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Zeta functions and topology of Heisenberg cycles for linear ergodic flows

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Abstract. Placing a Dirac–Schrödinger operator along the orbit of a flow on a compact manifold M defines an $\mathbb{R}$-equivariant spectral triple over the algebra of smooth functions on M. We study some of the properties of these triples, with special attention to their zeta functions. These zeta functions are defined for $\mathrm{Re}(s) > 1$ by $\mathrm{Trace}(f_p H^{-s})$, where f_p is the uniformly continuous function on the real line obtained by restricting the continuous or smooth function f on M to the orbit of a point $p \in M$, and $H = -\frac{\partial^2}{\partial x^2} + x^2$ is the harmonic oscillator. The meromorphic continuation property and pole structure of these zeta functions are related to ergodic time averages in dynamics. In the case of the periodic flow on the circle, one obtains a spectral triple over the smooth irrational torus $A_\hbar^\infty \subset A_\hbar$ already studied by Lesch and Moscovici. We strengthen a result of these authors, showing that the zeta function $\mathrm{Trace}(aH^{-s})$ extends meromorphically to $\mathbb{C}$ for any element a of the C^*-algebra $A_\hbar$. Another variant of our construction yields a spectral cycle for $A_\hbar \otimes A_{1/\hbar}$ and a spectral triple over a suitable subalgebra with the meromorphic continuation property if $\hbar$ satisfies a Diophantine condition. The class of this cycle defines a fundamental class in the sense that it determines a KK-duality between $A_\hbar$ and $A_{1/\hbar}$. We employ the local index theorem of Connes and Moscovici in order to elaborate an index theorem of Connes for certain classes of differential operators on the line and compute the intersection form on K-theory induced by the fundamental class.

1. Introduction

The irrational rotation algebra $A_\hbar := C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}$, the crossed product of the C^*-algebra $C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z})$ by the action of $\mathbb{Z}$ by translation by $\hbar \in \mathbb{R} \setminus \mathbb{Q}$ mod $\mathbb{Z}$ on $\mathbb{T}$, is one of the key motivating examples in noncommutative geometry. Early results of Connes and Rieffel classified finitely generated projective modules over $A_\hbar$, or over its natural Schwartz subalgebra $A_\hbar^\infty$, by an analogue of the first Chern number of a line bundle over $\mathbb{T}^2$, defined for $e \in A_\hbar^\infty \subset A_\hbar$, by

$$c_1(e) := \frac{1}{2\pi i} \cdot \tau(e[\delta_1(e), \delta_2(e)]),$$

where δ_1, δ_2 are the derivations of $A_\hbar^\infty$ generating the natural $\mathbb{R}^2$-action, and τ is the trace. In fact, these numbers are *integers*, a fact related to the quantum Hall effect in solid state physics (see [1]).

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The reason for the integrality lies in the following. The densely defined operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ on $L^2(\mathbb{T}^2)$ assemble to the operator

$$\overline{\partial} := \begin{bmatrix} 0 & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & 0 \end{bmatrix}$$

on $L^2(\mathbb{T}^2) \oplus L^2(\mathbb{T}^2)$, and the representation of $C(\mathbb{T}^2)$ on $L^2(\mathbb{T}^2)$ by multiplication operators can be adjusted by introducing phase factors to give a representation $\lambda_\hbar \colon A_\hbar \to \mathbb{B}(L^2(\mathbb{T}^2))$ which makes the triple $(L^2(\mathbb{T}^2) \oplus L^2(\mathbb{T}^2), \lambda_\hbar, \overline{\partial})$ a 2-summable spectral triple over $A_\hbar^\infty$ whose Chern character may be computed using the local index formula of Connes and Moscovici to be the class of the cyclic cocycle

$$\tau_2(a^0, a^1, a^2) = \tau(a^0\delta_1(a^1)\delta_2(a^2) - a^0\delta_2(a^1)\delta_1(a^2)), \quad a^0, a^1, a^2 \in A_\hbar^\infty. \tag{1.1}$$

The integrality of the Chern numbers $\tau_2(e, e, e)$ follows from the Connes–Moscovici index theorem which implies that for any idempotent $e \in A_\hbar^\infty$,

$$c_1(e) = \tau_2(e, e, e) = \langle [e], [\overline{\partial}] \rangle \in \mathbb{Z},$$

where the right-hand side is the pairing between K-theory and K-homology. But it is a result going back to early direct computations of Connes [5] involving in particular a calculation of the cyclic cohomology of $A_\hbar^\infty$.

In this article, we study a slightly different method of constructing spectral triples, using the operators $x \pm \frac{d}{dx}$, the annihilation and creation operators of quantum mechanics. They assemble to form a spectral triple over a suitable smooth subalgebra of $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$, where $C_u(\mathbb{R})$ is the C^*-algebra of uniformly continuous, bounded functions on $\mathbb{R}$ and $\mathbb{R}_d$ is the group of real numbers with the *discrete* topology. The operator of the triple is

$$D = \begin{bmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{bmatrix}.$$

The closure of D is self-adjoint, and D^2 is essentially the direct sum of two copies of the harmonic oscillator

$$H = -\frac{d^2}{dx^2} + x^2$$

on $\mathbb{R}$, which has discrete spectrum consisting of the odd positive integers. We define a representation $\pi \colon C_u(\mathbb{R}) \rtimes \mathbb{R}_d \to \mathbb{B}(L^2(\mathbb{R}))$ by letting $f \in C_u(\mathbb{R})$ act by the corresponding multiplication operator $(f\xi)(x) = f(x)\xi(x)$, and a group element $t \in \mathbb{R}_d$ by the group translation unitary operator $(u_t\xi)(x) = \xi(x - t)$.

The triple just described gives a spectral (unbounded) cycle for $\mathrm{KK}_0(C_u(\mathbb{R}) \rtimes \mathbb{R}_d, \mathbb{C})$. We call it the *Heisenberg cycle*.

The Heisenberg cycle pulls back to any C^*-subalgebra of $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$, and in this article, we are most interested in subalgebras arising from ergodic flows. If α is a smooth flow on a compact manifold M and $p \in M$, then the function

$$f_p(t) := f(\alpha_t(p))$$

is bounded and uniformly continuous on $\mathbb{R}$ if f is continuous. We obtain an embedding

$$B_\alpha := C(M) \rtimes_\alpha \mathbb{R}_d \subset C_u(\mathbb{R}) \rtimes \mathbb{R}_d.$$

So one can associate a Heisenberg cycle to any smooth flow and corresponding class $[B_\alpha] \in \mathrm{KK}_0(C(M) \rtimes \mathbb{R}_d, \mathbb{C})$ or, in $\mathrm{KK}_0(C(M) \rtimes \Lambda, \mathbb{C})$, if one has a subgroup $\Lambda \subset \mathbb{R}_d$ of particular interest for the context. For example, if $\Lambda := \{0\}$ is the trivial subgroup, then the corresponding class in $\mathrm{KK}_0(C(M), \mathbb{C})$ is equal to the class in K-homology of the point $p \in M$, and so contains no interesting topological information (this follows from constructing a certain homotopy in KK, see [8]). However, simple examples show that for certain natural (non-trivial) choices of subgroup, the Heisenberg classes are non-trivial.

In the case of the periodic flow on $\mathbb{T}$, the C^*-algebra B_α is $C(\mathbb{T}) \rtimes \mathbb{R}_d$ which contains the irrational rotation algebra $A_\hbar = C(\mathbb{T}) \rtimes \hbar\mathbb{Z}$ for any $\hbar \in \mathbb{R} \setminus \mathbb{Q}$, by restricting to the subgroup $\Lambda := \hbar\mathbb{Z} \subset \mathbb{R}_d$. The Heisenberg cycle for this algebra has been studied by Connes [3,5], and Moscovici and Lesch [18]. The latter authors refer to *Heisenberg modules*. One can build a Heisenberg module by twisting the Dirac–Dolbeault cycle of Connes by a Morita bimodule; such bimodules come from compact transversals to the Kronecker flow. One obtains thus a family of such cycles (for $A_\hbar$) all having a somewhat similar form, and involving the Dirac–Schrödinger operators $x \pm \frac{d}{dx}$ on $L^2(\mathbb{R})$, or a finite sum of copies of $L^2(\mathbb{R})$. But as observed above, our Heisenberg cycles are defined over a much larger algebra $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$ than $A_\hbar$. One gets cycles for $C(M) \rtimes \mathbb{R}_d$ or $C(M) \rtimes \Lambda$ for flows on manifolds M and arbitrary subgroups $\Lambda \subset \mathbb{R}$, and such cycles may be a source of topological invariants of flows. We restrict ourselves in this note to examining linear flows, e.g., Kronecker flows on $\mathbb{T}^2$, with Diophantine periods. For the subgroup $\Lambda \subset \mathbb{R}_d$ generated by 1, $\hbar$, the crossed-product $B_\hbar := C(\mathbb{T}^2) \rtimes \Lambda$ is isomorphic to $A_\hbar \otimes A_{1/\hbar}$. In the second part of the paper, we compute some topological invariants of these Heisenberg cycles, using the local index theorem of Connes and Moscovici (and the exposition of it in [14]). The Heisenberg cycle for $A_\hbar \otimes A_{1/\hbar} = B_\hbar$ induces a KK-duality between $A_\hbar$ and $A_{1/\hbar}$ and we compute the index pairing $\mathrm{K}_0(A_\hbar) \times \mathrm{K}_0(A_{1/\hbar}) \to \mathbb{Z}$ and show that it has matrix

$$\begin{bmatrix} 1 & -\lfloor \frac{1}{\hbar} \rfloor \\ -\lfloor \hbar \rfloor & 1 \end{bmatrix}$$

with respect to the bases consisting of the unit and the respective Rieffel projections. This strengthens an index calculation of Connes in [5] for classes of differential operators on the real line. This is based on our computation of the Chern character of the Heisenberg cycle over $A_\hbar$, which we show is given by the mixed degree cyclic cochain

$$\tau - \hbar\tau_2,$$

where τ_2 is as in (1.1).

The main technical contribution of this note concerns the meromorphic extension problem of the zeta functions

$$\zeta(a, s) := \mathrm{Trace}(aH^{-s})$$

for $a \in C_u(\mathbb{R}) \rtimes \mathbb{R}_d$. Establishing such meromorphic extensions is necessary to apply the local index theorem, at least in the presentation [14], as the cyclic cocycles involved in the local Chern character formula are obtained as poles of such zeta functions.

If Δ_M is the Laplacian on a compact Riemannian manifold, and $f \in C^\infty(M)$, then $\mathrm{Trace}(f\Delta_M^{-s})$ extends meromorphically to $\mathbb{C}$, with certain poles. This fact is proved by the theory of asymptotic expansions, specifically of the kernel of $fe^{-t\Delta_M}$, because the Mellin transform transforms the meromorphic extension problem into a problem about the asymptotics of the heat kernel as $t \to 0$. Such asymptotic expansions are also available in the situation of the Schwartz algebra of the irrational rotation algebra $A_\hbar$, as noted by Lesch and Moscovici [18], who used them to deduce the meromorphic extendibility of $\mathrm{Trace}(aH^{-s})$ for $a \in A_\hbar^\infty$ in the *smooth* irrational torus, with H the harmonic oscillator. Using a different technique, we prove here that $\mathrm{Trace}(aH^{-s})$ meromorphically extends for a in the *C^*-algebra* $A_\hbar$. Actually, there is a connection between the zeta functions $\mathrm{Trace}(f_pH^{-s})$ for $f \in C(M)$, $f_p(t) := f(\alpha_t(p))$ for a smooth flow α on M, and ergodic time averages in dynamics. For example, we show that if the flow is ergodic, then

$$\lim_{s\to 1^+}(s-1)\cdot \mathrm{Trace}(f_pH^{-s}) = \frac{1}{2}\int_M f\,d\mu$$

for a.e. $p \in M$, $f \in C(M)$, and μ any α-invariant measure. Therefore, the residue trace, defined spectrally as the pole at $s = 1$ of $\mathrm{Trace}(f_pH^{-s})$, recovers the invariant measure μ. The proof is based on an integral formula for $\mathrm{Trace}(fH^{-s})$ following a fairly well-established route using the heat equation. In fact, as we show more generally that if $f \in C_u(\mathbb{R})$ and if $\lim_{T\to\pm\infty}\frac{1}{T}\int_0^T f(t)dt$ exists, then the limit equals $\lim_{s\to 1^+}(s-1)\cdot \mathrm{Trace}(fH^{-s})$. The statement above regarding ergodic flows follows from these observations and the Birkhoff ergodic theorem.

The meromorphic extension property from this point of view, for a given smooth flow, requires a strengthening of the Birkhoff ergodic theorem for that situation which gives a finer estimate for the deviation $\int_0^T f(t)d\mu - T\int_M f\,d\mu$. From our integral formula for the zeta function, it is apparent that the meromorphic extension property of $\mathrm{Trace}(f_pH^{-s})$ would follow, for example, for any $f \in C^\infty(M)$, if one was guaranteed smooth solvability of the *cohomological equation* $Xu = f$ for a smooth flow with generating vector field X (see, e.g., [9, 11, 16, 17].) The condition $\int_M f\,d\mu = 0$ of f is an obvious obstruction to $Xu = f$ being continuously solvable for any α-invariant μ. For the standard periodic flow on the circle, this is the *only* obstruction. This is because if f is continuous and ρ-periodic and $\int_0^\rho f\,d\mu = 0$, then the anti-derivative $F(T) := \int_0^T f(t)dt$ is also ρ-periodic, so F solves the equation continuously. As we show, then the zeta function can be meromorphically extended to $\mathrm{Re}(s) > 1 - \frac{n}{2}$ by solving the equation n times, and so meromorphically extended to $\mathbb{C}$. For the Kronecker flow on $\mathbb{T}^2$, the cohomological equation $Xu = f$ is smoothly solvable for smooth f of zero Lebesgue mean if α satisfies a Diophantine condition, and it follows that $\mathrm{Trace}(fH^{-s})$ extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ in this case as well, if f is *smooth*.

Our methods thus prove that any smooth *cohomology free* vector field on a compact manifold M (in the sense of [10]) determines a zeta function $\mathrm{Trace}(f_p H^{-s})$ ($p \in M$, $f \in C^\infty(M)$) with the meromorphic continuation property, and hence a spectral triple over $C^\infty(M) \rtimes_{\mathrm{alg}} \mathbb{R}_d$ with the meromorphic continuation property. (For the class of *hypoelliptic* fields, it is apparently an open conjecture that a hypoelliptic vector field is cohomology free.) Another current conjecture is to the effect that any globally hypoelliptic field is smoothly conjugate to a linear Diophantine flow on a torus. So there appears to be not much more generality achieved here than that of linear Diophantine flows on tori. It would be interesting to see more complicated examples of (parabolic) flows with zeta functions with the meromorphic continuation property, and, perhaps, a richer pole structure, involving invariant distributions on M which are not invariant measures.

The noncommutative geometry of the irrational torus $A_\hbar^\infty$ and various interesting variations of it involving changing the metric, has been intensely studied recently. See, e.g., [7, 13]. The problem of constructing spectral triples in connection with crossed products and other C^*-algebras from dynamics remains an important one in noncommutative geometry, after the seminal work of Connes and Moscovici [6] on 'diffeomorphism invariant' geometry. Some recent examples of spectral triples from dynamics are [12, 15, 23]. Dirac–Schrödinger operators as giving non-standard spectral cycles for $C(\mathbb{T}^2)$ and connections with KK-duality and Baum–Connes are studied in [21]. See [13] for information about the noncommutative torus $A_\hbar^\infty$ and its noncommutative pseudodifferential calculus. The short article [22] is a very good source for the local index theorem (although we have used [14] as our main source for the index theorem in this article).

