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Uniform in time propagation of chaos for the 2D vortex model and other singular stochastic systems

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Abstract. We adapt the work of Jabin and Wang (2018) to show the first result of uniform in time propagation of chaos for a class of singular interaction kernels. In particular, our models contain the Biot–Savart kernel on the torus and thus the 2D vortex model.

Keywords. Propagation of chaos, relative entropy, logarithmic Sobolev inequality, 2D vortex equation, singular kernels

1. Introduction

1.1. Framework

Our main subject is the convergence of the law of a stochastic particle system with mean field singular interactions towards its non-linear limit. More precisely, we will establish the first quantitative bounds on the distance in the number of particles uniformly in time. Let $K : \mathbb{T}^d \to \mathbb{R}^d$ be an *interaction kernel* on the d-dimensional ($d \geq 2$) 1-periodic torus $\mathbb{T}^d$ (represented as $[-1/2, 1/2]^d$), on which we will specify some assumptions later. In this paper, we consider the non-linear stochastic differential equation of *McKean–Vlasov type*

$$\begin{cases} dX_t = \sqrt{2}\, dB_t + K * \bar{\rho}_t(X_t)dt, \\ \bar{\rho}_t = \text{density of Law}(X_t), \end{cases} \tag{1.1}$$

where $X_t \in \mathbb{T}^d$, $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion and

$$f * g(x) = \int_{\mathbb{T}^d} f(x-y)g(y)\, dy$$

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stands for the convolution operation on the torus. The density $\bar{\rho}_t$ satisfies

$$\partial_t \bar{\rho}_t = -\nabla \cdot (\bar{\rho}_t \, (K * \bar{\rho}_t)) + \Delta \bar{\rho}_t. \tag{1.2}$$

In other words, the non-linear equation (1.2) has the following natural probabilistic interpretation: the solution $\bar{\rho}_t$ is the density of the law at time t of the $\mathbb{T}^d$-valued process $(X_t)_{t\geq 0}$ evolving according to (1.1). As we understand (1.1) to be the motion of a particle interacting with its own law, (1.2) thus describes the dynamic of a cloud of charged particles (where $(X_t)_{t\geq 0}$ would be one particle). In particular, it is of importance in plasma physics [31]. We also consider the associated system of particles, describing the motion of N particles interacting with one another through the interaction kernel K:

$$dX_t^i = \sqrt{2}\, dB_t^i + \frac{1}{N} \sum_{j=1}^{N} K(X_t^i - X_t^j) dt, \tag{1.3}$$

where $X_t^i \in \mathbb{T}^d$ is the position at time t of the i-th particle, and $(B_t^i, 1 \leq i \leq N)$ are independent Brownian motions in $\mathbb{T}^d$. We assume that $(X_0^i)_{i=1,\dots,N}$ are exchangeable, i.e. have a law which is invariant under permutation of the particles, so that this property is true for all times. We denote by ρ_N the density of the law of the system of particles, formally satisfying

$$\partial_t \rho_N = -\sum_{i=1}^{N} \nabla_{x_i} \cdot \left(\left(\frac{1}{N} \sum_{j=1}^{N} K(x_i - x_j) \right) \rho_N \right) + \sum_{i=1}^{N} \Delta_{x_i} \rho_N. \tag{1.4}$$

We define ρ_N^k to be the density of the law of the first k marginals of the N-particle system,

$$\rho_N^k(t, x_1, \dots, x_k) = \int_{\mathbb{T}^{(N-k)d}} \rho_N(t, x_1, \dots, x_N)\, dx_{k+1} \dots dx_N,$$

which is also, thanks to the exchangeability of particles, the density of the law of any k marginals. More precisely, in this work, we focus on equation (1.4) and we will not address the question of well-posedness of the stochastic equation (1.3).

Here, although we will consider general assumptions on K, the main example motivating our work is the singular interaction kernel known as the *Biot–Savart* kernel, defined in $\mathbb{R}^2$ by

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} = \frac{1}{2\pi} \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right). \tag{1.5}$$

Consider the 2D incompressible Navier–Stokes system on $\mathbb{R}^2$,

$$\begin{aligned} \partial_t u &= -u \cdot \nabla u - \nabla p + \Delta u, \\ \nabla \cdot u &= 0, \end{aligned}$$

where p is the local pressure. Taking the curl of the equation above, we find that $\omega(t,x) = \nabla \times u(t,x)$ satisfies (1.2) with K given by (1.5) (see for instance [24, Chapter 1]).

One can see equation (1.3) as an approximation of equation (1.1), where the law $\bar\rho_t$ is replaced by the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$. It is well known, at least in a setting where the interaction kernel K is Lipschitz continuous [25, 30], that, under some mild conditions on K, for all fixed $k \in \mathbb{N}$ and all $t \geq 0$, $\rho_N^k(t,\cdot)$ converges toward $\bar\rho_k(t,\cdot) = \bar\rho_t^{\otimes k}$ as $N \to \infty$, where $\bar\rho_t$ is the density of the law of X_t solving (1.1). Thus, provided the particles start independent, they will stay (more or less) independent, as the law of any k-uplet of particles converges toward a tensorized law. The expression *propagation of chaos* to describe this behavior was coined by Kac [20]), and we refer to Sznitman [30] for a landmark study of the phenomenon. Of course there is a huge literature on propagation of chaos, but limited to uniform in time results, and always when the interaction potential is regular; see Malrieu [23] for an example treated by a coupling approach under convexity conditions and the recent work of Durmus et al. [12] via reflection coupling allowing non-convexity but where the interaction is considered small and acts mainly as a perturbation. For more recent results we refer to [21] (and its uniform in time extension in [22]) for a nice new approach to propagation of chaos furnishing better speed but under strong assumptions on the interactions (regularity, integrability), including a nice survey of the existing results, and [11] using Lions derivatives for uniform in time results on the torus but also under regularity assumptions on the interaction kernel.

