Propositions 1.22.2 and 1.22.3]). The elements of $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ are identified with continuous functions on $\mathbb{R}^d \times \mathbb{S}^{d-1}$. This $*$-homomorphism is called a principal symbol mapping. It properly extends the notion of the principal symbol of the classical pseudodifferential operator.

It is natural to ask whether C^*-algebraic approach works in the general setting of smooth compact manifolds. It makes sense to de-manifoldize the question and reformulate it in a purely Euclidean fashion. We begin with the natural question on the properties of the C^*-algebra Π.

Question 1.2. *The natural unitary action of the group of diffeomorphisms on $\mathbb{R}^d$ is defined as follows. Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism. Let $U_\Phi \in B(L_2(\mathbb{R}^d))$ be a unitary operator given by setting*

$$U_\Phi \xi = \big|\det(J_\Phi)\big|^{\frac{1}{2}} \cdot (\xi \circ \Phi), \quad \xi \in L_2(\mathbb{R}^d).$$

Here, J_Φ is the Jacobian matrix of Φ.

Is the C^*-algebra Π invariant under the action $T \to U_\Phi^{-1} T U_\Phi$? Does the $*$-homomorphism sym *behave equivariantly under this action?*

Theorem 3.5 provides a positive answer to Question 1.2 (under the additional requirement that Φ is affine outside of some ball). This additional assumption yields, in particular, that Φ extends to a diffeomorphism of the projective space $P^d(\mathbb{R})$. We emphasise that Question 1.2 in full generality remains open. Furthermore, Theorem 3.11 proves an invariance of Π and equivariance of sym under local diffeomorphisms.

The resolution of Question 1.2 has opened an avenue for the definition of the C^*-algebra Π_X associated with an arbitrary compact smooth manifold X. This C^*-algebra has a remarkable property: it admits a $*$-homomorphism $\mathrm{sym}_X : \Pi_X \to C(S^*X)$, where S^*X is the cosphere bundle of X (see Section 2.7). If $X = \mathbb{R}^d$, then sym_X coincides with the mapping sym above. Every *classical* order 0 pseudodifferential operator T on X belongs to Π_X and its principal symbol in the sense of pseudodifferential operators equals $\mathrm{sym}_X(T)$. On the other hand, not every element of Π_X is pseudodifferential (e.g. because the principal symbol of a pseudodifferential operator is necessarily smooth, while that of the element of Π_X is only continuous). An approach to pseudodifferential calculi based on C^*-algebras theory was first suggested by H. O. Cordes [8] (see [20] for the case of a closed manifold).

Below, we briefly describe the construction of Π_X via the patching process (see a more precise description in Section 7.2).

Let X be a compact smooth manifold with an atlas $(\mathcal{U}_i, h_i)_{i\in\mathbb{I}}$. We will fix a sufficiently good measure ν on X, given by a continuous positive density (see Definition 2.20). If $T \in B(L_2(X,\nu))$ is compactly supported in some chart $(\mathcal{U}_i, h_i)$ (i.e., there exists $\phi \in C_c^\infty(\mathcal{U}_i)$ such that $T = TM_\phi = M_\phi T$), then, by composing with h_i, we can transfer T to an operator on $L_2(\mathbb{R}^d)$.

Definition 1.3. Let X be a compact smooth manifold equipped with a continuous positive density ν and let $T \in B(L_2(X,\nu))$. We say that $T \in \Pi_X$ if

(1) for every $i \in \mathbb{I}$ and for every $\phi \in C_c(\mathcal{U}_i)$, the operator $M_\phi T M_\phi$ transferred to an operator on $L_2(\mathbb{R}^d)$ belongs to Π;

(2) for every $\psi \in C(X)$, the operator $[T, M_\psi]$ is compact.

Theorem 1.4. *If X is a smooth compact manifold and if ν is a continuous positive density on X, then Π_X is a C^*-algebra and there exists (see Definition 7.8) a surjective $*$-homomorphism*

$$\mathrm{sym}_X : \Pi_X \to C(S^*X)$$

such that

$$\ker(\mathrm{sym}_X) = \mathcal{K}\big(L_2(X,\nu)\big).$$

In other words, we have a short exact sequence

$$0 \to \mathcal{K}\big(L_2(X,\nu)\big) \xrightarrow{\mathrm{id}} \Pi_X \xrightarrow{\mathrm{sym}_X} C(S^*X) \to 0.$$

This short exact sequence first appeared in [3] (see Proposition 5.2, p. 512) and plays an important role in index theory (see, for instance, [5, Section 24.1.8] or [4, Section 2]). It is essentially equivalent to the fact that for any operator $T \in \Pi_X$ with principal symbol $a \in C(S^*X)$,

$$\inf\big\{\|T + K\|_\infty : K \in \mathcal{K}\big(L_2(X,\nu)\big)\big\} = \|a\|_{C(S^*X)}.$$

For singular integral operators, this result was proved by Gohberg [10] and Seeley [25]. Proofs in the language of pseudodifferential operators have been given in [12,14]. It should be noted that the definition given in [3] is somewhat imprecise (see [20], in particular, a discussion on p. 329).

As a corollary of Theorem 1.4, we provide a version of Connes trace theorem (see Theorem 1.5 below). As stated, it extends [7, Theorem 1]. Connes trace theorem is ubiquitous in non-commutative geometry. It serves as a ground for defining a general notion of the non-commutative integral and non-commutative Yang–Mills action (that is, [7, Theorem 14] is taken as a definition in the non-commutative setting).

We now compare our Theorem 1.5 with various versions of Connes trace theorem available in the literature. The original proof of Connes was according to [11] "somewhat telegraphic". For example, it was not mentioned in [7] that the manifold is Riemannian and that the pseudodifferential operator featuring in [7, Theorem 1] is classical. Two proofs are given in [11] (Theorem 7.18, p. 293) and both of them rely on the assumption of ellipticity of the underlying pseudo-differential operator (this assumption is redundant as demonstrated in our approach). Despite their critique of Connes exposition, the authors of [11] also do not mention the classicality of their pseudodifferential operator. Another two proofs are given in [2]. As the authors of [2] admit, their proofs are quite sketchy; however, they provide a correct statement. The advantage of our approach is threefold: (a) we consider a strictly larger class of operators; (b) we consider a strictly larger class of traces; (c) we work in a convenient category of C^*-algebras (i.e., non-commutative topological spaces) and not in a category of classical pseudodifferential operators which does not have a natural counterpart in non-commutative geometry.

Theorem 1.5. *Let φ be a normalised continuous trace on $\mathcal{L}_{1,\infty}$. Let (X, G) be a compact Riemannian manifold and let ν be the Riemannian volume. If $T \in \Pi_X$, then*

$$\varphi\big(T(1-\Delta_G)^{-\frac{d}{2}}\big) = c_d \int_{T^*X} \mathrm{sym}_X(T) e^{-q_X}\, d\lambda,$$

where λ is the Liouville measure on T^*X and e^{-q_X} is the canonical weight of the Riemannian manifold (as defined in Section 2.8).

When T is a classical pseudodifferential operator, the right-hand side coincides with the Wodzicki residue of $T(1-\Delta_G)^{-\frac{d}{2}}$. We refer the reader to the extensive discussion of this matter in [17].

One should note a sharp contrast between the setting of Theorem 1.5 and that of Theorem 1.4. Indeed, in the latter theorem, the (smooth compact) manifold is rather arbitrary, while in the former it is Riemannian. The Riemannian structure of X in Theorem 1.5 is needed in two places: (a) there is no natural measure on the cosphere bundle of an arbitrary smooth manifold (but such a measure arises naturally if the manifold is Riemannian); (b) Riemannian structure provides us with a natural second-order differential operator (i.e, Laplace–Beltrami operator). In the setting of a general smooth manifold, the second issue can be circumvented by replacing Δ_G with an arbitrary elliptic second-order differential operator (whose resolvent falls into the ideal $\mathcal{L}_{d,\infty}$). However, the lack of a natural measure on S^*X prevents us from stating Theorem 1.5 in that generality.

We now briefly describe the structure of the paper. Section 2 collects known facts used further in the text. Theorems 3.5 and 3.11 in Section 3 assert the equivariant behavior of the principal symbol mapping in Euclidean setting under the action of diffeomorphisms. Theorem 3.5 is proved in Section 5. Theorem 3.11 is proved in Section 6. Our main result, Theorem 1.4, is proved in Section 7 with the help of Globalisation Theorem from Section 7.1 (proved in the appendix). Finally, Connes trace theorem on compact Riemannian manifolds (that is, Theorem 1.5) is proved in Section 8.

2. Preliminaries and notations

As usual, $B(H)$ denotes the $*$-algebra of all bounded operators on the Hilbert space H and $\mathcal{K}(H)$ denotes the ideal of all compact operators in $B(H)$. As usual, the Euclidean length of a vector $t \in \mathbb{R}^d$ is denoted by $|t|$.

We frequently use the equality

$$[D_k, M_f] = M_{D_k f}, \quad f \in C^\infty(\mathbb{R}^d). \tag{2.1}$$

2.1. Principal ideals in $B(H)$

It is well known that every ideal in $B(H)$ consists of compact operators.

Undoubtedly, the most important ideals are the principal ones. Among them, a special role is played by the ideal $\mathcal{L}_{p,\infty}$, a principal ideal generated by the diagonal operator $\mathrm{diag}(((k+1)^{-\frac{1}{p}})_{k\geq 0})$. We frequently use the following property (related to the Hölder inequality) of this scale of ideals:

$$\mathcal{L}_{p,\infty} \cdot \mathcal{L}_{q,\infty} = \mathcal{L}_{r,\infty}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

We mention in passing that $\mathcal{L}_{p,\infty}$ is quasi-Banach for every $p > 0$ (however, we do not need the quasi-norms in this text).

2.2. Traces on $\mathcal{L}_{1,\infty}$

Definition 2.1. If $\mathcal{I}$ is an ideal in $B(H)$, then a unitarily invariant linear functional $\varphi : \mathcal{I} \to \mathbb{C}$ is said to be a trace.

Since $U^{-1}TU - T = [U^{-1}, TU]$ for all $T \in \mathcal{I}$ and for all unitaries $U \in B(H)$, and since the unitaries span $B(H)$, it follows that traces are precisely the linear functionals on $\mathcal{I}$ satisfying the condition

$$\varphi(TS) = \varphi(ST), \quad T \in \mathcal{I},\ S \in B(H).$$

The latter may be reinterpreted as the vanishing of the linear functional φ on the commutator subspace which is denoted $[\mathcal{I}, B(H)]$ and defined to be the linear span of all commutators $[T,S] : T \in \mathcal{I}, S \in B(H)$. Note that $\varphi(T_1) = \varphi(T_2)$ whenever $0 \leq T_1, T_2 \in \mathcal{I}$

are such that the singular value sequences $\mu(T_1)$ and $\mu(T_2)$ coincide. For $p > 1$, the ideal $\mathcal{L}_{p,\infty}$ does not admit a non-zero trace, while for $p = 1$, there exists a plethora of traces on $\mathcal{L}_{1,\infty}$ (see e.g. [18]). An example of a trace on $\mathcal{L}_{1,\infty}$ is the Dixmier trace introduced in [9] that we now explain.

Example 2.2. Let ω be an extended limit. Then, the functional $\mathrm{Tr}_\omega : \mathcal{L}_{1,\infty}^+ \to \mathbb{R}_+$ defined by setting

$$\mathrm{Tr}_\omega(A) = \omega\left(\left\{\frac{1}{\log(2+n)}\sum_{k=0}^{n}\mu(k,A)\right\}_{n\geq 0}\right), \quad 0 \leq A \in \mathcal{L}_{1,\infty},$$

is additive and, therefore, extends to a trace on $\mathcal{L}_{1,\infty}$. We call such traces *Dixmier traces*. These traces clearly depend on the choice of the functional ω on l_∞.

An extensive discussion of traces, and more recent developments in the theory, may be found in [18] including a discussion of the following facts.

(1) All Dixmier traces on $\mathcal{L}_{1,\infty}$ are positive.

(2) All positive traces on $\mathcal{L}_{1,\infty}$ are continuous in the quasi-norm topology.

(3) There exist positive traces on $\mathcal{L}_{1,\infty}$ which are not Dixmier traces.

(4) There exist traces on $\mathcal{L}_{1,\infty}$ which fail to be continuous.

We are mostly interested in *normalised traces* $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$, that is, satisfying $\varphi(T) = 1$ whenever $0 \leq T$ is such that $\mu(k,T) = \frac{1}{k+1}$ for all $k \geq 0$.

Traces on $\mathcal{L}_{1,\infty}$ play a fundamental role in non-commutative geometry. For example, they allow writing Connes character formula (we refer the reader to [17, Section 5.3] and references therein).

2.3. Sobolev spaces

Sobolev space $W^{m,2}(\mathbb{R}^d)$, $m \in \mathbb{Z}_+$, consists of all distributions $f \in L_2(\mathbb{R}^d)$ such that every distributional derivative $D^\alpha f$, $\alpha \in \mathbb{Z}_+^d$, of order $|\alpha|_1 = \sum_{k=1}^d \alpha_k \leq m$ also belongs to $L_2(\mathbb{R}^d)$.

The importance of Sobolev spaces in the theory of differential operators can be seen e.g. from the fact that $W^{1,2}(\mathbb{R}^d)$ is the domain of the self-adjoint tuple ∇. Also, $W^{2,2}(\mathbb{R}^d)$ is the domain of the self-adjoint positive operator $-\Delta$.

We refer the reader to the books [1, 30] for further information on Sobolev spaces.

Further, we need the following standard result (see e.g. [30, p. 322]).

Theorem 2.3. *Sobolev space $W^{m,2}(\mathbb{R}^d)$, $m \in \mathbb{Z}_+$, is invariant under diffeomorphisms which are affine outside of some ball.*

2.4. Pseudodifferential operators

If $p \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ (that is, bounded smooth function whose derivatives are also bounded functions), then the Calderon–Vaillancourt theorem (see e.g. unnumbered proposition on

[27, p. 282]) asserts that the operator $\mathrm{Op}(p)$ defined by the formula (here, $\mathcal{F}$ is Fourier transform on $\mathbb{R}^d$)

$$\big(\mathrm{Op}(p)\xi\big)(t) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle t,s\rangle} p(t,s)(\mathcal{F}\xi)(s)ds, \quad \xi \in L_2(\mathbb{R}^d), \tag{2.2}$$

is bounded in $L_2(\mathbb{R}^d)$.

If $m \in \mathbb{Z}$, $m \leq 0$, is such that

$$\sup_{t,s\in\mathbb{R}^d} \big(1+|s|^2\big)^{\frac{|\beta|_1-m}{2}} \big|D_t^\alpha D_s^\beta p(t,s)\big| < \infty, \quad \alpha,\beta \in \mathbb{Z}_+^d, \tag{2.3}$$

then we say that $\mathrm{Op}(p) \in \Psi^m(\mathbb{R}^d)$. For $m > 0$, the class $\Psi^m(\mathbb{R}^d)$ is defined by the same formula. The difference is that, for $m > 0$, operators in $\Psi^m(\mathbb{R}^d)$ are no longer bounded as operators from $L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$; instead, they are bounded operators from $W^{m,2}(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$.

The key property is that

$$\Psi^{m_1}(\mathbb{R}^d) \cdot \Psi^{m_2}(\mathbb{R}^d) \subset \Psi^{m_1+m_2}(\mathbb{R}^d), \quad m_1, m_2 \in \mathbb{Z}. \tag{2.4}$$

Moreover, by [23, Theorem 2.5.1], we have

$$\mathrm{Op}(p_1)\cdot \mathrm{Op}(p_2) \in \mathrm{Op}(p_1p_2) + \Psi^{m_1+m_2-1}(\mathbb{R}^d), \tag{2.5}$$

whenever $\mathrm{Op}(p_1) \in \Psi^{m_1}(\mathbb{R}^d)$ and $\mathrm{Op}(p_2) \in \Psi^{m_2}(\mathbb{R}^d)$. The next lemma follows immediately from (2.4) and (2.5).

Lemma 2.4. *If $m_1, m_2 \in \mathbb{Z}$,*

$$T_l \in \mathrm{Op}(p_l) + \Psi^{m_l-1}(\mathbb{R}^d), \quad \mathrm{Op}(p_l) \in \Psi^{m_l}(\mathbb{R}^d), \quad l = 1,2,$$

then

$$T_1T_2 - \mathrm{Op}(p_1p_2) \in \Psi^{m_1+m_2-1}(\mathbb{R}^d).$$

Let $T \in \Psi^m(\mathbb{R}^d)$, $m < 0$, and let $\psi \in C_c^\infty(\mathbb{R}^d)$ be such that $T = M_\psi T$. Recall that the operator $M_\psi(1-\Delta)^{\frac{m}{2}}$ is compact (see e.g. [26, Theorem 4.1]). Thus,

$$T = M_\psi(1-\Delta)^{\frac{m}{2}} \cdot (1-\Delta)^{-\frac{m}{2}} T \in \mathcal{K}\big(L_2(\mathbb{R}^d)\big)\cdot B\big(L_2(\mathbb{R}^d)\big) = \mathcal{K}\big(L_2(\mathbb{R}^d)\big). \tag{2.6}$$

Differential operators of order $m \geq 0$ with smooth bounded coefficients (all derivatives of the coefficients are also assumed bounded) belong to $\Psi^m(\mathbb{R}^d)$. Indeed, it follows directly from (2.2) that

$$\sum_{|\alpha|_1\leq m} M_{f_\alpha} D^\alpha = \mathrm{Op}(p), \quad p(t,s) = \sum_{|\alpha|_1 \leq m} f_\alpha(t)s^\alpha, \quad t,s \in \mathbb{R}^d. \tag{2.7}$$

The following standard result is available, e.g., in [16, Theorem 1.6.20].

Theorem 2.5. *Let $T \in \Psi^m(\mathbb{R}^d)$, $m \geq 0$, extend to a self-adjoint positive operator $T : W^{m,2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$. For every $z \in \mathbb{C}$, we have $(T+1)^z \in \Psi^{m\Re(z)}(\mathbb{R}^d)$.*

If, in addition, T is a differential operator with positive principal symbol p, then

$$(T+1)^z - \mathrm{Op}\big((p+1)^z\big) \in \Psi^{m\Re(z)-1}(\mathbb{R}^d).$$

2.5. Pseudodifferential-like operators in Π

If $q \in C_c^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$, then we set

$$(T_q\xi)(t) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle t,s\rangle} q\Big(t, \frac{s}{|s|}\Big)(\mathcal{F}\xi)(s)ds, \quad \xi \in L_2(\mathbb{R}^d). \tag{2.8}$$

Lemma 2.6 ([29, Lemma 8.1]). *For every $q \in C_c^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$, we have $T_q \in \Pi$ and* $\mathrm{sym}(T_q) = q$.

Lemma 2.7 ([29, Lemma 8.2]). *Let $q \in C_c^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$. If $\psi \in C_c^\infty(\mathbb{R}^d)$ equals 1 near 0, then*

$$\mathrm{Op}(p) - T_q \in \mathcal{K}\big(L_2(\mathbb{R}^d)\big),$$

where

$$p(t,s) = q\Big(t, \frac{s}{|s|}\Big) \cdot \big(1 - \psi(s)\big), \quad t, s \in \mathbb{R}^d.$$

2.6. Cotangent bundle

Notation 2.8. Let X be a smooth d-dimensional manifold with atlas $(\mathcal{U}_i, h_i)_{i\in\mathbb{I}}$, where $\mathbb{I}$ is an arbitrary set of indices.

(1) We denote

$$\Omega_i = h_i(\mathcal{U}_i) \subset \mathbb{R}^d, \quad \Omega_{i,j} = h_i(\mathcal{U}_i \cap \mathcal{U}_j) \subset \mathbb{R}^d, \quad i, j \in \mathbb{I}.$$

(2) We denote by $\Phi_{i,j} : \Omega_{i,j} \to \Omega_{j,i}$ the diffeomorphism given by the formula

$$\Phi_{i,j}(t) = h_j\big(h_i^{-1}(t)\big), \quad t \in \Omega_{i,j}.$$

In the next fact, we recall the manifold structure of T^*X.

Fact 2.9. *Let X be a d-dimensional manifold with an atlas $\{(\mathcal{U}_i, h_i)\}_{i\in\mathbb{I}}$. Let T^*X be the cotangent bundle of X and let $\pi : T^*X \to X$ be the canonical projection. There exists an atlas $\{\pi^{-1}(\mathcal{U}_i), H_i\}_{i\in\mathbb{I}}$ of T^*X such that*

(1) *for every $i \in \mathbb{I}$, $H_i : \pi^{-1}(\mathcal{U}_i) \to \Omega_i \times \mathbb{R}^d$ is a homeomorphism;*

(2) *for every $i, j \in \mathbb{I}$ such that $\mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$, we have*

$$(H_j \circ H_i^{-1})(t,s) = \Big(\Phi_{i,j}(t), \big(J^*_{\Phi_{i,j}}(t)\big)^{-1} s\Big), \quad t \in \Omega_{i,j},\ s \in \mathbb{R}^d.$$

In the next fact, we identify functions on the T^*X and their local representations. It is important that this identification preserves continuity.