2. Spectral cycles from the canonical anti-commutation relations

The Heisenberg group

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

has Lie algebra $\mathfrak{h}$ the 3-by-3 strictly upper triangular matrices under matrix commutator. Let X, Y be the elements

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

of $\mathfrak{h}$. Then

$$[X, Y] = Z := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

while Z is central in $\mathfrak{h}$. It follows that if π is any *irreducible* representation of H, $\pi(Z) = \pi([X,Y]) = [\pi(X),\pi(Y)]$ is a multiple of the identity operator $[\pi(X),\pi(Y)] = \hbar$ for some $\hbar \in \mathbb{R}$, a 'Planck constant'.

The name *Heisenberg group* originates in these relations, which have the same form as the canonical commutation relations in quantum mechanics, where x and $\frac{d}{dx}$ model position and momentum operators.

From the above remarks, we obtain a classification of irreducible representations of H. Either $\hbar = 0$, in which case $\pi(Z) = 0$ and hence $\pi(X)$ and $\pi(Y)$ commute, which implies the representation is 1-dimensional, and is completely determined by the ordered pair of real numbers $(\pi(X),\pi(Y))$, or $\hbar \neq 0$, in which case one can show that the representation is isomorphic to the following interesting representation $\pi_\hbar$ of $\mathfrak{h}$ by unbounded operators on $L^2(\mathbb{R})$. Let

$$\pi_\hbar(X) = x \quad \text{and} \quad \pi_\hbar(Y) = \hbar\frac{d}{dx}.$$

Then $[x, \hbar\frac{d}{dx}] = \hbar$ as required.

Application of functional calculus to the operators x and $\frac{d}{dx}$ produces the operators

$$u = e^{2\pi i x}, \quad v_\hbar := e^{-\hbar d/dx},$$

where u is multiplication by the periodic function $e^{2\pi i x}$ and

$$(v_\hbar)\xi(x) = \xi(x-\hbar).$$

We have

$$uv_\lambda = e^{-2\pi i\hbar}v_\lambda u.$$

If $\hbar \in \mathbb{R}\setminus\mathbb{Q}$, then the *irrational rotation algebra* is the C^*-algebra

$$A_\hbar := C(\mathbb{T}) \rtimes_\hbar \mathbb{Z},$$

where $\mathbb{Z}$ acts on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ with generator the automorphism induced by translation by $\hbar$ mod $\mathbb{Z}$. If $U \in C(\mathbb{T}) \rtimes_h \mathbb{Z}$ is the generator $U(t) = e^{2\pi i t}$ of $C(\mathbb{T})$ and V the generator of the $\mathbb{Z}$-action in the crossed-product, then a quick computation shows that

$$UV = e^{-2\pi i\hbar}VU \in A_\hbar, \tag{2.1}$$

and it follows that we obtain, for each $\hbar$, a representation

$$\pi_\hbar\colon A_\hbar \to \mathbb{B}(L^2(\mathbb{R})) \tag{2.2}$$

of $A_\hbar$ on $L^2(\mathbb{R})$. Note that $\pi_\hbar$ depends on $\hbar$ *as a real number* while $A_\hbar$ only depends on the class of $\hbar$ mod $\mathbb{Z}$.

We are going to fit these representations into a spectral cycle for $\mathrm{KK}_0(A_\hbar,\mathbb{C})$, using the properties of the *harmonic oscillator*

$$H := -\frac{d^2}{dx^2} + x^2, \tag{2.3}$$

a second-order elliptic operator on $\mathbb{R}$, whose domain we will take initially to be the Schwartz space $\mathcal{S}(\mathbb{R})$. Actually, the construction is more general, and produces a spectral cycle for the C^*-algebra crossed product $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$, with $\mathbb{R}_d$ denoting $\mathbb{R}$ with the discrete topology.

Let $A = x + \frac{d}{dx}$, initial domain the Schwartz space $\mathcal{S}(\mathbb{R})$, and $A^* = x - \frac{d}{dx}$. The relations

$$\begin{gathered} AA^* = H + 1, \quad A^*A = H - 1, \\ [A, A^*] = 2, \quad [H, A] = -2A, \quad [H, A^*] = 2A^*. \end{gathered} \tag{2.4}$$

hold as operators on $\mathcal{S}$. (See [24].)

Now set $\psi_0 := \pi^{-1/4} \cdot e^{-x^2/2} \in L^2(\mathbb{R})$. In quantum mechanics, ψ_0 is called the *ground state*, and the states inductively defined by $\psi_k := (2k)^{-1/2} \cdot A^*\psi_{k-1}$ the 'excited states'. Observe that due to $HA^* = A^*H + 2A^*$, from (2.4), we see by induction that ψ_k is a unit-length eigenvector of H with eigenvalue $2k + 1$,

$$\begin{aligned} H\psi_k &= (2k)^{-1/2} \cdot HA^*\psi_{k-1} = (2k)^{-1/2} \cdot (A^*H + 2A^*)\xi_{k-1} \\ &= (2k)^{-1/2} \cdot ((2k-1) \cdot A^*\psi_{k-1} + 2A^*\psi_{k-1}) = (2k+1) \cdot \psi_k. \end{aligned}$$

It follows from $[H, A] = -2A$ that

$$A\psi_k = \sqrt{2k} \cdot \psi_{k-1}, \quad A^*\psi_k = \sqrt{2k+2} \cdot \psi_{k+1}.$$

The eigenvectors of H are given by $\xi_k = H_k(x)e^{-x^2/2}$, where H_k is the kth *Hermite polynomial*. This follows from induction using the recurrence

$$H_k(x) = (2k)^{-1/2} \cdot (2xH_{k-1}(x) - H'_{k-1}(x))$$

to define the polynomials.

See [24] for the proof of the following statement(s).

Lemma 2.1. *In the above notation,*

(a) *The vectors $\{\psi_k\}$ form an orthonormal basis for $L^2(\mathbb{R})$, and each ψ_k is in the Schwartz class $\mathcal{S}(\mathbb{R})$.*

(b) *The harmonic oscillator H has a canonical extension to a self-adjoint unbounded operator on $L^2(\mathbb{R})$. Moreover, H is invertible, and H^{-1} is compact, equivalently, $f(H)$ is a compact operator for all $f \in C_0(\mathbb{R})$.*

(c) *With respect to the basis $\{\psi_k\}$ of* (a)*, H is diagonal with eigenvalues the odd positive integers*

$$H = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 3 & 0 & \cdots & 0 \\ 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2k-1 \end{bmatrix}.$$

(d) *Given $f \in L^2(\mathbb{R})$, let $(\widehat{f}(n))$ denote the l^2 sequence of its Fourier coefficients with respect to the spectral decomposition of $L^2(\mathbb{R})$ into the eigenspaces* $\ker(H - (2n+1))$, $n = 0, 1, \dots$ *Then $f \in \mathcal{S}(\mathbb{R})$ if and only if $(\widehat{f}(n))$ is a rapidly decreasing sequence of integers.*

Let D be the unbounded operator

$$D = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$$

on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, defined initially on Schwartz functions; it admits a canonical extension to a densely defined self-adjoint operator on $L^2(\mathbb{R})$. Since $D^2 = \left[\begin{smallmatrix} H-1 & 0 \\ 0 & H+1 \end{smallmatrix}\right]$, $1 + D^2 = \left[\begin{smallmatrix} H & 0 \\ 0 & H+2 \end{smallmatrix}\right]$, which is now diagonal with respect to the basis described above, and invertible as an unbounded operator.

If $f \in C_b^\infty(\mathbb{R})$ is a smooth bounded function with bounded first derivative, acting by a multiplication operator on $L^2(\mathbb{R})$, then the commutator $[f, D] = \left[\begin{smallmatrix} 0 & -f' \\ f' & 0 \end{smallmatrix}\right]$ is a bounded operator. Let

$$C_u(\mathbb{R}) := \{f \in C_b(\mathbb{R}) \mid f \text{ is uniformly continuous}\}$$

be the (non-separable) C^*-algebra of bounded uniformly continuous functions on $\mathbb{R}$. The group $\mathbb{R}_d$ of real numbers *with the discrete topology*, acts by translation on $\mathbb{R}$ and then by automorphisms of $C_u(\mathbb{R})$. Let $\pi\colon C_u(\mathbb{R}) \rtimes \mathbb{R}_d \to \mathbb{B}(L^2(\mathbb{R}))$ be the representation determined by letting $C_u(\mathbb{R})$ act by multiplication operators and $\mathbb{R}_d$ by translation unitaries. We will refer to the *-subalgebra $C_u(\mathbb{R})[\mathbb{R}_d]$, or $C_u^\infty(\mathbb{R})[\mathbb{R}_d]$, meaning the corresponding (twisted) group algebra of *finite* sums $\sum_{t \in \mathbb{R}} f_t u_t$ in $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$.

Proposition 2.2. *The triple*

$$\left(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}),\ \pi \oplus \pi,\ D = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}\right)$$

is a spectral triple over $C_u^\infty(\mathbb{R})[\mathbb{R}^d] \subset C_u(\mathbb{R}) \rtimes \mathbb{R}_d$; it is 2-dimensional in the sense that $|D|^{-2} \in \mathcal{L}^{(1,\infty)}$.

We refer to the cycle of Proposition 2.2 as the *Heisenberg cycle*.

As $C_u^\infty(\mathbb{R})[\mathbb{R}_d]$ is dense in $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$, the proposition implies that the associated Fredholm module

$$\left(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \pi \oplus \pi, F := \chi(D) = \begin{bmatrix} 0 & A^*(H+2)^{-1/2} \\ AH^{-1/2} & 0 \end{bmatrix}\right)$$

obtained by applying a normalizing function χ, here chosen to be $\chi(x) = x(1 + x^2)^{-1/2}$, defines a cycle for $\mathrm{KK}_0(C_u(\mathbb{R}) \rtimes \mathbb{R}_d, \mathbb{C})$.

The corresponding class in $\mathrm{KK}_0(C_u(\mathbb{R}) \rtimes \mathbb{R}_d, \mathbb{C})$ is non-zero: the unital inclusion $\mathbb{C} \to C_u(\mathbb{R}) \rtimes \mathbb{R}_d$ pulls it back to the class $1 \in \mathrm{KK}_0(\mathbb{C}, \mathbb{C})$ because D has index $+1$.

Since the spectrum of H grows linearly, $\pi(a)H^{-s}$ is trace-class for $\mathrm{Re}(s) > 1$ and the zeta function $\mathrm{Trace}(\pi(a)H^{-s})$ is analytic for $\mathrm{Re}(s) > 1$ and $a \in C_u(\mathbb{R}) \rtimes \mathbb{R}_d$. One of our main interests is in the possible meromorphic continuation properties of such zeta functions.

The irrational rotation algebra $A_\hbar$ is a subalgebra of $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$ and the restriction of the representation π above to $A_\hbar$ lets $f \in C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z})$ act by multiplication on $L^2(\mathbb{R})$ by the corresponding periodic function, and the group $\mathbb{Z}$ by $n \mapsto u_{n\hbar}$, with, recall, $u_t\xi(x) = \xi(x-t)$. However, $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$ contains numerous other subalgebras of related interest. We first point out a generic example of such a subalgebra, arising from dynamics.

Lemma 2.3. *If M is a compact manifold, $\{\alpha_t\}_{t\in\mathbb{R}}$ is a smooth flow on M, and if $p \in M$, then mapping $f \in C(M)$ to the bounded, uniformly continuous function $f_p(t) := f(\alpha_t p)$ on $\mathbb{R}$, and mapping $t \in \mathbb{R}_d$ to u_t, determine a C^*-algebra homomorphism*

$$\mu\colon C(M) \rtimes_\alpha \mathbb{R}_d \to C_u(\mathbb{R}) \rtimes \mathbb{R}_d,$$

*where $\mathbb{R}_d$ is the group of real numbers with the discrete topology. It is injective if the flow is minimal, and restricts to a *-algebra homomorphism $C^\infty(M)[\mathbb{R}_d] \to C_u^\infty(\mathbb{R})[\mathbb{R}_d]$.*

The proof is the observation that if the vector field X generates the flow, then $f \in C^\infty(M)$ implies $X(f) \in C^\infty(M)$, and hence that f_p' is bounded, so f_p is Lipschitz and hence uniformly continuous (and bounded) on $\mathbb{R}$.

In particular, the Heisenberg cycle pulls back to a cycle for $C(M) \rtimes \mathbb{R}_d$, and a spectral triple over the subalgebra $C^\infty(M)[\mathbb{R}_d]$.

Returning to irrational rotation, fix $\hbar \in \mathbb{R}$, so that we have the representation $\pi_\hbar$ (see (2.2)) of $A_\hbar$ determined by $C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z})$ acting by multiplication operators by $\mathbb{Z}$-periodic functions, and the subgroup $\hbar\mathbb{Z} \subset \mathbb{R}$.

Lemma 2.4. *Let $\pi^\hbar\colon A_{1/\hbar} \to \mathbb{B}(L^2(\mathbb{R}))$ be the representation obtained by letting $f \in C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z})$ act by multiplication by $x \mapsto f(\frac{x}{\hbar})$ and $n \in \mathbb{Z}$ by translation by n. Then $\pi_\hbar(A_\hbar)$ and $\pi^\hbar(A_{1/\hbar})$ commute. Hence the tensor product*

$$\rho_\hbar(a \otimes b) := \pi_\hbar(a)\pi^\hbar(b)$$

defines a representation of $B_\hbar$ on $L^2(\mathbb{R})$, which is injective if $\hbar \in \mathbb{R} \setminus \mathbb{Q}$.

We may consider the C^*-algebra $A_\hbar \otimes A_{1/\hbar} =: B_\hbar$ as the crossed product of $C(\mathbb{T}^2)$ by the group $\mathbb{Z}^2$ with action

$$(n,m)\cdot(x,y) = \Big(x + n\hbar,\, y + \frac{m}{\hbar}\Big),$$

and the homomorphism $\rho_\hbar$ embeds $B_\hbar$ into $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$ by letting

$$f \in C(\mathbb{T}^2) = C(\mathbb{R}^2/\mathbb{Z}^2)$$

map to $f_p(t) := f(t, \frac{t}{\hbar})$, and embedding $\mathbb{Z}^2$ isomorphically to the dense subgroup

$$\Lambda := \{n\hbar + m \mid n, m \in \mathbb{Z}\} \subset \mathbb{R}.$$

We can consider the $\mathbb{Z}^2$-action on $\mathbb{T}^2$ as factoring through the translation action of the subgroup $\mathbb{Z}^2 \cong \Lambda \subset \mathbb{R}$ acting through the Kronecker flow

$$\alpha_t(x, y) = \Big(x + t, y + \frac{t}{\hbar}\Big)$$

along lines of slope $\frac{1}{\hbar}$, because

$$\alpha_{n\hbar+m}(x, y) = \Big(x + n\hbar, y + \frac{m}{\hbar}\Big).$$

The restriction of $\rho_\hbar$ to $C(\mathbb{T}^2)$ is thus a special case of the construction of Lemma 2.3, as the following lemma shows.