Hence, neither these classical nor the recent results apply to the Biot–Savart kernel, which is singular at 0. For convergence without rate, and specific to the vortex 2D equation, a first striking result appeared in [15], relying on proving that close encounters of particles are rare and that the possible limits of the particle system are made up of solutions of the non-linear SDE. As a second step, in the recent work [18], Jabin and Wang have proven that propagation of chaos still holds in this case with a *quantitative* rate. The goal of the present paper is to extend their works and show a *quantitative propagation of chaos uniform in time*. We refer to [6–8, 15, 18] for detailed discussions on the literature concerning propagation of chaos with singular kernels, which is still at its beginnings as regards quantitative rates. Shortly after this work was submitted, an alternative approach to global in time estimates was developed in [28]; see also the very recent preprint [10].

Obtaining uniform in time estimates for propagation of chaos is an important challenge. One of its applications concerns the use of the particle system, which can easily be simulated numerically, to approximate the solution of a non-linear physics motivated problem, such as here the vorticity equation arising from fluid mechanics. Likewise, it provides a framework for studying noisy gradient descent used in machine learning (see the recent [9]) and thus attracts some attention.

The approach of Jabin and Wang [18] is to compute the time evolution of the relative entropy of ρ_N with respect to $\bar\rho_N$ and then to use integration by parts to deal with the singularity of K thanks to the regularity of the probability density $\bar\rho_t$. In order to improve this argument to get uniform in time propagation of chaos, our main contribution is the proof of time-uniform bounds for $\bar\rho_t$, in Lemma 2.2, from which a time-uniform logarithmic Sobolev inequality is deduced. From the latter, in the spirit of the work of Malrieu [23] in the smooth convex case, the Fisher information appearing in the entropy dissipation yields control on the relative entropy itself, inducing time uniformity. However, a major difficulty is that these quantities are expressed in terms of the solution of a non-linear equation. We then have to prove a logarithmic Sobolev inequality, uniformly in time, for $\bar{\rho}_t$, and sufficient decay of the derivatives of $\bar{\rho}_t$. This requires new estimates on regularity and a priori bounds of the solutions of a non-linear 2D vortex equation. Indeed, we prove that the bounds on the derivative of $\bar{\rho}_t$ decay sufficiently fast (see again Lemma 2.2) to ensure uniform in time convergence without smallness assumption on the interaction. Finally, the remaining error term in the entropy evolution due to the difference between (1.2) and (1.4) is tackled thanks to a law of large numbers already used in [18]. Compared to [15] we thus obtain a quantitative and uniform in time result.

The organization of the article is as follows. In the remainder of this section, we state the main theorem as well as the various assumptions on both the initial condition and the interaction kernel K. In Section 2 we gather various tools that will be useful later on: we state the regularity of the solutions, the existence of uniform in time bounds on the density and its derivatives, and we prove a logarithmic Sobolev inequality. Finally, in Section 3, we prove the uniform in time propagation of chaos following the method of [18].

1.2. Main results

First, let us describe the assumptions on the initial condition. Unless otherwise specified, L^p and H^p respectively refer to the spaces $L^p(\mathbb{T}^d)$ and $H^p(\mathbb{T}^d)$. Given $\lambda > 1$, we denote by $\mathcal{C}^\infty_\lambda(\mathcal{X})$ the set of functions f in $\mathcal{C}^\infty(\mathcal{X})$ such that $0 < 1/\lambda \leq f \leq \lambda < \infty$, and $\mathcal{C}^\infty_{>0}(\mathcal{X}) = \bigcup_{\lambda>1} \mathcal{C}^\infty_\lambda(\mathcal{X})$, which is simply the set of positive smooth functions when $\mathcal{X}$ is compact. We make the following assumptions on $\bar{\rho}_0$:

Assumption 1.1. • There is $\lambda > 1$ such that $\bar{\rho}_0 \in \mathcal{C}^\infty_\lambda(\mathbb{T}^d)$.

• For all $n \geq 1$, $C^0_n := \|\nabla^n \bar{\rho}_0\|_{L^\infty} < \infty$.

Remark 1.2. Let us discuss the smoothness assumption on the initial condition. Via Theorem 2.1 below, which follows from the result of [3], this will ensure the smoothness of $\bar{\rho}_t$. This fact (and the fact that we consider, as we will see later, a smooth solution ρ_N of (1.4)) allows us to justify all calculations in a comfortable way. This could however be improved. First, as in [18], the calculations should hold for any entropy solution of (1.4). Second, it is also shown in [3], in the case of the vorticity equation, that an initial condition in L^1 yields existence and uniqueness of a solution of (1.2) which is smooth for positive times. One could thus think of using the non-uniform in time result of [18] on a small time interval $[0, \varepsilon]$, and then complete the proof on $[\varepsilon, \infty[$ with our result. We would then require some bounds on $\bar{\rho}_\varepsilon$ and its derivatives of a sufficient order (depending on the Sobolev embedding – see the proof of Lemma 2.2 below) that we could propagate in time.

For the sake of clarity and conciseness, however, we choose not to go in this direction.

Let us describe the assumptions on the interaction kernel K. Below, $\nabla\cdot$ stands for the divergence operator. We make the following assumptions on K:

Assumption 1.3. • $\|K\|_{L^1} < \infty$.

• In the sense of distributions, $\nabla \cdot K = 0$.

- There is a matrix field $V \in L^\infty$ such that $K = \nabla \cdot V$, i.e. $K_\alpha = \sum_{\beta=1}^d \partial_\beta V_{\alpha,\beta}$ for $1 \le \alpha \le d$.

The problem of finding a matrix field $V \in L^\infty(\mathbb{T}^d)$ such that $K = \nabla \cdot V$ for a given K is a complex mathematical question. We refer to [5, 27] and the references therein for a more detailed discussion of the literature. As noted in [18, Proposition 2], such a matrix V exists for any kernel $K \in L^d$ (by the results of [5]), and for any kernel K such that

$$\exists M > 0,\ \forall x \in \mathbb{T}^d, \quad |K(x)| \le M/|x|$$

(in view of the results of [27]).