Fact 2.10. *Let $F_i : \Omega_i \times \mathbb{R}^d \to \mathbb{C}$ for every $i \in \mathbb{I}$. If*

$$F_i \circ H_i = F_j \circ H_j \quad \text{on } \pi^{-1}(\mathcal{U}_i \cap \mathcal{U}_j) \quad \text{for every } i, j \in \mathbb{I},$$

then there exists a unique function $F : T^*X \to \mathbb{C}$ such that

$$F_i = F \circ H_i^{-1} \quad \text{on } \Omega_i \times \mathbb{R}^d, \quad i \in \mathbb{I}.$$

2.7. Cosphere bundle

Definition 2.11. Define a dilation action $\lambda \to \sigma_\lambda$ of $(0, \infty)$ on each $\Omega_i \times \mathbb{R}^d$ by setting

$$\sigma_\lambda : (t, s) \to (t, \lambda s), \quad t \in \Omega_i,\ s \in \mathbb{R}^d.$$

This action lifts down to an action on T^*X (also denoted by σ_λ).

A function on T^*X invariant with respect to this action is called dilation invariant.

Definition 2.12. Let X be a compact manifold. C^*-algebra of all continuous dilation invariant functions on $T^*X \backslash 0_{T^*X}$ (here, 0_{T^*X} is the zero section of T^*X) is denoted by $C(S^*X)$ and is called the algebra of continuous functions on the cosphere bundle of X.

2.8. Canonical weight of Riemannian manifold

If X is a smooth d-dimensional manifold, then T^*X has a canonical symplectic structure. The corresponding Liouville measure λ on T^*X satisfies the following property (see, for instance, [6]):

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f dm = \int_{T^*X} (f \circ H_i) d\lambda, \quad f \in C_c(\Omega_i \times \mathbb{R}^d),\ i \in \mathbb{I}. \tag{2.9}$$

Here, $f \circ H_i$ denotes a function on T^*X which equals $f \circ H_i$ on $\pi^{-1}(\mathcal{U}_i)$ and which vanishes outside $\pi^{-1}(\mathcal{U}_i)$.

However, there is no canonical way to equip the cosphere bundle S^*X of a smooth manifold X with a measure. The following class of measures is of particular interest: if $w \in L_1(T^*X, \lambda)$, then the functional

$$f \to \int_{T^*X} f w d\lambda, \quad f \in C(S^*X),$$

generates a measure on S^*X by the Riesz–Markov theorem. However, there is no canonical way to select an integrable function w on T^*X. This choice becomes possible if we assume in addition a Riemannian structure on X.

Let G be a Riemannian metric on X. For any $i \in \mathbb{I}$, the components of the metric G in the chart $(\mathcal{U}_i, h_i)$ give rise to a smooth mapping $G_i : \mathcal{U}_i \to \mathrm{GL}^+(d, \mathbb{R})$. (In what follows, $\mathrm{GL}^+(d, \mathbb{R})$ stands for the set of all positive elements in $\mathrm{GL}(d, \mathbb{R})$.) For any $i, j \in \mathbb{I}$ such that $\mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$, we have

$$G_j(t) = J^*_{\Phi_{j,i}}\big(h_j(t)\big) \cdot G_i(t) \cdot J_{\Phi_{j,i}}\big(h_j(t)\big), \quad t \in \mathcal{U}_i \cap \mathcal{U}_j.$$

Here, $\Phi_{j,i}$ are given in Notation 2.8.

Notation 2.13. For every $i \in \mathbb{I}$, let Ω_i be as in Notation 2.8 and set $g_i = G_i \circ h_i^{-1} : \Omega_i \to \mathrm{GL}^+(d, \mathbb{R})$. We also set

$$q_i(t, s) = \langle g_i(t)^{-1} s, s \rangle, \quad t \in \Omega_i,\ s \in \mathbb{R}^d.$$

It can be easily verified by a direct calculation that, for every $i, j \in \mathbb{I}$, we have

$$q_i \circ H_i = q_j \circ H_j \text{ on } \pi^{-1}(\mathcal{U}_i \cap \mathcal{U}_j).$$

By Fact 2.10, there exists a function q_X on T^*X such that

$$q_i = q_X \circ H_i^{-1} \text{ on } \Omega_i \times \mathbb{R}^d, \quad i \in \mathbb{I}. \tag{2.10}$$

The function q_X is the square of the length function on T^*X defined by the induced Riemannian metric on the cotangent bundle T^*X.

Definition 2.14. The function e^{-q_X} on T^*X is called the canonical weight of the Riemannian manifold (X, G).

If X is compact, then $e^{-q_X} \in L_1(T^*X, \lambda)$. The functional on $C(S^*X)$ given by the formula

$$f \to \int_{T^*X} f e^{-q_X} d\lambda$$

plays a crucial role. It defines the natural measure on S^*X. We note that the latter functional coincides (modulo a constant factor) with integration with respect to the kinematic density on S^*X (see [6, p. 318]).

2.9. Laplace–Beltrami operator on compact Riemannian manifold

Notation 2.15. Let $\Omega \subset \mathbb{R}^d$ be connected and open. Let $g : \Omega \to \mathrm{GL}^+(d, \mathbb{R})$ be a smooth mapping. Laplace–Beltrami operator $\Delta_g : C_c^\infty(\Omega) \to C_c^\infty(\Omega)$ is defined by the formula

$$\Delta_g = M_{\det(g)^{-\frac{1}{2}}} \sum_{k,l=1}^{d} D_k M_{\det(g)^{\frac{1}{2}} \cdot (g^{-1})_{k,l}} D_l.$$

Definition 2.16. Let $(\phi_n)_{n=1}^N \subset C^\infty(X)$ be a finite partition of unity. We call it good if each ϕ_n is compactly supported in some chart.

Obviously, good partitions of unity exist only on compact manifolds.

Definition 2.17. Let (X, G) be a compact Riemannian manifold. Let Ω_i be as in Notation 2.8 and let $g_i : \Omega_i \to \mathrm{GL}^+(d, \mathbb{R})$ be as in Notation 2.13. Let $\Delta_{g_i} : C_c^\infty(\Omega_i) \to C_c^\infty(\Omega_i)$, $i \in \mathbb{I}$, be the Laplace–Beltrami operator as in Notation 2.15. Let $(\phi_n)_{n=1}^N$ be a good partition of unity.

Laplace–Beltrami operator $\Delta_G : C^\infty(X) \to C^\infty(X)$ is defined by the formula

$$\Delta_G f = \sum_{n=1}^{N} \left(\Delta_{g_{i_n}}\left((f\phi_n) \circ h_{i_n}^{-1}\right)\right) \circ h_{i_n}, \quad f \in C^\infty(X).$$

Here, $i_n \in \mathbb{I}$ is chosen such that ϕ_n is compactly supported in $\mathcal{U}_{i_n}$.

Though Definition 2.17 involves good partition of unity, the operator Δ_G does not actually depend on the particular choice of a good partition of unity.

Theorem 2.4 in [28] yields the following results. The first one is of conceptual importance. The second one is used in the proof of Theorem 1.5.

Theorem 2.18. *Let (X, G) be a compact Riemannian manifold. Laplace–Beltrami operator admits a self-adjoint extension $\Delta_G : W^{2,2}(X) \to L_2(X)$.*

Theorem 2.19. *Let $g : \mathbb{R}^d \to \mathrm{GL}^+(d, \mathbb{R})$. Suppose that*

(1) $g \in C^\infty(\mathbb{R}^d, M_d(\mathbb{C}))$ *(that is, g is smooth and all derivatives are bounded);*

(2) $\det(g) \geq c$ *for some* $0 < c \in \mathbb{R}$.

The operator $\Delta_g : W^{2,2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is self-adjoint.

2.10. Density on a manifold

Let $\mathfrak{B}$ be the Borel σ-algebra on the manifold X. We need the notion of density on a manifold available, e.g., in [21, p. 87].

Definition 2.20. Let ν be a countably additive measure on $\mathfrak{B}$. We assume that, for every $i \in \mathbb{I}$, the measure $\nu \circ h_i^{-1}$ on Ω_i is absolutely continuous with respect to the Lebesgue measure on Ω_i, and its Radon–Nikodym derivative a_i is strictly positive and continuous on Ω_i.

In this case, we say that ν is a continuous positive density on X.

3. Invariance of principal symbol under diffeomorphisms

In this section, we formulate a theorem which provides a partial positive answer to Question 1.2. This result is stated in two versions: Theorem 3.5 for the diffeomorphisms of $\mathbb{R}^d$ (which a core technical difficulty) and Theorem 3.11 for local diffeomorphisms (the result which would be actually used).

3.1. Invariance under diffeomorphisms of $\mathbb{R}^d$

We need the following notations. Recall that $\mathrm{GL}(d, \mathbb{R})$ stands for the group of invertible real $d \times d$ matrices.

Notation 3.1. Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism.

(1) Let $J_\Phi : \mathbb{R}^d \to \mathrm{GL}(d, \mathbb{R})$ be the Jacobian matrix of Φ.

(2) Let unitary operator $U : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ be defined by setting

$$(U_\Phi \xi)(t) = \big|\det(J_\Phi)\big|^{\frac{1}{2}}(t)\xi\big(\Phi(t)\big), \quad \xi \in L_2(\mathbb{R}^d),\ t \in \mathbb{R}^d.$$

(3) Let $\Theta_\Phi : \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}^d \times \mathbb{S}^{d-1}$ be defined by setting

$$\Theta_\Phi(t, s) = \big(\Phi^{-1}(t), O_{J_\Phi^*(\Phi^{-1}(t))} s\big), \quad t \in \mathbb{R}^d,\ s \in \mathbb{S}^{d-1},$$

where J_Φ^* is the adjoint to the Jacobi matrix.

Here, for $A \in \mathrm{GL}(d,\mathbb{R})$, we set

$$O_A s = \frac{As}{|As|}, \quad s \in \mathbb{S}^{d-1}.$$

(4) Let $\Xi_\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ be defined by setting

$$\Xi_\Phi(t,s) = \big(\Phi^{-1}(t), J_\Phi^*\big(\Phi^{-1}(t)\big)s\big), \quad t,s \in \mathbb{R}^d.$$

We frequently need the following compatibility lemma.

Lemma 3.2. *Let $\Phi_1, \Phi_2 : \mathbb{R}^d \to \mathbb{R}^d$ be diffeomorphisms. We have*

$$\Theta_{\Phi_1\circ\Phi_2} = \Theta_{\Phi_2} \circ \Theta_{\Phi_1}.$$

Proof. Indeed,

$$\Theta_\Phi(t,s) = \big(\Phi^{-1}(t), O_{A^\Phi(t)} s\big),$$

where

$$A^\Phi(t) = J_\Phi^*\big(\Phi^{-1}(t)\big).$$

We have

$$(\Theta_{\Phi_2} \circ \Theta_{\Phi_1})(t,s) = \Theta_{\Phi_2}(t',s'), \quad (t',s') = \big(\Phi_1^{-1}(t), O_{A^{\Phi_1}(t)} s\big).$$

Thus,

$$(\Theta_{\Phi_2} \circ \Theta_{\Phi_1})(t,s) = \big(\Phi_2^{-1}(t'), O_{A^{\Phi_2}(t')} s'\big) = \big(\Phi_2^{-1}\big(\Phi_1^{-1}(t)\big), O_{A^{\Phi_2}(\Phi_1^{-1}(t))} O_{A^{\Phi_1}(t)} s\big).$$

Note that

$$(O_{A_1} \circ O_{A_2})s = \frac{A_1(O_{A_2}s)}{\big|A_1(O_{A_2}s)\big|} = \frac{\frac{A_1A_2s}{|A_2s|}}{\frac{|A_1A_2s|}{|A_2s|}} = \frac{A_1A_2s}{|A_1A_2s|} = O_{A_1\cdot A_2}s, \quad s \in \mathbb{S}^{d-1}.$$

Since $O_{A_1} \circ O_{A_2} = O_{A_1\cdot A_2}$, it follows that

$$(\Theta_{\Phi_2} \circ \Theta_{\Phi_1})(t,s) = \big((\Phi_1 \circ \Phi_2)^{-1}(t), O_{A^{\Phi_2}(\Phi_1^{-1}(t))\cdot A^{\Phi_1}(t)} s\big).$$

At the same time,

$$\Theta_{\Phi_1\circ\Phi_2} = \big((\Phi_1 \circ \Phi_2)^{-1}(t), O_{A^{\Phi_1\circ\Phi_2}(t)} s\big).$$

It suffices to show that

$$A^{\Phi_1\circ\Phi_2}(t) = A^{\Phi_2}\big(\Phi_1^{-1}(t)\big) \cdot A^{\Phi_1}(t).$$

The latter equality is written as

$$J_{\Phi_1\circ\Phi_2}^*\big((\Phi_1 \circ \Phi_2)^{-1}(t)\big) = J_{\Phi_2}^*\big(\Phi_2^{-1}\big(\Phi_1^{-1}(t)\big)\big) \cdot J_{\Phi_1}^*\big(\Phi_1^{-1}(t)\big).$$

Replacing t with $(\Phi_1 \circ \Phi_2)(t)$, we need to verify that

$$J^*_{\Phi_1 \circ \Phi_2}(t) = J^*_{\Phi_2}(t) \cdot J^*_{\Phi_1}\big(\Phi_2(t)\big).$$

In other words,

$$J_{\Phi_1 \circ \Phi_2}(t) = J_{\Phi_1}\big(\Phi_2(t)\big) \cdot J_{\Phi_2}(t).$$

This is the chain rule property. ■

Corollary 3.3. $\Theta_\Phi : \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}^d \times \mathbb{S}^{d-1}$ *is a diffeomorphism.*

Proof. Obviously, $J^*_\Phi \circ \Phi^{-1} : \mathbb{R}^d \to \mathrm{GL}(d, \mathbb{R})$ is a smooth mapping. For every smooth mapping $A : \mathbb{R}^d \to \mathrm{GL}(d, \mathbb{R})$, the mapping

$$(t, s) \to O_{A(t)} s, \quad t \in \mathbb{R}^d,\ s \in \mathbb{S}^{d-1},$$

is smooth. Thus, Θ_Φ is smooth.

By Lemma 3.2, its inverse is $\Theta_{\Phi^{-1}}$ which is also a smooth mapping. ■

We are now ready to state the main result in this subsection.

Definition 3.4. We say that $T \in \Pi$ is compactly supported if there exists $\phi \in C^\infty_c(\mathbb{R}^d)$ such that $T = T\pi_1(\phi) = \pi_1(\phi)T$.

Theorem 3.5. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that Φ is affine outside of some ball. If $T \in \Pi$ is compactly supported, then $U_\Phi^{-1} T U_\Phi \in \Pi$. Furthermore,*

$$\mathrm{sym}(U_\Phi^{-1} T U_\Phi) = \mathrm{sym}(T) \circ \Theta_\Phi.$$

If we view symbols as homogeneous functions[1] *on $\mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$, then*

$$\mathrm{sym}(U_\Phi^{-1} T U_\Phi) = \mathrm{sym}(T) \circ \Xi_\Phi.$$

We prove Theorem 3.5 in Section 5.

There are two reasons for us to require that Φ is affine outside of some ball. The first reason is that having the equivariant behavior of the principal symbol under such diffeomorphisms is sufficient in the proof of Theorem 3.11 below. The second reason is that in the proof of Theorem 3.5 we conjugate the Laplacian with U_Φ. Hence, it is of crucial importance that U_Φ preserves the domain of Laplacian. Recall that the domain of Laplacian is Sobolev spaces $W^{2,2}(\mathbb{R}^d)$ and, by Theorem 2.3, U_Φ leaves the domain of Laplacian invariant.

[1] A function $f : \mathbb{R}^d \times \mathbb{S}^{d-1}$ can be uniquely extended to a homogeneous function on $\mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$ by setting

$$f(t, s) \overset{\text{def}}{=} f\Big(t, \frac{s}{|s|}\Big), \quad t, s \in \mathbb{R}^d.$$

If f is homogeneous, then

$$(f \circ \Theta_\Phi)(t, s) = f\left(\Phi^{-1}(t), \frac{J^*_\Phi(\Phi^{-1}(t))s}{\big|J^*_\Phi(\Phi^{-1}(t))s\big|}\right) = f\big(\Phi^{-1}(t), J^*_\Phi(\Phi^{-1}(t))s\big) = (f \circ \Xi_\Phi)(t, s).$$

3.2. Invariance under local diffeomorphisms

One may ask how the algebra and the principal symbol mapping (locally) behave under the change of coordinates. We need the following notations.

Notation 3.6. Let H be a Hilbert space and let $p \in B(H)$ be a projection.

(1) If $T \in B(H)$ is such that $T = pTp$, then we define the operator $\mathrm{Rest}_p(T) \in B(pH)$ by setting $\mathrm{Rest}_p(H) = T|_{pH}$.

(2) If $T \in B(pH)$, then we define $\mathrm{Ext}_p(T) \in B(H)$ by setting $\mathrm{Ext}_p(T) = T \circ p$.

Notation 3.7. Let $\Omega \subset \mathbb{R}^d$. If $T \in B(L_2(\mathbb{R}^d))$ is such that $T = M_{\chi_\Omega} T M_{\chi_\Omega}$, then

$$\mathrm{Rest}_\Omega(T) \in B\big(L_2(\Omega)\big)$$

is a shorthand for $\mathrm{Rest}_{M_{\chi_\Omega}}(T)$. If $T \in B(L_2(\Omega))$, then $\mathrm{Ext}_\Omega(T) \in B(L_2(\mathbb{R}^d))$ is a shorthand for $\mathrm{Ext}_{M_{\chi_\Omega}}(T)$.

Notation 3.8. Let $\Omega, \Omega' \subset \mathbb{R}^d$ be open sets and let $\Phi : \Omega \to \Omega'$ be a diffeomorphism.

(1) Let $J_\Phi : \Omega \to \mathrm{GL}(d, \mathbb{R})$ be the Jacobian matrix of Φ.

(2) Let unitary operator $U_\Phi : L_2(\Omega') \to L_2(\Omega)$ be defined by setting

$$(U_\Phi \xi)(t) = \big|\det(J_\Phi)\big|^{\frac{1}{2}}(t)\xi\big(\Phi(t)\big), \quad \xi \in L_2(\Omega'),\ t \in \Omega.$$

(3) Let $\Theta_\Phi : \Omega' \times \mathbb{S}^{d-1} \to \Omega \times \mathbb{S}^{d-1}$ be defined by setting

$$\Theta_\Phi(t, s) = \big(\Phi^{-1}(t), O_{J_\Phi^*(\Phi^{-1}(t))} s\big), \quad t \in \Omega',\ s \in \mathbb{S}^{d-1}.$$

(4) Let $\Xi_\Phi : \Omega' \times \mathbb{R}^d \to \Omega \times \mathbb{R}^d$ be defined by setting

$$\Xi_\Phi(t, s) = \big(\Phi^{-1}(t), J_\Phi^*\big(\Phi^{-1}(t)\big)s\big), \quad t \in \Omega',\ s \in \mathbb{R}^d.$$

Lemma 3.9. *Let $\Phi_1 : \Omega \to \Omega'$ and $\Phi_2 : \Omega'' \to \Omega$ be diffeomorphisms. We have*

$$\Theta_{\Phi_1 \circ \Phi_2} = \Theta_{\Phi_2} \circ \Theta_{\Phi_1}.$$

Proof. The proof is identical to that of Lemma 3.2. ∎

Corollary 3.10. *$\Theta_\Phi : \Omega' \times \mathbb{S}^{d-1} \to \Omega \times \mathbb{S}^{d-1}$ is a diffeomorphism.*

Proof. Obviously, $J_\Phi^* \circ \Phi^{-1} : \Omega' \to \mathrm{GL}(d, \mathbb{R})$ is a smooth mapping. For every smooth mapping $A : \Omega \to \mathrm{GL}(d, \mathbb{R})$, the mapping

$$(t, s) \to O_{A(t)} s, \quad t \in \Omega,\ s \in \mathbb{S}^{d-1},$$

is smooth. Thus, Θ_Φ is smooth. By Lemma 3.9, its inverse is $\Theta_{\Phi^{-1}}$ which is also a smooth mapping. ∎

Theorem 3.11. *Let $\Omega, \Omega' \subset \mathbb{R}^d$ be open sets and let $\Phi : \Omega \to \Omega'$ be a diffeomorphism. If $T \in \Pi$ is compactly supported in Ω, then*

$$\mathrm{Ext}_{\Omega'}\big(U_\Phi^{-1} \cdot \mathrm{Rest}_\Omega(T) \cdot U_\Phi\big) \in \Pi.$$

Furthermore,

$$\mathrm{sym}\,\big(\mathrm{Ext}_{\Omega'}\big(U_\Phi^{-1} \cdot \mathrm{Rest}_\Omega(T) \cdot U_\Phi\big)\big) = \mathrm{sym}(T) \circ \Theta_\Phi.$$

If we view symbols as homogeneous functions on $\mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$, then

$$\mathrm{sym}\,\big(\mathrm{Ext}_{\Omega'}\big(U_\Phi^{-1} \cdot \mathrm{Rest}_\Omega(T) \cdot U_\Phi\big)\big) = \mathrm{sym}(T) \circ \Xi_\Phi.$$

Theorem 3.11 is proved in Section 6 as a corollary of Theorem 3.5.

4. Conjugation of differential operators with U_Φ

In this section, we examine the operators

$$U_\Phi^{-1} D_k U_\Phi : W^{1,2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d) \quad \text{and} \quad U_\Phi^{-1} \Delta U_\Phi : W^{2,2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$$

and show that they may be viewed as differential operators.