Proposition 2.5. *Let $\gamma\colon \mathbb{R} \to \mathbb{T}^2$ be the group homomorphism $\gamma(t) = (\hbar t, t)$, let U be its image. Then*

(a) *U is dense if $\hbar \notin \mathbb{Q}$.*

(b) *An element $(x, y) \in \mathbb{R}^2$ projects to an element of U if and only if $\hbar y = x + n + m\hbar$ for some integers n, m if and only if $\hbar y = x \bmod \Lambda \subset \mathbb{R}$.*

(c) *The subgroup Λ is contained in U for all integers n, m.*

(d) *If $U' := \gamma'(\mathbb{R})$ with $\gamma'(t) = (t, \hbar t)$, then $\Lambda = U \cap U'$.*

Proof. (a)–(c) are routine. The (dense) subgroup $\Lambda \subset \mathbb{T}^2$ is obtained as follows. The line of slope $\frac{1}{\hbar}$ in $\mathbb{R}^2$ through the origin intersects the vertical lines $x = n$, for $n \in \mathbb{Z}$, in the points $\mathbb{Z}^2$-congruent to $(n\hbar, 0)$, and through the horizontal lines $y = m$, in the points congruent to $(0, \frac{m}{\hbar})$. Summing all of these points in $\mathbb{T}^2$ gives Λ, which is contained in U by (c). The flip $\mathbb{R}^2 \to \mathbb{R}^2$ interchanges lines of slope $\hbar$ and of $\frac{1}{\hbar}$ and leaves $\mathbb{Z}^2$ invariant interchanging vertical and horizontal lines. The last statement follows from symmetry. ∎

The subgroup Λ consists therefore of all points of $\mathbb{T}^2$ in the intersection of the two dense subgroups U and U', projections of lines of slope $\hbar$ and $\frac{1}{\hbar}$. We call Λ the *homoclinic subgroup*.

Definition 2.6. The *Heisenberg bi-cycle* is the spectral triple over $C^\infty(\mathbb{T}^2)[\Lambda] \subset A_\hbar \otimes A_{1/\hbar}$ given by the triple

$$\left(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}),\ \rho_\hbar \oplus \rho_\hbar,\ D = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}\right),$$

where $\rho_\hbar$ is as in Lemma 2.4.

The class of the Heisenberg cycle is denoted $\Delta_\hbar \in \mathrm{KK}_0(A_\hbar \otimes A_{1/\hbar}, \mathbb{C})$.

In the next section, we will show that the zeta functions $\mathrm{Trace}(\rho_\hbar(a)H^{-s})$ extend meromorphically to $\mathbb{C}$ for $a \in C^\infty(\mathbb{T}^2)[\Lambda]$, provided that $\hbar$ satisfies a Diophantine condition.

The inclusion $A_\hbar \to A_\hbar \otimes A_{1/\hbar}$ pulls the Heisenberg bi-cycle back to a spectral cycle for $\mathrm{KK}_0(A_\hbar, \mathbb{C})$, which is a spectral triple over the smooth subalgebra $A_\hbar^\infty$. As it is of special interest to us, we single it out in a definition.

Definition 2.7. The *Heisenberg cycle* is the even, 2-dimensional spectral cycle

$$\left(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}),\ \pi_\hbar \oplus \pi_\hbar,\ D = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}\right),$$

for $\mathrm{KK}_0(A_\hbar, \mathbb{C})$, defining a spectral triple over the Schwartz subalgebra $A_\hbar^\infty$ of the rotation algebra $A_\hbar := C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}$.

The class in $\mathrm{KK}_0(C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}, \mathbb{C})$ of the Heisenberg cycle is denoted by $[D_\hbar]$.

Remark 2.8. The injection $A_{1/\hbar}$ into $A_\hbar \otimes A_{1/\hbar}$ pulls the Heisenberg bi-cycle back to a cycle and class $[D^\hbar]$ for $\mathrm{KK}_0(A_{1/\hbar}, \mathbb{C})$. But the unitary $U\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $U\xi(x) = \sqrt{\hbar}\xi(\hbar x)$ conjugates the representation $\pi_{1/\hbar}$ to the representation $\pi^\hbar$ (in notation of Lemma 2.4). This effects the operator by a homotopically trivial re-scaling, and hence $[D^\hbar] = [D_{1/\hbar}] \in \mathrm{KK}_0(A_{1/\hbar}, \mathbb{C})$.

If $0 < h < 1$, then the spectral triple describing $[D_\hbar]$ is studied in [18]. It is related in an exact way to Connes' Dolbeault class (for any $\hbar$) as we now establish, although the result is already proved in [18] (for $0 < \hbar < 1$, and in slightly different language). Let $L^2(A_\hbar)$ denote the GNS (Gelfand–Naimark–Segal) Hilbert space associated to the trace $\tau\colon A_\hbar \to \mathbb{C}$. In the notation of the discussion around (2.1), U and V are the standard generators of $A_\hbar$, with $U(x) = e^{2\pi i x}$ and $VUV^* = e^{-2\pi i\hbar}U$. On $L^2(A_\hbar)$ the derivations δ_1, δ_2 are defined $\delta_1(U) = 2\pi i U$, $\delta_1(V) = 0$, $\delta_2(U) = 0$, $\delta_2(V) = 2\pi i V$. Then the derivations assemble to give

$$\bar\partial := \begin{bmatrix} 0 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & 0 \end{bmatrix}.$$

The GNS representation $\lambda\colon A_\hbar \to \mathbb{B}(L^2(A_\hbar))$ then fits into a spectral triple over $A_\hbar^\infty$ with Hilbert space $L^2(A_\hbar) \oplus L^2(A_\hbar)$ and representation $\lambda \oplus \lambda$, because, as one verifies without difficulty, that the commutators $[\lambda(a), \bar\partial]$ are bounded for $a \in A_\hbar^\infty$.

We let $[\bar\partial] \in \mathrm{KK}_0(A_\hbar, \mathbb{C})$ be its class. Since τ is not just a state, but a trace, the right multiplication operation of $A_\hbar$ on itself determines another, commuting representation $\lambda^{\mathrm{op}}\colon A_\hbar^{\mathrm{op}} \to \mathbb{B}(L^2(A_\hbar))$. As $A_\hbar$ is isomorphic to its opposite algebra by the map sending U to U and V to V^*, we obtain a pair of commuting representations of $A_\hbar$ on $L^2(A_\hbar)$ and on $L^2(A_\hbar) \oplus L^2(A_\hbar)$. These observations determine a cycle and class

$$\Delta_{\bar\partial} \in \mathrm{KK}_0(A_\hbar \otimes A_\hbar, \mathbb{C}).$$

Connes proves (see [5]) that cup-cap product

$$\mathrm{PD}_{\overline{\partial}}\colon \mathrm{KK}_*(D_1, A_\hbar \otimes D_2) \to \mathrm{KK}_*(A_\hbar \otimes D_1, D_2) \tag{2.5}$$

(for any D_i) with $\Delta_{\overline{\partial}}$ induces an isomorphism, that is, yields a self KK-duality for $A_\hbar$. The result is refined in [8].

The most important basic example of Morita equivalence of groupoids involves two commuting proper, free locally compact group actions on a locally compact space. This determines a Morita equivalence between a suitable pair of crossed products, see [20]. The lemma below presents a special case.

Lemma 2.9. *Define on $C_c(\mathbb{R})$ the inner products*

$$_{C(\mathbb{R}/\hbar\mathbb{Z})\rtimes\mathbb{Z}}\langle \xi, \eta\rangle(x, m) = \sum_{n\in\mathbb{Z}} \xi(x - n\hbar)\cdot\overline{\eta(x - n\hbar - m)}, \quad x \in \mathbb{R}/\hbar\mathbb{Z},\ m \in \mathbb{Z},$$

and

$$\langle \xi, \eta\rangle_{C(\mathbb{R}/\mathbb{Z})\rtimes_\hbar\mathbb{Z}}(x, m) = \sum_{n\in\mathbb{Z}} \overline{\xi(x - n)}\eta(x - n - m\hbar), \quad x \in \mathbb{R}/\mathbb{Z},\ m \in \mathbb{Z}.$$

Give $C_c(\mathbb{R})$ the $C(\mathbb{R}/\hbar\mathbb{Z}) \rtimes \mathbb{Z}$- $C(\mathbb{R}/\mathbb{Z}) \rtimes_\hbar \mathbb{Z}$ bimodule structure with

$$(n\xi)(x) = \xi(x - n), \qquad (f\xi)(x) = f(x)\xi(x),$$
$$(\xi n)(x) = \xi(x + n\hbar), \quad (\xi f)(x) = f(x)\xi(x).$$

Then $C_c(\mathbb{R})$ completes to a Morita equivalence $C(\mathbb{R}/\hbar\mathbb{Z}) \rtimes \mathbb{Z}$- $C(\mathbb{R}/\mathbb{Z}) \rtimes_\hbar \mathbb{Z}$ bimodule $\mathcal{E}_\hbar$, that is, to a Morita equivalence $A_{1/\hbar}$-$A_\hbar$-bimodule.

The relation between Connes' Dolbeault class $[\overline{\partial}]$ and the Heisenberg $[D_\hbar]$ is that $D_\hbar$ is, roughly speaking, obtained by twisting $\overline{\partial}$ by $\mathcal{E}_\hbar$.

Lemma 2.10. *The tensor product of Hilbert modules $\mathcal{E}_\hbar \otimes_{A_\hbar} L^2(A_\hbar)$ over the representation $\lambda\colon A_\hbar \to \mathbb{B}(L^2(A_\hbar))$, is naturally isomorphic to $L^2(\mathbb{R})$ as a Hilbert space.*

Under this identification,

(a) *The representation λ of $A_{1/\hbar}$ on $\mathcal{E}_\hbar \otimes_{A_\hbar} L^2(A_\hbar)$ induced by its representation on $\mathcal{E}_\hbar$ corresponds to the representation $\pi^\hbar$ on $L^2(\mathbb{R})$ of Lemma 2.4.*

(b) *The representation λ^{op} of $A_\hbar$ on $L^2(A_\hbar)$ commutes with the representation λ involved in the tensor product. Hence $A_\hbar$ is also represented on $\mathcal{E}_\hbar \otimes_{A_\hbar} L^2(A_\hbar)$ by $1 \otimes \lambda^{\mathrm{op}}$. This representation identifies with $\pi_\hbar$ on $L^2(\mathbb{R})$ of Lemma 2.4.*

Proof. If $f_1, f_2 \in C_c(\mathbb{R})$, then their $A_\hbar = C(\mathbb{R}/\mathbb{Z}) \rtimes_\hbar \mathbb{Z}$-valued inner product is given in the above lemma. Let $\delta_0 \in L^2(A_\hbar)$ the vector corresponding to $1 \in A_\hbar$ and consider the elements $f_i \otimes \delta_0 \in \mathcal{E}_\hbar \otimes_{A_\hbar} L^2(A_\hbar)$. Their inner product is given by

$$\begin{aligned}\langle f_1 \otimes \delta_0, f_2 \otimes \delta_0\rangle &= \langle \delta_0, \langle f_1, f_2\rangle_{A_\hbar}\delta_0\rangle = \tau(\langle f_1, f_2\rangle) \\ &= \int_0^1 \langle f_1, f_2\rangle_{A_\hbar}(x, 0)dx = \langle f_1, f_2\rangle_{L^2(\mathbb{R})},\end{aligned}$$

where $\tau \colon A_\hbar \to \mathbb{C}$ is the trace. It follows that $f \mapsto f \otimes \delta_0$ induces a Hilbert space isometry $L^2(\mathbb{R}) \to \mathcal{E}_\hbar \otimes_{A_\hbar} L^2(A_\hbar)$. Since elements of the form $\xi \otimes \delta_0$, $\xi \in \mathcal{E}_\hbar$, are dense in the tensor product (because the GNS representation is cyclic), this isometry is actually a unitary. The other statements are easy to check. ∎

Corollary 2.11. *Let $[\mathcal{E}_\hbar] \in \mathrm{KK}_0(A_{1/\hbar}, A_\hbar)$ be the class of the Morita equivalence bimodule $\mathcal{E}_\hbar$, $\Delta_\hbar$ be the class of the Heisenberg bi-cycle (Definition 2.6) and $\mathrm{PD}_{\bar\partial}$ be Connes' Poincaré duality (2.5). Then*

(a) $\mathrm{PD}_{\bar\partial}([\mathcal{E}_\hbar]) = [\Delta_\hbar] \in \mathrm{KK}_0(A_\hbar \otimes A_{1/\hbar}, \mathbb{C})$.

(b) *The class $\Delta_\hbar \in \mathrm{KK}_0(A_\hbar \otimes A_{1/\hbar}, \mathbb{C})$ determines a KK-duality between $A_\hbar$ and $A_{1/\hbar}$.*

(c) *If $[p_\hbar] \in \mathrm{K}_0(A_\hbar)$ denotes the class of the Rieffel projection, then $\mathrm{PD}_{\bar\partial}([p_\hbar]) = [D_h]$.*

Proof. We have $\mathrm{PD}_{\bar\partial}([\mathcal{E}_\hbar]) = (1_{A_\hbar} \otimes [\mathcal{E}_\hbar]) \otimes_{A_\hbar \otimes A_\hbar} \Delta_{\bar\partial} \in \mathrm{KK}_0(A_\hbar \otimes A_{1/\hbar}, \mathbb{C})$ by definition. The module composition involved in the Kasparov product results in (two copies of) $L^2(\mathbb{R})$ with (two copies of) the Heisenberg representation $\rho_\hbar$ of Theorem 2.4, by Lemma 2.10. The operator D satisfies the connection condition for the axiomatic approach to the product by [18].

Let $\mathrm{PD}_\hbar$ denote the analogue of (2.5) using $\Delta_\hbar$ in place of $\Delta_{\bar\partial}$. Then for $x \in \mathrm{K}_*(A_{1/\hbar})$, $y \in \mathrm{K}_*(A_\hbar)$,

$$\begin{aligned}\langle \mathrm{PD}_\hbar(x), y\rangle &= \langle y \otimes_{\mathbb{C}} x, \Delta_\hbar\rangle = \langle y \otimes_{\mathbb{C}} x, (1_{A_\hbar} \otimes [\mathcal{E}_\hbar]) \otimes_{A_\hbar \otimes A_\hbar} \Delta_{\bar\partial}\rangle \\ &= \langle y \otimes_{\mathbb{C}} \mathcal{E}_\hbar^*(x), \Delta_{\bar\partial}\rangle.\end{aligned}$$

Since $[\mathcal{E}_\hbar]$ is an equivalence in KK, the intersection form for $\Delta_\hbar$ is obtained by twisting the form for $\Delta_{\bar\partial}$ by an isomorphism, and hence is non-degenerate, since Connes' is.

By definition,

$$\begin{aligned}\mathrm{PD}_{\bar\partial}([p_\hbar]) &= ([p_\hbar] \otimes 1_{A_\hbar}) \otimes_{A_\hbar \otimes A_\hbar} \Delta_{\bar\partial} = (u \otimes 1_{A_\hbar})^*(\mathcal{E}_\hbar \otimes 1_{A_\hbar}) \otimes_{A_\hbar \otimes A_\hbar} \Delta_{\bar\partial} \\ &= (u \otimes 1_{A_\hbar})^*(\Delta_\hbar) = [D_\hbar],\end{aligned}$$

where $u \colon \mathbb{C} \to A_{1/\hbar}$ is the unital inclusion, where the non-trivial step was the penultimate one, which used (c). ∎

We are going to show using cyclic cohomology calculations that $\langle [p_\hbar], [D_\hbar]\rangle = -\lfloor \hbar \rfloor$.

This is enough to describe the intersection form induced by $\Delta_\hbar$. We note the result for the record here.