Remark 1.4. If a function a satisfies $\nabla \cdot a = 0$, then for $\psi : \mathbb{T}^d \to \mathbb{R}$ we have

$$\nabla \cdot (a\psi) = (a \cdot \nabla)\psi.$$

Suppose $\tilde{K}$ is an interaction kernel in $\mathbb{R}^d$ (such as the Biot–Savart kernel). One can periodize $\tilde{K}$ on the torus as follows. For a function f on the torus (identified with a 1-periodic function on $\mathbb{R}^d$), writing $f *_{\mathcal{X}} g(x) = \int_{\mathcal{X}} f(x-y)g(y)\,dy$ for the convolution on a space $\mathcal{X}$, we have

$$\begin{aligned}\tilde{K} *_{\mathbb{R}^d} f(x) = \int_{\mathbb{R}^d} \tilde{K}(x-y)f(y)\,dy &= \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{T}^d} \tilde{K}(x-y+k)f(y-k)\,dy \\ &= \int_{\mathbb{T}^d} \Big(\sum_{k\in\mathbb{Z}^d} \tilde{K}(x-y+k) \Big) f(y)\,dy,\end{aligned}$$

and thus $\tilde{K} *_{\mathbb{R}^d} f(x) = K *_{\mathbb{T}^d} f(x)$, where $K(x) = \sum_{k\in\mathbb{Z}^d} \tilde{K}(x+k)$. In particular, the periodized Biot–Savart kernel obtained by taking $\tilde{K}$ from (1.5) reads

$$K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} + \frac{1}{2\pi} \sum_{k\in\mathbb{Z}^2, k\neq 0} \frac{(x-k)^\perp}{|x-k|^2} =: \tilde{K}(x) + K_0(x). \tag{1.6}$$

It has been shown that the sum defining K_0 converges (in the sense that $K_0(x) = \lim_{N\to\infty} \sum_{|k|^2\le N, k\neq 0} \frac{(x-k)^\perp}{|x-k|^2}$) in $\mathcal{C}^\infty$ (see for instance [29]). It is straightforward to check that K is periodic, bounded in L^1, and divergence free. Finally, Proposition 2 of [18] yields the existence of $V \in L^\infty$ such that $K = \nabla \cdot V$. As a consequence, Assumption 1.3 holds for the periodized Biot–Savart kernel.

Remark 1.5. Notice that, for the Biot–Savart kernel on the whole space $\mathbb{R}^2$,

$$\tilde{K}(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2},$$

a matrix field $\tilde{V}$ such that $\tilde{K} = \nabla \cdot \tilde{V}$ can be chosen explicitly:

$$V(x) = \frac{1}{2\pi}\begin{pmatrix} -\arctan(x_1/x_2) & 0 \\ 0 & \arctan(x_2/x_1) \end{pmatrix}.$$

One could also consider collision-like interactions, as mentioned in [18]. Let $\phi \in L^1$ be a function on the torus, M be a smooth antisymmetric matrix field and consider the kernel $K = \nabla \cdot (M \mathbb{1}_{\phi(x) \leq 0})$. By construction, K is the divergence of an L^∞ matrix field, and since M is antisymmetric, K is divergence free.

Example 1.6. Consider in dimension 2 the function $\phi : x \mapsto |x|^2 - (2R)^2$ for a given radius $R > 0$ and the matrix

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which yield

$$K(x) = 2x^\perp \delta_{\phi(x)=0}.$$

This interaction kernel models particles, seen as balls of radius R, interacting via some form of collision.

The well-posedness of equations (1.2) and (1.4) under Assumptions 1.1 and 1.3 will be discussed respectively in Sections 2.1 and 3.5. In particular, we will see in Theorem 2.1 that $\bar{\rho}_t$ is in $\mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^d)$.

The comparison between the law of the system of N interacting particles and the law of N independent particles satisfying the non-linear equation (1.1) is stated in terms of relative entropy.

Definition 1.7. Let μ and ν be two probability densities on $\mathbb{T}^{dN}$. We consider the rescaled relative entropy

$$\mathcal{H}_N(\nu, \mu) = \begin{cases} \frac{1}{N}\mathbb{E}_\mu\left(\frac{\nu}{\mu} \log \frac{\nu}{\mu}\right) & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.7}$$

For brevity, for all $k \in \mathbb{N}$ and $t \geq 0$, we denote $\rho_N(t) : \mathbb{T}^{dN} \ni \mathbf{x} \mapsto \rho_N(t, \mathbf{x})$ and $\bar{\rho}_N(t) : \mathbb{T}^{dN} \ni \mathbf{x} \mapsto \bar{\rho}_t^{\otimes N}(\mathbf{x})$. The main result is the following.

Theorem 1.8. *Under Assumptions 1.1 and 1.3, there are constants C_1, C_2 and C_3 such that for all $N \in \mathbb{N}$ and every exchangeable density probability $\rho_N(0) \in \mathcal{C}^\infty_{>0}(\mathbb{T}^{dN})$ there exists a weak solution ρ_N of (1.4) such that for all $t \geq 0$,*

$$\mathcal{H}_N(\rho_N(t), \bar{\rho}_N(t)) \leq C_1 e^{-C_2 t} \mathcal{H}_N(\rho_N(0), \bar{\rho}_N(0)) + C_3/N. \tag{1.8}$$

In particular, if $\rho_N(0) = \bar{\rho}_N(0)$, the first term of the right-hand side vanishes, and this property has been called *entropic propagation of chaos*; see for example [17].

1.3. Strong propagation of chaos

We show that Theorem 1.8 yields strong propagation of chaos, uniform in time. For μ and ν two probability measures on $\mathbb{T}^{dk}$, denote by $\Pi(\mu, \nu)$ the set of couplings of μ and ν, i.e. the set of probability measures Γ on $\mathbb{T}^{dk} \times \mathbb{T}^{dk}$ with $\Gamma(A \times \mathbb{T}^{dk}) = \mu(A)$ and $\Gamma(\mathbb{T}^{dk} \times A) = \nu(A)$ for all Borel subsets A of $\mathbb{T}^{dk}$. Let us define the usual L^2-Wasserstein distance by

$$\mathcal{W}_2(\mu,\nu) = \left(\inf_{\Gamma \in \Pi(\mu,\nu)} \int_{\mathbb{T}^{dk}} d_{\mathbb{T}^{dk}}(x,y)^2 \, \Gamma(dxdy) \right)^{1/2},$$

where $d_{\mathbb{T}^{dk}}$ is the usual distance on the torus. For $\mathbf{x} = (x_i)_{i \in [\![1,N]\!]} \in \mathbb{T}^{dN}$, we write $\pi(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ for the associated empirical measure.