Lemma 4.1. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that Φ is affine outside of some ball. Mapping V_Φ by setting $V_\Phi : \xi \to \xi \circ \Phi$ is bounded on $L_2(\mathbb{R}^d)$ and so is V_Φ^{-1}.*

Proof. This is a special case of Theorem 2.3 (or it can be verified by hands). ∎

Lemma 4.2. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that Φ is affine outside of some ball. We have*

$$U_\Phi^{-1} \pi_1(f) U_\Phi = V_\Phi^{-1} \pi_1(f) V_\Phi = \pi_1(f \circ \Phi^{-1}).$$

Proof. By definition of U_Φ (in Notation 3.1) and of V_Φ (in Lemma 4.1), we have

$$U_\Phi = M_{|\det(J_\Phi)|^{\frac{1}{2}}} V_\Phi.$$

It is immediate that

$$U_\Phi^{-1} M_f U_\Phi = V_\Phi^{-1} M_{|\det(J_\Phi)|^{-\frac{1}{2}}} M_f M_{|\det(J_\Phi)|^{\frac{1}{2}}} V_\Phi.$$

Since

$$M_{|\det(J_\Phi)|^{-\frac{1}{2}}} M_f M_{|\det(J_\Phi)|^{\frac{1}{2}}} = M_f,$$

it follows that

$$U_\Phi^{-1} M_f U_\Phi = V_\Phi^{-1} M_f V_\Phi = M_{f \circ \Phi^{-1}}.$$

The assertion of the lemma now follows from Definition 1.1. ∎

Notation 4.3. Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism. Denote

$$(a^{\Phi}_{k,l})^d_{k,l=1} = J^*_{\Phi} \circ \Phi^{-1}, \quad (b^{\Phi}_{k,l})^d_{k,l=1} = |J^*_{\Phi} \circ \Phi^{-1}|^2.$$

Lemma 4.4. *Functions*

$$a^{\Phi}_k = \big(|\det(J_{\Phi})|^{-\frac{1}{2}} \cdot D_k\big(\big|\det(J_{\Phi})\big|^{\frac{1}{2}}\big)\big) \circ \Phi^{-1}, \quad 1 \le k \le d,$$

$$b^{\Phi}_l = 2\sum_{k=1}^d \Re(\bar{a}^{\Phi}_k \cdot a^{\Phi}_{k,l}), \quad 1 \le l \le d,$$

$$b^{\Phi} = \sum_{k=1}^d \sum_{l=1}^d D_l(\bar{a}^{\Phi}_{k,l} \cdot a^{\Phi}_k) + \sum_{k=1}^d |a^{\Phi}_k|^2$$

belong to $C^{\infty}_c(\mathbb{R}^d)$.

Proof. Since Φ is a diffeomorphism, it follows that all those functions are smooth. Since Φ is affine outside of some ball, it follows that J_{Φ} is constant outside of some ball. Thus, $D_k(|\det(J_{\Phi})|^{\frac{1}{2}}) = 0$ outside of some ball. Using the definition of a^{Φ}_k, we now see that it vanishes outside of some ball. Using the definition of b^{Φ}_l and b^{Φ}, we now see that it vanishes outside of some ball. ■

Lemma 4.5. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that Φ is affine outside of some ball. We have*

$$V_{\Phi}^{-1} D_k V_{\Phi} = \sum_{l=1}^d M_{a^{\Phi}_{k,l}} D_l, \quad 1 \le k \le d.$$

Here, equalities are understood as equalities of linear differential operators acting from $W^{1,2}(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$.

Proof. By the chain rule, we have

$$D_k V_{\Phi}\xi = D_k(\xi \circ \Phi) = \sum_{l=1}^d \big((D_l\xi) \circ \Phi\big) \cdot i D_k \Phi_l, \quad \xi \in W^{1,2}(\mathbb{R}^d).$$

Using the notations for V_{Φ} (in Lemma 4.1) and for the multiplication operator, we can rewrite this formula as follows:

$$D_k V_{\Phi} = \sum_{l=1}^d M_{iD_k\Phi_l} V_{\Phi} D_l.$$

Thus,

$$V_{\Phi}^{-1} D_k V_{\Phi} = \sum_{l=1}^d V_{\Phi}^{-1} M_{iD_k\Phi_l} V_{\Phi} \cdot D_l \overset{\text{Lem. 4.2}}{=} \sum_{l=1}^d M_{i(D_k\Phi_l)\circ\Phi^{-1}} D_l = \sum_{l=1}^d M_{a^{\Phi}_{k,l}} D_l,$$

where the last equality follows from the definition of $a^{\Phi}_{k,l}$ (in Notation 4.3) and the fact that $J_{\Phi} = (iD_l\Phi_k)^d_{k,l=1}$. ■

Lemma 4.6. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that Φ is affine outside of some ball. We have $U_\Phi : W^{m,2}(\mathbb{R}^d) \to W^{m,2}(\mathbb{R}^d)$, $m \in \mathbb{Z}_+$.*

Proof. By definition of U_Φ (in Notation 3.1) and of V_Φ (in Lemma 4.1), we have

$$U_\Phi = M_{h_\Phi} V_\Phi, \quad h_\Phi = \left|\det(J_\Phi)\right|^{\frac{1}{2}}.$$

By Theorem 2.3, $V_\Phi : W^{m,2}(\mathbb{R}^d) \to W^{m,2}(\mathbb{R}^d)$. Since Φ is a diffeomorphism and since Φ is affine outside of some ball, it follows that h_Φ is a smooth function on $\mathbb{R}^d$ which is constant outside of some ball. It follows that $M_{h_\Phi} : W^{m,2}(\mathbb{R}^d) \to W^{m,2}(\mathbb{R}^d)$. A combination of those mappings yields the assertion. ∎

By Lemma 4.6, we have

$$W^{1,2}(\mathbb{R}^d) \xrightarrow{U_\Phi} W^{1,2}(\mathbb{R}^d) \xrightarrow{D_k} L_2(\mathbb{R}^d) \xrightarrow{U_\Phi^{-1}} L_2(\mathbb{R}^d),$$
$$W^{2,2}(\mathbb{R}^d) \xrightarrow{U_\Phi} W^{2,2}(\mathbb{R}^d) \xrightarrow{\Delta} L_2(\mathbb{R}^d) \xrightarrow{U_\Phi^{-1}} L_2(\mathbb{R}^d).$$

Hence, we may view $U_\Phi^{-1} D_k U_\Phi$ (respectively, $U_\Phi^{-1} \Delta U_\Phi$) as operators from $W^{1,2}(\mathbb{R}^d)$ (respectively, from $W^{2,2}(\mathbb{R}^d)$) to $L_2(\mathbb{R}^d)$.

Lemma 4.7. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that Φ is affine outside of some ball. We have*

$$U_\Phi^{-1} D_k U_\Phi = \sum_{l=1}^{d} M_{a_{k,l}^\Phi} D_l + M_{a_k^\Phi}, \tag{4.1}$$

$$-U_\Phi^{-1} \Delta U_\Phi = \sum_{l_1,l_2=1}^{d} D_{l_1} M_{b_{l_1,l_2}^\Phi} D_{l_2} + \sum_{l=1}^{d} M_{b_l^\Phi} D_l + M_{b^\Phi}. \tag{4.2}$$

Here, equalities are understood as equalities of linear operators acting from $W^{1,2}(\mathbb{R}^d)$ (respectively, from $W^{2,2}(\mathbb{R}^d)$) to $L_2(\mathbb{R}^d)$.

Proof. Repeating the beginning of the proof of Lemma 4.6, we write

$$U_\Phi = M_{h_\Phi} V_\Phi, \quad h_\Phi = \left|\det(J_\Phi)\right|^{\frac{1}{2}}.$$

It is immediate that

$$U_\Phi^{-1} D_k U_\Phi = V_\Phi^{-1} M_{h_\Phi^{-1}} D_k M_{h_\Phi} V_\Phi. \tag{4.3}$$

Clearly,

$$M_{h_\Phi^{-1}} D_k M_{h_\Phi} = D_k + M_{h_\Phi^{-1}} \cdot [D_k, M_{h_\Phi}] = D_k + M_{h_\Phi^{-1} \cdot D_k h_\Phi}. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$U_\Phi^{-1} D_k U_\Phi = V_\Phi^{-1} D_k V_\Phi + V_\Phi^{-1} M_{h_\Phi^{-1} \cdot D_k h_\Phi} V_\Phi. \tag{4.5}$$

It follows from Lemma 4.2 and the definition of a_k^Φ (in Lemma 4.4) that

$$V_\Phi^{-1} M_{h_\Phi^{-1}\cdot D_k h_\Phi} V_\Phi = M_{a_k^\Phi}. \tag{4.6}$$

Equality (4.1) follows by combining Lemma 4.5, (4.5), and (4.6).

Taking the adjoint of (4.1), we write

$$U_\Phi^{-1} D_k U_\Phi = \sum_{l=1}^d D_l M_{\bar{a}_{k,l}^\Phi} + M_{\bar{a}_k^\Phi}.$$

Thus,

$$\begin{aligned} U_\Phi^{-1} D_k^2 U_\Phi &= \Big(\sum_{l=1}^d D_l M_{\bar{a}_{k,l}^\Phi} + M_{\bar{a}_k^\Phi}\Big) \cdot \Big(\sum_{l=1}^d M_{a_{k,l}^\Phi} D_l + M_{a_k^\Phi}\Big) \\ &= \sum_{l_1,l_2=1}^d D_{l_1} M_{\bar{a}_{k,l_1}^\Phi} M_{a_{k,l_2}^\Phi} D_{l_2} + \sum_{l_1=1}^d D_{l_1} M_{\bar{a}_{k,l_1}^\Phi} M_{a_k^\Phi} \\ &\quad + \sum_{l_2=1}^d M_{\bar{a}_k^\Phi} M_{a_{k,l_2}^\Phi} D_{l_2} + M_{|a_k^\Phi|^2}. \end{aligned}$$

Clearly,

$$\begin{aligned} &\sum_{l_1=1}^d D_{l_1} M_{\bar{a}_{k,l_1}^\Phi} M_{a_k^\Phi} + \sum_{l_2=1}^d M_{\bar{a}_k^\Phi} M_{a_{k,l_2}^\Phi} D_{l_2} \\ &\quad = \sum_{l=1}^d D_l M_{\bar{a}_{k,l}^\Phi \cdot a_k^\Phi} + \sum_{l=1}^d M_{\bar{a}_k^\Phi \cdot a_{k,l}^\Phi} D_l \\ &\quad = \sum_{l=1}^d M_{\bar{a}_{k,l}^\Phi \cdot a_k^\Phi} D_l + \sum_{l=1}^d M_{\bar{a}_k^\Phi \cdot a_{k,l}^\Phi} D_l + \sum_{l=1}^d [D_l, M_{\bar{a}_{k,l}^\Phi \cdot a_k^\Phi}] \\ &\quad \overset{(2.1)}{=} 2\sum_{l=1}^d M_{\Re(\bar{a}_k^\Phi \cdot a_{k,l}^\Phi)} D_l + \sum_{l=1}^d M_{D_l(\bar{a}_{k,l}^\Phi \cdot a_k^\Phi)}. \end{aligned}$$

Thus,

$$\begin{aligned} -U_\Phi^{-1} \Delta U_\Phi &= \sum_{k=1}^d U_\Phi^{-1} D_k^2 U_\Phi = \sum_{k=1}^d \sum_{l_1,l_2=1}^d D_{l_1} M_{\bar{a}_{k,l_1}^\Phi \cdot a_{k,l_2}^\Phi} D_{l_2} \\ &\quad + 2\sum_{k=1}^d \sum_{l=1}^d M_{\Re(\bar{a}_k^\Phi \cdot a_{k,l}^\Phi)} D_l + \sum_{k=1}^d \sum_{l=1}^d M_{D_l(\bar{a}_{k,l}^\Phi \cdot a_k^\Phi)} + \sum_{k=1}^d M_{|a_k^\Phi|^2}. \end{aligned}$$

By the definition on b_l^Φ and b^Φ (in Lemma 4.4), we have

$$-U_\Phi^{-1} \Delta U_\Phi = \sum_{k=1}^d \sum_{l_1,l_2=1}^d D_{l_1} M_{\bar{a}_{k,l_1}^\Phi \cdot a_{k,l_2}^\Phi} D_{l_2} + \sum_{l=1}^d M_{b_l^\Phi} D_l + M_{b^\Phi}.$$

Consider now the highest order term. Recalling Notation 4.3, we write

$$\sum_{k=1}^{d} \bar{a}^{\Phi}_{k,l_1} a^{\Phi}_{k,l_2} = \big(\big|(a^{\Phi}_{k,l})^{d}_{k,l=1}\big|^2\big)_{l_1,l_2} = \big(|J^*_{\Phi} \circ \Phi^{-1}|^2\big)_{l_1,l_2} = b^{\Phi}_{l_1,l_2}.$$

This delivers (4.2). ∎

5. Proof of Theorem 3.5

The proof of Theorem 3.5 is somewhat technical and is presented below in the series of lemmas. The strategy is as follows:

(1) to show that every compact operator on $L_2(\mathbb{R}^d)$ belongs to Π;

(2) to show that the conjugation of $M_\phi \frac{D_k}{\sqrt{1-\Delta}}$, $\phi \in C^\infty_c(\mathbb{R}^d)$, by U_Φ belongs to Π modulo compact operators;

(3) to conclude that the conjugation of $M_\phi \frac{D_k}{\sqrt{-\Delta}}$, $\phi \in C^\infty_c(\mathbb{R}^d)$, by U_Φ belongs to Π;

(4) to conclude the argument in Theorem 3.5.

The following assertion is well known (see e.g. [9, Corollary 4.1.10]).

Lemma 5.1. *Let $\mathcal{A}$ be a C^*-algebra. Let $\pi : \mathcal{A} \to B(H)$ be an irreducible representation. One of the following mutually exclusive options holds:*

(1) *$\pi(\mathcal{A})$ does not contain any compact operator (except for 0);*

(2) *$\pi(\mathcal{A})$ contains every compact operator.*

We now apply Lemma 5.1 to the C^*-algebra $\mathcal{A} = \Pi$ and infer that Π contains the ideal $\mathcal{K}(L_2(\mathbb{R}^d))$.

Lemma 5.2. *The algebra $\mathcal{K}(L_2(\mathbb{R}^d))$ is contained in Π and coincides with the kernel of the homomorphism* sym.

Proof. Since Π contains $\pi_1(\mathcal{A}_1)$, it follows (here, X' denotes the commutant of the set $X \subset B(L_2(\mathbb{R}^d))$) that

$$\Pi' \subset \big(\pi_1(\mathcal{A}_1)\big)' = \big(\pi_1\big(L_\infty(\mathbb{R}^d)\big)\big)' = \pi_1\big(L_\infty(\mathbb{R}^d)\big).$$

Define $g_{n,k} \in C(\mathbb{S}^{d-1})$, $1 \le k \le d$, $n \in \mathbb{N}$, by setting $g_k(s) = s_k^{\frac{1}{2n+1}}$, $s \in \mathbb{S}^{d-1}$. Clearly, $\pi_2(g_{n,k}) \to \mathrm{sgn}(D_k)$ as $n \to \infty$ in weak operator topology. Thus, $\mathrm{sgn}(D_k)$ belongs to the weak closure of $\pi_2(\mathcal{A}_2)$ and, hence, to the weak closure of Π. Therefore,

$$\Pi' \subset \big(\mathrm{sgn}(D_k)\big)', \quad 1 \le k \le d.$$

Thus,

$$\Pi' \subset \Big(\bigcap_{1\le k\le d} \big(\mathrm{sgn}(D_k)\big)'\Big) \cap \pi_1\big(L_\infty(\mathbb{R}^d)\big).$$

For $t \in \mathbb{R}^d$, denote by $\check{t}_k \in \mathbb{R}^{d-1}$ the vector obtained by eliminating the k-th component of t. If $f \in L_\infty(\mathbb{R}^d)$ is such that $\pi_1(f)$ commutes with $\mathrm{sgn}(D_k)$, then, for almost every $\check{t}_k \in \mathbb{R}^{d-1}$, the function $f(\check{t}_k, \cdot)$ commutes with the Hilbert transform. This easily implies that, for almost every $\check{t}_k \in \mathbb{R}^{d-1}$, the function $f(\check{t}_k, \cdot)$ is constant. If $f \in L_\infty(\mathbb{R}^d)$ is such that $\pi_1(f)$ commutes with *every* $\mathrm{sgn}(D_k)$, $1 \leq k \leq d$, then $f = \mathrm{const}$. Hence, Π' is trivial.

By [5, Proposition II.6.1.8], representation $\mathrm{id} : \Pi \to B(L_2(\mathbb{R}^d))$ is irreducible.

We now demonstrate that Π contains a non-zero compact operator. As proved above, for every non-zero $f \in C_c^\infty(\mathbb{R}^d)$, there exists $1 \leq k \leq d$ such that $\pi_1(f)$ does not commute with $\mathrm{sgn}(D_k)$. Since $\pi_2(g_{n,k}) \to \mathrm{sgn}(D_k)$ as $n \to \infty$ in weak operator topology, it follows that $\pi_1(f)$ does not commute with $\pi_2(g_{n,k})$ for some n, k. Thus, the operator $[\pi_1(f), \pi_2(g_{n,k})]$ is a non-zero compact operator, which belongs to Π. The first assertion of the lemma follows now from Lemma 5.1.

Let $q : B(L_2(\mathbb{R}^d)) \to B(L_2(\mathbb{R}^d))/\mathcal{K}(L_2(\mathbb{R}^d))$ be the canonical quotient map. Recall (see the proof of Theorem 3.3 in [19]) that sym is constructed as a composition

$$\mathrm{sym} = \theta^{-1} \circ q,$$

where θ^{-1} is some linear isomorphism (its definition and properties are irrelevant at the current proof). It follows that the kernel of sym coincides with the kernel of q, which is $\mathcal{K}(L_2(\mathbb{R}^d))$. ■

Notation 5.3. Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that Φ is affine outside of some ball. Denote

$$p^\Phi(t,s) = \sum_{l_1,l_2=1}^d b^\Phi_{l_1,l_2}(t) s_{l_1} s_{l_2}, \quad r^\Phi_k(t,s) = \sum_{l=1}^d a^\Phi_{k,l}(t) s_l, \quad 1 \leq k \leq d.$$

Here, $(a^\Phi_{k,l})_{k,l=1}^d$ and $(b^\Phi_{l_1,l_2})_{l_1,l_2=1}^d$ are as in Notation 4.3.

The following two lemmas form the core of our computation.

Lemma 5.4. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that Φ is affine outside of some ball. We have*

$$U_\Phi^{-1} \frac{D_k}{\sqrt{1-\Delta}} U_\Phi \in \mathrm{Op}\Big(\frac{r_k^\Phi}{(1+p^\Phi)^{\frac{1}{2}}}\Big) + \Psi^{-1}(\mathbb{R}^d).$$

Proof. Lemma 4.7 asserts that

$$U_\Phi^{-1} D_k U_\Phi = \mathrm{Op}(r_k^\Phi) + M_{a_k^\Phi}. \tag{5.1}$$

It is immediate that

$$\mathrm{Op}(r_k^\Phi) \in \Psi^1(\mathbb{R}^d), \quad M_{a_k^\Phi} \in \Psi^0(\mathbb{R}^d). \tag{5.2}$$

Lemma 4.6 yields that $-U_\Phi^{-1} \Delta U_\Phi$ is a self-adjoint positive operator with the domain $U_\Phi^{-1}(W^{2,2}(\mathbb{R}^d)) = W^{2,2}(\mathbb{R}^d)$. Lemma 4.7 and (2.7) yield that $-U_\Phi^{-1} \Delta U_\Phi$ is a differential operator of order 2 with principal symbol $p^\Phi \geq 0$.

By Theorem 2.5 applied with $T=-U_\Phi^{-1}\Delta U_\Phi$ and $z=-\frac{1}{2}$, we have

$$(1-U_\Phi^{-1}\Delta U_\Phi)^{-\frac{1}{2}}-\mathrm{Op}\big((p^\Phi+1)^{-\frac{1}{2}}\big)\in\Psi^{-2}(\mathbb{R}^d), \tag{5.3}$$

$$\mathrm{Op}\big((p^\Phi+1)^{-\frac{1}{2}}\big)\in\Psi^{-1}(\mathbb{R}^d). \tag{5.4}$$

Equations (5.1), (5.2), (5.3), and (5.4) yield that the operators

$$T_1=U_\Phi^{-1}D_kU_\Phi,\quad T_2=U_\Phi^{-1}(1-\Delta)^{-\frac{1}{2}}U_\Phi$$

satisfy the assumptions in Lemma 2.4. By Lemma 2.4, we have

$$U_\Phi^{-1}\frac{D_k}{\sqrt{1-\Delta}}U_\Phi=T_1T_2\in\mathrm{Op}\big(r_k^\Phi\cdot(1+p^\Phi)^{-\frac{1}{2}}\big)+\Psi^{-1}(\mathbb{R}^d). \qquad \blacksquare$$

In our next lemma, we approximate the operators on the left-hand side with pseudo-differential-like operators on the right-hand side. The latter is defined in (2.8) in Section 2.5.