Proposition 2.12. *For any $\hbar$, give $\mathrm{K}_0(A_\hbar)$ the ordered free abelian group basis $\{[1], [p_\hbar]\}$. Then the matrix of the intersection form induced by $\Delta_\hbar$ is*

$$\begin{bmatrix} 1 & -\lfloor \frac{1}{\hbar} \rfloor \\ -\lfloor \hbar \rfloor & 1 \end{bmatrix}.$$

Proof. By the definitions, $\langle \mathrm{PD}_\hbar([p_\hbar]), [p_{1/\hbar}]\rangle = \langle [p_\hbar] \otimes_{\mathbb{C}} [p_{1/\hbar}], \Delta_\hbar\rangle$. As noted above,

$$\Delta_\hbar = \mathrm{PD}_{\bar\partial}([\mathcal{E}_\hbar]) := (1_{A_\hbar} \otimes [\mathcal{E}_\hbar]) \otimes_{A_\hbar \otimes A_\hbar} \Delta_{\bar\partial},$$

so this may be written

$$\langle [p_\hbar] \otimes_{\mathbb{C}} [p_{1/\hbar}], (1_{A_\hbar} \otimes_{\mathbb{C}} [\mathcal{E}_\hbar])^*(\Delta_{\bar\partial})\rangle.$$

Moving $[\mathcal{E}_\hbar]$ to the other side and noting that $[p_{1/\hbar}] \otimes_{A_{1/\hbar}} [\mathcal{E}_\hbar] = [1] \in \mathrm{K}_0(A_\hbar)$ gives that

$$\begin{aligned}\langle \mathrm{PD}_\hbar([p_\hbar]), [p_{1/\hbar}]\rangle &= \langle [p_\hbar] \otimes_{\mathbb{C}} [p_\hbar], \Delta_{\bar\partial}\rangle = \langle \mathrm{PD}_{\bar\partial}([p_\hbar]), [1]\rangle \\ &= \langle [D_\hbar], [1]\rangle = 1.\end{aligned}$$

∎

We end this section by noting that Heisenberg cycles, over $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$, are only *topologically* interesting if G is non-trivial. Lück and Rosenberg construct a homotopy in KK-theory by considering the operators $\lambda x + \frac{d}{dx}$ for $\lambda \in [1, \infty)$. This field can be continuously extended to $[1, \infty]$ by adding a copy of $\mathbb{C}$ to $L^2(\mathbb{R})$ at infinity, and extending the operator by the direct sum of the multiplication operator $\frac{x}{|x|}$ on $L^2(\mathbb{R})$, and 0 on the 1-dimensional summand. Their argument implies the following.

Proposition 2.13. *The class in* $\mathrm{KK}_0(C_u(\mathbb{R}), \mathbb{C})$ *of the Heisenberg cycle over* $C_u(\mathbb{R})$ *is equal to the class* $[\mathrm{ev}_0] \in \mathrm{KK}_0(C_u(\mathbb{R}), \mathbb{C})$ *of the point-evaluation homomorphism* $C_u(\mathbb{R}) \to \mathbb{C}$, $f \mapsto f(0)$.

In particular, if $[D_\alpha]$ *is the Heisenberg class for an ergodic flow on* M*, a point* $p \in M$*, and the trivial group acting, then* $[D_\alpha]) = [\mathrm{ev}_p] \in \mathrm{KK}_0(C(M), \mathbb{C})$.

This shows that it is essential to consider the crossed products $C_u(\mathbb{R}) \rtimes \Gamma$, for suitable *non*-trivial groups $G \subset \mathbb{R}$, in order to see interesting topological phenomena.

However, the *geometry* of the Heisenberg cycles is by contrast interesting, even without taking into account a group action, as we discuss in the next section.

3. Zeta functions and ergodic flows

Let $f \in C_u(\mathbb{R})$ be a bounded, uniformly continuous function.

We are going to be studying the zeta functions $\mathrm{Trace}(fH^{-s})$, where H is the harmonic oscillator (2.3) and $f \in C_u(\mathbb{R})$. This function is analytic for $\mathrm{Re}(s) > 1$. The goal of this section is to establish classes of functions for which $\mathrm{Trace}(fH^{-s})$ meromorphically extends to $\mathbb{C}$.

Theorem 3.1. *If* $f \in C_u(\mathbb{R})$*, then*

$$\Gamma(s) \cdot \mathrm{Trace}(fH^{-s}) = \frac{1}{2\sqrt{\pi}} \cdot \int_0^1 \int_{\mathbb{R}} t^{s-1} \operatorname{csch} t \cdot f(x\sqrt{\coth t}) \cdot e^{-x^2}\, dx\, dt + \epsilon(s) \quad (3.1)$$

holds for $\mathrm{Re}(s) > 1$*, where* ϵ *is entire.*

Remark 3.2. (1) If $f = 1$ is constant, (3.1) gives that

$$\Gamma(s) \cdot \mathrm{Trace}(H^{-s}) = \frac{1}{2}\int_0^1 t^{s-1} \operatorname{csch} t\, dt,$$

and considering the Laurent series expansion $\operatorname{csch} t \sim \frac{1}{t} + \cdots$ at $t = 0$, we see that the

$$\frac{1}{2}\int_0^1 t^{s-1} \operatorname{csch} t\, dt \sim \frac{1}{2}\Big(\frac{1}{s-1}\Big) + R(s),$$

where $R(s)$ extends analytically to $\mathrm{Re}(s) > 0$.

Hence

$$\mathrm{Res}_{s=1}\Gamma(s) \cdot \mathrm{Trace}(H^{-s}) = \Gamma(1) \cdot \mathrm{Res}_{s=1}(H^{-s}) = \frac{1}{2}.$$

(2) A change of variables in the expression in (3.1) results in

$$\Gamma(s) \cdot \mathrm{Trace}(fH^{-s}) = \frac{1}{\sqrt{2\pi}}\int_0^1 t^{s-1}\sqrt{\operatorname{csch} t}\int_{\mathbb{R}} f(x)e^{-\tanh t x^2}\,dx\,dt.$$

If $f(x) = e^{-i\alpha x}$ for $\alpha \in \mathbb{R}$, $\alpha \neq 0$, then, using the Gaussian integral formula

$$\int_{\mathbb{R}} e^{-ax^2/2+ibx}\,dx = \sqrt{\frac{2\pi}{a}} \cdot e^{-b^2/(2a)},$$

we obtain

$$\Gamma(s) \cdot \mathrm{Trace}(fH^{-s}) = \int_0^1 t^{s-1}\sqrt{\operatorname{sech} t} \cdot e^{-\frac{\alpha^2}{\tanh t}}\,dt,$$

and $e^{-\alpha^2/\tanh t} \to 0$ exponentially fast as $t \to 0$. This implies that $\Gamma(s) \cdot \mathrm{Trace}(fH^{-s})$ extends to an entire function if $f(x) = e^{i\alpha x}$ with $\alpha \neq 0$. This argument is used in [25] to establish the meromorphic extension property for smooth periodic functions.

We defer the proof of Theorem 3.1 and first use it. If $f \in C_u(\mathbb{R})$, suppose that $F(T) = \int_0^T f(t)dt$. Then F is uniformly continuous, not necessarily bounded, but $|F(T)| = \mathrm{O}(T)$ as $T \to \infty$.

Lemma 3.3. *If $f \in C_u(\mathbb{R})$ admits n successive bounded anti-derivatives, ${}^1\!f, {}^2\!f, \ldots, {}^n\!f$, then* $\mathrm{Trace}(fH^{-s})$ *extends analytically to* $\mathrm{Re}(s) > 1 - \frac{n}{2}$.

Proof. By Theorem 3.1,

$$\Gamma(s) \cdot \mathrm{Trace}(fH^{-s}) \sim \frac{1}{2\sqrt{\pi}} \cdot \int_0^1\int_{\mathbb{R}} t^{s-1} \operatorname{csch} t \cdot \int_{\mathbb{R}} f(x\sqrt{\coth t}) \cdot e^{-x^2}\,dx\,dt, \quad (3.2)$$

where $\sim$ means up to an entire function. Let $F = {}^1\!f$, then integration by parts gives

$$\begin{aligned}\text{r.h.s. of (3.2)} &= \frac{1}{\sqrt{\pi}} \cdot \int_0^1\int_{\mathbb{R}} t^{s-1} \operatorname{csch} t\sqrt{\tanh t} \cdot F(x\sqrt{\coth t})xe^{-x^2}\,dx\,dt \\ &= \frac{1}{\sqrt{\pi}}\int_0^1 t^{s-1} \operatorname{csch} t\sqrt{\tanh t} \cdot \phi(t)\,dt\end{aligned}$$

with $\phi(t) = \int_{\mathbb{R}} F(x\sqrt{\coth t})xe^{-x^2}dx$. The function $t^{s-1}\operatorname{csch} t\sqrt{\tanh t}\cdot\phi(t)$ is equivalent to $t^{s-3/2}\cdot\phi(t)$ as $t\to 0$, and is integrable over $[0,1]$ for $\mathrm{Re}(s) > \frac{1}{2}$ if ϕ is continuous and bounded as $t \to 0$. In particular, this holds if F is bounded on $\mathbb{R}$. So we have verified analyticity for $\mathrm{Re}(s) > \frac{1}{2}$. Similarly, if ${}^2f = {}^1F$ is the second anti-derivative, then the previous expression can be written

$$\frac{1}{\sqrt{\pi}}\cdot\int_0^1 t^{s-1}\operatorname{csch} t\tanh t\cdot\int_{\mathbb{R}} {}^2f(x\sqrt{\coth t})(1-2x)e^{-x^2}dxdt$$

which is analytic now for $\mathrm{Re}(s) > 0$ if ${}^2f$ is also bounded. One repeats this argument n times and the statement follows. ■

The *cohomological equation* in dynamics refers to the differential equation

$$Xu = f,$$

where X is a generating vector field for a smooth flow α on a compact manifold M. Let $p\in M$, $f\in C(M)$ and

$$f_p(t) = f(\alpha_t(p)).$$

Then $f_p\in C_u(\mathbb{R})$. If $f\in C^\infty(M)$, then $f_p\in C_u^\infty(\mathbb{R})$.

An obstruction to solving the cohomological equation for given f is the mean of f with respect to any α-invariant probability measure μ. This follows from differentiating the equation $\int_M u\circ\alpha_t\, d\mu = \int_M u d\mu$, which gives that $\int_M Xu\, d\mu = 0$, that is, $\int_M f d\mu = 0$ if $Xu = f$ has a smooth solution.

Conversely, if one can solve $Xu = f$ for given f, then $u_p(t) := u(\alpha_t(p))$ supplies a bounded anti-derivative of f_p. Hence Lemma 3.3 and Remark 3.2 give the following.

Proposition 3.4. *Let α be a smooth flow on M with generator X and μ any α-invariant measure. If $f\in C^\infty(M)$ and $\int_M f d\mu = 0$ implies that $Xu = f$ for some $u\in C^\infty(M)$, then for any $p\in M$,* Trace(f_pH^{-s}) *extends meromorphically to $\mathbb{C}$ and*

$$\mathrm{Res}_{s=1}\,\mathrm{Trace}(f_pH^{-s}) = \frac{1}{2}\int_M f d\mu$$

for any $p\in M$.

In some simple situations of *elliptic dynamics*, e.g., the periodic flow on the circle, having mean zero is the only obstruction to solving the cohomological equation for f: indeed, in this case one can make an extremely strong statement not even requiring smoothness.

Lemma 3.5. *Let f be continuous and ρ-periodic on $\mathbb{R}$ with zero mean: $\int_0^\rho f(t)dt = 0$. Then $F(T)$ is also continuous, ρ-periodic, with zero mean.*

Corollary 3.6. *If f is continuous and ρ-periodic on $\mathbb{R}$, then* Trace(fH^{-s}) *meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$ and*

$$\mathrm{Res}_{s=1}\,\mathrm{Trace}(fH^{-s}) = \frac{1}{2\rho}\int_0^\rho f(t)dt.$$

Proof. The function $\overline{f} := f - \mu(f)$ has zero mean. Applying the previous lemma gives that f has bounded anti-derivatives of all orders; the result follows from Lemma 3.3. and Remark 3.2. ∎

Definition 3.7. Let α be a smooth, ergodic Riemannian flow on a compact Riemannian manifold M ($\alpha_t\colon M \to M$ is a Riemannian isometry for all t). Let Δ be the Laplacian on M, $0 = \lambda_0 < \lambda_1 < \cdots$ its eigenvalues, μ normalized volume measure on M and $L^2(M) = \bigoplus_{n=0}^{\infty} H_n$ the Δ-spectral decomposition of $L^2(M)$, $H_n = \ker(\lambda_n - \Delta)$.

The vector field X commutes with Δ as an operator on $C^\infty(M)$ and so leaves each Hilbert subspace H_n invariant. For $n > 0$, let $\epsilon_n := \|X|_{H_n}\|$ (the operator norm). Since α is ergodic, the kernel of X consists of constant functions, so equals the zero eigenspace H_0 of Δ.

A Riemannian flow α satisfies a *Diophantine condition* if there exist $C \geq 0$ and $\gamma > 0$ such that $\epsilon_n \geq C n^{-\gamma}$.

Corollary 3.8. *If α is a smooth, Riemannian, ergodic flow on M satisfying a Diophantine condition, and $f \in C^\infty(M)$, then* Trace$(f_p H^{-s})$ *extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$, and*

$$\operatorname{Res}_{s=1} \operatorname{Trace}(f_p H^{-s}) = \frac{1}{2} \int_M f d\mu$$

for any α-invariant measure μ and any $p \in M$.

Proof. Proceeding as in the discussion above, let f be smooth on M, then $f \in L^2(M)$ and $f = \sum_{n=0}^{\infty} f_n s_n$, where s_n are λ_n- eigenvectors for Δ. The linear operators $e_n = X|_{H_n}$ have no kernel for $n > 0$ because the flow is ergodic, and $e_0 = 0$. Note that $f_0 = \int_M f d\mu$. Assuming that this is zero, we can set

$$u := \sum_{n=1}^{\infty} e_n^{-1} f_n s_n.$$

If f is smooth, the sequence $\{\|f_n\|\}$ has rapid decay. The Diophantine assumption implies that $\{e_n^{-1} f_n\}$ also has rapid decay, and hence defines a smooth function on M.

This shows that the only obstruction to solving the cohomological equation $Xu = f$ for f smooth, is $\int_M f d\mu = 0$. The result follows from Lemma 3.4. ∎

The hypothesis holds if $\hbar \in \mathbb{R} \setminus \mathbb{Q}$ is an irrational number satisfying a Diophantine condition, and $f \in C^\infty(\mathbb{T}^2)$, $f_p(t) = f(\alpha_t p)$ with $\alpha_t(x, y) = (x + t, y + \hbar t)$ Kronecker flow. Then $X = \frac{\partial}{\partial x} + \hbar \frac{\partial}{\partial y}$ acts on the eigenfunctions $z^n z^m$ for Δ on $\mathbb{T}^2$ by the constant $n + \hbar m$. The usual Diophantine condition on an irrational number gives γ such that

$$|n + \hbar m| \geq C(n^2 + m^2)^{-\gamma/2},$$

and this implies the flow is Diophantine in the sense above.

Corollary 3.9. *Let $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$, where $\omega_1, \ldots, \omega_n$ are rationally independent. Assume the following Diophantine condition: there exist $C > 0$ and $\gamma > 0$ such that $|\sum_{i=1}^n k_i \omega_i| \geq C|k|^{-\gamma}$, with $|k|$ the word length of $k = (k_1, \ldots, k_n)$ in $\mathbb{Z}^n$. Then if $\alpha_t(x) = x + t\omega$ is the corresponding linear flow on $\mathbb{T}^n$, $f \in C^\infty(\mathbb{T}^n)$, $p \in \mathbb{T}^n$, and $f_p(t) := f(x + t\omega)$, then $\operatorname{Trace}(f_p H^{-s})$ extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ and*

$$\operatorname{Res}_{s=1} \operatorname{Trace}(f H^{-s}) = \frac{1}{2} \int_{\mathbb{T}^n} f \, d\mu,$$

μ (normalized) Lebesgue measure on $\mathbb{T}^n$.