Corollary 1.9. *Under Assumptions 1.1 and 1.3, assuming moreover that $\rho_N(0) = \bar{\rho}_N(0)$, there is a constant C such that for all $k \leq N$ in $\mathbb{N}$ and all $t \geq 0$,*

$$\|\rho_N^k(t) - \bar{\rho}_k(t)\|_{L^1} + \mathcal{W}_2(\rho_N^k(t), \bar{\rho}_k(t)) \leq C(\lfloor N/k \rfloor)^{-1/2}$$

and

$$\mathbb{E}_{\rho_N(t)}(\mathcal{W}_2(\pi(X), \bar{\rho}_t)) \leq C\alpha(N),$$

where $\alpha(N) = N^{-1/2}\log(1+N)$ if $d = 2$ and $\alpha(N) = N^{-1/d}$ if $d > 2$.

As shown in [4], the last result yields a confidence interval in uniform norm when estimating $\bar{\rho}_t$ with $\pi(\mathbf{X}_t^N)$ convoluted with a smooth kernel.

We postpone the proof as it will rely on results shown later. It will, however, be a direct corollary of Theorem 1.8 and of the logarithmic Sobolev inequality proven in Corollary 2.6, which is a crucial ingredient in the proof of Theorem 1.8.

2. Preliminary work

2.1. *First results on the non-linear PDE*

We have the following result concerning the solution of (1.2).

Theorem 2.1. *Under Assumption 1.3, let $\mu_0 \in \mathcal{C}_\lambda^\infty(\mathbb{T}^d)$. Then the system*

$$\begin{cases} \partial_t \bar{\rho}_t = -\nabla \cdot ((K * \bar{\rho}_t)\bar{\rho}_t) + \Delta \bar{\rho}_t & \text{in } \mathbb{R}^+ \times \mathbb{T}^d, \\ \bar{\rho}_0 = \mu_0, \end{cases} \tag{2.1}$$

has a unique bounded solution $\bar{\rho}(t,x) \in \mathcal{C}_\lambda^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$.

Proof. The existence, uniqueness and smoothness can be proven by following closely the proof of Ben-Artzi [3]. For the sake of completeness, this is detailed in Appendix A. Note that a similar result has also been recently proven in [32], where the $\mathcal{C}^k$ regularity of $\bar{\rho}_t$ for any given k and t is shown. The proof relies heavily on the fact that the kernel K is divergence free, that the convolution operation tends to keep the regularity of the most regular term, and that the Fokker–Planck equation has a smoothing effect.

Let us now prove the time-uniform bounds on $\bar{\rho}_t$. Assume that $\mu_0 \in \mathcal{C}_\lambda^\infty(\mathbb{T}^d)$, which by definition implies $1/\lambda \leq \mu_0 \leq \lambda$, and consider the unique solution $\bar{\rho}_t$ of (2.1). We start by proving that $K * \bar{\rho}_t$ is in $\mathcal{C}^\infty$. By definition

$$K * \bar{\rho}_t(x) = \int_{\mathbb{T}^d} K(x-y)\bar{\rho}_t(y)\,dy = -\int_{\mathbb{T}^d} K(y)\bar{\rho}_t(x-y)\,dy.$$

Then

$$K * \bar{\rho}_t(x) = -\int_{\mathbb{T}^d} \nabla \cdot V(y)\bar{\rho}_t(x-y)\,dy = -\int_{\mathbb{T}^d} V(y)\nabla_y\bar{\rho}_t(x-y)\,dy.$$

Since $V \in L^\infty(\mathbb{T}^d)$ and $\bar{\rho} \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$, we easily deduce that $K * \bar{\rho}$, and all its derivatives, are Lipschitz continuous on $[0,T] \times \mathbb{T}^d$ for all $T > 0$. Hence $K * \bar{\rho}$ is $\mathcal{C}^\infty$. Moreover, using $\nabla \cdot K = 0$ (in the sense of distributions), we immediately get $\nabla \cdot (K * \bar{\rho}_t) = 0$ for all $t \geq 0$.

For $t \geq 0$ and $x \in \mathbb{T}^d$, let Z_s be the strong solution of the following stochastic differential equation for $s \in [0,t]$:

$$dZ_s = \sqrt{2}\,dB_s - K * \bar{\rho}_{t-s}(Z_s)\,ds, \quad Z_0 = x,$$

which exists, is unique and non-explosive since $K * \bar{\rho}_{t-s}$ is smooth and bounded. Then

$$\bar{\rho}(t,x) = \mathbb{E}_x(\bar{\rho}_0(Z_t)),$$

and the bounds on $\bar{\rho}_t$ follow. ∎

2.2. Higher order estimates

We have already established that $\bar{\rho}_t$ is bounded uniformly in time. In this section, we extend this result to all derivatives.

Lemma 2.2. *For all $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in [\![1,d]\!]$, there exist $C_n^u, C_n^\infty > 0$ such that for all $t \geq 0$,*

$$\|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_t\|_{L^\infty} \leq C_n^u \quad \text{and} \quad \int_0^t \|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_s\|_{L^\infty}^2\,ds \leq C_n^\infty.$$

Proof. Thanks to Morrey's inequality and Sobolev embeddings, it is sufficient to prove such bounds in the Sobolev space H^m for all m, in other words it is sufficient to prove similar bounds for $\|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_s\|_{L^2}^2$ for all multi-indices α. The proof is by induction on the order of the derivatives; we only detail the first iterations. We write $f = \nabla \cdot ((K * \bar{\rho}_t)\bar{\rho}_t) = (K * \bar{\rho}_t) \cdot \nabla\bar{\rho}_t$.