Lemma 5.5. *Let $\Phi:\mathbb{R}^d\to\mathbb{R}^d$ be a diffeomorphism such that Φ is affine outside of some ball. If $\phi\in C_c^\infty(\mathbb{R}^d)$, then*

$$U_\Phi^{-1}M_\phi\frac{D_k}{\sqrt{1-\Delta}}U_\Phi\in T_{(\phi\circ\Phi^{-1}\otimes 1)\cdot q_k^\Phi}+\mathcal{K}\big(L_2(\mathbb{R}^d)\big),$$

where

$$q_k^\Phi(t,s)=(O_{J_\Phi^*(\Phi^{-1}(t))}s)_k,\quad t\in\mathbb{R}^d,\ s\in\mathbb{S}^{d-1}.$$

Proof. For every $f\in C^\infty(\mathbb{R}^d)$ and for every $p\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, we have

$$M_f\cdot\mathrm{Op}(p)=\mathrm{Op}\big((f\otimes 1)p\big).$$

Also,

$$U_\Phi^{-1}M_\phi\frac{D_k}{\sqrt{1-\Delta}}U_\Phi=M_{\phi\otimes\Phi^{-1}}\cdot U_\Phi^{-1}\frac{D_k}{\sqrt{1-\Delta}}U_\Phi.$$

It follows now from Lemma 5.4 that

$$U_\Phi^{-1}M_\phi\frac{D_k}{\sqrt{1-\Delta}}U_\Phi\in\mathrm{Op}\Bigg((\phi\circ\Phi^{-1}\otimes 1)\frac{r_k^\Phi}{(1+p^\Phi)^{\frac{1}{2}}}\Bigg)+\Psi^{-1}(\mathbb{R}^d). \tag{5.5}$$

Fix a function $\psi\in C_c^\infty(\mathbb{R}^d)$ such that $\psi=1$ near 0 and such that $(\phi\circ\Phi^{-1})\cdot\psi=\phi\circ\Phi^{-1}$. Set

$$e_k(t,s)=\phi\big(\Phi^{-1}(t)\big)\cdot r_k^\Phi(t,s)\cdot\big(1+p^\Phi(t,s)\big)^{-\frac{1}{2}},\quad t,s\in\mathbb{R}^d,$$

$$f_k(t,s)=\phi\big(\Phi^{-1}(t)\big)\cdot r_k^\Phi(t,s)\cdot\big(p^\Phi(t,s)\big)^{-\frac{1}{2}}\cdot\big(1-\psi(s)\big),\quad t,s\in\mathbb{R}^d.$$

We have $e_k-f_k=g_k\cdot h$, where

$$g_k(t,s)=\phi\big(\Phi^{-1}(t)\big)\cdot r_k^\Phi(t,s),\quad t,s\in\mathbb{R}^d,$$

$$h(t,s)=\big(1+p^\Phi(t,s)\big)^{-\frac{1}{2}}-\big(p^\Phi(t,s)\big)^{-\frac{1}{2}}\cdot\big(1-\psi(s)\big),\quad t,s\in\mathbb{R}^d.$$

An elementary computation shows that

$$\sup_{t,s\in\mathbb{R}^d}\big(1+|s|^2\big)^{\frac{|\beta|_1+2}{2}}\big|D_t^{\alpha}D_s^{\beta}h(t,s)\big|<\infty,\quad \alpha,\beta\in\mathbb{Z}_+^d.$$

By the Leibniz rule, we have

$$D_t^{\alpha}D_s^{\beta}(g_k\cdot h)=\sum_{0\leq\gamma\leq\alpha}\sum_{0\leq\delta\leq\beta}c(\alpha,\gamma)c(\beta,\delta)D_t^{\gamma}D_s^{\delta}g_k\cdot D_t^{\alpha-\gamma}D_s^{\beta-\delta}h.$$

This implies

$$\sup_{t,s\in\mathbb{R}^d}\big(1+|s|^2\big)^{\frac{|\beta|_1+1}{2}}\big|D_t^{\alpha}D_s^{\beta}(e_k-f_k)(t,s)\big|<\infty,\quad \alpha,\beta\in\mathbb{Z}_+^d,$$

so that $\mathrm{Op}(e_k-f_k)\in\Psi^{-1}(\mathbb{R}^d)$ (see (2.3)). It follows now from (5.5) that

$$U_\Phi^{-1}M_\phi\frac{D_k}{\sqrt{1-\Delta}}U_\Phi-\mathrm{Op}(f_k)\in\Psi^{-1}(\mathbb{R}^d). \tag{5.6}$$

Denote for brevity the left-hand side of (5.6) by T (so that $T\in\Psi^{-1}(\mathbb{R}^d)$). Due to the choice of ψ, we have that $T=M_\psi T$. It follows now from (2.6) (applied with $m=-1$) that T is compact. In other words, we have

$$U_\Phi^{-1}M_\phi\frac{D_k}{\sqrt{1-\Delta}}U_\Phi-\mathrm{Op}(f_k)\in\mathcal{K}\big(L_2(\mathbb{R}^d)\big). \tag{5.7}$$

Appealing to the definition of r_k^Φ and p^Φ, we note that

$$r_k^\Phi(t,s)=\big(J_\Phi^*(\Phi^{-1}(t))s\big)_k,\quad p^\Phi(t,s)=\big|J_\Phi^*(\Phi^{-1}(t))s\big|^2,\quad t,s\in\mathbb{R}^d.$$

Therefore,

$$r_k^\Phi(t,s)\cdot\big(p^\Phi(t,s)\big)^{-\frac12}=\Big(O_{J_\Phi^*(\Phi^{-1}(t))}\frac{s}{|s|}\Big)_k,\quad t,s\in\mathbb{R}^d.$$

Thus,

$$f_k(t,s)=\phi\big(\Phi^{-1}(t)\big)\cdot q_k^\Phi\Big(t,\frac{s}{|s|}\Big)\cdot\big(1-\psi(s)\big),\quad t,s\in\mathbb{R}^d.$$

By Lemma 2.7 applied with $q=(\phi\circ\Phi^{-1}\otimes 1)\cdot q_k^\Phi$, we have

$$\mathrm{Op}(f_k)-T_{(\phi\circ\Phi^{-1}\otimes1)\cdot q_k^\Phi}\in\mathcal{K}\big(L_2(\mathbb{R}^d)\big). \tag{5.8}$$

Combining (5.7) and (5.8), we complete the proof. ∎

Lemma 5.6. *Let Φ be a diffeomorphism such that Φ is affine outside of some ball. If $\phi\in C_c^\infty(\mathbb{R}^d)$, then*

$$U_\Phi^{-1}M_\phi\frac{D_k}{\sqrt{-\Delta}}U_\Phi\in\Pi$$

and

$$\mathrm{sym}\Big(U_\Phi^{-1}M_\phi\frac{D_k}{\sqrt{-\Delta}}U_\Phi\Big)=(\phi\circ\Phi^{-1}\otimes1)\cdot q_k^\Phi.$$

Proof. Applying bounded Borel function

$$t \to (1+|t|^2)\cdot\Big(\frac{t}{|t|}-\frac{t}{\sqrt{1+|t|^2}}\Big), \quad t\in\mathbb{R}^d,$$

to the tuple ∇, we obtain that

$$(1-\Delta)\cdot\Big(\frac{D_k}{\sqrt{-\Delta}}-\frac{D_k}{\sqrt{1-\Delta}}\Big)\in B\big(L_2(\mathbb{R}^d)\big).$$

Recall that (see e.g. [26, Theorem 4.1]) $M_\phi(1-\Delta)^{-1}\in\mathcal{K}(L_2(\mathbb{R}^d))$. Since the product of bounded and compact operators is compact, it follows that

$$M_\phi\Big(\frac{D_k}{\sqrt{-\Delta}}-\frac{D_k}{\sqrt{1-\Delta}}\Big)\in\mathcal{K}\big(L_2(\mathbb{R}^d)\big)$$

and

$$U_\Phi^{-1}M_\phi\Big(\frac{D_k}{\sqrt{-\Delta}}-\frac{D_k}{\sqrt{1-\Delta}}\Big)U_\Phi\in\mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

Combining with Lemma 5.5, we obtain

$$U_\Phi^{-1}M_\phi\frac{D_k}{\sqrt{-\Delta}}U_\Phi - T_{(\phi\circ\Phi^{-1}\otimes 1)\cdot q_k^\Phi}\in\mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

By Lemma 2.6, we have

$$T_{(\phi\circ\Phi^{-1}\otimes 1)\cdot q_k^\Phi}\in\Pi,\quad \mathrm{sym}(T_{(\phi\circ\Phi^{-1}\otimes 1)\cdot q_k^\Phi})=(\phi\circ\Phi^{-1}\otimes 1)\cdot q_k^\Phi.$$

The assertion follows by combining the last two equations and Lemma 5.2. ∎

Lemma 5.7. *If $T_1,T_2\in\Pi$, then*

$$[T_1,T_2]\in\mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

Proof. Since sym is a $*$-homomorphism, it follows that $\mathrm{sym}([T_1,T_2])=0$. The assertion is now an immediate consequence of Lemma 5.2. ∎

Lemma 5.8. *Let $(f_k)_{k=1}^m\subset\mathcal{A}_1$ and $(g_k)_{k=1}^m\subset\mathcal{A}_2$. We have*

$$\prod_{k=1}^m\pi_1(f_k)\pi_2(g_k)\in\pi_1\Big(\prod_{k=1}^m f_k\Big)\pi_2\Big(\prod_{k=1}^m g_k\Big)+\mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

Proof. We prove the assertion by induction on m. For $m=1$, there is nothing to prove. So, we only have to prove the step of induction.

Let us prove the assertion for $m=2$. We have

$$\begin{aligned}\pi_1(f_1)\pi_2(g_1)\pi_1(f_2)\pi_2(g_2)&=\big[\pi_2(g_1),\pi_1(f_1f_2)\big]\cdot\pi_2(g_2)\\&\quad+\big[\pi_1(f_1),\pi_2(g_1)\big]\cdot\pi_1(f_2)\pi_2(g_2)\\&\quad+\pi_1(f_1f_2)\pi_2(g_1g_2).\end{aligned}$$

By Lemma 5.7, we have

$$\big[\pi_1(f_1),\pi_2(g_1)\big],\big[\pi_2(g_1),\pi_1(f_1f_2)\big]\in\mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

Therefore,

$$\pi_1(f_1)\pi_2(g_1)\pi_1(f_2)\pi_2(g_2)\in\pi_1(f_1f_2)\pi_2(g_1g_2)+\mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

This proves the assertion for $m=2$.

It remains to prove the step of induction. Suppose the assertion holds for $m\geq 2$ and let us prove it for $m+1$. Clearly,

$$\prod_{k=1}^{m+1}\pi_1(f_k)\pi_2(g_k)=\pi_1(f_1)\pi_2(g_1)\cdot\prod_{k=2}^{m+1}\pi_1(f_k)\pi_2(g_k).$$

Using the inductive assumption, we obtain

$$\prod_{k=1}^{m+1}\pi_1(f_k)\pi_2(g_k)\in\pi_1(f_1)\pi_2(g_1)\cdot\pi_1\Bigg(\prod_{k=2}^{m+1}f_k\Bigg)\pi_2\Bigg(\prod_{k=2}^{m+1}g_k\Bigg)+\mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

Using the assertion for $m=2$, we obtain

$$\pi_1(f_1)\pi_2(g_1)\cdot\pi_1\Bigg(\prod_{k=2}^{m+1}f_k\Bigg)\pi_2\Bigg(\prod_{k=2}^{m+1}g_k\Bigg)\in\pi_1\Bigg(\prod_{k=1}^{m+1}f_k\Bigg)\pi_2\Bigg(\prod_{k=1}^{m+1}g_k\Bigg)+\mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

Combining the last two equations, we obtain

$$\prod_{k=1}^{m+1}\pi_1(f_k)\pi_2(g_k)\in\pi_1\Bigg(\prod_{k=1}^{m+1}f_k\Bigg)\pi_2\Bigg(\prod_{k=1}^{m+1}g_k\Bigg)+\mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

This establishes the step of induction and, hence, completes the proof of the lemma. ∎

Lemma 5.9. *Let Φ be a diffeomorphism which is affine outside of some ball. If $g\in C(\mathbb{S}^{d-1})$ and $f\in C_c(\mathbb{R}^d)$, then*

$$U_\Phi^{-1}\pi_1(f)\pi_2(g)U_\Phi\in\Pi$$

and

$$\mathrm{sym}\,\big(U_\Phi^{-1}\pi_1(f)\pi_2(g)U_\Phi\big)=(f\circ\Phi^{-1}\otimes 1)\cdot g(q_1^\Phi,\dots,q_d^\Phi).$$

Proof. Let $\mathrm{Poly}(\mathbb{S}^{d-1})$ be the algebra of polynomials on $\mathbb{S}^{d-1}$.

Suppose first that $g\in\mathrm{Poly}(\mathbb{S}^{d-1})$ is monomial. Let $g(s)=\prod_{l=1}^d s_l^{n_l}$. Let $\phi\in C_c^\infty(\mathbb{R}^d)$ be such that $f\cdot\phi=f$. Obviously,

$$\pi_1(f)\pi_2(g)=\pi_1(f\cdot\phi^{\sum_{l=1}^d n_l})\cdot\pi_2(g)=\pi_1(f)\cdot\pi_1(\phi^{\sum_{l=1}^d n_l})\pi_2(g).$$

Setting $m = \sum_{l=1}^{d} n_l$, $f_k = \phi$, $1 \le k \le m$,

$$g_k(s) = s_l, \quad s \in \mathbb{S}^{d-1}, \ \sum_{i=1}^{l-1} n_i < k \le \sum_{i=1}^{l} n_i.$$

In this notation,

$$\pi_1(\phi^{\sum_{l=1}^{d} n_l})\pi_2(g) = \pi_1\Big(\prod_{k=1}^{m} f_k\Big)\pi_2\Big(\prod_{k=1}^{m} g_k\Big).$$

By Lemma 5.8, we have

$$\pi_1(\phi^{\sum_{l=1}^{d} n_l})\pi_2(g) \in \prod_{k=1}^{m} \pi_1(f_k)\pi_2(g_k) + \mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

Since

$$\pi_1(f_k)\pi_2(g_k) = \pi_1(\phi)\frac{D_l}{\sqrt{-\Delta}}, \quad \sum_{i=1}^{l-1} n_i < k \le \sum_{i=1}^{l} n_i,$$

it follows that

$$\pi_1(f)\pi_2(g) \in \pi_1(f) \cdot \prod_{l=1}^{d} \Big(\pi_1(\phi)\frac{D_l}{\sqrt{-\Delta}}\Big)^{n_l} + \mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

Thus,

$$U_\Phi^{-1}\pi_1(f)\pi_2(g)U_\Phi \in U_\Phi^{-1}\pi_1(f)U_\Phi \cdot \prod_{l=1}^{d} \Big(U_\Phi^{-1}\pi_1(\phi)\frac{D_l}{\sqrt{-\Delta}}U_\Phi\Big)^{n_l} + \mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

By Lemma 5.6, we have

$$U_\Phi^{-1}\pi_1(f)U_\Phi \cdot \prod_{l=1}^{d} \Big(U_\Phi^{-1}\pi_1(\phi)\frac{D_l}{\sqrt{-\Delta}}U_\Phi\Big)^{n_l} \in \Pi$$

and

$$\begin{aligned}
&\mathrm{sym}\Big(U_\Phi^{-1}\pi_1(f)U_\Phi \cdot \prod_{l=1}^{d} \Big(U_\Phi^{-1}\pi_1(\phi)\frac{D_l}{\sqrt{-\Delta}}U_\Phi\Big)^{n_l}\Big) \\
&= \mathrm{sym}\big(U_\Phi^{-1}\pi_1(f)U_\Phi\big) \cdot \prod_{l=1}^{d} \Big(\mathrm{sym}\Big(U_\Phi^{-1}\pi_1(\phi)\frac{D_l}{\sqrt{-\Delta}}U_\Phi\Big)\Big)^{n_l} \\
&= (f \circ \Phi^{-1} \otimes 1) \cdot \prod_{l=1}^{d} \big((\phi \circ \Phi^{-1} \otimes 1) \cdot q_l^\Phi\big)^{n_l} = (f \circ \Phi^{-1} \otimes 1) \cdot \prod_{l=1}^{d} (q_l^\Phi)^{n_l}.
\end{aligned}$$

By Lemma 5.2, we have

$$U_\Phi^{-1}\pi_1(f)\pi_2(g)U_\Phi \in \Pi$$

and

$$\mathrm{sym}\left(U_\Phi^{-1}\pi_1(f)\pi_2(g)U_\Phi\right) = (f \circ \Phi^{-1} \otimes 1)\cdot g(q_1^\Phi,\dots,q_d^\Phi).$$

By linearity, the same assertion holds if $g \in \mathrm{Poly}(\mathbb{S}^{d-1})$. To prove the assertion in general, let $g \in C(\mathbb{S}^{d-1})$ and consider a sequence

$$\{g_n\}_{n\geq 1} \subset \mathrm{Poly}(\mathbb{S}^{d-1})$$

such that $g_n \to g$ in the uniform norm. We have

$$U_\Phi^{-1}\pi_1(f)\pi_2(g_n)U_\Phi \to U_\Phi^{-1}\pi_1(f)\pi_2(g)U_\Phi$$

in the uniform norm. Since

$$U_\Phi^{-1}\pi_1(f)\pi_2(g_n)U_\Phi \in \Pi, \quad n \geq 1,$$

it follows that

$$U_\Phi^{-1}\pi_1(f)\pi_2(g)U_\Phi \in \Pi$$

and

$$\mathrm{sym}\left(U_\Phi^{-1}\pi_1(f)\pi_2(g_n)U_\Phi\right) \to \mathrm{sym}\left(U_\Phi^{-1}\pi_1(f)\pi_2(g)U_\Phi\right)$$

in the uniform norm. In other words,

$$(f \circ \Phi^{-1} \otimes 1)\cdot g_n(q_1^\Phi,\dots,q_d^\Phi) \to \mathrm{sym}\left(U_\Phi^{-1}\pi_1(f)\pi_2(g)U_\Phi\right)$$

in the uniform norm. Thus,

$$\mathrm{sym}\left(U_\Phi^{-1}\pi_1(f)\pi_2(g)U_\Phi\right) = (f \circ \Phi^{-1} \otimes 1)\cdot g(q_1^\Phi,\dots,q_d^\Phi). \qquad \blacksquare$$

Proof of Theorem 3.5. By the definition of the C^*-algebra Π, for every $T \in \Pi$, there exists a sequence $(T_n)_{n\geq 1}$ in the $*$-algebra generated by $\pi_1(\mathcal{A}_1)$ and $\pi_2(\mathcal{A}_2)$ such that $T_n \to T$ in the uniform norm. We can write

$$T_n = \sum_{l=1}^{l_n}\prod_{k=1}^{k_n}\pi_1(f_{n,k,l})\pi_2(g_{n,k,l}).$$

By Lemma 5.8, we have

$$T_n \in \sum_{l=1}^{l_n}\pi_1\left(\prod_{k=1}^{k_n} f_{n,k,l}\right)\pi_2\left(\prod_{k=1}^{k_n} g_{n,k,l}\right) + \mathcal{K}\left(L_2(\mathbb{R}^d)\right).$$

Denote, for brevity,

$$f_{n,l} = \prod_{k=1}^{k_n} f_{n,k,l} \in \mathcal{A}_1, \quad g_{n,l} = \prod_{k=1}^{k_n} g_{n,k,l} \in \mathcal{A}_2.$$

We have

$$T_n = S_n + \sum_{l=1}^{l_n} \pi_1(f_{n,l})\pi_2(g_{n,l}), \quad S_n \in \mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

By Lemma 5.2, we have

$$\mathrm{sym}(T_n) = \sum_{l=1}^{l_n} f_{n,l} \otimes g_{n,l}.$$

Suppose in addition that T is compactly supported. In particular, $T = M_\phi T$ for some $\phi \in C_c^\infty(\mathbb{R}^d)$. Replacing S_n with $M_\phi S_n$ and $f_{n,l}$ with $\phi \cdot f_{n,l}$ if necessary, we may assume without loss of generality that $f_{n,l} \in C_c^\infty(\mathbb{R}^d)$ for every $n \geq 1$ and for every $1 \leq l \leq l_n$.

By Lemma 5.9, we have

$$\sum_{l=1}^{l_n} U_\Phi^{-1}\pi_1(f_{n,l})\pi_2(g_{n,l})U_\Phi \in \Pi$$

and

$$\mathrm{sym}\left(\sum_{l=1}^{l_n} U_\Phi^{-1}\pi_1(f_{n,l})\pi_2(g_{n,l})U_\Phi\right) = \left(\sum_{l=1}^{l_n} f_{n,l} \otimes g_{n,l}\right) \circ \Theta_\Phi,$$

where Θ_Φ is introduced in Notation 3.1.

By Lemma 5.2, we have

$$U_\Phi^{-1} S_n U_\Phi \in \Pi$$

and

$$\mathrm{sym}(U_\Phi^{-1} S_n U_\Phi) = 0.$$

Thus, $U_\Phi^{-1} T_n U_\Phi \in \Pi$ and

$$\mathrm{sym}(U_\Phi^{-1} T_n U_\Phi) = \left(\sum_{l=1}^{l_n} f_{n,l} \otimes g_{n,l}\right) \circ \Theta_\Phi = \mathrm{sym}(T_n) \circ \Theta_\Phi.$$

Since Π is a C^*-algebra and since $U_\Phi^{-1} T_n U_\Phi \to U_\Phi^{-1} T U_\Phi$ in the uniform norm, it follows that $U_\Phi^{-1} T U_\Phi \in \Pi$ and

$$\mathrm{sym}(U_\Phi^{-1} T_n U_\Phi) \to \mathrm{sym}(U_\Phi^{-1} T U_\Phi)$$

in the uniform norm. In other words,

$$\mathrm{sym}(T_n) \circ \Theta_\Phi \to \mathrm{sym}(U_\Phi^{-1} T U_\Phi)$$

in the uniform norm. Since $\mathrm{sym}(T_n) \to \mathrm{sym}(T)$ in the uniform norm, it follows that

$$\mathrm{sym}(U_\Phi^{-1} T U_\Phi) = \mathrm{sym}(T) \circ \Theta_\Phi. \qquad \blacksquare$$

6. Invariance of principal symbol under local diffeomorphisms

Theorem 3.11 is supposed to be a corollary of Theorem 3.5. To demonstrate this is indeed the case, we need an extension result for diffeomorphisms.