We now proceed to the proof of Theorem 3.1. A computation of the heat kernel of e^{-tH} follows from solving a differential equation: the heat kernel k_t satisfies $(\frac{\partial}{\partial t} + H) \cdot \phi_t = 0$, where $\phi_t(x) = \int_{\mathbb{R}} k_t(x, y)\phi(y)dy$, for $\phi \in \mathcal{S}(\mathbb{R})$, and $t \geq 0$, together with the initial condition $\lim_{t\to 0} \phi_t = \phi$. Consider the *ansatz*

$$k_t(x, y) = \exp\Big(\frac{a_t}{2}x^2 + b_t xy + \frac{a_t}{2}y^2 + c_t\Big).$$

Setting this equal to 0 and solving for coefficients gives the ordinary differential equations

$$\frac{\dot{a}_t}{2} = a_t^2 - 1 = b_t^2, \quad \dot{c}_t^2 = a_t.$$

Solving these gives

$$a_t = -\coth(2t + C), \quad b_t = \operatorname{csch}(2t + C), \quad c_t = -\frac{1}{2} \log \sinh(2t + C) + D.$$

Using the initial conditions, we get $C = 0$ and $D = \log(2\pi)^{-1/2}$. See [2].

We obtain the following, called *Mehler's formula* [19].

Lemma 3.10. *We have*

$$k_t(x, y) = \frac{1}{\sqrt{2\pi \sinh 2t}} \exp\Big(-\tanh t \cdot \frac{(x + y)^2}{4} - \coth t \cdot \frac{(x - y)^2}{4}\Big). \tag{3.3}$$

Proof of Theorem 3.1. The operator H^{-s} is trace-class for $\operatorname{Re}(s) > 1$, and the operator-valued integral $\int_0^\infty t^{s-1} e^{-tH} dt$ converges in norm to $\Gamma(s) \cdot H^{-s}$. Hence if $a \in \mathbb{B}(L^2\mathbb{R})$,

$$\Gamma(s) \cdot a H^{-s} = \int_0^\infty t^{s-1} a e^{-tH} dt,$$

and taking traces gives

$$\Gamma(s) \cdot \operatorname{Trace}(a H^{-s}) = \int_0^\infty t^{s-1} \operatorname{Trace}(a e^{-tH}) dt.$$

Furthermore, if a is any bounded operator, then

$$\int_1^\infty t^{s-1} \operatorname{Trace}(a e^{-tH}) dt$$

clearly extends to an analytic function on $\mathbb{C}$. Hence

$$\Gamma(s)\cdot \operatorname{Trace}(aH^{-s}) - \int_0^1 t^{s-1}\operatorname{Trace}(ae^{-tH})dt$$

extends analytically to C.

Now let $f \in C_u(\mathbb{R})$, set $a = f$. Then fe^{-tH} is an integral operator with kernel $f(x)k_t(x,y)$, and hence $\operatorname{Trace}(fe^{-tH}) = \int_{\mathbb{R}} f(x)k_t(x,x)dx$. Applying Mehler's formula, Lemma 3.10 gives

$$\Gamma(s)\cdot \operatorname{Trace}(aH^{-s}) = \int_0^\infty t^{s-1}\frac{1}{\sqrt{2\pi\sinh 2t}}\int_{\mathbb{R}} f(x)e^{-x^2\tanh t}dxdt.$$

Making the change of variables $x \mapsto \frac{x}{\sqrt{\tanh(t)}}$ gives

$$\Gamma(s)\cdot \operatorname{Trace}(aH^{-s}) = \int_0^\infty t^{s-1}\frac{\sqrt{\coth t}}{\sqrt{2\pi\sinh 2t}}\int_{\mathbb{R}} f(x\sqrt{\coth t})e^{-x^2}dxdt.$$

The result follows from the identity $\frac{\coth t}{\sinh 2t} = \operatorname{csch}^2 t$. ∎

4. The residue trace

If $\operatorname{Trace}(fH^{-s})$ meromorphically extends past $\operatorname{Re}(s) = 1$, then (up to the factor of $\frac{1}{2}$) the residue of the pole at $s = 1$ defines kind of asymptotic mean of $f \in C_u(\mathbb{R})$. In certain examples of flows where $f = g_p$ for $p \in M$, and $g_p(t) := g(\alpha_t(p))$, we have noted (Proposition 3.4) that this spectrally defined mean agrees with the geometric mean $\int_M f d\mu$ over the manifold.

The spectrally defined mean, which we will denote by Res Trace(f), does not necessarily require meromorphic continuation to define it, but only existence of the limit $\lim_{s\to 1^+}(s-1)\operatorname{Trace}(fH^{-s})$, which is a weaker condition.

Definition 4.1. Let $\mathcal{D} \subset C_u(\mathbb{R})$ be the closed linear subspace of all f such that

$$\operatorname{Res}\operatorname{Trace}(f) := 2\lim_{s\to 1^+}(s-1)\cdot \operatorname{Trace}(fH^{-s})$$

exists.

The residue trace Res Trace defines a positive linear functional of norm 1 on $\mathcal{D}$. This follows from the following geometric description of Res Trace.

Theorem 4.2. *If $f \in C_u(\mathbb{R})$, then $f \in \mathcal{D}$ if and only if*

$$\mu_u(f) := \lim_{\lambda\to\infty}\frac{1}{2\sqrt{\pi}}\int_0^1\int_{\mathbb{R}} f(xt^{-\lambda})e^{-x^2}dxdt \tag{4.1}$$

exists, and if this holds, then Res Trace$(f) = \mu_u(f)$.

Proof. Choose $\epsilon > 0$. Since $\Gamma(1) = 1$, by Theorem 3.1 we have for $\mathrm{Re}(s) > 1$,

$$\lim_{s\to 1+} (s-1)\,\mathrm{Trace}(fH^{-s}) = \lim_{s\to 1+} \frac{s-1}{2\sqrt{\pi}} \int_0^1 \int_{\mathbb{R}} t^{s-1} \operatorname{csch} t\, f(x\sqrt{\coth t}) e^{-x^2}\, dx\, dt. \tag{4.2}$$

The part of the integral corresponding to $t \geq \delta$ extends analytically to $\mathbb{C}$. Hence it contributes zero to the limit, and we may choose $\delta > 0$ small enough that $|t \operatorname{csch} t - 1| < \epsilon$ for $0 < t < \delta$, so that $|\operatorname{csch} t - \frac{1}{t}| < \frac{\epsilon}{t}$ for $t < \delta$. Let $\phi_f(t) = \int_{\mathbb{R}} f(x\sqrt{\coth t}) \cdot e^{-x^2} dx$, then

$$\left| \int_0^\delta t^{s-1} \Big(\operatorname{csch} t - \frac{1}{t} \Big) \phi(t) dt \right| < \frac{\epsilon}{s-1} \cdot \|f\|$$

by a brief computation. Letting $\epsilon \to 0$, we see that the limit on the right-hand side of (4.2), if it exists, equals the limit

$$\lim_{s\to 1+} \frac{s-1}{2\sqrt{\pi}} \cdot \int_0^1 \int_{\mathbb{R}} t^{s-2} f(x\sqrt{\coth t}) \cdot e^{-x^2} dx\, dt.$$

Let $\lambda = \frac{1}{s-1}$ and substitute $t \to t^\lambda$ in the above expression, and, noting $\delta^{1/\lambda} \to 1$ as $\lambda \to \infty$, we deduce that

$$\mu_u(f) = \lim_{s\to 1+} (s-1)\,\mathrm{Trace}(fH^{-s}) = \lim_{\lambda\to\infty} \frac{1}{2\sqrt{\pi}} \cdot \int_0^1 \int_{\mathbb{R}} f(x\sqrt{\coth t^\lambda}) \cdot e^{-x^2} dx\, dt.$$

Since Lipschitz functions are dense in $C_u^\infty(\mathbb{R})$ and $\mathcal{D}$ is closed, we may assume f is Lipschitz, and it follows that

$$\left| \int_0^1 \int_{\mathbb{R}} (f(x\sqrt{\coth t^\alpha}) - f(xt^{-\alpha/2})) e^{-x^2} dx\, dt \right| \leq \mathrm{const} \cdot \lim_{\alpha\to\infty} \int_0^1 |\sqrt{\coth t^\alpha} - t^{-\alpha/2}| dt$$

which converges to zero as $\lambda \to \infty$. This proves the result. ∎

The theorem can be expressed this way.

Theorem 4.3. *Let $\mu_0 = \frac{1}{2\sqrt{\pi}} e^{-x^2} dx$, the Gaussian probability measure on $\mathbb{R}$. For $t \in \mathbb{R}_+^*$, let $\rho_t : \mathbb{R} \to \mathbb{R}$, $\rho_t(x) = tx$, and $\mu_t := (\rho_t)_* \mu_0$. Then*

$$\lim_{\lambda\to\infty} \int_0^1 \mu_{t^\lambda} dt = \mathrm{Res\ Trace} \in \mathcal{D}'.$$

We deduce the following.

Corollary 4.4. *If $f \in C_u(\mathbb{R})$ and $\mu_\pm(f) := \lim_{T\to\pm\infty} \frac{1}{T}\int_0^T f(t)dt$ each exists, then $f \in \mathfrak{D}$ and*

$$\text{Res Trace}(f) = \frac{\mu_-(f) + \mu_+(f)}{2}. \tag{4.3}$$

In particular, if α is an ergodic flow on a compact smooth manifold M, μ an α-invariant probability measure, $f \in C(M)$, $f_p(t) := f(\alpha_t(p))$, then $f_p \in \mathfrak{D}$ and

$$\text{Res Trace}(f_p) = \int_M f d\mu$$

for a.e. $p \in M$.

Proof. Integration by parts, the change of variables $u \to ut^\lambda$, and a slight re-arrangement give

$$\begin{aligned}\int_0^1 \int_{\mathbb{R}} f(xt^{-\lambda})e^{-x^2}dxdt &= 2\int_0^1 \int_{\mathbb{R}} \int_0^x f(ut^{-\lambda})xe^{-x^2}dudxdt \\ &= 2\int_0^1 \int_{\mathbb{R}} t^\lambda \int_0^{xt^{-\lambda}} f(u)xe^{-x^2}dudxdt \\ &= 2\int_0^1 \int_{\mathbb{R}} \frac{1}{xt^{-\lambda}} \int_0^{xt^{-\lambda}} f(u)x^2e^{-x^2}dudxdt.\end{aligned}$$

Now letting $\lambda \to \infty$ and using the hypothesis that $L := \lim_{T\to\pm\infty} \frac{1}{T}\int_0^T f(t)dt$ exists, we obtain

$$\lim_{\lambda\to\infty} \int_0^1 \int_{\mathbb{R}} f(xt^{-\lambda})e^{-x^2}dxdt = L\sqrt{\pi}.$$

By Theorem 4.2,

$$\text{Res Trace}(f) = \lim_{\lambda\to\infty} \frac{1}{2\sqrt{\pi}} \int_0^1 \int_{\mathbb{R}} f(xt^{-\lambda})e^{-x^2}dxdt$$

giving (4.3).

The second statement follows from combining the first with the Birkhoff ergodic theorem. ∎

Remark 4.5. By a slightly more elaborate argument, the assumption of Corollary 4.4 can be weakened as follows: there exists $0 \le \beta < 1$ such that

$$L := \lim_{x\to\pm\infty} \frac{1}{\int_0^x u^{-\beta}} \int_0^x f(u)u^{-\beta}du$$

exists. Then $f \in \mathfrak{D}$ and $\text{Res Trace}(f) = L$ remains true.

We next produce some estimates related to group translation operators on $L^2(\mathbb{R})$.

Lemma 4.6. *Let U_α be the unitary induced by translation on the real line by $\alpha \neq 0$. Then if $f \in C_u(\mathbb{R})$ and $a = fU_\alpha \in C_u(\mathbb{R}) \rtimes \mathbb{R}_d \subset \mathbb{B}(L^2(\mathbb{R}))$, then*

$$\Gamma(s) \cdot \operatorname{Trace}(fU_\alpha H^{-s}) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty t^{s-1} \operatorname{csch} t \exp\Big(-\frac{\alpha^2}{4} \coth t\Big) \cdot \mu_{\alpha,t}(f) dt, \quad (4.4)$$

where $\mu_{\alpha,t}(f) = \int_{\mathbb{R}} f(x\sqrt{\coth t} + \alpha) e^{-x^2} dx$.

Proof. The argument proceeds as in the proof of Lemma 3.10. The operator $fU_\alpha e^{-tH}$ is a compact integral operator with kernel

$$k_t'(x, y) = f(x) k_t(x - \alpha, y),$$

where k_t is the harmonic oscillator heat kernel (3.3). Hence for $\operatorname{Re}(s) > 1$,

$$\begin{aligned} &\Gamma(s) \cdot \operatorname{Trace}(fU_\alpha H^{-s}) \\ &\quad = \int_0^\infty \int_{\mathbb{R}} t^{s-1} k_t(x - \alpha, x) dx dt \\ &\quad = \int_0^\infty \int_{\mathbb{R}} t^{s-1} (2\pi \sinh 2t)^{-1/2} f(x) \exp\Big(-\frac{(2x - \alpha)^2}{4} \tanh t - \frac{\alpha^2}{4} \coth t\Big) dx dt. \end{aligned}$$

Basic manipulations yield (4.4). ■

Lemma 4.7. *If $f \in C_u(\mathbb{R})$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, then the function*

$$\phi_{f,\alpha}(s) := \Gamma(s) \cdot \operatorname{Trace}(fU_\alpha H^{-s}), \quad \operatorname{Re}(s) > 1,$$

extends to an analytic function on $\mathbb{C}$. There are constants C_s' and C_s'' depending holomorphically on s such that

$$|\phi_{f,\alpha}(s)| \leq (C_s' \alpha^{-2\operatorname{Re}(s)} + C_s'') e^{-\alpha^2/4} \cdot \|f\| \quad \textit{for all } s \in \mathbb{C}.$$

Proof. As shown above, for the family of states $\mu_{\alpha,t}$ on $C_u(\mathbb{R})$,

$$\begin{aligned} 2\sqrt{2\pi}\,\Gamma(s) \cdot \operatorname{Trace}(fU_\alpha H^{-s}) &= \int_0^\infty t^{s-1} \operatorname{csch} t \exp\Big(-\frac{\alpha^2}{4} \coth t\Big) \mu_{\alpha,t}(f) dt \\ &= \int_0^1 t^{s-1} \operatorname{csch} t \exp\Big(-\frac{\alpha^2}{4} \coth t\Big) \mu_{\alpha,t}(f) dt \\ &\quad + \int_1^\infty t^{s-1} \operatorname{csch} t \exp\Big(-\frac{\alpha^2}{4} \coth t\Big) \mu_{\alpha,t}(f) dt \\ &= \zeta_1(s) + \zeta_2(s). \end{aligned}$$

Consider first $\zeta_1(s)$. Since $\frac{\tanh t}{t}$ and $\frac{\sinh t}{t}$ are bounded on $[0, 1]$, we can bound the integrand of $\zeta_1(s)$ by

$$t^{s-2} e^{-\beta/t} \|f\|, \quad \beta = \frac{\alpha^2}{4}.$$

A change of variables gives

$$\int_0^1 t^{s-2} e^{-\beta/t}\, dt = \int_1^\infty t^{-s} e^{-\beta t}\, dt.$$

If $A_s = \int_0^\infty t^{-s} e^{-\lambda t}\, dt$, then $A_s = \frac{e^{-\beta}}{\beta} + \frac{1}{\beta} A_{s+1}$, by integration by parts, and it follows that $|\int_0^1 t^{s-2} e^{-\beta t}\, dt| \leq \text{const} \cdot \beta^{-\operatorname{Re}(s)} e^{-\beta}$, where the constant does not depend on β or s, and hence that

$$|\zeta_1(s)| \leq C_s' \cdot \|f\| \cdot \alpha^{-2\operatorname{Re}(s)} e^{-\alpha^2/4}.$$

We can bound $\zeta_2(s)$ as follows:

$$\begin{aligned} \left| \int_1^\infty t^{s-1} \operatorname{csch} t\, e^{-(\alpha^2/4)\coth t} \mu_{\alpha,t}(f)\, dt \right| &\leq \|f\| e^{-\alpha^2/4} \cdot \int_1^\infty t^{s-1} \operatorname{csch} t\, dt \\ &= C_s'' e^{-\alpha^2/4} \|f\|. \end{aligned} \tag{4.5}$$

This completes the proof. ∎

The significance of Lemma 4.7 is that it sheds light on when $\operatorname{Trace}(fH^{-s})$ meromorphically extends, when $a = \sum_{\alpha \in \Gamma} f_\alpha U_\alpha$ is an element of $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$, and $\Gamma \subset \mathbb{R}_d$ is a finitely generated subgroup.