Integrated bound for $\|\nabla\bar{\rho}_t\|_{L^2}^2$. We have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} |\bar{\rho}_t|^2 = \int_{\mathbb{T}^d} \bar{\rho}_t\partial_t\bar{\rho}_t = \int_{\mathbb{T}^d} \bar{\rho}_t\Delta\bar{\rho}_t - \int_{\mathbb{T}^d} \bar{\rho}_t f.$$

On the one hand,

$$\int_{\mathbb{T}^d} \bar{\rho}_t\Delta\bar{\rho}_t = -\int_{\mathbb{T}^d} |\nabla\bar{\rho}_t|^2.$$

On the other hand,

$$\int_{\mathbb{T}^d} \bar{\rho}_t f = \int_{\mathbb{T}^d} \bar{\rho}_t\nabla \cdot ((K * \bar{\rho}_t)\bar{\rho}_t) = -\int_{\mathbb{T}^d} \nabla\bar{\rho}_t \cdot (K * \bar{\rho}_t)\bar{\rho}_t = -\int_{\mathbb{T}^d} \bar{\rho}_t f = 0.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}_t\|_{L^2}^2 + \|\nabla\bar{\rho}_t\|_{L^2}^2 = 0.$$

By integrating the equality above, we get

$$\int_0^t \|\nabla\bar{\rho}_t\|_{L^2}^2 = \frac{\|\bar{\rho}_0\|_{L^2}^2 - \|\bar{\rho}_t\|_{L^2}^2}{2} \leq \frac{\lambda^2}{2} = C_1^\infty.$$

Integrated bound for $\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2$ and uniform bound for $\|\nabla\bar{\rho}_t\|_{L^2}^2$. Similarly, we calculate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}|\partial_{\alpha_1}\bar{\rho}_t|^2 &= \int_{\mathbb{T}^d}\partial_{\alpha_1}\bar{\rho}_t\partial_{\alpha_1}(\partial_t\bar{\rho}_t)\\ &= \int_{\mathbb{T}^d}\partial_{\alpha_1}\bar{\rho}_t\partial_{\alpha_1}(\Delta\bar{\rho}_t - f)\\ &= -\sum_{\alpha_2}\int_{\mathbb{T}^d}|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t|^2 + \int_{\mathbb{T}^d}\partial_{\alpha_1,\alpha_1}\bar{\rho}_t f.\end{aligned}$$

Bounding

$$\begin{aligned}\int_{\mathbb{T}^d}\partial_{\alpha_1,\alpha_1}\bar{\rho}_t f &\leq \|\partial_{\alpha_1,\alpha_1}\bar{\rho}_t\|_{L^2}\|f\|_{L^2}\\ &\leq \frac{1}{2}\sum_{\alpha_2}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 + \frac{1}{2}\|f\|_{L^2}^2,\end{aligned}$$

and

$$\begin{aligned}\|f\|_{L^2}^2 = \int_{\mathbb{T}^d}\left|\sum_{\gamma=1}^d (K_\gamma * \bar{\rho}_t)\partial_\gamma\bar{\rho}_t\right|^2 &\leq \|K * \bar{\rho}_t\|_{L^\infty}^2\|\nabla\bar{\rho}_t\|_{L^2}^2\\ &\leq \|K\|_{L^1}^2\|\bar{\rho}_t\|_{L^\infty}^2\|\nabla\bar{\rho}_t\|_{L^2}^2,\end{aligned}$$

where we have used Young's convolution inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1}\bar{\rho}_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_2}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 \leq \frac{1}{2}\|K\|_{L^1}^2\|\bar{\rho}_t\|_{L^\infty}^2\|\nabla\bar{\rho}_t\|_{L^2}^2.$$

By integrating the equality above and using Theorem 2.1, we get

$$\begin{aligned}\frac{\|\partial_{\alpha_1}\bar{\rho}_t\|_{L^2}^2 - \|\partial_{\alpha_1}\bar{\rho}_0\|_{L^2}^2}{2} + \frac{1}{2}\int_0^t\sum_{\alpha_2}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_s\|_{L^2}^2\,ds &\leq \frac{1}{2}\|K\|_{L^1}^2\lambda^2\int_0^t\|\nabla\bar{\rho}_s\|_{L^2}^2\,ds\\ &\leq \frac{1}{2}\|K\|_{L^1}^2\lambda^2 C_1^\infty.\end{aligned}$$

This provides both the existence of C_2^∞ such that for all $t \geq 0$, $\int_0^t\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_s\|_{L^2}^2\,ds \leq C_2^\infty$, and the existence of C_1^u such that for all $t \geq 0$, $\|\partial_{\alpha_1}\bar{\rho}_t\|_{L^2}^2 \leq C_1^u$.

Integrated bound bound for $\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar\rho_t\|^2_{L^2}$ and uniform bound for $\|\partial_{\alpha_1,\alpha_2}\bar\rho_t\|^2_{L^2}$. We have

$$\partial_\alpha f = \sum_\gamma (\partial_\alpha K_\gamma * \bar\rho_t)\partial_\gamma\bar\rho_t + \sum_\gamma (K_\gamma * \bar\rho_t)\partial_{\alpha,\gamma}\bar\rho_t,$$

and

$$\begin{aligned}
\partial_\alpha K_\gamma * \bar\rho_t &= \int_{\mathbb{T}^d} \partial_\alpha K_\gamma(x-y)\bar\rho_t(y)\,dy = -\int_{\mathbb{T}^d} \partial_\alpha K_\gamma(y)\bar\rho_t(x-y)\,dy \\
&= -\int_{\mathbb{T}^d} K_\gamma(y)\partial_\alpha\bar\rho_t(x-y)\,dy = -\sum_\beta \int_{\mathbb{T}^d} V_{\gamma,\beta}(y)\partial_{\alpha,\beta}\bar\rho_t(x-y)\,dy \\
&= \sum_\beta V_{\gamma,\beta} * \partial_{\alpha,\beta}\bar\rho_t.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_\gamma (\partial_\alpha K_\gamma * \bar\rho_t)\partial_\gamma\bar\rho_t &= \sum_\gamma\Big(\sum_\beta V_{\gamma,\beta} * \partial_{\alpha,\beta}\bar\rho_t\Big)\partial_\gamma\bar\rho_t \\
&= (V * \partial_\alpha\nabla\bar\rho_t)\nabla\bar\rho_t,
\end{aligned}$$