The following fundamental result is due to Palais [22] (see Corollary 4.3 there).

Theorem 6.1. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth mapping. Necessary and sufficient conditions for Φ to be a diffeomorphism are as follows:*

(1) *for every $t \in \mathbb{R}^d$, we have $\det(J_\Phi(t)) \neq 0$;*

(2) *we have $\Phi(t) \to \infty$ as $|t| \to \infty$.*

The next lemma is also due to Palais [22]. We provide a proof for the convenience of the reader. Note that $B(t,r)$ is the open ball with radius r centered at t.

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\Phi : \Omega \to \mathbb{R}^d$ be a smooth mapping. If $t \in \Omega$ is such that $\det(J_\Phi(t)) \neq 0$, then there exists a diffeomorphism $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ such that*

(1) *$\Phi_t = \Phi$ on $B(t, r_1(t))$ with some $r_1(t) > 0$;*

(2) *Φ_t is affine outside $B(t, r_2(t))$ for some $r_2(t) < \infty$.*

Proof. Without loss of generality, $t = 0$, $\Phi(0) = 0$, and $J_\Phi(0) = 1_{M_d(\mathbb{R})}$, the unity in the algebra of real $d \times d$ matrices. Let $\theta \in C_c^\infty(\mathbb{R}^d)$ be such that $\theta = 1$ on the unit ball. Set

$$\Psi_r(u) = u + \theta\Big(\frac{u}{r}\Big) \cdot \big(\Phi(u) - u\big), \quad u \in \mathbb{R}^d.$$

It is clear that Ψ_r is a well-defined smooth mapping for every sufficiently small $r > 0$. A direct calculation shows that $\det(J_{\Psi_r}) \to 1$ in the uniform norm as $r \to 0$. In particular, for sufficiently small $r > 0$, $\det(J_{\Psi_r})$ never vanishes. It follows from Theorem 6.1 that, for sufficiently small $r > 0$, $\Psi_r : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism. Choose any such r and denote it by $r(0)$. Set $\Phi_0 = \Psi_{r(0)}$. This diffeomorphism obviously satisfies the required properties. ∎

Lemma 6.3. *Let $\Omega, \Omega' \subset \mathbb{R}^d$ and let $\Phi : \Omega \to \Omega'$ be a diffeomorphism. Let $B \subset \Omega$ be a ball and let $\Phi_0 : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that $\Phi_0 = \Phi$ on B. If $T \in B(L_2(\mathbb{R}^d))$ is supported on B, then*

$$\mathrm{Ext}_{\Omega'}\big(U_\Phi^{-1} \cdot \mathrm{Rest}_\Omega(T) \cdot U_\Phi\big) = U_{\Phi_0}^{-1} T U_{\Phi_0}.$$

Proof. Indeed, since both sides are continuous in weak operator topology, it suffices to prove the assertion for the case when T is rank 1 operator.

Let

$$T\xi = \langle \xi, \xi_1 \rangle \xi_2, \quad \xi \in L_2(\mathbb{R}^d).$$

where $\xi_1, \xi_2 \in L_2(\mathbb{R}^d)$ are supported in B. It is immediate that

$$(U_{\Phi_0}^{-1} T U_{\Phi_0})\xi = \langle \xi, U_{\Phi_0}^{-1} \xi_1 \rangle \cdot U_{\Phi_0^{-1}} \xi_2, \quad \xi \in L_2(\mathbb{R}^d),$$

$$\big(U_\Phi^{-1}\cdot \mathrm{Rest}_\Omega(T)\cdot U_\Phi\big)\xi = \langle \xi, U_\Phi^{-1}\xi_1\rangle \cdot U_{\Phi^{-1}}\xi_2, \quad \xi\in L_2(U'),$$
$$\big(\mathrm{Ext}_{\Omega'}\big(U_\Phi^{-1}\cdot \mathrm{Rest}_\Omega(T)\cdot U_\Phi\big)\big)\xi = \langle \xi\cdot\chi_{\Omega'}, U_\Phi^{-1}\xi_1\rangle \cdot U_{\Phi^{-1}}\xi_2\cdot\chi_{\Omega'}, \quad \xi\in L_2(\mathbb{R}^d).$$

Since ξ_1 and ξ_2 are supported in B, it follows that expressions in the first and last displays coincide. This proves the assertion for every rank 1 operator T and, therefore, for every T. ∎

Proof of Theorem 3.11. Let the operator T be supported on a compact set $K\subset\Omega$. Let $t\in K$. Let diffeomorphism $\Phi_t:\mathbb{R}^d\to\mathbb{R}^d$ and numbers $r_1(t)$ and $r_2(t)$ be as in Lemma 6.2.

The collection $\{B(t,r_1(t))\}_{t\in K}$ is an open cover of K. By compactness, one can choose a finite sub-cover. So, let $\{t_n\}_{n=1}^N$ be such that $\{B(t_n,r_1(t_n))\}_{n=1}^N$ is a finite sub-cover. Let $\{\Phi_{t_n}\}_{n=1}^N$ be diffeomorphisms from $\mathbb{R}^d$ to $\mathbb{R}^d$ given by Lemma 6.2 so that

$$\Phi(t)=\Phi_{t_n}(t), \quad t\in B\big(t_n,r_1(t_n)\big).$$

Let $\{\phi_n\}_{n=1}^N$ be such that $\phi_n\in C_c^\infty(B(t_n,r_1(t_n)))$ and

$$\sum_{n=1}^N \phi_n^2 = 1 \quad \text{on } K.$$

Set

$$T_0=\sum_{m=1}^N M_{\phi_m}[M_{\phi_m},T], \quad T_n = M_{\phi_n}TM_{\phi_n}, \quad 1\le n\le N.$$

We write

$$T=\sum_{n=1}^N M_{\phi_n^2}T=\sum_{n=0}^N T_n. \tag{6.1}$$

If $1\le n\le N$, then T_n is supported on the ball $B(t_n,r_1(t_n))$. By Lemma 6.3, we have

$$\mathrm{Ext}_{\Omega'}\big(U_\Phi^{-1}\cdot\mathrm{Rest}_\Omega(T_n)\cdot U_\Phi\big) = U_{\Phi_{t_n}}^{-1}T_nU_{\Phi_{t_n}}.$$

By Theorem 3.5, we have $U_{\Phi_{t_n}}^{-1}T_nU_{\Phi_{t_n}}\in\Pi$ and

$$\mathrm{sym}(U_{\Phi_{t_n}}^{-1}T_nU_{\Phi_{t_n}}) = \mathrm{sym}(T_n)\circ\Theta_{\Phi_{t_n}}.$$

Thus,

$$\mathrm{Ext}_{\Omega'}\big(U_\Phi^{-1}\cdot\mathrm{Rest}_\Omega(T_n)\cdot U_\Phi\big)\in\Pi \tag{6.2}$$

and

$$\mathrm{sym}\,\big(\mathrm{Ext}_{\Omega'}\big(U_\Phi^{-1}\cdot\mathrm{Rest}_\Omega(T_n)\cdot U_\Phi\big)\big) = \mathrm{sym}(T_n)\circ\Theta_\Phi. \tag{6.3}$$

If $n=0$, then T_0 is compact by Lemma 5.7. Clearly, T_0 is compactly supported in Ω. If $A\in B(L_2(\Omega'))$ is compact, then $\mathrm{Ext}_{\Omega'}(A)\in B(L_2(\mathbb{R}^d))$ is also compact (see Notation 3.6 and recall that a composition of bounded and compact operators is compact). Therefore,

$$\mathrm{Ext}_{\Omega'}\big(U_\Phi^{-1}\cdot\mathrm{Rest}_\Omega(T_0)\cdot U_\Phi\big)\in\mathcal{K}\big(L_2(\mathbb{R}^d)\big)\subset\Pi \tag{6.4}$$

and

$$\mathrm{sym}\left(\mathrm{Ext}_{\Omega'}\left(U_\Phi^{-1}\cdot \mathrm{Rest}_\Omega(T_0)\cdot U_\Phi\right)\right)=0. \tag{6.5}$$

Combining (6.1), (6.2), and (6.4), we obtain

$$\mathrm{Ext}_{\Omega'}\left(U_\Phi^{-1}\cdot \mathrm{Rest}_\Omega(T)\cdot U_\Phi\right)=\sum_{n=0}^{N}\mathrm{Ext}_{\Omega'}\left(U_\Phi^{-1}\cdot \mathrm{Rest}_\Omega(T_n)\cdot U_\Phi\right)\in\Pi.$$

Now, combining (6.1), (6.3), and (6.5), we obtain

$$\begin{aligned}&\mathrm{sym}\left(\mathrm{Ext}_{\Omega'}\left(U_\Phi^{-1}\cdot \mathrm{Rest}_\Omega(T)\cdot U_\Phi\right)\right)\\ &\quad=\sum_{n=0}^{N}\mathrm{sym}\left(\mathrm{Ext}_{\Omega'}\left(U_\Phi^{-1}\cdot \mathrm{Rest}_\Omega(T_n)\cdot U_\Phi\right)\right)=\sum_{n=1}^{N}\mathrm{sym}(T_n)\circ\Theta_\Phi\\ &\quad=\mathrm{sym}(T)\circ\Theta_\Phi-\mathrm{sym}(T_0)\circ\Theta_\Phi=\mathrm{sym}(T)\circ\Theta_\Phi.\end{aligned}$$

■

7. Principal symbol on compact manifolds

7.1. Globalisation theorem

Globalisation theorem is a folklore. We provide its proof in the appendix for the convenience of the reader.

Definition 7.1. Let X be a compact manifold with an atlas $\{(\mathcal{U}_i,h_i)\}_{i\in\mathbb{I}}$. Let $\mathfrak{B}$ be the Borel σ-algebra on X and let ν be a countably additive measure on $\mathfrak{B}$. We say that $\{\mathcal{A}_i\}_{i\in\mathbb{I}}$ are local algebras if

1. for every $i\in\mathbb{I}$, $\mathcal{A}_i$ is a $*$-subalgebra in $B(L_2(X,\nu))$;
2. for every $i\in\mathbb{I}$, the elements of $\mathcal{A}_i$ are compactly supported[2] in $\mathcal{U}_i$;
3. for every $i,j\in\mathbb{I}$, if $T\in\mathcal{A}_i$ is compactly supported in $\mathcal{U}_i\cap\mathcal{U}_j$, then $T\in\mathcal{A}_j$;
4. for every $i\in\mathbb{I}$, if $T\in\mathcal{K}(L_2(X,\nu))$ is compactly supported in $\mathcal{U}_i$, then $T\in\mathcal{A}_i$;
5. for every $i\in\mathbb{I}$, if $\phi\in C_c(\mathcal{U}_i)$, then $M_\phi\in\mathcal{A}_i$;
6. for every $i\in\mathbb{I}$, if $\phi\in C_c(\mathcal{U}_i)$, then the closure of $M_\phi\mathcal{A}_iM_\phi$ in the uniform norm is contained in $\mathcal{A}_i$;
7. for every $i\in\mathbb{I}$, if $T\in\mathcal{A}_i$ and if $\phi\in C_c(\mathcal{U}_i)$, then $[T,M_\phi]\in\mathcal{K}(L_2(X,\nu))$.

Definition 7.2. In the setting of Definition 7.1, we say that $T\in\mathcal{A}$ if

1. for every $i\in\mathbb{I}$ and for every $\phi\in C_c(\mathcal{U}_i)$, we have $M_\phi TM_\phi\in\mathcal{A}_i$;
2. for every $\psi\in C(X)$, the commutator $[T,M_\psi]$ is compact.

Definition 7.3. Let $\mathcal{B}$ be a $*$-algebra. In the setting of Definition 7.1, $\{\mathrm{hom}_i\}_{i\in\mathbb{I}}$ are called local homomorphisms if

1. for every $i\in\mathbb{I}$, $\mathrm{hom}_i:\mathcal{A}_i\to\mathcal{B}$ is a $*$-homomorphism;

---

[2] This notion is introduced immediately before Definition 1.3.

(2) for every $i, j \in \mathbb{I}$, we have $\mathrm{hom}_i = \mathrm{hom}_j$ on $\mathcal{A}_i \cap \mathcal{A}_j$;

(3) $T \in \mathcal{A}_i$ is compact iff $\mathrm{hom}_i(T) = 0$;

(4) there exists a $*$-homomorphism $\mathrm{Hom} : C(X) \to \mathcal{B}$ such that

$$\mathrm{hom}_i(M_\phi) = \mathrm{Hom}(\phi), \quad \phi \in C_c(\mathcal{U}_i),\ i \in \mathbb{I}.$$

Theorem 7.4. *In the setting of Definitions 7.1, 7.3, and 7.2, we have that*

(1) *$\mathcal{A}$ is a unital C^*-subalgebra in $B(L_2(X,\nu))$ which contains $\mathcal{A}_i$ for every $i \in \mathbb{I}$ and $\mathcal{K}(L_2(X,\nu))$;*

(2) *there exists a $*$-homomorphism* $\mathrm{hom} : \mathcal{A} \to \mathcal{B}$ *such that*

 (a) $\mathrm{hom} = \mathrm{hom}_i$ *on $\mathcal{A}_i$ for every $i \in \mathbb{I}$;*

 (b) $\ker(\mathrm{hom}) = \mathcal{K}(L_2(X,\nu))$*;*

(3) *$*$-homomorphism as in (2) is unique.*

7.2. Construction of the principal symbol mapping

Let $\mathfrak{B}$ be the Borel σ-algebra on the manifold X and let $\nu : \mathfrak{B} \to \mathbb{R}$ be a continuous positive density.

It is immediate that the mapping $h_i : (\mathcal{U}_i, \nu) \to (\Omega_i, \nu \circ h_i^{-1})$ preserves the measure. Define an isometry

$$W_i : L_2(\mathcal{U}_i, \nu) \to L_2(\Omega_i, \nu \circ h_i^{-1})$$

by setting

$$W_i f = f \circ h_i^{-1}, \quad f \in L_2(\mathcal{U}_i, \nu).$$

If T is compactly supported in $\mathcal{U}_i$, then $W_i T W_i^{-1}$ is understood as an element of the algebra $B(L_2(\Omega_i, \nu \circ h_i^{-1}))$. The latter operator is compactly supported in Ω_i. By Definition 2.20, exactly the same operator also belongs to $B(L_2(\Omega_i))$ and, therefore, can be extended to an element $\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1})$ of $B(L_2(\mathbb{R}^d))$. For the notion Ext_{Ω_i}, we refer to Notation 3.7.

Definition 7.5. Let X be a smooth compact manifold and let ν be a continuous positive density on X. For every $i \in \mathbb{I}$, let Π_i consist of the operators $T \in B(L_2(X,\nu))$ compactly supported in $\mathcal{U}_i$ such that

$$\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1}) \in \Pi.$$

For example, every operator M_ϕ, $\phi \in C_c(\mathcal{U}_i)$ belongs to Π_i.

For notation $C(S^*X)$ below, we refer to Definition 2.12. For the notion sym, we refer to (1.1).

Definition 7.6. Let X be a smooth compact manifold and let ν be a continuous positive density on X. For every $i \in \mathbb{I}$, the mapping $\mathrm{sym}_i : \Pi_i \to C(S^*X)$ is defined by the formula

$$\mathrm{sym}_i(T) = \mathrm{sym}\left(\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1})\right) \circ H_i, \quad T \in \Pi_i.$$

Theorem 7.7. *Let X be a smooth compact manifold and let ν be a continuous positive density on X.*

(1) *Collection $\{\Pi_i\}_{i\in\mathbb{I}}$ introduced in Definition 7.5 satisfies all the conditions in Definition 7.1.*

(2) *Collection $\{\mathrm{sym}_i\}_{i\in\mathbb{I}}$ introduced in Definition 7.6 satisfies all the conditions in Definition 7.3.*

That is, the collection $\{\Pi_i\}_{i\in\mathbb{I}}$ of $*$-algebras and the collection $\{\mathrm{sym}_i\}_{i\in\mathbb{I}}$ of $*$-homomorphisms satisfy the conditions in Theorem 7.4.

Definition 7.8 below is the culmination of the paper. Having this definition at hands, we easily prove Theorem 1.4.

Definition 7.8. Let X be a smooth compact Riemannian manifold and let ν be a continuous positive density on X.

(1) The domain Π_X of the principal symbol mapping is the C^*-algebra constructed in Theorem 7.4 from the collection $\{\Pi_i\}_{i\in\mathbb{I}}$.

(2) The principal symbol mapping $\mathrm{sym}_X : \Pi_X \to C(S^*X)$ is the $*$-homomorphism constructed in Theorem 7.4 from the collection $\{\mathrm{sym}_i\}_{i\in\mathbb{I}}$.

7.3. Proof of Theorem 7.7

Lemma 7.9 below delivers verification of condition (1) in Definitions 7.1 and 7.3.

Lemma 7.9. *For every $i\in\mathbb{I}$, we have that*

(1) *Π_i is a $*$-subalgebra in $B(L_2(X,\nu))$;*

(2) *$\mathrm{sym}_i : \Pi_i \to C(S^*X)$ is a $*$-homomorphism.*

Proof. It is immediate that Π_i is a subalgebra in $B(L_2(X,\nu))$ and that

$$\mathrm{sym}_i : \Pi_i \to C(S^*X)$$

is a homomorphism. We need to show that Π_i is closed with respect to taking adjoints and that sym_i is invariant with respect to this operation.

Let $T\in\Pi_i$ and let us show that $T^*\in\Pi_i$. Recall that, due to Condition 2.20, $\nu\circ h_i^{-1}$ is absolutely continuous and that its density denoted by a_i and its inverse a_i^{-1} are assumed to be continuous in Ω_i. The following equality[3] is easy to verify directly:

$$W_iT^*W_i^{-1} = M_{a_i^{-1}}\cdot(W_iTW_i^{-1})^*\cdot M_{a_i}. \tag{7.1}$$

[3] The operators M_{a_i} and $M_{a_i^{-1}}$ are unbounded. The equality should be understood as $\mathrm{LHS}\xi=\mathrm{RHS}\xi$ for every compactly supported $\xi\in L_2(\Omega_i)$.

Indeed, for such ξ, we have $\xi_1 = M_{a_i}\xi\in L_2(\Omega_i)$. Since T is compactly supported in $\mathcal{U}_i$, it follows that $(W_iTW_i^{-1})^*$ is compactly supported in Ω_i. Hence, the function $\xi_2=(W_iTW_i^{-1})^*\xi_1$ is compactly supported in Ω_i. Hence, the function $M_{a_i^{-1}}\xi_2$ belongs to $L_2(\Omega_i)$ and the right-hand side of (7.1) makes sense.

However, by Definition 7.5, the operator T is compactly supported in $\mathcal{U}_i$. Hence, the operator $(W_iTW_i^{-1})^*$ is compactly supported in Ω_i. Choose $\phi\in C_c(\Omega_i)$ such that

$$(W_iTW_i^{-1})^* = M_\phi\cdot(W_iTW_i^{-1})^* = (W_iTW_i^{-1})^*\cdot M_\phi.$$

Thus,

$$W_iT^*W_i^{-1} = M_{a_i^{-1}\phi}\cdot(W_iTW_i^{-1})^*\cdot M_{a_i\phi}.$$

Thus,

$$\mathrm{Ext}_{\Omega_i}(W_iT^*W_i^{-1}) = M_{a_i^{-1}\phi}\cdot\big(\mathrm{Ext}_{\Omega_i}(W_iTW_i^{-1})\big)^*\cdot M_{a_i\phi}. \tag{7.2}$$

Since $a_i\phi, a_i^{-1}\phi\in C_c(\mathbb{R}^d)$, it follows that every factor in the right-hand side of (7.2) belongs to Π. Hence, so is the expression on the left-hand side. In other words, $T^*\in\Pi_i$. Thus, Π_i is closed with respect to taking adjoints.

Recall that (by [29]) sym is a $*$-homomorphism. Applying sym to equality (7.2), we obtain

$$\begin{aligned}\mathrm{sym}_i(T^*) &= \mathrm{sym}\,\big(\mathrm{Ext}_{\Omega_i}(W_iT^*W_i^{-1})\big)\\ &= \mathrm{sym}(M_{a_i^{-1}\phi})\cdot\mathrm{sym}\,\big((\mathrm{Ext}_{\Omega_i}(W_iTW_i^{-1}))^*\big)\cdot\mathrm{sym}(M_{a_i\phi})\\ &= \mathrm{sym}(M_{\phi^2})\cdot\mathrm{sym}\,\big(\mathrm{Ext}_{\Omega_i}(W_iTW_i^{-1})\big)^*\\ &= \mathrm{sym}(\mathrm{Ext}_{\Omega_i}\big(M_{\phi^2}\cdot W_iTW_i^{-1})\big)^*.\end{aligned}$$

It is clear that

$$M_{\phi^2}\cdot W_iTW_i^{-1} = W_iTW_i^{-1}.$$

Thus,

$$\mathrm{sym}_i(T^*) = \mathrm{sym}\,\big(\mathrm{Ext}_{\Omega_i}(W_iTW_i^{-1})\big)^* = \mathrm{sym}_i(T)^*.$$ ∎

In the following lemma, $\Xi_{\Phi_{i,j}}$ is defined according to Notation 3.8.