The easiest case is that of a cyclic subgroup, and this produces a very strong result.

If $f \in C_u(\mathbb{R}) \rtimes \Gamma$ for a subgroup $\Gamma \subset \mathbb{R}$, let f_0 be the coefficient of f at the identity $0 \in \Gamma$. Let $\mathcal{D}^\infty$ denote the subspace of $\mathcal{D} \subset C_u(\mathbb{R})$ of f such that $\operatorname{Trace}(fH^{-s})$ meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$.

Theorem 4.8. *Let $f \in C_u(\mathbb{R}) \rtimes_\hbar \mathbb{Z}$, where $\hbar \in \mathbb{R}$ is non-zero. Then given $f_0 \in \mathcal{D}^\infty$, we have that* $\operatorname{Trace}(fH^{-s})$, $\operatorname{Re}(s) > 1$, *meromorphically extends to $\mathbb{C}$, with a simple pole at $s = 1$, and*

$$\operatorname{Res}_{s=1} \operatorname{Trace}(fH^{-s}) = \mu_u(f_0),$$

where μ_u is the uniform mean, see (4.1).

Proof. Suppose first that f has expansion $f = \sum f_n U_{n\hbar}$ with $f_0 = 0$. Then

$$\Gamma(s) \cdot \operatorname{Trace}(fH^{-s}) = \sum_n \operatorname{Trace}(f_n U_{n\hbar} H^{-s}) = \sum_n \phi_n(s),$$

where $\phi_n(s)$ abbreviates $\phi_{f_n, n\hbar}(s)$ of Lemma 4.7. The series converges absolutely and uniformly on compact subsets of $\mathbb{C}$ because of the bound

$$|\phi_n(s)| \leq (C_s' \hbar^{-2\operatorname{Re}(s)} n^{-2\operatorname{Re}(s)} + C_s'') e^{-(\hbar^2/4)n^2},$$

due to the lemma, shows that $\phi_n \to 0$ exponentially fast as $n \to \pm\infty$.

In the general case, $f = f - f_0$, $\operatorname{Trace}((f - f_0)H^{-s})$ extends to an entire function for arbitrary $f \in C_u(\mathbb{R})$, and $\operatorname{Trace}(f_0 H^{-s})$ to a meromorphic function with the stated pole structure if $f_0 \in \mathcal{D}^\infty$ by definition. ∎

Corollary 4.9. *If $a \in A_\hbar := C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}$, then* $\operatorname{Trace}(aH^{-s})$ *meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$, and* $\operatorname{Res}_{s=1} \operatorname{Trace}(aH^{-s}) = \tau(a)$*, where τ is the standard trace on $A_\hbar$.*

In particular, the Heisenberg cycle Definition 2.7 over the irrational rotation algebra $A_\hbar$ defines a spectral triple over $A_\hbar^\infty$ with the meromorphic continuation property over the whole C^*-algebra $A_\hbar$.

Definition 4.10. A finitely generated subgroup $\Gamma \subset \mathbb{R}$ with word length function $|\cdot|_\Gamma$ satisfies a *Diophantine property* if

$$|\alpha| \geq C|\alpha|_\Gamma^{-\gamma}$$

for some $\gamma > 0$ and $C > 0$.

Definition 4.11. Let $B_\Gamma := C_u(\mathbb{R}) \rtimes \Gamma$, where $\Gamma \subset \mathbb{R}$ is a finitely generated subgroup. Then B_Γ^∞ denotes the completion of the (twisted) group algebra $C_u^\infty(\mathbb{R})[\Gamma]$ with respect to the family of semi-norms $p_{s,m}(f) = \sum_{\alpha\in\Gamma} \|f_\gamma^{(m)}\| \cdot |\gamma|_\Gamma^s$.

Remark 4.12. Note that B_Γ^∞ consists of operators in $C_u(\mathbb{R}) \rtimes \Gamma$ whose expansions $f = \sum_{\alpha\in\Gamma} f_\alpha U_\alpha$ have rapid decay in the sense that

$$\sum_{\alpha\in\Gamma} \|f_\alpha^{(m)}\| \cdot |\alpha|_\Gamma^n < \infty \quad \forall m, n \geq 0. \tag{4.6}$$

It is not difficult to prove that B_Γ^∞ is closed under holomorphic functional calculus.

Theorem 4.13. *Suppose $\Gamma \subset \mathbb{R}$ has a Diophantine property, let $f \in B_\Gamma^\infty$, and assume $f_0 \in \mathcal{D}$ (Definition 4.1). Then* $\operatorname{Trace}(fH^{-s})$ *meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$ and* $\operatorname{Res}_{s=1} \operatorname{Trace}(fH^{-s}) = \mu_u(f_0)$.

Proof. Write $f = \sum_{\alpha\in\Gamma} f_\alpha U_\alpha \in B_\Gamma^\infty \subset \mathbb{B}(L^2(\mathbb{R}))$ and assume $f_0 = 0$. It suffices to prove that $\operatorname{Trace}(fH^{-s})$ extends to an analytic function on $\mathbb{C}$. This equals

$$\sum_{\alpha\in\Gamma} \operatorname{Trace}(f_\alpha U_\alpha H^{-s}) = \sum_{\alpha\in\Gamma} \phi_{f_\alpha,\alpha}(s),$$

where $\phi_{f,\alpha}$ is notation as in Lemma 4.7, and it suffices to show that this is an absolutely summable sequence of analytic functions, uniformly on compact subsets of $\mathbb{C}$. Shorten notation $\phi_\alpha := \operatorname{Trace}(f_\alpha U_\alpha H^{-s})$. By the same lemma,

$$|\phi_\alpha(s)| \leq (C_s'\alpha^{-2\operatorname{Re}(s)} + C_s'')e^{-|\alpha|^2/4} \cdot \|f_\alpha\|$$

for all $s \in \mathbb{C}$. Since there are potentially infinitely many α with small absolute value, the exponential term is no longer of any use, and we discard it, obtaining a polynomial bound for $\phi_\alpha(s)$ of order $|\alpha|^\mu$ for $\mu = -2\operatorname{Re}(s) \in \mathbb{R}$. Since Γ is finitely generated, there exists a constant C_Γ such that $|\alpha| \leq C_\Gamma \cdot |\alpha|_\Gamma$ for all $\alpha \in \Gamma$. Combining with the Diophantine assumption gives that

$$C|\alpha|_\Gamma^{-\gamma} \leq |\alpha| \leq C'|\alpha|_\Gamma.$$

If $\mu \geq 0$, we get

$$C''|\alpha|_\Gamma^{-\mu\gamma} \leq |\alpha|^\mu \leq C'''|\alpha|_\Gamma^\mu.$$

Hence

$$\sum_{\alpha\in\Gamma} \| f_\alpha \| \cdot |\alpha|^\mu \leq C \sum_{\alpha\in\Gamma} \| f_\alpha \| \cdot |\alpha|_\Gamma^\mu,$$

and the last term is finite by (4.6).

If $\mu < 0$, then we use the bound

$$|\alpha|^\mu \leq \text{const} \cdot |\alpha|_\Gamma^{-\mu\gamma}.$$

Again, $\sum_{\alpha\in\Gamma} \| f_\alpha \| \cdot |\alpha|_\Gamma^{-\mu\gamma}$ is finite by assumption on f (4.6). ■

Remark 4.14. We make two comments about the proof.

(a) The Diophantine condition on Γ only seems relevant for the zone $0 < \mathrm{Re}(s) < 1$.

(b) A weaker condition on f than (4.6) still seems to ensures the result. It suffices to assume that

$$\sum_{\alpha\in\Gamma} \| f_\alpha \| \cdot |\alpha|_\Gamma^s \cdot e^{-|\alpha|^2/4} < \infty$$

for all real $s > 0$.

Let α be a smooth flow on M compact. Let $p \in M$, and let $\pi_p\colon C(M) \rtimes \mathbb{R}_d \to C_u(\mathbb{R}) \rtimes \mathbb{R}_d \subset \mathbb{B}(L^2(\mathbb{R}))$ be the *-homomorphism induced by restriction of functions to the orbit of p. Pulling back the Heisenberg cycle for $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$, we obtain a cycle for $C(M) \rtimes \mathbb{R}_d$, and for $C(M) \rtimes \Gamma$ for any $\Gamma \subset \mathbb{R}$ a subgroup.

From the results above, if both the flow and the group satisfy Diophantine conditions (Definitions 3.7 and 4.10, respectively), then the pulled-back cycle determines a spectral triple over a suitable smooth subalgebra of $C(M) \rtimes \Gamma$.

Actually, we are mainly interested in the situation of Kronecker flow $\alpha^\hbar$ on $\mathbb{T}^2$, where both Diophantine properties are implied by a Diophantine condition on the irrational number $\hbar$.

Theorem 4.15. *Let $B_\hbar := A_\hbar \otimes A_{1/\hbar} \cong C(\mathbb{T}^2) \rtimes \Gamma$, with $\Gamma \subset \mathbb{R}$ the group generated by $1, \hbar$. Let $B_\hbar^\infty$ be the Schwartz subalgebra of $B_\hbar$ of all $\sum_{\alpha\in\Gamma} f_\alpha U_\alpha$ with*

$$\sum_{\alpha\in\Gamma} \| X^n f \| \cdot |\alpha|_\Gamma^m < \infty,$$

where X generates the flow.

Then the Heisenberg bi-cycle of Definition 2.6 determines a spectral triple over $B_\hbar^\infty$ with the meromorphic extension property.

This follows from Corollary 3.9, Theorem 4.13 (and see the discussion prior to Proposition 2.5).

5. Topology of Heisenberg cycles

We now use cyclic cohomology to perform some K-theory and index-pairing calculations with the Heisenberg cycles over $A_\hbar$ and $B_\hbar = A_\hbar \otimes A_{1/\hbar}$.

We will focus on $A_\hbar$. Let $[D_\hbar] \in \mathrm{KK}_0(A_\hbar, \mathbb{C})$ be the Heisenberg class of Definition 2.7, involving

(a) the unitary action of $\mathbb{Z}$ on $L^2(\mathbb{R})$ with n acting by $U_\hbar^n$;

(b) the action of $C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z}) = C(\mathbb{R})^{\mathbb{Z}}$ of $\mathbb{Z}$-periodic functions by multiplication operators;

(c) the operator

$$D = \begin{bmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{bmatrix}.$$

From (a) and (b), we get the representation $\pi_\hbar \colon A_\hbar \to \mathbb{B}(L^2(\mathbb{R}))$.

It is important to note that $A_\hbar$ only depends, of course, on the class of $\hbar$ mod $\mathbb{Z}$, but $\pi_\hbar$ depends on $\hbar$, not just its equivalence class. (This is a difference between our set-up and that of [18], as they only allow $0 < \hbar < 1$.)

In particular, fixing $0 < \hbar < 1$, then the $[D_{\hbar+b}]$, for $b \in \mathbb{Z}$, define a $\mathbb{Z}$-family of spectral cycles over the single $A_\hbar$.

In fact, due to the identity $[\mathcal{E}_\hbar] \otimes_{A_\hbar} [\overline{\partial}] = [D_\hbar]$, this is accounted for by a similar fact about the Morita equivalences $\mathcal{E}_\hbar$ of Theorem 2.9. Namely, that for $\hbar$ varying within a $\mathbb{Z}$-coset of $\mathbb{R}$, we obtain a $\mathbb{Z}$-parameterized family of Morita $A_{1/\hbar}$-$A_\hbar$-bimodules: note that $A_{1/\hbar}$ changes as an integer is added to $\hbar$, and so does the bimodule, but the ring of scalars $A_\hbar$ in the right multiplication remains the same. Twisting $\overline{\partial}$ by these bimodules and forgetting the left action gives a $\mathbb{Z}$-parameterized family of spectral triples that are exactly the $[D_{\hbar+b}]$.

Actually, there is a certain internal symmetry of $A_\hbar$, which we call the 'Heisenberg twist' (which appeared already in [8]) whose iterates act transitively on these sets of data in both the K-theory and K-homology picture. The Heisenberg twist is a KK-morphism determined by the *bundle* of cycles $D_\hbar$, over $\mathbb{R}$, as we explain below.

We are going to use the local index theorem of [6] in dimension 2 to compute the pairing of the class $[D_\hbar]$ with $\mathrm{K}_0(A_\hbar)$; the result, and its proof, effectively computes the index theory of the class $\Delta_\hbar$ as well. We will use the development of the local index formula by Higson in [14] and our results on the harmonic oscillator residue trace of the previous section.

Remark 5.1. The index formula of Corollary 5.9 we arrive at is very similar to one due to Connes (see [3, 5]), who discovered it using his computation of the cyclic theory of $A_\hbar^\infty$. There are some minor differences, as Connes fixes $0 < \hbar < 1$ and computes the index map $\mathrm{K}_0(A_{1/\hbar}) \to \mathbb{Z}$ induced by $D_\hbar$, and we are allowing arbitrary $\hbar$ and computing the index pairing with $\mathrm{K}_0(A_\hbar)$.

In the paper [3], Connes described an invariant of a finitely generated projective module over $A_\hbar$, generalizing the first Chern number of a complex vector bundle over $\mathbb{T}^2$. In a sense, this was the starting point of noncommutative geometry.

Connes' construction was the following. Let A be any C^*-algebra endowed with an action of $\mathbb{R}^2$ by automorphisms with (s,t) acting by $\alpha_s \circ \beta_t$.

Let $\delta_i\colon A \to A$ be the densely defined derivations

$$\delta_1(a) := \lim_{t\to 0} \frac{\alpha_t(a) - a}{t}, \quad \delta_2(a) := \lim_{t\to 0} \frac{\beta_t(a) - a}{t}, \quad a \in A^\infty,$$

where $A^\infty = \bigcap_{n,m} \operatorname{dom}(\delta^n) \cap \operatorname{dom}(\delta^m)$, the *-subalgebra of elements such that $(s,t) \mapsto \alpha_s(\beta_t(a))$ is smooth.

In addition, let $\tau\colon A \to \mathbb{C}$ be an $\mathbb{R}^2$-invariant tracial state. Then Connes' invariant of a f.g.p. (finitely generated projective) module eA^∞, where e is a projection in A^∞, is given by

$$c_1(e) := \frac{1}{2\pi i} \tau(e[\delta_1(e), \delta_2(e)]).$$

We call $c_1(e)$ the *first Chern number of* e. The number $c_1(e)$ only depends on the equivalence class of e in $\mathrm{K}_0(A^\infty)$ (see [4]).

Moreover, $c_1(\mathcal{E} \oplus \mathcal{E}') = c_1(\mathcal{E}) + c_1(\mathcal{E}')$ and c_1 thus determines a group homomorphism $\mathrm{K}_0(A) \to \mathbb{R}$.