and thus

$$\begin{aligned}
\Big\|\sum_\gamma (\partial_\alpha K_\gamma * \bar\rho_t)\partial_\gamma\bar\rho_t\Big\|_{L^2} &\leq \|V * \partial_\alpha\nabla\bar\rho_t\|_{L^\infty}\|\nabla\bar\rho_t\|_{L^2} \\
&\leq \|V\|_{L^\infty}\|\partial_\alpha\nabla\bar\rho_t\|_{L^1}\|\nabla\bar\rho_t\|_{L^2}.
\end{aligned}$$

Therefore

$$\|\partial_\alpha f\|^2_{L^2} \leq 2\|V\|^2_{L^\infty}\|\partial_\alpha\nabla\bar\rho_t\|^2_{L^1}\|\nabla\bar\rho_t\|^2_{L^2} + 2\|K\|^2_{L^1}\|\bar\rho_t\|^2_{L^\infty}\|\partial_\alpha\nabla\bar\rho_t\|^2_{L^2}.$$

Similarly to the previous computations,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}|\partial_{\alpha_1,\alpha_2}\bar\rho_t|^2 &= \int_{\mathbb{T}^d}\partial_{\alpha_1,\alpha_2}\bar\rho_t\,\partial_{\alpha_1,\alpha_2}(\Delta\bar\rho_t - f) \\
&= -\sum_{\alpha_3}\int_{\mathbb{T}^d}|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar\rho_t|^2 + \int_{\mathbb{T}^d}\partial_{\alpha_1,\alpha_2,\alpha_2}\bar\rho_t\,\partial_{\alpha_1} f \\
&\leq -\sum_{\alpha_3}\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar\rho_t\|^2_{L^2} + \|\partial_{\alpha_1,\alpha_2,\alpha_2}\bar\rho_t\|_{L^2}\|\partial_{\alpha_1} f\|_{L^2} \\
&\leq -\sum_{\alpha_3}\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar\rho_t\|^2_{L^2} + \frac{1}{2}\sum_{\alpha_3}\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar\rho_t\|^2_{L^2} \\
&\quad + \|V\|^2_{L^\infty}\|\partial_{\alpha_1}\nabla\bar\rho_t\|^2_{L^1}\|\nabla\bar\rho_t\|^2_{L^2} + \|K\|^2_{L^1}\|\bar\rho_t\|^2_{L^\infty}\|\partial_{\alpha_1}\nabla\bar\rho_t\|^2_{L^2} \\
&\leq -\frac{1}{2}\sum_{\alpha_3}\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar\rho_t\|^2_{L^2} + \|V\|^2_{L^\infty}\|\nabla\bar\rho_t\|^2_{L^2}\|\partial_{\alpha_1}\nabla\bar\rho_t\|^2_{L^2} \\
&\quad + \|K\|^2_{L^1}\|\bar\rho_t\|^2_{L^\infty}\|\partial_{\alpha_1}\nabla\bar\rho_t\|^2_{L^2},
\end{aligned}$$

and thus

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_3}\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar{\rho}_t\|_{L^2}^2 \leq \|V\|_{L^\infty}^2\|\partial_{\alpha_1}\nabla\bar{\rho}_t\|_{L^2}^2\|\nabla\bar{\rho}_t\|_{L^2}^2 + \|K\|_{L^1}^2\|\bar{\rho}_t\|_{L^\infty}^2\|\partial_{\alpha_1}\nabla\bar{\rho}_t\|_{L^2}^2.$$

Integrating over time and using Theorem 2.1 gives

$$\begin{aligned}\frac{\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 - \|\partial_{\alpha_1,\alpha_2}\bar{\rho}_0\|_{L^2}^2}{2} &+ \frac{1}{2}\sum_{\alpha_3}\int_0^t \|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar{\rho}_s\|_{L^2}^2\,ds \\ &\leq \|V\|_{L^\infty}^2 dC_1^u \int_0^t \|\partial_{\alpha_1}\nabla\bar{\rho}_s\|_{L^2}^2\,ds + \|K\|_{L^1}^2\lambda^2\int_0^t \|\partial_{\alpha_1}\nabla\bar{\rho}_s\|_{L^2}^2\,ds \\ &\leq d(d\|V\|_{L^\infty}^2 C_1^u + \|K\|_{L^1}^2\lambda^2)C_2^\infty.\end{aligned}$$

This provides both the existence of C_3^∞ such that for all $t \geq 0$,

$$\int_0^t \|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar{\rho}_s\|_{L^2}^2\,ds \leq C_3^\infty,$$

and the existence of C_2^u such that for all $t \geq 0$, $\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 \leq C_2^u$.

The proof is then by induction on the order of the derivative, iterating the same method. ■

2.3. *Logarithmic Sobolev inequality*

We now establish a logarithmic Sobolev inequality (LSI) for $\bar{\rho}_t$ solving (1.2). To this end, we use the fact that the uniform distribution u on $\mathbb{T}^d$ satisfies a LSI and that $\bar{\rho}_t$ is bounded (below and above) uniformly in time. Recall the following Holley–Stroock perturbation lemma, from [2, Propostion 5.1.6].