Lemma 7.10. *For every $i,j\in\mathbb{I}$, we have*

$$\Xi_{\Phi_{i,j}} = H_i\circ H_j^{-1}$$

on $\Omega_{j,i}\times\mathbb{R}^d$.

Proof. By Notation 3.8,

$$\Xi_{\Phi_{i,j}}(t,s) = \big(\Phi_{i,j}^{-1}(t), J_{\Phi^*_{i,j}(\Phi_{i,j}^{-1}(t))}s\big).$$

By the chain rule, we have

$$J_{\Phi_{i,j}}\big(\Phi_{i,j}^{-1}(t)\big)\cdot J_{\Phi_{i,j}^{-1}}(t) = J_{\Phi_{i,j}\circ\Phi_{i,j}^{-1}}(t) = 1_{M_d(\mathbb{C})}.$$

Taking into account that $\Phi_{i,j}^{-1}=\Phi_{j,i}$, we write

$$J_{\Phi_{i,j}(\Phi_{i,j}^{-1}(t))} = (J_{\Phi_{j,i}})^{-1}(t) \quad\text{and}\quad J^*_{\Phi_{i,j}(\Phi_{i,j}^{-1}(t))} = (J^*_{\Phi_{j,i}})^{-1}(t).$$ ∎

The following lemma verifies condition (3) in Definition 7.1 and condition (2) in Definition 7.3.

Lemma 7.11. *Let $(\mathcal{U}_i, h_i)$ and $(\mathcal{U}_j, h_j)$ be charts. Let $T \in B(L_2(X,\nu))$ be compactly supported in $\mathcal{U}_i \cap \mathcal{U}_j$.*

(1) *If $T \in \Pi_i$, then $T \in \Pi_j$.*

(2) *We have $\mathrm{sym}_i(T) = \mathrm{sym}_j(T)$.*

Proof. Let $V_\Phi \xi = \xi \circ \Phi$ (provided that the image of the mapping Φ is contained in the domain of the function ξ). Since $W_j = V^{-1}_{\Phi_{i,j}} W_i$, it follows that (using Notation 3.8)

$$W_j T W_j^{-1} = V^{-1}_{\Phi_{i,j}} \cdot W_i T W_i^{-1} \cdot V_{\Phi_{i,j}}$$
$$= U^{-1}_{\Phi_{i,j}} \cdot M_{|J_{\Phi_{i,j}}|^{\frac12}} \cdot W_i T W_i^{-1} \cdot M_{|J_{\Phi_{i,j}}|^{-\frac12}} \cdot U_{\Phi_{i,j}}.$$

Since T is compactly supported in $\mathcal{U}_i \cap \mathcal{U}_j$, it follows that $W_i T W_i^{-1} \in \mathrm{Rest}_{\Omega_i}(\Pi)$ is compactly supported in $\Omega_{i,j}$. Let $A \subset \Omega_{i,j}$ be compact such that

$$M_{\chi_A} \cdot W_i T W_i^{-1} \cdot M_{\chi_A} = W_i T W_i^{-1}.$$

Using Tietze extension theorem, choose $\phi \in C(\mathbb{R}^d)$ such that $\phi^{-1} \in C(\mathbb{R}^d)$ and such that $\phi = |J_{\Phi_{i,j}}|^{\frac12}$ on A. It follows that

$$M_{|J_{\Phi_{i,j}}|^{\frac12}} \cdot W_i T W_i^{-1} \cdot M_{|J_{\Phi_{i,j}}|^{-\frac12}}$$
$$= M_\phi \cdot W_i T W_i^{-1} \cdot M_{\phi^{-1}}$$
$$= W_i T W_i^{-1} + [M_\phi, W_i T W_i^{-1}] \cdot M_{\phi^{-1}} \overset{\text{Lem. 5.7}}{\in} W_i T W_i^{-1} + \mathcal{K}\big(L_2(\Omega_i)\big).$$

Combining the preceding paragraphs, we conclude that

$$W_j T W_j^{-1} \in U^{-1}_{\Phi_{i,j}} \cdot W_i T W_i^{-1} \cdot U_{\Phi_{i,j}} + \mathcal{K}\big(L_2(\Omega_j)\big).$$

Denote for brevity

$$T_i = \mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1}), \quad T_j = \mathrm{Ext}_{\Omega_j}(W_j T W_j^{-1}).$$

The preceding display can be now re-written as

$$\mathrm{Rest}_{\Omega_j}(T_j) \in U^{-1}_{\Phi_{i,j}} \cdot \mathrm{Rest}_{\Omega_i}(T_i) \cdot U_{\Phi_{i,j}} + \mathcal{K}\big(L_2(\Omega_j)\big).$$

Thus,

$$T_j \in \mathrm{Ext}_{\Omega_j}\big(U^{-1}_{\Phi_{i,j}} \cdot \mathrm{Rest}_{\Omega_i}(T_i) \cdot U_{\Phi_{i,j}}\big) + \mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

By Theorem 3.11, we have

$$\mathrm{Ext}_{\Omega_j}\big(U^{-1}_{\Phi_{i,j}} \cdot \mathrm{Rest}_{\Omega_i}(T_i) \cdot U_{\Phi_{i,j}}\big) \in \Pi$$

and

$$\mathrm{sym}\left(\mathrm{Ext}_{\Omega_j}\left(U_{\Phi_{i,j}}^{-1}\cdot \mathrm{Rest}_{\Omega_i}(T_i)\cdot U_{\Phi_{i,j}}\right)\right) = \mathrm{sym}(T_i)\circ \Xi_{\Phi_{i,j}}.$$

By Lemma 5.2, compact operators belong to Π. Therefore, $T_j \in \Pi$ and

$$\mathrm{sym}(T_j) = \mathrm{sym}(T_i)\circ \Xi_{\Phi_{i,j}} \overset{\text{Lem. 7.10}}{=} \mathrm{sym}(T_i)\circ H_i \circ H_j^{-1}.$$

Finally,

$$\begin{aligned}\mathrm{sym}_j(T) &= \mathrm{sym}(T_j)\circ H_j = \mathrm{sym}(T_i)\circ H_i\circ H_j^{-1}\circ H_j\\ &= \mathrm{sym}(T_i)\circ H_i = \mathrm{sym}_i(T).\end{aligned}$$

■

Proof of Theorem 7.7 (1). Condition (1) in Definition 7.1 is verified in Lemma 7.9. Condition (2) in Definition 7.1 is immediate. Condition (3) in Definition 7.1 is verified in Lemma 7.11.

Let us verify condition (4) in Definition 7.1. If $i \in \mathbb{I}$ and if $T \in \mathcal{K}(L_2(X,\nu))$ is compactly supported in $\mathcal{U}_i$, then $\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1}) \in \mathcal{K}(L_2(\mathbb{R}^d))$. Using Lemma 5.2, we conclude that $\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1}) \in \Pi$. In other words, $T \in \Pi_i$.

Condition (5) in Definition 7.1 is immediate.

Let us verify condition (6) in Definition 7.1. Let $i \in \mathbb{I}$ and let $\phi \in C_c(\mathcal{U}_i)$. Suppose $\{T_n\}_{n\geq 1} \subset M_\phi \Pi_i M_\phi$ are such that $T_n \to T$ in the uniform norm. It follows that T is compactly supported in $\mathcal{U}_i$ and

$$\mathrm{Ext}_{\Omega_i}(W_i T_n W_i^{-1}) \to \mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1}),\quad n\to\infty,$$

in the uniform norm. The sequence on the left-hand side is in Π. Hence, so is its limit. In other words, $T \in \Pi_i$.

Let us verify condition (7) in Definition 7.1. Let $i \in \mathbb{I}$ and let $T \in \Pi_i$ and $\phi \in C_c(\mathcal{U}_i)$. Let $\psi = \phi\circ h_i^{-1} \in C_c(\mathbb{R}^d)$. We have

$$\mathrm{Ext}_{\Omega_i}\left(W_i[T, M_\phi]W_i^{-1}\right) = \left[\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1}), M_\psi\right].$$

Since $\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1}) \in \Pi$, it follows that the commutator on the right-hand side is compact by Lemma 5.7. Therefore, the operator on the left-hand side is compact and, therefore, so is $[T, M_\phi]$. ■

Proof of Theorem 7.7 (2). Condition (1) in Definition 7.3 is verified in Lemma 7.9. Condition (2) in Definition 7.3 is verified in Lemma 7.11.

Let us verify condition (3) in Definition 7.3. If an operator $T \in \Pi_i$ is compact, then so is $\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1})$. Since sym vanishes on compact operators, it follows that

$$\mathrm{sym}_i(T) \overset{\text{Def. 7.6}}{=} \mathrm{sym}\left(\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1})\right)\circ H_i = 0\circ H_i = 0.$$

Conversely, if $T \in \Pi_i$ is such that $\mathrm{sym}_i(T) = 0$, then

$$\mathrm{sym}\left(\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1})\right) = 0.$$

Since $\ker(\mathrm{sym}) = \mathcal{K}(L_2(\mathbb{R}^d))$, it follows that

$$\mathrm{Ext}_{\Omega_i}(W_i T W_i^{-1}) \in \mathcal{K}\big(L_2(\mathbb{R}^d)\big).$$

Thus, $T \in \mathcal{K}(L_2(X,\nu))$.

Condition (4) in Definition 7.3 is immediate if we take Hom to be the natural embedding $C(X) \to C(S^*X)$. ∎

7.4. Proof of Theorem 1.4

Proof of Theorem 1.4. By Definition 7.8, Π_X is a C^*-algebra and the mapping $\mathrm{sym}_X : \Pi_X \to C(S^*X)$ is a $*$-homomorphism. By Definition 7.8 and Theorem 7.4 (2),

$$\ker(\mathrm{sym}_X) = \mathcal{K}\big(L_2(X,\nu)\big).$$

Let us show that sym_X is surjective. Denote the image of sym_X by A and note that A is a C^*-subalgebra in $C(S^*X)$. Let $F \in C^\infty(S^*X)$. Let $(\phi_n)_{n=1}^N$ be a good[4] partition of unity so that $\phi_n \in C_c^\infty(\mathcal{U}_{i_n})$ for $1 \leq n \leq N$. It follows that

$$q_n = (F\phi_n) \circ H_{i_n}^{-1} \in C_c(\Omega_{i_n} \times \mathbb{R}^d).$$

By Lemma 2.6, we have $T_{q_n} \in \Pi$ and $\mathrm{sym}(T_{q_n}) = q_n$. Let $\psi_n \in C_c(\Omega_{i_n})$ be such that $\phi_n = \phi_n\psi_n$. We have $T_n = M_{\psi_n} T_{q_n} M_{\psi_n} \in \Pi$ and $\mathrm{sym}(T_n) = q_n\psi_n^2 = q_n$. Since T_n is (bounded and) compactly supported in Ω_{i_n}, it follows that $S_n = W_{i_n}^{-1}\mathrm{Rest}_{\Omega_i}(T_n)W_{i_n}$ is bounded and compactly supported in $\mathcal{U}_{i_n}$. It is clear that $S_n \in \Pi_{i_n} \subset \Pi_X$ and that

$$\mathrm{sym}_X(S_n) = \mathrm{sym}\left(\mathrm{Ext}_{\Omega_{i_n}}(W_{i_n} T_n W_{i_n}^{-1})\right) \circ H_{i_n} = \mathrm{sym}(T_n) \circ H_{i_n} = q_n \circ H_{i_n} = F\phi_n.$$

Thus, $S = \sum_{n=1}^N S_n \in \Pi_X$ and

$$\mathrm{sym}_X(S) = \sum_{n=1}^N \mathrm{sym}_X(S_n) = \sum_{n=1}^N F\phi_n = F.$$

Hence, every $F \in C^\infty(S^*X)$ belongs to A. In other words, $C^\infty(S^*X) \subset A$. Since A is a C^*-subalgebra in $C(S^*X)$, it follows that $A = C(S^*X)$. Hence, sym_X is surjective. ∎

8. Proof of the Connes trace theorem

Lemma 8.1. *Let g be as in Theorem 2.19. Let $\phi \in C_c^\infty(\mathbb{R}^d)$. We have*

$$M_\phi(1-\Delta_g)^{-\frac{r}{2}} \in \mathcal{L}_{\frac{d}{r},\infty}.$$

[4] See Definition 2.16.

Proof. By definition, the principal symbol of $1-\Delta_g$ is

$$p_0 : (t,s) \to \langle g(t)^{-1}s, s\rangle, \quad t,s \in \mathbb{R}^d.$$

By Theorem 2.5, we have

$$(1-\Delta_g)^{-\frac{r}{2}} \in \Psi^{-r}(\mathbb{R}^d).$$

We now write

$$M_\phi(1-\Delta_g)^{-\frac{r}{2}} = M_\phi(1-\Delta)^{-\frac{r}{2}} \cdot (1-\Delta)^{\frac{r}{2}}(1-\Delta_g)^{-\frac{r}{2}}.$$

When $r > \frac{d}{2}$, the first factor belongs to $\mathcal{L}_{\frac{d}{r},\infty}$ by [15, Theorem 1.4]. When $r = \frac{d}{2}$, the first factor belongs to $\mathcal{L}_{\frac{d}{r},\infty}$ by Theorem 1.3 in [15]. When $r < \frac{d}{2}$, the first factor belongs to $\mathcal{L}_{\frac{d}{r},\infty}$ by [15, Theorem 1.1] (applied with $r < \frac{d}{2}$). The second factor belongs to $\Psi^0(\mathbb{R}^d)$ and is, therefore, bounded. ∎

Lemma 8.2. *Let g be as in Theorem 2.19. We have*

$$M_\psi(1-\Delta_g)^{-\frac{d}{2}}(1-\Delta)^{\frac{d}{2}} \in \Pi,$$
$$\mathrm{sym}\left(M_\psi(1-\Delta_g)^{-\frac{d}{2}}(1-\Delta)^{\frac{d}{2}}\right)(t,s) = \psi(t)\langle g(t)^{-1}s, s\rangle^{-\frac{d}{2}}.$$

Proof. By definition, the principal symbol of $1-\Delta_g$ is

$$p_0 : (t,s) \to \langle g(t)^{-1}s, s\rangle, \quad t,s \in \mathbb{R}^d.$$

By Theorem 2.5, we have

$$(1-\Delta_g)^{-\frac{d}{2}} = \mathrm{Op}\left((p_0+1)^{-\frac{d}{2}}\right) + \mathrm{Err}_0, \quad \mathrm{Err}_0 \in \Psi^{-d-1}(\mathbb{R}^d).$$

Let

$$p_1 : (t,s) \to \psi(t)\left(1+\langle g(t)^{-1}s, s\rangle\right)^{-\frac{d}{2}}\left(1+|s|^2\right)^{-\frac{d}{2}}, \quad t,s \in \mathbb{R}^d.$$

Clearly,

$$M_\psi \cdot \mathrm{Op}\left((p_0+1)^{-\frac{d}{2}}\right) \cdot (1-\Delta)^{\frac{d}{2}} = \mathrm{Op}(p_1).$$

Thus,

$$M_\psi(1-\Delta_g)^{-\frac{d}{2}}(1-\Delta)^{\frac{d}{2}} = \mathrm{Op}(p_1) + \mathrm{Err}_1, \quad \mathrm{Err}_1 \in \Psi^{-1}(\mathbb{R}^d).$$

Since

$$M_\psi(1-\Delta_g)^{-\frac{d}{2}}(1-\Delta)^{\frac{d}{2}} \quad \text{and} \quad \mathrm{Op}(p_1)$$

are compactly supported from the left, it follows that so is Err_1. Thus, Err_1 is a compact operator. Consequently, $\mathrm{Err}_1 \in \Pi$ and $\mathrm{sym}(\mathrm{Err}_1) = 0$.

Let

$$p_2(t,s) = \psi(t)\langle g(t)^{-1}s, s\rangle^{-\frac{d}{2}}, \quad t \in \mathbb{R}^d,\ s \in \mathbb{S}^{d-1}.$$

By Lemma 2.7, we have that $\mathrm{Op}(p_1) - T_{p_2}$ is compact. So, our operator belongs to Π and its symbol equals that of T_{p_2}, i.e., equals p_2. ∎

Lemma 8.3. *Let (X, G) be a compact Riemannian manifold. Let $\psi \in C^\infty(X)$ be compactly supported in the chart $(\mathcal{U}_i, h_i)$. Let $\hat{g}_i : \mathbb{R}^d \to \mathrm{GL}^+(d, \mathbb{R})$ be as in Theorem 2.19 such that $\hat{g}_i = g_i$ in the neighborhood of the support of $\psi \circ h_i^{-1}$. We have*

$$\mathrm{Ext}_{\Omega_i}\big(W_i M_\psi^2 (1-\Delta_G)^{-1} M_\psi^2 W_i^{-1}\big) - M^2_{\psi\circ h_i^{-1}}(1-\Delta_{\hat{g}_i})^{-1} M^2_{\psi\circ h_i^{-1}} \in \mathcal{L}_{\frac{d}{3},\infty}.$$

Proof. Let $\Omega_i' \subset \Omega_i$ be a compact set such that $\psi \circ h_i^{-1}$ is supported in Ω_i' and such that $g_i = \hat{g}_i$ on Ω_i'. Let $\phi \in C_c(\mathbb{R}^d)$ be supported in Ω_i' such that $\phi \cdot (\psi \circ h_i^{-1}) = \psi \circ h_i^{-1}$. We write

$$\begin{aligned}
&\mathrm{Ext}_{\Omega_i}\big(W_i M_\psi^2 (1-\Delta_G)^{-1} M_\psi^2 W_i^{-1}\big)\\
&\quad = M_\phi^2 \cdot \mathrm{Ext}_{\Omega_i}\big(W_i M_\psi^2 (1-\Delta_G)^{-1} M_\psi^2 W_i^{-1}\big)\\
&\quad = M_\phi (1-\Delta_{\hat{g}_i})^{-1} \cdot (1-\Delta_{\hat{g}_i}) M_\phi \mathrm{Ext}_{\Omega_i}\big(W_i M_\psi^2 (1-\Delta_G)^{-1} M_\psi^2 W_i^{-1}\big).
\end{aligned}$$

It follows directly from Definition 2.17 that

$$(1-\Delta_{\hat{g}_i}) M_\phi = \mathrm{Ext}_{\Omega_i}\big(W_i (1-\Delta_G) M_{\phi\circ h_i} W_i^{-1}\big).$$

Thus,

$$\begin{aligned}
&(1-\Delta_{\hat{g}_i}) M_\phi \mathrm{Ext}_{\Omega_i}\big(W_i M_\psi^2 (1-\Delta_G)^{-1} M_\psi^2 W_i^{-1}\big)\\
&\quad = \mathrm{Ext}_{\Omega_i}\big(W_i (1-\Delta_g) M_{\phi\circ h_i} W_i^{-1}\big) \cdot \mathrm{Ext}_{\Omega_i}\big(W_i M_\psi^2 (1-\Delta_G)^{-1} M_\psi^2 W_i^{-1}\big)\\
&\quad = \mathrm{Ext}_{\Omega_i}\big(W_i (1-\Delta_G) M_\psi^2 (1-\Delta_G)^{-1} M_\psi^2 W_i^{-1}\big).
\end{aligned}$$

Combining these equalities, we obtain

$$\begin{aligned}
&\mathrm{Ext}_{\Omega_i}\big(W_i M_\psi^2 (1-\Delta_g)^{-1} M_\psi^2 W_i^{-1}\big)\\
&\quad = M_\phi (1-\Delta_{\hat{g}_i})^{-1} \cdot \mathrm{Ext}_{\Omega_i}\big(W_i (1-\Delta_G) M_\psi^2 (1-\Delta_G)^{-1} M_\psi^2 W_i^{-1}\big).
\end{aligned}$$

Now,

$$(1-\Delta_G) M_\psi^2 (1-\Delta_G)^{-1} M_\psi^2 = M_\psi^4 - [\Delta_G, M_\psi^2](1-\Delta_G)^{-1} M_\psi^2.$$

Thus,

$$\begin{aligned}
&\mathrm{Ext}_{\Omega_i}\big(W_i M_\psi^2 (1-\Delta_G)^{-1} M_\psi^2 W_i^{-1}\big)\\
&= M_\phi (1-\Delta_{\hat{g}_i})^{-1} M^4_{\psi\circ h_i^{-1}} - M_\phi (1-\Delta_{\hat{g}_i})^{-1} \cdot \mathrm{Ext}_{\Omega_i}\big(W_i [\Delta_G, M_\psi^2](1-\Delta_G)^{-1} M_\psi^2 W_i^{-1}\big).
\end{aligned}$$

Since X is compact, it follows that

$$(1-\Delta_G)^{-1} : L_2(X) \to W^{2,2}(X), \quad [\Delta_G, M_\psi^2] : W^{2,2}(X) \to W^{1,2}(X)$$

are bounded operators. We now write

$$[\Delta_G, M_\psi^2](1-\Delta_G)^{-1} = (1-\Delta_G)^{-\frac{1}{2}} \cdot (1-\Delta_G)^{\frac{1}{2}} [\Delta_G, M_\psi^2](1-\Delta_G)^{-1},$$

where the first factor is in $\mathcal{L}_{d,\infty}$ and the second factor is bounded. By Lemma 8.1, we have

$$M_\phi(1-\Delta_{\hat{g}_i})^{-1} \in \mathcal{L}_{\frac{d}{2},\infty}.$$

By Hölder inequality, we have

$$M_\phi(1-\Delta_{\hat{g}_i})^{-1}\cdot \mathrm{Ext}_{\Omega_i}\big(W_i[\Delta_g, M_\psi^2](1-\Delta_G)^{-1}M_\psi^2 W_i^{-1}\big) \in \mathcal{L}_{\frac{d}{3},\infty}.$$

Thus,

$$\mathrm{Ext}_{\Omega_i}\big(W_i M_\psi^2(1-\Delta_G)^{-1}M_\psi^2 W_i^{-1}\big) - M_\phi(1-\Delta_{\hat{g}_i})^{-1}M^4_{\psi\circ h_i^{-1}} \in \mathcal{L}_{\frac{d}{3},\infty}.$$

Note that

$$\begin{aligned} M_\phi(1-\Delta_{\hat{g}_i})^{-1}M^4_{\psi\circ h_i^{-1}} &= M^2_{\psi\circ h_i^{-1}}(1-\Delta_{\hat{g}_i})^{-1}M^2_{\psi\circ h_i^{-1}} \\ &\quad + M_\phi\cdot\big[(1-\Delta_{\hat{g}_i})^{-1}, M^2_{\psi\circ h_i^{-1}}\big]\cdot M^2_{\psi\circ h_i^{-1}}. \end{aligned}$$

Let $\theta\in C_c^\infty(\mathbb{R}^d)$ be such that $\theta\cdot(\psi\circ h_i^{-1}) = \psi\circ h_i^{-1}$. We have

$$\begin{aligned} &\big[(1-\Delta_{\hat{g}_i})^{-1}, M^2_{\psi\circ h_i^{-1}}\big] \\ &\quad = (1-\Delta_{\hat{g}_i})^{-1}[\Delta_{\hat{g}_i}, M^2_{\psi\circ h_i^{-1}}](1-\Delta_{\hat{g}_i})^{-1} \\ &\quad = (1-\Delta_{\hat{g}_i})^{-1}M_\theta\cdot[\Delta_{\hat{g}_i}, M^2_{\psi\circ h_i^{-1}}](1-\Delta_{\hat{g}_i})^{-1} \overset{\text{Lem. 8.1}}{\in} \mathcal{L}_{\frac{d}{2},\infty}\cdot\mathcal{L}_{d,\infty} \\ &\quad = \mathcal{L}_{\frac{d}{3},\infty}. \end{aligned}$$

Combining the last three formulae, we complete the proof. ∎

Lemma 8.4. *Let (X,G) be a compact Riemannian manifold. If $0\le\psi\in C^\infty(X)$, then*

$$M_\psi^d(1-\Delta_G)^{-\frac{d}{2}}M_\psi^d - \big(M_\psi^2(1-\Delta_G)^{-1}M_\psi^2\big)^{\frac{d}{2}} \in \mathcal{L}_{\frac{2d}{2d+1},\infty}.$$

The same assertion holds for Δ_g, for g as in Theorem 2.19 and for $\psi\in C_c^\infty(\mathbb{R}^d)$.