Let $\hbar \in \mathbb{R}$ and $A_\hbar = C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}$ be the corresponding rotation algebra, with $u \in A_\hbar$ the generator of the $\mathbb{Z}$-action. Then the $\mathbb{R}^2$-action with $\alpha_t(f) = f(x-t)$, $\alpha_t(u) = u$ and $\beta_t(u^n) = e^{2\pi i n t} u^n$, $\beta_t(f) = f$, gives rise to the derivations

$$\delta_1\Big(\sum_n f_n[n]\Big) = \sum_n f_n'[n], \quad \delta_2\Big(\sum_n f_n[n]\Big) = \sum_n 2\pi i n \cdot f_n[n].$$

Recall the Rieffel modules described in Lemma 2.9. For any $\hbar$, $\mathcal{E}_\hbar$ is the completion of $C_c(\mathbb{R})$ with respect to the $A_\hbar$-valued inner product

$$\langle \xi, \eta \rangle_{A_\hbar}(x,m) = \sum_{n\in\mathbb{Z}} \overline{\xi(x-n)}\eta(x-n-m\hbar), \quad x \in \mathbb{R}/\mathbb{Z},\ m \in \mathbb{Z}.$$

Give $C_c(\mathbb{R})$ the right $C(\mathbb{R}/\mathbb{Z}) \rtimes_\hbar \mathbb{Z} = A_\hbar$ module structure with

$$(\xi n)(x) = \xi(x + n\hbar), \quad (\xi f)(x) = f(x)\xi(x).$$

This extends to a multiplication $\mathcal{E}_\hbar \times A_\hbar \to \mathcal{E}_\hbar$ giving a finitely generated and projective Hilbert $A_\hbar$-module. To find a projection $p_\hbar$ such that $p_\hbar A_\hbar \cong \mathcal{E}_\hbar$, recall that the $A_{1/\hbar}$-valued inner product is given by

$${}_{C(\mathbb{R}/\hbar\mathbb{Z})\rtimes\mathbb{Z}}\langle \xi, \eta \rangle(x,m) = \sum_{n\in\mathbb{Z}} \xi(x-n\hbar) \cdot \overline{\eta(x-n\hbar-m)}, \quad x \in \mathbb{R}/\hbar\mathbb{Z},\ m \in \mathbb{Z}.$$

If we can find $\xi \in \mathcal{E}_\hbar$ such that ${}_{C(\mathbb{R}/\hbar\mathbb{Z})\rtimes\mathbb{Z}}\langle \xi, \xi\rangle = 1$, then the map $\eta \mapsto \langle \eta, \xi\rangle_{A_\hbar}$ embeds $\mathcal{E}_\hbar$ in the trivial rank-one $A_\hbar$-module, and the required projection is $\langle \xi, \xi\rangle_{A_\hbar}$. We thus seek ξ such that

$$\sum_{n\in\mathbb{Z}} \xi(x-n\hbar)\cdot \overline{\xi(x-n\hbar-m)} = \delta_{0m}$$

in Kronecker notation. In particular, ξ should have support contained within an interval, e.g., [0, 1], of length 1, for the outcome to be zero if $m \neq 0$. But this means that if $\hbar > 1$, the function $\sum_{n\in\mathbb{Z}} \xi(x-n\hbar)^2$ will have zeros. A necessary condition therefore to find such $p_\hbar$ is that $0 < \hbar < 1$. For larger $\hbar$, one must find a finite set $\xi_1, \dots, \xi_m$ of vectors in $\mathcal{E}_\hbar$, and use them to embed $\mathcal{E}_\hbar$ in $A_\hbar^m$. One obtains projections $p_\hbar$ in matrix algebras over $A_\hbar$. This makes the computation of curvature more complicated.

If $0 < \hbar < 1$, then the problem can be solved: the ensuing projection is of the form

$$p_\hbar = f + gu + g^{-\hbar}u^*,$$

where f and g are suitably chosen functions. For $a \in (0,1)$ and $\epsilon > 0$ small, f equals zero on $[0, a]$ and on $[a+\hbar+\epsilon, 2\pi]$, and $f = 1$ on $[a+\epsilon, a+\hbar]$. We choose f so that $f(x) + f(x+\hbar) = 1$. We set $g = \sqrt{f - f^2}$ on $[a+\hbar, a+\hbar+\epsilon]$ and zero otherwise.

The following calculation from [4] is reproduced below for the benefit of the reader.

Lemma 5.2. *Let $p_\hbar \in A_\hbar = C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}$ be the Rieffel projection. Then $c_1(p_\hbar) = +1$.*

Proof. For brevity, for $f \in C(\mathbb{T})$ understood as a $\mathbb{Z}$-periodic function on $\mathbb{R}$, let $f^\hbar(x) := f(x-\hbar)$ denote the action.

The Rieffel projection is given by $p_\hbar = f + gu + g^{-\hbar}u^*$ as above. We then have to compute $c_1(e) = \frac{1}{2\pi i}\tau([\delta_1(p_\hbar), \delta_2(p_\hbar)])$. We first compute

$$\frac{1}{2\pi i}[\delta_1(p_\hbar), \delta_2(p_\hbar)] = u^*(gf' - gf'^\hbar) + 2((gg')^{-\hbar} - gg') + (gf' - gf'^\hbar)u.$$

Multiplying this on the left by $p_\hbar$ produces a terrific mess, but we are only interested in its trace, so the only part that is relevant is

$$g^2(f' - f'^\hbar) + 2f((gg')^{-\hbar} - gg') + (g^2(f' - f'^\hbar))^{-\hbar}$$

which we want to integrate over $\mathbb{T}$.

Set $w = g^2$, $v = f - f^\hbar$. The integral is given by

$$\int wv' + f(w'^{-\hbar} - w') + (wv')^{-\hbar}.$$

The middle term is

$$\int fw'^{-\hbar} - \int fw' = \int f^\hbar w' - \int fw' = -\int vw'.$$

Hence we are reduced to computing

$$\int wv' - vw' + (wv')^{-h} = \int 2wv' - vw' = 3\int wv'$$

by integration by parts. Next, since $f^{\hbar} = 1 - f$ on $\mathrm{supp}(g)$, we can replace $f' - f'^{\hbar}$ by $-2f'$ and get

$$-6\int f'(f - f^2) = -6\int f'f + 6\int f'f^2 = -3\int (f^2)' + 2\int (f^3)' = 3 - 2 = 1,$$

where the integration is understood to be restricted to the support of g. This completes the calculation. ∎

The first Chern class of *any* projection in $A_{\hbar}^{\infty}$ is an integer, a fact related to the quantum Hall effect (see [5]). This is due to agreement of the first Chern number with the index pairing (an integer) of the projection and the Dirac–Dolbeault spectral cycle. This a consequence of the local index formula of Connes and Moscovici, which we are going to work out in the case of the Heisenberg cycles, but first state in low dimensions.

Theorem 5.3 (Connes–Moscovici, [6]). *Let (H, π, D) be an even, 2-dimensional spectral triple over $A^{\infty} \subset A$, for a C^*-algebra A, regular and with the meromorphic continuation property over A^{∞}. Let $[D] \in \mathrm{KK}_0(A, \mathbb{C})$ be the class of the triple. Let $\Delta := D^2$, and let ϵ be the grading operator on H.*

Define functionals

(a) $\psi_0 \colon A^{\infty} \to \mathbb{C}$,

$$\psi_0(a) := \mathrm{Res}_{s=0}\, \Gamma(s) \cdot \mathrm{Trace}(\epsilon a(\Delta + \mathrm{proj}_{\ker D})^{-s}),$$

(b) $\psi_2 \colon A^{\infty} \otimes A^{\infty} \otimes A^{\infty} \to \mathbb{C}$,

$$\psi_2(a^0, a^1, a^2) := \frac{1}{2}\mathrm{Res}_{s=1}\, \mathrm{Trace}(\epsilon a^0 [D, a^1][D, a^2]\Delta^{-s}).$$

Then if $e \in A^{\infty}$ is a projection, then

$$\langle [e], [D] \rangle = \Psi_0(e) - \Psi_2\Big(e - \frac{1}{2}, e, e\Big),$$

where $\langle [e], [D] \rangle \in \mathbb{Z}$ is the pairing between the $\mathrm{K}_0(A)$ class $[e]$ and the $\mathrm{KK}_0(A, \mathbb{C})$ class $[D]$.

We are going to apply the local index formula to some examples of Heisenberg cycles.

Lemma 5.4. *Let $a \in C_u(\mathbb{R}) \rtimes \mathbb{R}_d \subset \mathbb{B}(L^2(\mathbb{R}))$, and assume that $\mathrm{Trace}(aH^{-s})$ meromorphically extends to $\mathbb{C}$. Then the function $\Psi_0(a)_s := \Gamma(s) \cdot \mathrm{Trace}(\epsilon a(\Delta + \mathrm{proj}_{\ker D})^{-s})$ of Theorem 5.3 meromorphically extends to $\mathbb{C}$, has a simple pole at $s = 0$ and*

$$\Psi_0(a) := \mathrm{Res}_{s=0}\Psi_0(a)_s = 2\mathrm{Res}_{s=1}\, \mathrm{Trace}(aH^{-s}) =: \mathrm{Res}\, \mathrm{Trace}(a).$$

Proof. We refer to Theorem 5.3. The Hilbert space for the Heisenberg triple is the direct sum of two copies of $L^2(\mathbb{R})$, and

$$D = \begin{bmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{bmatrix}.$$

The kernel of D is the same as the kernel of $x + \frac{d}{dx}$, and is spanned by the ground state $\psi_0(x) = \pi^{-1/4} e^{-x^2/2}$, and

$$\Delta = \begin{bmatrix} H-1 & 0 \\ 0 & H+1 \end{bmatrix},$$

where H is the harmonic oscillator. Hence

$$\Delta + \mathrm{proj}_{\ker D} = \begin{bmatrix} H - 1 + \mathrm{pr}_{\ker D} & 0 \\ 0 & H+1 \end{bmatrix}.$$

The first copy of the Hilbert space $L^2(\mathbb{R})$ is even in the grading, the second is odd. Hence we need to compute the residue at $s = 0$ of the difference

$$\Gamma(s) \cdot \mathrm{Trace}(a(H - 1 + \mathrm{pr}_{\ker D})^{-s}) - \Gamma(s) \cdot \mathrm{Trace}(a \cdot (H+1)^{-s}) \tag{5.1}$$

for $a \in A_\hbar = C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}$. By the Mellin transform, (5.1) equals

$$\begin{aligned} &\int_0^\infty t^{s-1}\, \mathrm{Trace}(a(e^{-t(H-1+\mathrm{pr}_{\ker D})} - e^{-t(H+1)}))dt \\ &\quad = \int_0^\infty t^{s-1}\, \mathrm{Trace}(ae^{-tH}(e^{t(1-\mathrm{pr}_{\ker D})} - e^{-t}))dt. \end{aligned}$$

Using $e^{t(1-\mathrm{pr}_{\ker D})} - e^{-t} = 2\sinh t + (1 - e^t)\mathrm{pr}_{\ker D}$, we get

$$\begin{aligned} &\Gamma(s) \cdot \mathrm{Trace}(a(H - 1 + \mathrm{pr}_{\ker D})^{-s}) - \Gamma(s) \cdot \mathrm{Trace}(a(H+1)^{-s}). \\ &\quad = \int_0^\infty t^{s-1} 2 \sinh t\, \mathrm{Trace}(ae^{-tH})dt + \int_0^\infty t^{s-1}(1 - e^t)\, \mathrm{Trace}(ae^{-tH}\mathrm{pr}_{\ker D})dt \\ &\quad = 2\int_0^\infty t^{s-1} \sinh t\, \mathrm{Trace}(ae^{-tH})dt + \langle a\psi_0, \psi_0\rangle \int_0^\infty t^{s-1}(e^{-t} - 1). \end{aligned}$$

The second term extends analytically to $\mathrm{Re}(s) > -1$ and hence is irrelevant for the pole at $s = 0$.

As $\sinh t = t + \frac{1}{6}t^3 + \cdots$, we have

$$\begin{aligned} &\int_0^\infty t^{s-1} \sinh t\, \mathrm{Trace}(ae^{-tH})dt \\ &\quad = \int_0^\infty t^s\, \mathrm{Trace}(ae^{-tH})dt + \frac{1}{6}\int_0^\infty t^{s+3}\, \mathrm{Trace}(ae^{-tH})dt + \cdots. \end{aligned}$$

As a consequence of our earlier work, $\mathrm{Trace}(ae^{-tH}) = O(\frac{1}{t})$ as $t \to 0$. It follows that

$$2\int_0^\infty t^{s-1}\sinh t\,\mathrm{Trace}(ae^{-tH})dt - 2\int_0^\infty t^s\,\mathrm{Trace}(ae^{-tH})dt$$
$$= 2\Gamma(s+1)\,\mathrm{Trace}(aH^{-(s+1)})$$

extends analytically to $\mathrm{Re}(s) > -2$. In particular, if one knows that $\mathrm{Trace}(aH^{-s})$ meromorphically extends to $\mathbb{C}$, then so does this term, and putting everything together gives

$$\Gamma(s)\cdot\mathrm{Trace}(a(H-1+\mathrm{pr}_{\ker D})^{-s}) - \Gamma(s)\cdot\mathrm{Trace}(a\cdot(H+1)^{-s})$$
$$= 2\mathrm{Res}_{s=0}\int_0^\infty t^s\,\mathrm{Trace}(ae^{-tH})dt = 2\mathrm{Res}_{s=1}\,\mathrm{Trace}(aH^{-s}) = \mathrm{Res}\,\mathrm{Trace}(a),$$

as claimed. ∎

Lemma 5.5. *Let* $a \in C_u(\mathbb{R}) \rtimes \mathbb{R}_d \subset \mathbb{B}(L^2(\mathbb{R}))$, *and assume that* $\mathrm{Trace}(aH^{-s})$ *meromorphically extends to* $\mathbb{C}$. *Then the function*

$$\Psi_2(a^0, a^1, a^2 1)_s := \mathrm{Trace}(\epsilon a^0[D, a^1][D, a^2]\Delta^{-s})$$

meromorphically extends to $\mathbb{C}$ *and*

$$\Psi_2(a^0, a^1, a^2) := \frac{1}{2}\mathrm{Res}_{s=1}\Psi_2(a^0, a^1, a^2)_s$$
$$= \frac{1}{2\pi i}\mathrm{Res}\,\mathrm{Trace}(a^0\delta_1(a^1)\delta_2(a^2) - a^0\delta_2(a^1)\delta_1(a^2)).$$

Proof. Expanding $[D, a^1][D, a^2]$ as a block matrix

$$[D, a^1][D, a^2] = \begin{bmatrix} [x - \frac{d}{dx}, a^1][x + \frac{d}{dx}, a^2] & 0 \\ 0 & [x + \frac{d}{dx}, a^1][x - \frac{d}{dx}, a^2] \end{bmatrix}$$

gives

$$\mathrm{Res}_{s=0}\Psi_2(a^0, a^1, a^2)_s = \frac{1}{2}\mathrm{Res}\,\mathrm{Trace}(\epsilon a^0[D, a^1][D, a^2]\Delta^{-1})$$
$$= \mathrm{Res}\,\mathrm{Trace}\Big(a^0[x, a^1]\Big[\frac{d}{dx}, a^2\Big]\Big) - \mathrm{Res}\,\mathrm{Trace}\Big(a^0\Big[\frac{d}{dx}, a^1\Big][x, a^2]\Big).$$

Now $[x, a] = -\frac{1}{2\pi i}\cdot\delta_2(a)$ and $[\frac{d}{dx}, a] = \delta_1(a)$. Hence

$$\mathrm{Res}\,\mathrm{Trace}\Big(a^0[x, a^1]\Big[\frac{d}{dx}, a^2\Big]\Big) - \mathrm{Res}\,\mathrm{Trace}\Big(a^0\Big[\frac{d}{dx}, a^1\Big]\Big)$$
$$= \frac{1}{2\pi i}\mathrm{Res}\,\mathrm{Trace}(a^0\delta_1(a^1)\delta_2(a^2) - a^0\delta_2(a^1)\delta_1(a^2)).$$ ∎

Let $B = C(M) \rtimes \Lambda$ for a flow α on a smooth compact manifold M, and $\Lambda \subset \mathbb{R}$ be a Diophantine subgroup. Let μ be an α-invariant measure, let the vector field X generate the flow, and assume that $Xu = f$ is smoothly solvable for any smooth f such that $\int_M f d\mu = 0$. Fix $p \in M$ and let

$$\pi\colon C(M) \rtimes \Lambda \to \mathbb{B}(L^2(\mathbb{R}))$$

be the corresponding representation, with $\pi(f) = f_p$ as a multiplication operator, where $f_p(t) = f(\alpha_t(p))$.