Lemma 2.3. *Assume that ν is a probability measure on $\mathbb{T}^d$ satisfying a logarithmic Sobolev inequality with constant C_ν^{LS}, i.e. for all $f \in \mathcal{C}_{>0}^\infty(\mathbb{T}^d)$,*

$$\mathrm{Ent}_\nu(f) := \int_{\mathbb{T}^d} f\log f\,d\nu - \int_{\mathbb{T}^d} f\,d\nu \log\left(\int_{\mathbb{T}^d} f\,d\nu\right) \leq C_\nu^{\mathrm{LS}}\int_{\mathbb{T}^d}\frac{|\nabla f|^2}{f}\,d\nu.$$

Let μ be a probability measure with density h with respect to ν such that $1/\lambda \leq h \leq \lambda$ for some constant $\lambda > 0$. Then μ satisfies a logarithmic Sobolev inequality with constant $C_\mu^{\mathrm{LS}} = \lambda^2 C_\nu^{\mathrm{LS}}$, i.e. for all $f \in \mathcal{C}_{>0}^\infty(\mathbb{T}^d)$,

$$\mathrm{Ent}_\mu(f) \leq \lambda^2 C_\nu^{\mathrm{LS}}\int_{\mathbb{T}^d}\frac{|\nabla f|^2}{f}\,d\mu.$$

We also know that the uniform distribution u (i.e. the Lebesgue measure) on $\mathbb{T}^d$ satisfies a LSI. See for instance [2, Proposition 5.7.5], or [16] for a proof in dimension 1, the results in higher dimension being a consequence of tensorization properties.

Lemma 2.4. *Let u be the uniform distribution on $\mathbb{T}^d$. Then u satisfies a logarithmic Sobolev inequality: for all $f \in \mathcal{C}^\infty_{>0}(\mathbb{T}^d)$,*

$$\mathrm{Ent}_u(f) \leq \frac{1}{8\pi^2} \int_{\mathbb{T}^d} \frac{|\nabla f|^2}{f} du. \tag{2.2}$$

A direct consequence of Lemmas 2.3 and 2.4 and the bounds on $\bar{\rho}_t$ given in Theorem 2.1 is the following theorem, as well as its corollary. It establishes a uniform in time logarithmic Sobolev inequality for $\bar{\rho}_t$, crucial for the uniform control of the Fisher information appearing in the study of the dissipation of the entropy between the law of the particle system and the non-linear particles.

Theorem 2.5. *Under Assumptions 1.1 and 1.3, for all $t \geq 0$ and all $f \in \mathcal{C}^\infty_{>0}(\mathbb{T}^d)$,*

$$\mathrm{Ent}_{\bar{\rho}_t}(f) \leq \frac{\lambda^2}{8\pi^2} \int_{\mathbb{T}^d} \frac{|\nabla f|^2}{f} d\bar{\rho}_t.$$

Corollary 2.6. *Under Assumptions 1.1 and 1.3, for all $N \in \mathbb{N}$ and $t \geq 0$ and all probability densities $\mu_N \in \mathcal{C}^\infty_{>0}(\mathbb{T}^{dN})$,*

$$\mathcal{H}_N(\mu_N, \bar{\rho}_N(t)) \leq \frac{\lambda^2}{8\pi^2} \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{T}^d} \mu_N \left| \nabla_{x_i} \log \frac{\mu_N}{\bar{\rho}_N(t)} \right|^2.$$

Proof. By tensorization of the logarithmic Sobolev inequality (see for instance [2, Proposition 5.2.7]), since $\bar{\rho}$ satisfies a LSI with constant $\frac{\lambda^2}{8\pi^2}$, so does $\bar{\rho}_N$. Using Theorem 2.5 for $f = \frac{\mu_N}{\bar{\rho}_N}$ we thus get

$$\mathcal{H}_N(\mu_N, \bar{\rho}_N(t)) = \frac{1}{N} \mathrm{Ent}_{\bar{\rho}_N(t)}\left(\frac{\mu_N}{\bar{\rho}_N(t)} \right) \leq \frac{\lambda^2}{8\pi^2} \frac{1}{N} \mathbb{E}_{\bar{\rho}_N(t)}\left(\left| \nabla_x \frac{\mu_N}{\bar{\rho}_N(t)} \right|^2 \frac{\bar{\rho}_N(t)}{\mu_N} \right),$$

which yields the result. ■

3. Proofs of the main results

From now on and up to Section 3.5 (excluded), in addition to Assumptions 1.1 and 1.3, we suppose that there exists a solution $\rho_N \in \mathcal{C}^\infty_{>0}(\mathbb{R}^+ \times \mathbb{T}^{dN})$ of (1.4). This justifies the validity of the various calculations conducted in this part of the proof. The question of lifting this assumption (by taking a limit in a regularized problem) is addressed in Section 3.5.

3.1. Time evolution of the relative entropy

We write

$$\mathcal{H}_N(t) = \mathcal{H}_N(\rho_N(t), \bar{\rho}_N(t)), \quad \mathcal{I}_N(t) = \frac{1}{N} \sum_i \int_{\mathbb{T}^{dN}} \rho_N(t) \left| \nabla_{x_i} \log \frac{\rho_N(t)}{\bar{\rho}_N(t)} \right|^2 d\mathbf{x}.$$

as shorthands for the relative entropy and relative Fisher information. We start by calculating the time evolution of the relative entropy.

Lemma 3.1. *For all $t \geq 0$,*

$$\frac{d}{dt}\mathcal{H}_N(t) \leq A_N(t) + \frac{1}{2}B_N(t) - \frac{1}{2}\mathcal{I}_N(t) \tag{3.1}$$

with

$$A_N(t) := \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_N (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \frac{\nabla^2_{x_i}\bar{\rho}_N}{\bar{\rho}_N}\, d\mathbf{x},$$

$$B_N(t) := \frac{1}{N}\sum_{i}\int_{\mathbb{T}^{dN}} \rho_N \frac{|\nabla_{x_i}\bar{\rho}_N|^2}{\bar{\rho}_N^2}\left|\frac{1}{N}\sum_j V(x_i - x_j) - V * \bar{\rho}_t(x_i)\right|_f^2 d\mathbf{x}.$$

Here, $|\cdot|_f^2$ denotes the sum of the squares of the coefficients of the matrix.