Proof. The proof consists of two steps.

Step 1. We prove by induction that

$$M_\psi^{2n}(1-\Delta_G)^{-n}M_\psi^{2n} - \big(M_\psi^2(1-\Delta_G)^{-1}M_\psi^2\big)^n \in \mathcal{L}_{\frac{d}{2n+1},\infty}, \quad n\ge 0. \tag{8.1}$$

The base of induction (i.e., the case $n=1$) is obvious. It remains to prove the step of induction. Suppose (8.1) holds for n and let us prove it for $n+1$. We write

$$\begin{aligned} &\big(M_\psi^2(1-\Delta_G)^{-1}M_\psi^2\big)^{n+1} - M_\psi^{2n+2}(1-\Delta_G)^{-n-1}M_\psi^{2n+2} \\ &\quad = M_\psi^2(1-\Delta_G)^{-1}M_\psi^2\cdot\Big(\big(M_\psi^2(1-\Delta_G)^{-1}M_\psi^2\big)^n - M_\psi^{2n}(1-\Delta_G)^{-n}M_\psi^{2n}\Big) \\ &\qquad + M_\psi^2(1-\Delta_G)^{-\frac{3}{2}}\cdot(1-\Delta_G)^{\frac{1}{2}}[\Delta_G, M_\psi^{2n}](1-\Delta_G)^{-1}\cdot M_\psi^2(1-\Delta_G)^{-n}M_\psi^{2n} \\ &\qquad - M_\psi^{2n+2}(1-\Delta_G)^{-n-1}\cdot\big[M_\psi^2,(1-\Delta_G)^n\big](1-\Delta_G)^{\frac{1}{2}-n}\cdot(1-\Delta_G)^{-\frac{1}{2}}M_\psi^{2n}. \end{aligned}$$

The first term on the right-hand side belongs to $\mathcal{L}_{\frac{d}{2n+3},\infty}$ by inductive assumption and Hölder inequality. Note that the operators

$$(1-\Delta_G)^{\frac{1}{2}}[\Delta_G, M_\psi^{2n}](1-\Delta_G)^{-1}, \quad \big[M_\psi^2,(1-\Delta_G)^n\big](1-\Delta_G)^{\frac{1}{2}-n}$$

are bounded. Hence, the second and third terms on the right-hand side belong to $\mathcal{L}_{\frac{d}{2n+3},\infty}$ by Hölder inequality. This establishes the step of induction and, hence, proves the claim in Step 1.

Step 2. Note that

$$\begin{aligned} &M_\psi^d(1-\Delta_G)^{-\frac{d}{2}}M_\psi^d - M_\psi^{2d}(1-\Delta_G)^{-\frac{d}{2}} \\ &\qquad = M_\psi^d\cdot\big[M_\psi^d,(1-\Delta_G)^{-\frac{d}{2}}\big](1-\Delta_G)^{\frac{d+1}{2}}\cdot(1-\Delta_G)^{-\frac{d+1}{2}}. \end{aligned}$$

Since the operator

$$\big[M_\psi^d,(1-\Delta_G)^{-\frac{d}{2}}\big](1-\Delta_G)^{\frac{d+1}{2}}$$

is bounded, it follows that

$$M_\psi^d(1-\Delta_G)^{-\frac{d}{2}}M_\psi^d - M_\psi^{2d}(1-\Delta_G)^{-\frac{d}{2}} \in \mathcal{L}_{\frac{d}{d+1},\infty}.$$

Taking adjoints, we obtain

$$M_\psi^d(1-\Delta_G)^{-\frac{d}{2}}M_\psi^d - (1-\Delta_G)^{-\frac{d}{2}}M_\psi^{2d} \in \mathcal{L}_{\frac{d}{d+1},\infty}.$$

Therefore,

$$\big(M_\psi^d(1-\Delta_G)^{-\frac{d}{2}}M_\psi^d\big)^2 - M_\psi^{2d}(1-\Delta_G)^{-d}M_\psi^{2d} \in \mathcal{L}_{\frac{d}{2d+1},\infty}. \tag{8.2}$$

Applying (8.1) with $n=d$ and using (8.2), we obtain

$$\big(M_\psi^d(1-\Delta_G)^{-\frac{d}{2}}M_\psi^d\big)^2 - \big(M_\psi^2(1-\Delta_G)^{-1}M_\psi^2\big)^d \in \mathcal{L}_{\frac{d}{2d+1},\infty}.$$

The assertion follows now from Birman–Koplienko–Solomyak inequality. ∎

We remind the reader of the following version of the Connes trace theorem on the Euclidean space established in [29].

Theorem 8.5. *Let φ be a normalised continuous trace on $\mathcal{L}_{1,\infty}$. If $T\in\Pi$ is compactly supported from the right (i.e., there exists $\phi\in C_c^\infty(\mathbb{R}^d)$ such that $T=T\pi_1(\phi)$), then*

$$\varphi\big(T(1-\Delta)^{-\frac{d}{2}}\big) = c_d'\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\mathrm{sym}(T)\,dm,$$

where m is the product of Lebesgue measure on $\mathbb{R}^d$ and Haar measure on $\mathbb{S}^{d-1}$.

Lemma 8.6. *Let (X,G) be a compact Riemannian manifold. Let $T\in\Pi_X$ be compactly supported in the chart $(\mathcal{U}_i,h_i)$. Let φ be a continuous normalised trace on $\mathcal{L}_{1,\infty}$. We have*

$$\varphi\big(T(1-\Delta_G)^{-\frac{d}{2}}\big)=c_d\int_{\Omega_i\times\mathbb{R}^d}\mathrm{sym}(T_i)\Big(t,\frac{s}{|s|}\Big)e^{-q_i(t,s)}dt\,ds,\quad T_i=\mathrm{Ext}_{\Omega_i}(W_iTW_i^{-1}).$$

Here, q_i is as in Notation 2.13.

Proof. Fix $0\le\psi\in C^\infty(X)$ such that $T=M_\psi TM_\psi$ and such that ψ is compactly supported in $\mathcal{U}_i$.

By the tracial property, we have

$$\varphi\big(T(1-\Delta_G)^{-\frac{d}{2}}\big)=\varphi\big(M_\psi^dTM_\psi^d(1-\Delta_G)^{-\frac{d}{2}}\big)=\varphi\big(TM_\psi^d(1-\Delta_G)^{-\frac{d}{2}}M_\psi^d\big).$$

Since φ vanishes on $\mathcal{L}_{\frac{2d}{2d+1},\infty}$, it follows from Lemma 8.4 that

$$\varphi\big(T(1-\Delta_G)^{-\frac{d}{2}}\big)=\varphi(TA^{\frac{d}{2}}),\quad A=M_\psi^2(1-\Delta_G)^{-1}M_\psi^2.$$

Since both operators T and A are compactly supported in the chart $(\mathcal{U}_i,h_i)$, it follows that

$$\begin{aligned}\varphi(TA^{\frac{d}{2}})&=\varphi\big(W_iTW_i^{-1}\cdot(W_iAW_i^{-1})^{\frac{d}{2}}\big)\\&=\varphi\big(\mathrm{Ext}_{\Omega_i}(W_iTW_i^{-1})\cdot\big(\mathrm{Ext}_{\Omega_i}(W_iAW_i^{-1})\big)^{\frac{d}{2}}\big).\end{aligned}$$

Denote for brevity

$$B_i=M^2_{\psi\circ h_i^{-1}}(1-\Delta_{\hat{g}_i})^{-1}M^2_{\psi\circ h_i^{-1}}.$$

By Lemma 8.3 and Birman–Koplienko–Solomyak inequality, we have

$$\big(\mathrm{Ext}_{\Omega_i}(W_iAW_i^{-1})\big)^{\frac{d}{2}}-B_i^{\frac{d}{2}}\in\mathcal{L}_{\frac{2d}{2d+1},\infty}.$$

Since φ vanishes on $\mathcal{L}_{\frac{2d}{2d+1},\infty}$, it follows that

$$\varphi(TA^{\frac{d}{2}})=\varphi(T_iB_i^{\frac{d}{2}}).$$

Using the second assertion in Lemma 8.4, we obtain

$$\begin{aligned}\varphi(TA^{\frac{d}{2}})&=\varphi\big(T_iM^d_{\psi\circ h_i^{-1}}(1-\Delta_{\hat{g}_i})^{-\frac{d}{2}}M^d_{\psi\circ h_i^{-1}}\big)\\&=\varphi\big(T_i(1-\Delta)^{-\frac{d}{2}}X_i\big)=\varphi\big(X_iT_i(1-\Delta)^{-\frac{d}{2}}\big),\end{aligned}$$

where

$$X_i=(1-\Delta)^{\frac{d}{2}}(1-\Delta_{\hat{g}_i})^{-\frac{d}{2}}M^d_{\psi\circ h_i^{-1}}.$$

By Lemma 8.2, we have $X_i \in \Pi$. Hence, the operator $X_i T_i \in \Pi$ is compactly supported from the right. By Theorem 8.5, we have

$$\varphi\big(T(1-\Delta_g)^{-\frac{d}{2}}\big) = \varphi(TA^{\frac{d}{2}}) = c'_d \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \operatorname{sym}(X_i T_i)\, dm,$$

where m is the product of Lebesgue measure on $\mathbb{R}^d$ and Haar measure on $\mathbb{S}^{d-1}$. By Lemma 8.2, we have

$$\begin{aligned}\operatorname{sym}(X_i T_i)(t,s) &= \operatorname{sym}(T_i)(t,s)\cdot \psi\big(h_i^{-1}(t)\big)^d \cdot \big\langle \hat{g}_i(t)^{-1}s, s\big\rangle^{-\frac{d}{2}}\\ &= \operatorname{sym}(T_i)(t,s)\cdot \big\langle g_i(t)^{-1}s, s\big\rangle^{-\frac{d}{2}}.\end{aligned}$$

Thus,

$$\varphi\big(T(1-\Delta_g)^{-\frac{d}{2}}\big) = c'_d \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \operatorname{sym}(T_i)(t,s)\cdot \big\langle g_i(t)^{-1}s, s\big\rangle^{-\frac{d}{2}}\, dt\, ds.$$

By passing to spherical coordinates, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \operatorname{sym}(T_i)(t,s)\cdot \big\langle g_i(t)^{-1}s, s\big\rangle^{-\frac{d}{2}}\, dt\, ds\\ &\qquad = c''_d \int_{\mathbb{R}^d \times \mathbb{R}^d} \operatorname{sym}(T_i)\Big(t, \frac{s}{|s|}\Big) e^{-q_i(t,s)}\, dt\, ds.\end{aligned}$$

Combining these two equalities, we complete the proof. ∎

Proof of Theorem 1.5. Suppose first that $T \in \Pi_X$ is compactly supported in the chart $(\mathcal{U}_i, h_i)$. By Lemma 8.6, we have

$$\varphi\big(T(1-\Delta_G)^{-\frac{d}{2}}\big) = c_d \int_{\Omega_i \times \mathbb{R}^d} \operatorname{sym}(T_i)\Big(t, \frac{s}{|s|}\Big)\cdot e^{-q_i(t,s)}\, dt\, ds.$$

By (2.9) and (2.10), we have

$$\int_{\Omega_i \times \mathbb{R}^d} \operatorname{sym}(T_i)\Big(t, \frac{s}{|s|}\Big)\cdot e^{-q_i(t,s)}\, dt\, ds = \int_{T^*X} \operatorname{sym}_X(T) e^{-q_X}\, d\lambda.$$

A combination of these two equalities proves the assertion for T compactly supported in some $\mathcal{U}_i$.

Let now $T \in \Pi_X$ be arbitrary. Let $(\phi_n)_{n=1}^N$ be a fixed good partition of unity.

We write

$$T = \sum_{n=0}^N T_n, \quad T_0 = \sum_{m=1}^N M_{\phi_m}^{\frac{1}{2}} \cdot [M_{\phi_m}^{\frac{1}{2}}, T], \quad T_n = M_{\phi_n}^{\frac{1}{2}} T M_{\phi_n}^{\frac{1}{2}}, \quad n \geq 1.$$

By assumption, $[T, M_\psi]$ is compact for every $\psi \in C(X)$. In particular, T_0 is compact. Thus,

$$\varphi\big(T_0(1-\Delta_G)^{-\frac{d}{2}}\big) = 0, \quad \operatorname{sym}_X(T_0) = 0.$$

By the first paragraph, we have

$$\begin{aligned}\varphi\big(T(1-\Delta_G)^{-\frac{d}{2}}\big) &= \sum_{n=0}^{N} \varphi\big(T_n(1-\Delta_G)^{-\frac{d}{2}}\big)\\ &= \sum_{n=1}^{N} c_d \int_{T^*X} \mathrm{sym}_X(T_n) e^{-qx}\, d\lambda\\ &= c_d \int_{T^*X} \mathrm{sym}_X(T) e^{-qx}\, d\lambda. \qquad \blacksquare\end{aligned}$$

Appendix: Proof of the globalisation theorem

We prove Theorem 7.4 in the following series of lemmas.

Lemma A.1. *In the setting of Definitions 7.1 and 7.2, $\mathcal{A}$ is a unital $*$-subalgebra in $B(L_2(X,\nu))$.*

Proof. Suppose $T, S \in \mathcal{A}$. It is immediate that $1, T+S, T^* \in \mathcal{A}$. It suffices to show that also $TS \in \mathcal{A}$.

Let $i \in \mathbb{I}$ and let $0 \le \phi \in C_c(\mathcal{U}_i)$. We write

$$\begin{aligned}M_\phi TSM_\phi = M_{\phi^{\frac12}} TM_{\phi^{\frac12}} \cdot M_{\phi^{\frac12}} SM_{\phi^{\frac12}} + M_{\phi^{\frac12}} \cdot [M_\phi^{\frac12}, T] \cdot SM_\phi\\ + M_\phi^{\frac12} TM_\phi^{\frac12} \cdot [S, M_\phi^{\frac12}] M_\phi^{\frac12}.\end{aligned}$$

By Definition 7.2 (1), we have

$$M_{\phi^{\frac12}} TM_{\phi^{\frac12}}, M_{\phi^{\frac12}} SM_{\phi^{\frac12}} \in \mathcal{A}_i.$$

Since $\mathcal{A}_i$ is a subalgebra, it follows that

$$M_{\phi^{\frac12}} TM_{\phi^{\frac12}} \cdot M_{\phi^{\frac12}} SM_{\phi^{\frac12}} \in \mathcal{A}_i.$$

By Definition 7.2 (2), the operators $[M_\phi^{\frac12}, T]$ and $[S, M_\phi^{\frac12}]$ are compact. Therefore,

$$M_{\phi^{\frac12}} \cdot [M_\phi^{\frac12}, T] \cdot SM_\phi + M_\phi^{\frac12} TM_\phi^{\frac12} \cdot [S, M_\phi^{\frac12}] M_\phi^{\frac12}$$

is compact. However, the latter operator is compactly supported in $\mathcal{U}_i$. By Definition 7.1 (4), we have

$$M_{\phi^{\frac12}} \cdot [M_\phi^{\frac12}, T] \cdot SM_\phi + M_\phi^{\frac12} TM_\phi^{\frac12} \cdot [S, M_\phi^{\frac12}] M_\phi^{\frac12} \in \mathcal{A}_i.$$

Therefore,

$$M_\phi TSM_\phi \in \mathcal{A}_i, \quad \phi \in C_c(\mathcal{U}_i),\ i \in \mathbb{I}.$$

Since also

$$[TS, M_\psi] = T \cdot [S, M_\psi] + [T, M_\psi] \cdot S$$

is compact for every $\psi \in C(X)$, it follows that $TS \in \mathcal{A}$. ■

Lemma A.2. *In the setting of Definitions 7.1 and 7.2, $\mathcal{A}$ is a unital C^*-subalgebra in $B(L_2(X,\nu))$.*

Proof. It is established in Lemma A.1 that $\mathcal{A}$ is a unital $*$-subalgebra in $B(L_2(X,\nu))$. It suffices to show that $\mathcal{A}$ is closed in the uniform norm.

Let $\{T_n\}_{n\geq 1} \subset \mathcal{A}$ and let $T \in B(L_2(X,\nu))$ be such that $T_n \to T$ in the uniform norm. Let us show that $T \in \mathcal{A}$.

Let $i \in \mathbb{I}$ and let $\phi \in C_c(\mathcal{U}_i)$. Take $\phi_0 \in C_c(\mathcal{U}_i)$ such that $\phi\phi_0 = \phi$. We have $M_\phi T_n M_\phi \in \mathcal{A}_i$ and, therefore,

$$M_\phi T_n M_\phi = M_{\phi_0} \cdot M_\phi T_n M_\phi \cdot M_{\phi_0} \in M_{\phi_0} \mathcal{A}_i M_{\phi_0}, \quad n \geq 1.$$

By assumption, $M_\phi T_n M_\phi \to M_\phi T M_\phi$ in the uniform norm. Hence, $M_\phi T M_\phi$ belongs to the closure of $M_{\phi_0}\mathcal{A}_i M_{\phi_0}$ in the uniform norm. By Definition 7.1 (6), $M_\phi T M_\phi \in \mathcal{A}_i$.

If $\psi \in C(X)$, then

$$[T, M_\psi] = \lim_{n\to\infty} [T_n, M_\psi]$$

is the limit of compact operators in the uniform norm and is, therefore, compact.

Combining the results in the preceding paragraphs, we conclude that $T \in \mathcal{A}$. This completes the proof. ■

Proof of Theorem 7.4 (1). We already demonstrated in Lemma A.2 that $\mathcal{A}$ is a unital C^*-subalgebra in $B(L_2(X,\nu))$. It remains to show that $\mathcal{A}_i \subset \mathcal{A}$ for every $i \in \mathbb{I}$ and that $\mathcal{K}(L_2(X,\nu)) \subset \mathcal{A}$.

Let $T \in \mathcal{A}_i$ and $\phi \in C_c(\mathcal{U}_j)$. By Definition 7.1 (2), T is compactly supported in $\mathcal{U}_i$. Choose $\phi_0 \in C_c(\mathcal{U}_i)$ such that $T = M_{\phi_0}T = TM_{\phi_0}$. Then, $\phi\phi_0 \in C_c(\mathcal{U}_i \cap \mathcal{U}_j)$ and $M_{\phi\phi_0} \in \mathcal{A}_i$. Since $\mathcal{A}_i$ is an algebra, it follows that $M_\phi T M_\phi = M_{\phi\phi_0} T M_{\phi\phi_0} \in \mathcal{A}_i$. Since $M_\phi T M_\phi$ is compactly supported in $\mathcal{U}_i \cap \mathcal{U}_j$, it follows from Definition 7.1 (3) that $M_\phi T M_\phi \in \mathcal{A}_j$. This verifies condition (1) in Definition 7.2 for the operator T. Let $\psi \in C(X)$. As above, choose $\phi_0 \in C_c(\mathcal{U}_i)$ such that $T = M_{\phi_0}T = TM_{\phi_0}$. It follows that $[T, M_\psi] = [T, M_{\psi\phi_0}]$. Since $\psi\phi_0 \in C_c(\mathcal{U}_i)$, it follows from condition (7) in Definition 7.1 that $[T, M_\psi]$ is compact. This verifies condition (2) in Definition 7.2 for the operator T. Hence, $T \in \mathcal{A}$ and, therefore, $\mathcal{A}_i \subset \mathcal{A}$.