We have shown that $B^\infty \subset B$ has the property that $\operatorname{Trace}(\pi(b)H^{-s})$ has the meromorphic extension property for all $b \in B^\infty$ and that

$$2\operatorname{Res}_{s=1} \operatorname{Trace}(\pi(b)H^{-s}) := \operatorname{Res}\operatorname{Trace}(b) = \tau_\mu(b),$$

where $\tau_\mu\colon B \to \mathbb{C}$ is the trace induced by μ, and $b \in B^\infty$.

Let $\delta_1^\alpha, \delta_2^\alpha$ be the derivations of B^∞ defined by

$$\delta_1^\alpha(f) = X(f), \quad \delta_1^\alpha(U_\alpha) = 0, \quad \delta_2^\alpha(f) = 0, \quad \delta_2^\alpha(U_\alpha) = \alpha.$$

We obtain the following assertion.

Corollary 5.6. *In the above notation, the functional Ψ_2 of Theorem 5.3* (b) *is given on B^∞ by*

$$\Psi_2(b^0, b^1, b^2) = \tau_\mu(a^0\delta_1^\alpha(a^1)\delta_2^\alpha(a^2) - a^0\delta_2^\alpha(a^1)\delta_1^\alpha(a^2))$$

for all $b_0, b_1, b_2 \in B^\infty$.

Corollary 5.7. *Let $B = C(M) \rtimes \Lambda$ for a smooth flow α on a compact manifold M. Let $\Lambda \subset \mathbb{R}$ a Diophantine subgroup. Let μ be an α-invariant measure, X generates the flow, and assume that $Xu = f$ is smoothly solvable for any smooth f such that $\int_M f d\mu = 0$. Then the Chern character of the Heisenberg cycle determined by a point $p \in M$ is given by $\tau_\mu - \tau_2^\alpha$, where τ_μ is the trace on B determined by μ, and τ_2^α is the cyclic 2-cocycle*

$$\tau_2^\alpha(b^0, b^1, b^2) = \tau_\mu(b^0\delta_1^\alpha(b^1)\delta_2^\alpha(b^2) - b^0\delta_2^\alpha(b^1)\delta_1^\alpha(b^2))$$

on B^∞.

Corollary 5.8. *Let $\hbar \in \mathbb{R}$ and $A_\hbar := C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}$ the corresponding rotation algebra. Let $[D_\hbar]$ be the class of the Heisenberg cycle (Definition 2.7)*

$$\left(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}),\, \pi_\hbar,\, D := \begin{bmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{bmatrix} \right).$$

Then the Chern character of $[D_\hbar]$ is given by $\tau - \hbar\tau_2$, where

$$\tau_2(a^0, a^1, a^2) = \tau(a^0\delta_1(a^1)\delta_2(a^2) - a^0\delta_2(a^1)\delta_1(a^2))$$

is the curvature cocycle of Connes.

Corollary 5.9. *Let $e \in A_\hbar^\infty$ be a projection, $[e] \in \mathrm{K}_0(A_\hbar)$ its class. Then*

$$\langle [e], [D_\hbar] \rangle = \tau(e) - \hbar c_1(e),$$

where $c_1(e)$ is the first Chern number of e. In particular, if $p_\hbar$ is the Rieffel projection for $A_\hbar$, then

$$\langle [p_\hbar], [D_h] \rangle = -\lfloor \hbar \rfloor,$$

where $\lfloor \hbar \rfloor$ is the greatest integer $< \hbar$.

The proof is immediate from the formula of Corollary 5.8, because $c_1(p_\hbar) = 1$ and $\tau(p_\hbar) = \hbar - \lfloor \hbar \rfloor$.

This yields the proof of Proposition 2.12.

We close with a remark. The integrality of Connes' first Chern number is due to the index result that

$$\langle [\overline{\partial}], [e] \rangle = c_1(e),$$

where $c_1(e)$ is the first Chern number of e. If one combines this with our index computation for $[D_\hbar]$, then we obtain the following 'gap-labelling' result. It is of course well known; but the argument which we give does not depend on computation of $\mathrm{K}_0(A_\hbar)$.

Corollary 5.10. *Suppose $\hbar \in \mathbb{R}$ is non-zero. Then if $\tau \colon A_\hbar \to \mathbb{C}$ is the trace,*

$$\tau_* \colon \mathrm{K}_0(A_\hbar) \to \mathbb{R}$$

the induced group homomorphism, then the range of $\tau_*(\mathrm{K}_0(A_\hbar))$ is the subgroup $\mathbb{Z} + \hbar\mathbb{Z} \subset \mathbb{R}$.

Proof. If $e \in A_\hbar^\infty$ is a projection, then application of our results above gives that $\tau(e) + \hbar \cdot c_1(e)$ is an integer. On the other hand, $c_1(e)$ is an integer. This implies $\tau(e) = m + n\hbar$ for a pair of integers m, n. Finally, $A_\hbar^\infty$ is dense and holomorphically closed in $A_\hbar$, so any projection in $A_\hbar$ is represented by a projection in $A_\hbar^\infty$. ∎

6. Transverse foliations

There is a well-known procedure for producing finitely generated projective (f.g.p.) modules over $A_\hbar$, using Morita equivalence. Fixing one of the standard linear loops in $\mathbb{T}^2$ determines a Morita equivalence of $A_\hbar$ with the C^*-algebra $B_\hbar = C(\mathbb{T}^2) \rtimes_\hbar \mathbb{R}$ of the Kronecker foliation $\mathcal{F}_\hbar$ into lines of slope $\hbar$. On the other hand, *any* linear loop in $\mathbb{T}^2$ is transverse to $\mathcal{F}_\hbar$. These linear loops, parameterized by pairs of relatively prime integers, determine therefore Morita equivalences between $A_\hbar$ and what turn out to be other rotation algebras, and in particular, unital algebras. Hence they determine f.g.p. modules over $A_\hbar$; these parameterize (the positive part of) the K-theory.

In [8], we showed that there is an analogue of this procedure using non-compact transversals: if $\hbar' \neq \hbar$, then the Kronecker foliations $\mathcal{F}_\hbar$ and $\mathcal{F}_{\hbar'}$ are transverse. Their product foliates $\mathbb{T}^2 \times \mathbb{T}^2$ and its restriction to the diagonal gives an equivalent étale groupoid. This reasoning produces for every $\hbar' \neq \hbar$ a f.g.p. module $\mathcal{L}_{\hbar,\hbar'}$ over $A_\hbar \otimes A_{\hbar'}$ and corresponding K-theory class $[\mathcal{L}_{\hbar,\hbar'}] \in \mathrm{KK}_0(\mathbb{C}, A_\hbar, \otimes A_{\hbar'})$. In particular, fixing $\hbar' = \hbar + b$ for any integer $b \neq 0$ gives a f.g.p. module over $A_\hbar \otimes A_\hbar$. We denote it by $\mathcal{L}_b$.

Let

$$\mathrm{PD}\colon \mathrm{KK}_0(\mathbb{C}, A_\hbar \otimes A_\hbar) \to \mathrm{KK}_0(A_\hbar, A_h)$$

be Connes' Poincaré duality map [5]. In [8], it is proved that

$$\mathrm{PD}([\mathcal{L}_b]) = \tau_b, \tag{6.1}$$

where τ_b is the Kasparov morphism defined in terms of Dirac–Schrödinger operators as follows. We take the standard right Hilbert $A_\hbar$-module $L^2(\mathbb{R}) \otimes A_\hbar$. We let $\mathbb{Z}$ act on the left by the following formula, where we designate a dense set of elements of our Hilbert module in the form $\sum_{n\in\mathbb{Z}} \xi_n \cdot [n]$, with $\xi_n \in L^2(\mathbb{R}) \otimes C(\mathbb{T})$:

$$k \cdot \Big(\sum_{n\in\mathbb{Z}} \xi_n \cdot [n]\Big) := \sum_{n\in\mathbb{Z}} k(\xi_n) \cdot [k+n],$$

where

$$k(\xi)([x], t) = \xi([x - k\hbar], t - k), \quad \xi \in L^2(\mathbb{R}) \otimes C(\mathbb{T}).$$

Let $f \in C(\mathbb{T})$ act by

$$f \cdot \Big(\sum_{n\in\mathbb{Z}} \xi_n \cdot [n]\Big) := \sum_{n\in\mathbb{Z}} f^b \cdot \xi_n \cdot [n],$$

where

$$f_b(t, [x]) = f([x + tb]).$$

These two assignments determine a covariant pair and representation

$$\pi_\hbar\colon A_\hbar \to \mathbb{B}(L^2(\mathbb{R}) \otimes A_\hbar).$$

Definition 6.1. The *Heisenberg twist* $\tau_b \in \mathrm{KK}_0(A_\hbar, A_h)$ is the class of the spectral cycle

$$(L^2(\mathbb{R}) \otimes A_\hbar \oplus L^2(\mathbb{R}) \otimes A_\hbar,\ \pi_h \oplus \pi_h,\ D \otimes 1_{A_\hbar})$$

with

$$D := \begin{bmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{bmatrix}.$$

Equality (6.1) gives rise to an explicit geometric cycle representing the unit in Connes' duality [8]. The relationship between the Heisenberg twist and the Heisenberg cycles is implied by the following lemma.

Lemma 6.2. *Let* $\hbar, \mu \in \mathbb{R}$.

(a) *The group multiplication* $m: \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ *intertwines the diagonal* $\mathbb{Z}$*-action on* $\mathbb{T} \times \mathbb{T}$ *by group addition of* $(\hbar, \mu)$ *and group addition on* $\mathbb{T}$ *of* $\hbar + \mu$*, and so determines a *-homomorphism*

$$\chi\colon A_{\hbar+\mu} \to A_{\hbar} \otimes A_{\mu}.$$

In this notation,

$$\chi^*([D_{\hbar}] \otimes_{\mathbb{C}} [D_{\mu}]) = [D_{\hbar+\mu}].$$

(b) *If* $\mu = b \in \mathbb{Z}$*, then*

$$\tau^b = \chi^*([D_b] \otimes 1_{A_{\hbar}}),$$

where τ^b *is the Heisenberg twist.*

(c) $\tau^b \otimes_{A\hbar} [D_{\hbar}] = [D_{\hbar+b}]$ *for any* $\hbar \in \mathbb{R}$, $b \in \mathbb{Z}$.

Proof. For (c), we have

$$\tau^b \otimes_{A\hbar} [D_{\hbar}] = \chi^*([D_b] \otimes 1_{A_{\hbar}}) \otimes_{A_{\hbar}} [D_{\hbar}] = \chi^*([D_b] \otimes_{\mathbb{C}} [D_h]) = [D_{b+\hbar}]$$

using first part (b) and then part (a).

Item (b) follows from an inspection at the level of cycles: they differ only in the representations, which are clearly homotopic.

We now prove (a).

Consider $\chi^*([D_{\hbar}] \otimes_{\mathbb{C}} [D_{\mu}])$, a class in $\mathrm{KK}(A_{\theta+\eta}, \mathbb{C})$. By the standard method of computing external products, it is represented by the following spectral cycle. The Hilbert space is $L^2(\mathbb{R}, \mathbb{C}^2) \otimes L^2(\mathbb{R}, \mathbb{C}^2)$ and operator $D \otimes 1 + 1 \otimes D$. With $u, v \in A_{\hbar}$ the standard unitary generators, $u = z$, $v = [1]$, the representation is given by

$$(u \cdot \phi)(x, y) = e^{2\pi i(x+y)}\phi(x, y), \quad (v \cdot \phi)(x, y) = \phi(x - \theta, y - \eta).$$

We apply a homotopy to the representation with $t \in [0, \frac{1}{2}]$,

$$(\pi_t(v) \cdot \phi)(x, y) = \phi(x - (1-t)\theta - t\eta, y - t\theta - (1-t)\eta).$$

The resulting cycle is $(L^2(\mathbb{R}, \mathbb{C}^2) \otimes L^2(\mathbb{R}, \mathbb{C}^2), \Delta, D \otimes 1 + 1 \otimes D)$, where Δ is the 'diagonal' representation of $A_{\theta+\eta}$,

$$(u\phi)(x, y) = e^{2\pi i(x+y)}\phi(x, y), \quad (v\phi)(x, y) = \phi\Big(x - \frac{\theta+\eta}{2}, y - \frac{\theta+\eta}{2}\Big).$$

Next consider the class $[D_{\hbar+\mu}]$. The unit $1_{\mathbb{C}} \in \mathrm{KK}(\mathbb{C}, \mathbb{C})$ can be represented by the cycle $(L^2(\mathbb{R}, \mathbb{C}^2), 1, D)$. Taking the intersection product of this with $[D_{\theta+\eta}]$ yields a cycle which is equivalent to $[D_{\theta+\eta}]$, but more closely resembles the cycle described in the previous paragraph: the Hilbert space is $L^2(\mathbb{R}, \mathbb{C}^2) \otimes L^2(\mathbb{R}, \mathbb{C}^2)$, the operator is $D \otimes 1 + 1 \otimes D$, and the representation is given by

$$(u \cdot \phi)(x, y) = e^{2\pi i x}\phi(x, y), \quad (v \cdot \phi)(x, y) = \phi(x - \theta - \eta, y).$$

From here, we take a homotopy by rotating this representation around $\mathbb{R}^2$ to lie along the diagonal (i.e., so that $(a \cdot \phi)(x, y)$ depends only on $x + y$), and the result follows.

By (b), the equation $\tau_b \otimes_{A_\hbar} [D_\hbar] = [D_{\hbar+b}]$ follows immediately. ■

Corollary 6.3. *The Heisenberg twist acts by the identity on $\mathrm{K}_1(A_\hbar)$. With respect to the ordered basis $\{[1], [p_\hbar]\}$ for $\mathrm{K}_0(A_\hbar)$, where $p_\hbar$ is the Rieffel projection, the morphism τ_b acts by matrix multiplication by $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$.*

Proof. The first statement follows from [8].

Consider $(\tau^b)_*([p_h]) \in \mathrm{K}_0(A_\hbar)$. Write

$$(\tau^b)_*([p_h]) = x[1] + y[p_\hbar].$$

Pairing both sides with $[D_h]$ and using Corollary 5.9 twice gives

$$x = (\tau^b)_*([p_h]) \otimes_{A_\hbar} [D_\hbar] = [p_\hbar] \otimes_{A_\hbar} (\tau^b)^*([D_h]) = [p_\hbar] \otimes_{A_\hbar} [D_{h+b}] = b.$$

Pairing the same equation with $[D_{\hbar+1}]$ and computing give that $y = 1$. ■

It follows from similar simple arguments that in the classical case, where $\hbar = b \in \mathbb{Z}$, the Heisenberg classes $[D_b] \in \mathrm{KK}_0(C(\mathbb{T}^2), \mathbb{C}) = \mathrm{K}_0(\mathbb{T}^2)$ are given by

$$[D_b] = [\mathrm{pt}] + b \cdot [\bar{\partial}] \in \mathrm{K}_0(\mathbb{T}^2).$$

For $b = 1$, we have noted that $[D_1] = [\bar{\partial} \cdot \mathcal{P}]$. This corresponds to $[\bar{\partial} \cdot \mathcal{P}] = [\mathrm{pt}] + [\bar{\partial}]$, which of course follows from the Riemann–Roch formula. We have

$$\langle [D_n], [E] \rangle = \dim E + n \cdot c_1(E)$$

for any complex vector bundle E over $\mathbb{T}^2$. Therefore, the classes $[D_n]$ taken together determine both the dimension and first Chern number, the two basic invariants of a complex vector bundle over $\mathbb{T}^2$.

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