Let $T \in \mathcal{K}(L_2(X,\nu))$. For every $i \in \mathbb{I}$ and for every $\phi \in C_c(\mathcal{U}_i)$, the operator $M_\phi T M_\phi$ is simultaneously compact and compactly supported in $\mathcal{U}_i$. By Definition 7.1 (4), we have $M_\phi T M_\phi \in \mathcal{A}_i$. Clearly, $[T, M_\psi] \in \mathcal{K}(L_2(X,\nu))$ for every $\psi \in C(X)$. Therefore, $T \in \mathcal{A}$. Since $T \in \mathcal{K}(L_2(X,\nu))$ is arbitrary, it follows that $\mathcal{K}(L_2(X,\nu)) \subset \mathcal{A}$. ■

The purpose of the next lemma is twofold: to establish Theorem 7.4 (3) and to provide a concrete form of hom used in the proof of Theorem 7.4 (2).

Lemma A.3. *Suppose we are in the setting of Theorem 7.4. Let $(\phi_n)_{n=1}^N$ be a good*[5] *partition of unity so that ϕ_n is compactly supported in $\mathcal{U}_{i_n}$ for $1 \le n \le N$. If* hom : $\mathcal{A} \to \mathcal{B}$ *is a $*$-homomorphism as in Theorem 7.4 (2), then*

$$\mathrm{hom}(T) = \sum_{n=1}^{N} \mathrm{hom}_{i_n}\big(M_{\phi_n^{\frac12}} T M_{\phi_n^{\frac12}}\big), \quad T \in \mathcal{A}. \tag{A.1}$$

In particular, hom *is unique.*

Proof. For every $T \in B(L_2(X,\nu))$, we write

$$T = \sum_{n=0}^{N} T_n, \quad T_n = M_{\phi_n^{\frac12}} T M_{\phi_n^{\frac12}}, \; 1 \le n \le N, \quad T_0 = \sum_{k=1}^{N} M_{\phi_k^{\frac12}} \cdot \big[M_{\phi_k^{\frac12}}, T\big].$$

Every T_n, $1 \le n \le N$, is compactly supported in the chart $(\mathcal{U}_{i_n}, h_{i_n})$. If $T \in \mathcal{A}$, then $T_n \in \mathcal{A}_{i_n}$ for $1 \le n \le N$. Hence,

$$\mathrm{hom}(T_n) = \mathrm{hom}_{i_n}(T_n), \quad 1 \le n \le N.$$

If $T \in \mathcal{A}$, then T_0 is compact by Definition 7.2 (2). Since hom vanish on compact operators, it follows that

$$\mathrm{hom}(T_0) = 0.$$

Thus,

$$\mathrm{hom}(T) = \sum_{n=0}^{N} \mathrm{hom}(T_n) = \sum_{n=1}^{N} \mathrm{hom}_{i_n}\big(M_{\phi_n^{\frac12}} T M_{\phi_n^{\frac12}}\big), \quad T \in \mathcal{A}. \qquad \blacksquare$$

We now fix a good partition of unity and prove that the concrete map hom introduced in Lemma A.3 is $*$-homomorphism.

Lemma A.4. *Let* hom *be the mapping on the right-hand side in (A.1). We have*

$$\mathrm{hom}(TM_\phi) = \mathrm{hom}(T) \cdot \mathrm{hom}(M_\phi), \quad T \in \mathcal{A}, \; \phi \in C(X).$$

Proof. Let $\psi_n \in C_c(\mathcal{U}_{i_n})$ be such that $\phi_n \psi_n = \phi_n$. We write

$$M_{\phi_n^{\frac12}} T M_\phi M_{\phi_n^{\frac12}} = M_{\phi_n^{\frac12}} T M_{\phi_n^{\frac12}} \cdot M_{\phi\psi_n}.$$

Since hom_{i_n} is a homomorphism, it follows that

$$\mathrm{hom}_{i_n}\big(M_{\phi_n^{\frac12}} T M_\phi M_{\phi_n^{\frac12}}\big) = \mathrm{hom}_{i_n}\big(M_{\phi_n^{\frac12}} T M_{\phi_n^{\frac12}}\big) \cdot \mathrm{hom}_{i_n}(M_{\phi\psi_n}).$$

It follows from Definition 7.3 (4) and (A.1) that

$$\mathrm{hom}_{i_n}(M_{\phi\psi_n}) = \mathrm{Hom}(\phi\psi_n) = \mathrm{Hom}(\phi) \cdot \mathrm{Hom}(\psi_n) = \mathrm{hom}_{i_n}(M_{\psi_n}) \cdot \mathrm{hom}(M_\phi).$$

[5] See Definition 2.16.

Again using the fact that $\hom_{i_n}$ is a homomorphism, we obtain

$$\hom_{i_n}\left(M_{\phi_n^{\frac{1}{2}}} T M_\phi M_{\phi_n^{\frac{1}{2}}}\right) = \hom_{i_n}\left(M_{\phi_n^{\frac{1}{2}}} T M_{\phi_n^{\frac{1}{2}}} \cdot M_{\psi_n}\right) \cdot \hom(M_\phi).$$

However, by the choice of ψ_n, we have

$$M_{\phi_n^{\frac{1}{2}}} T M_{\phi_n^{\frac{1}{2}}} \cdot M_{\psi_n} = M_{\phi_n^{\frac{1}{2}}} T M_{\phi_n^{\frac{1}{2}}}.$$

Therefore,

$$\hom_{i_n}\left(M_{\phi_n^{\frac{1}{2}}} T M_\phi M_{\phi_n^{\frac{1}{2}}}\right) = \hom_{i_n}\left(M_{\phi_n^{\frac{1}{2}}} T M_{\phi_n^{\frac{1}{2}}}\right) \cdot \hom(M_\phi), \quad 1 \le n \le N.$$

Summing over $1 \le n \le N$, we complete the proof. ∎

Lemma A.5. *Let* hom *be the mapping on the right-hand side in* (A.1)*. We have* $\hom = \hom_j$ *on* $\mathcal{A}_j$ *for every* $j \in \mathbb{I}$.

Proof. Let $T \in \mathcal{A}_j$ and let $(\phi_n)_{n=1}^N$ be as in Lemma A.3. For every $1 \le n \le N$, the operator $M_{\phi_n^{\frac{1}{2}}} T M_{\phi_n^{\frac{1}{2}}}$ is compactly supported both in the chart $(\mathcal{U}_j, h_j)$ and in the chart $(\mathcal{U}_{i_n}, h_{i_n})$. By Definition 7.3 (2), we have

$$\hom_{i_n}\left(M_{\phi_n^{\frac{1}{2}}} T M_{\phi_n^{\frac{1}{2}}}\right) = \hom_j\left(M_{\phi_n^{\frac{1}{2}}} T M_{\phi_n^{\frac{1}{2}}}\right).$$

Let $\phi \in C_c(\mathcal{U}_j)$ be such that $T = M_\phi T = T M_\phi$. Since $\hom_j$ is a homomorphism, it follows that

$$\begin{aligned}\hom_j\left(M_{\phi_n^{\frac{1}{2}}} T M_{\phi_n^{\frac{1}{2}}}\right) &= \hom_j\left(M_{\phi_n^{\frac{1}{2}}\phi} \cdot T \cdot M_{\phi\phi_n^{\frac{1}{2}}}\right)\\ &= \hom_j(T) \cdot \hom_j\left(M_{\phi_n^{\frac{1}{2}}\phi}\right) \cdot \hom_j\left(M_{\phi\phi_n^{\frac{1}{2}}}\right)\\ &= \hom_j(T) \cdot \hom_j(M_{\phi_n\phi^2}).\end{aligned}$$

Therefore,

$$\hom_{i_n}\left(M_{\phi_n^{\frac{1}{2}}} T M_{\phi_n^{\frac{1}{2}}}\right) = \hom_j(T) \cdot \hom_j(M_{\phi_n\phi^2}), \quad 1 \le n \le N.$$

Summing over $1 \le n \le N$, we obtain

$$\hom(T) = \hom_j(T) \cdot \left(\sum_{n=1}^N \hom_j(M_{\phi_n\phi^2})\right) = \hom_j(T) \cdot \hom_j(M_{\phi^2}).$$

Again using the fact that $\hom_j$ is a homomorphism, we obtain

$$\hom(T) = \hom_j(T M_{\phi^2}).$$

Taking into account that $T M_{\phi^2} = T$, we complete the proof. ∎

Lemma A.6. *We have*

$$\mathrm{hom}(TS) = \mathrm{hom}(T) \cdot \mathrm{hom}(S), \quad T, S \in \mathcal{A}_j, \ j \in \mathbb{I}.$$

Proof. Using Lemma A.5 and taking into account that hom_j is a homomorphism, we write

$$\mathrm{hom}(TS) = \mathrm{hom}_j(TS) = \mathrm{hom}_j(T) \cdot \mathrm{hom}_j(S) = \mathrm{hom}(T) \cdot \mathrm{hom}(S). \qquad \blacksquare$$

Lemma A.7. *If $T, S \in \mathcal{A}$ and if T is compactly supported in the chart $(\mathcal{U}_j, h_j)$, then*

$$\mathrm{hom}(TS) = \mathrm{hom}(T) \cdot \mathrm{hom}(S).$$

Proof. Let $\phi \in C_c(\mathcal{U}_j)$ be such that $T = TM_\phi$. We write

$$TS = TS_1 + TS_2, \quad S = M_{\phi^{\frac{1}{2}}} S M_{\phi^{\frac{1}{2}}}, \ S_2 = M_{\phi^{\frac{1}{2}}}[M_{\phi^{\frac{1}{2}}}, S].$$

By Definition 7.2 (2), S_2 is compact. By construction, hom vanishes on compact operators. It follows that

$$\mathrm{hom}(TS) = \mathrm{hom}(TS_1).$$

Since T and S_1 are compactly supported in the chart $(\mathcal{U}_j, h_j)$, it follows from Lemma A.6 that

$$\mathrm{hom}(TS) = \mathrm{hom}(T) \cdot \mathrm{hom}(S_1).$$

By Lemma A.4, we have

$$\begin{aligned}\mathrm{hom}(TS) &= \mathrm{hom}(T) \cdot \mathrm{hom}(S) \cdot \mathrm{hom}\left(M_{\phi^{\frac{1}{2}}}\right)^2 \\ &= \mathrm{hom}(T) \cdot \mathrm{hom}(M_\phi) \cdot \mathrm{hom}(S) \\ &= \mathrm{hom}(TM_\phi) \cdot \mathrm{hom}(S).\end{aligned}$$

Since $T = TM_\phi$, the assertion follows. $\blacksquare$

Proof of Theorem 7.4 (2). It is immediate that hom is a linear $*$-preserving mapping. We now prove that hom preserves multiplication.

Let $T, S \in \mathcal{A}$ and let $(\phi_n)_{n=1}^N$ be as in Lemma A.3. We write $T = \sum_{n=0}^N T_n$, where

$$T_n = M_{\phi_n^{\frac{1}{2}}} T M_{\phi_n^{\frac{1}{2}}}, \ 1 \le n \le N, \quad T_0 = \sum_{k=1}^N M_{\phi_k^{\frac{1}{2}}}\big[M_{\phi_k^{\frac{1}{2}}}, T\big].$$

Every T_n, $1 \le n \le N$, is compactly supported in the chart $(\mathcal{U}_{i_n}, h_{i_n})$. By Lemma A.7, we have

$$\mathrm{hom}(T_n S) = \mathrm{hom}(T_n) \cdot \mathrm{hom}(S), \quad 1 \le n \le N.$$

The operators T_0 and $T_0 S$ are compact. By construction, hom vanishes on compact operators. It follows that

$$\mathrm{hom}(T_0 S) = 0 = 0 \cdot \mathrm{hom}(S) = \mathrm{hom}(T_0) \cdot \mathrm{hom}(S).$$

By linearity, we have

$$\mathrm{hom}(TS) = \sum_{n=0}^{N} \mathrm{hom}(T_n S) = \sum_{n=0}^{N} \mathrm{hom}(T_n) \cdot \mathrm{hom}(S) = \mathrm{hom}(T) \cdot \mathrm{hom}(S).$$

Thus, hom is a $*$-homomorphism.

Let us now show that $\ker(\mathrm{hom}) = \mathcal{K}(L_2(X, \nu))$. If $T \in \mathcal{A}$ is such that $\mathrm{hom}(T) = 0$, then $\mathrm{hom}(T^*T) = 0$. By construction of hom, we have

$$\sum_{n=1}^{n} \mathrm{hom}_{i_n} \big(M_{\phi_n^{\frac{1}{2}}} T^* T M_{\phi_n^{\frac{1}{2}}} \big) = 0.$$

Every summand on the left-hand side is positive. Therefore,

$$\mathrm{hom}_{i_n} \big(M_{\phi_n^{\frac{1}{2}}} T^* T M_{\phi_n^{\frac{1}{2}}} \big) = 0, \quad 1 \le n \le N.$$

By Definition 7.3 (3), we have

$$M_{\phi_n^{\frac{1}{2}}} T^* T M_{\phi_n^{\frac{1}{2}}} \in \mathcal{K}\big(L_2(X, \nu)\big), \quad 1 \le n \le N.$$

In other words,

$$T M_{\phi_n^{\frac{1}{2}}} \in \mathcal{K}\big(L_2(X, \nu)\big), \quad 1 \le n \le N.$$

Multiplying on the right by $M_{\phi_n^{\frac{1}{2}}}$ and summing over $1 \le n \le N$, we conclude that $T \in \mathcal{K}(L_2(X, \nu))$. ∎

Acknowledgements. The authors sincerely thank Professor N. Higson for Question 1.2 which was the starting point of this article.

Special thanks are due to Professor A. Connes for his insightful comments on the earlier version of this manuscript. In particular, the observation that the diffeomorphisms of $\mathbb{R}^d$ considered in this text extend to the diffeomorphisms of $P^d(\mathbb{R})$ is due to Professor Connes.

Funding. The second and the third authors are supported by ARC grant DP 230100434.

References

[1] R. A. Adams, *Sobolev spaces*. Pure Appl. Math 65, Academic Press, New York, 1975 Zbl 0314.46030 MR 450957

[2] P. M. Alberti and R. Matthes, Connes' trace formula and Dirac realization of Maxwell and Yang-Mills action. In *Noncommutative geometry and the standard model of elementary particle physics (Hesselberg, 1999)*, pp. 40–74, Lecture Notes in Phys. 596, Springer, Berlin, 2002 Zbl 1042.81101 MR 1998530

[3] M. F. Atiyah and I. M. Singer, The index of elliptic operators. I. *Ann. of Math. (2)* **87** (1968), 484–530 Zbl 0164.24001 MR 236950

[4] P. Baum and R. G. Douglas, Toeplitz operators and Poincaré duality. In *Toeplitz centennial (Tel Aviv, 1981)*, pp. 137–166, Oper. Theory Adv. Appl. 4, Birkhäuser, Basel, 1982 Zbl 0517.55001 MR 669904

[5] B. Blackadar, *Operator algebras. Theory of C^*-algebras and von Neumann algebras*. Encyclopaedia Math. Sci. 122, Springer, Berlin, 2006 Zbl 1092.46003 MR 2188261

[6] I. Chavel, *Riemannian geometry. A modern introduction*. 2nd edn., Cambridge Stud. Adv. Math. 98, Cambridge University Press, Cambridge, 2006 Zbl 1099.53001 MR 2229062

[7] A. Connes, The action functional in noncommutative geometry. *Comm. Math. Phys.* **117** (1988), no. 4, 673–683 Zbl 0658.53068 MR 953826

[8] H. O. Cordes, *Spectral theory of linear differential operators and comparison algebras*. London Math. Soc. Lecture Note Ser. 76, Cambridge University Press, Cambridge, 1987 Zbl 0727.35092 MR 890743

[9] J. Dixmier, *C^*-algebras*. North-Holland Math. Library 15, North-Holland Publishing, Amsterdam, 1977 Zbl 0372.46058 MR 458185

[10] I. C. Gohberg, On the theory of multidimensional singular integral equations. *Dokl. Akad. Nauk SSSR* **133** (1960), 1279–1282; translation in *Soviet Math. Dokl.* **1** (1960), 960–963 Zbl 0161.32101 MR 124704

[11] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa, *Elements of noncommutative geometry*. Birkhäuser Adv. Texts Basler Lehrbücher, Birkhäuser, Boston, MA, 2001 Zbl 0958.46039 MR 1789831

[12] L. Hörmander, Pseudo-differential operators and hypoelliptic equations. In *Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966)*, pp. 138–183, American Mathematical Society, Providence, RI, 1967 Zbl 0167.09603 MR 383152

[13] M. S. Joshi, Lectures on pseudo-differential operators. 1999, arXiv:9906155

[14] J. J. Kohn and L. Nirenberg, An algebra of pseudo-differential operators. *Comm. Pure Appl. Math.* **18** (1965), 269–305 Zbl 0171.35101 MR 176362

[15] G. Levitina, F. Sukochev, and D. Zanin, Cwikel estimates revisited. *Proc. Lond. Math. Soc. (3)* **120** (2020), no. 2, 265–304 Zbl 1443.47023 MR 4008371

[16] S. Lord, E. McDonald, F. Sukochev, and D. Zanin, *Singular traces. Vol. 2. Trace formulas*. De Gruyter Stud. Math. 46/2, De Gruyter, Berlin, 2023 Zbl 07658350 MR 4604115

[17] S. Lord, F. Sukochev, and D. Zanin, Advances in Dixmier traces and applications. In *Advances in noncommutative geometry—on the occasion of Alain Connes' 70th birthday*, pp. 491–583, Springer, Cham, 2019 Zbl 1457.46064 MR 4300559

[18] S. Lord, F. Sukochev, and D. Zanin, *Singular traces. Vol. 1. Theory*. De Gruyter Stud. Math. 46/1, De Gruyter, Berlin, 2021 Zbl 1495.47001 MR 4304074

[19] E. McDonald, F. Sukochev, and D. Zanin, A C^*-algebraic approach to the principal symbol II. *Math. Ann.* **374** (2019), no. 1–2, 273–322 Zbl 1427.46048 MR 3961311

[20] S. T. Melo, Norm closure of classical pseudodifferential operators does not contain Hörmander's class. In *Geometric analysis of PDE and several complex variables*, pp. 329–336, Contemp. Math. 368, American Mathematical Society, Providence, RI, 2005 Zbl 1080.35183 MR 2126480

[21] P. W. Michor, *Topics in differential geometry*. Grad. Stud. Math. 93, American Mathematical Society, Providence, RI, 2008 Zbl 1175.53002 MR 2428390

[22] R. S. Palais, Natural operations on differential forms. *Trans. Amer. Math. Soc.* **92** (1959), 125–141 Zbl 0092.30802 MR 116352

[23] M. Ruzhansky and V. Turunen, *Pseudo-differential operators and symmetries. Background analysis and advanced topics*. Pseudo-Differential Oper. Theory Appl. 2, Birkhäuser, Basel, 2010 Zbl 1193.35261 MR 2567604

[24] S. Sakai, *C^*-algebras and W^*-algebras. Reprint of the 1971 edition*. Classics Math., Springer, Berlin, 1998 Zbl 1024.46001 MR 1490835

[25] R. T. Seeley, Integro-differential operators on vector bundles. *Trans. Amer. Math. Soc.* **117** (1965), 167–204 Zbl 0135.37102 MR 173174

[26] B. Simon, *Trace ideals and their applications*. 2nd edn., Math. Surveys Monogr. 120, American Mathematical Society, Providence, RI, 2005 Zbl 1074.47001 MR 2154153

[27] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Princeton Math. Ser. 43, Princeton University Press, Princeton, NJ, 1993 Zbl 0821.42001 MR 1232192

[28] R. S. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold. *J. Functional Analysis* **52** (1983), no. 1, 48–79 Zbl 0515.58037 MR 705991

[29] F. Sukochev and D. Zanin, A C^*-algebraic approach to the principal symbol. I. *J. Operator Theory* **80** (2018), no. 2, 481–522 Zbl 1424.46097 MR 3871833

[30] M. E. Taylor, *Partial differential equations I. Basic theory*. 2nd edn., Appl. Math. Sci. 115, Springer, New York, 2011 Zbl 1206.35002 MR 2744150

[31] F. Trèves, *Introduction to pseudodifferential and Fourier integral operators. Vol. 1: Pseudodifferential operators*. Univ. Ser. Math., Plenum Press, New York, 1980 Zbl 0453.47027 MR 597144

Received 18 October 2022.

Yuri Kordyukov
Institute of Mathematics, Ufa Federal Research Centre, Russian Academy of Sciences, 112 Chernyshevsky str., 450008 Ufa, Russia; yurikor@matem.anrb.ru

Fedor Sukochev
School of Mathematics and Statistics, University of New South Wales, Kensington, NSW 2052, Australia; f.sukochev@unsw.edu.au

Dmitriy Zanin
School of Mathematics and Statistics, University of New South Wales, Kensington, NSW 2052, Australia; d.zanin@unsw.edu.au