Proof. It has been shown in [18] that

$$\frac{d}{dt}\mathcal{H}_N(t) \leq -\mathcal{I}_N(t) - \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_N (K(x_i - x_j) - K * \bar{\rho}_t(x_i)) \cdot \nabla_{x_i}\log\bar{\rho}_N\, d\mathbf{x}$$

with

$$\begin{aligned}
-\frac{1}{N^2}&\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_N (K(x_i - x_j) - K * \bar{\rho}_t(x_i)) \cdot \nabla_{x_i}\log\bar{\rho}_N\, d\mathbf{x} \\
&= \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_N (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \frac{\nabla^2_{x_i}\bar{\rho}_N}{\bar{\rho}_N}\, d\mathbf{x} \\
&\quad + \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \nabla_{x_i}\bar{\rho}_N \otimes \nabla_{x_i}\frac{\rho_N}{\bar{\rho}_N}\, d\mathbf{x}.
\end{aligned}$$

Let us consider the last term:

$$\begin{aligned}
\frac{1}{N^2}&\sum_{i,j}\int_{\mathbb{T}^{dN}} (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \nabla_{x_i}\bar{\rho}_N \otimes \nabla_{x_i}\frac{\rho_N}{\bar{\rho}_N}\, d\mathbf{x} \\
&= \frac{1}{N}\sum_i\sum_{\alpha,\beta}\int_{\mathbb{T}^{dN}}\left(\frac{1}{N}\sum_j V(x_i - x_j) - V * \bar{\rho}_t(x_i)\right)_{\alpha,\beta}(\nabla_{x_i}\bar{\rho}_N)_\alpha\left(\nabla_{x_i}\frac{\rho_N}{\bar{\rho}_N}\right)_\beta d\mathbf{x}.
\end{aligned}$$

Let

$$y^i_\beta := \left(\nabla_{x_i}\frac{\rho_N}{\bar{\rho}_N}\right)_\beta \frac{\bar{\rho}_N}{\sqrt{\rho_N}}, \quad z^i_\alpha := (\nabla_{x_i}\bar{\rho}_N)_\alpha \frac{\sqrt{\rho_N}}{\bar{\rho}_N},$$

$$x^i_{\alpha,\beta} := \left(\frac{1}{N}\sum_j V(x_i - x_j) - V * \bar{\rho}(x_i)\right)_{\alpha,\beta}.$$

Then, using $xy \leq x^2/2 + y^2/2$ for all $x, y \in \mathbb{R}$, we get

$$\sum_{\alpha,\beta} x^i_{\alpha,\beta} z^i_\alpha y^i_\beta = \sum_\beta y^i_\beta\Big(\sum_\alpha x^i_{\alpha,\beta} z^i_\alpha\Big) \leq \frac{1}{2}\sum_\beta (y^i_\beta)^2 + \frac{1}{2}\sum_\beta\Big(\sum_\alpha x^i_{\alpha,\beta} z^i_\alpha\Big)^2,$$

and thus, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\sum_{\alpha,\beta} x^i_{\alpha,\beta} z^i_\alpha y^i_\beta &\leq \frac{1}{2}\sum_\beta (y^i_\beta)^2 + \frac{1}{2}\sum_\beta \Big(\sum_\alpha (x^i_{\alpha,\beta})^2\Big)\Big(\sum_\alpha (z^i_\alpha)^2\Big) \\ &= \frac{1}{2}\sum_\beta (y^i_\beta)^2 + \frac{1}{2}\Big(\sum_\alpha (z^i_\alpha)^2\Big)\Big(\sum_{\alpha,\beta} (x^i_{\alpha,\beta})^2\Big).\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} (V(x_i - x_j) - V * \bar\rho_t(x_i)) : \nabla_{x_i}\bar\rho_N \otimes \nabla_{x_i}\frac{\rho_N}{\bar\rho_N}\, d\mathbf{x} \\ &\quad\leq \frac{1}{2N}\sum_i \int \frac{\bar\rho_N^2}{\rho_N}\left|\nabla_{x_i}\frac{\rho_N}{\bar\rho_N}\right|^2 \\ &\qquad + \frac{1}{2N}\sum_i \int \rho_N \frac{|\nabla_{x_i}\bar\rho_N|^2}{\bar\rho_N^2}\left|\frac{1}{N}\sum_j V(x_i - x_j) - V * \bar\rho_t(x_i)\right|_f^2 \\ &\quad= \frac{1}{2}\mathcal{I}_N(t) + \frac{1}{2N}\sum_i \int_{\mathbb{T}^{dN}} \rho_N \frac{|\nabla_{x_i}\bar\rho_N|^2}{\bar\rho_N^2}\left|\frac{1}{N}\sum_j V(x_i - x_j) - V * \bar\rho_t(x_i)\right|_f^2 d\mathbf{x}.\end{aligned}$$

This yields the desired result. ∎

3.2. Change of reference measure and Law of Large Numbers

We now state three general results which will be useful in order to control the error terms A_N and B_N defined in Lemma 3.1. The first one will be used to perform a change of measure from ρ_N to $\bar\rho_N$.

Lemma 3.2. *Let $N \in \mathbb{N}$. For two probability densities μ and ν on $\mathbb{T}^{dN}$, and any $\Phi \in L^\infty(\mathbb{T}^{dN})$ and $\eta > 0$,*

$$\mathbb{E}^\mu \Phi \leq \eta \mathcal{H}_N(\mu, \nu) + \frac{\eta}{N}\log \mathbb{E}^\nu e^{N\Phi/\eta}.$$

Proof. Define

$$f = \frac{1}{\theta} e^{N\Phi/\eta}\nu, \quad \theta = \int_{\mathbb{T}^{dN}} e^{N\Phi/\eta}\nu\, d\mathbf{x}.$$

Notice that f is a probability density. By convexity of entropy,

$$\frac{1}{N}\int_{\mathbb{T}^{dN}} \mu \log f\, d\mathbf{x} \leq \frac{1}{N}\int_{\mathbb{T}^{dN}} \mu \log \mu\, d\mathbf{x}.$$

On the other hand,

$$\frac{1}{N}\int_{\mathbb{T}^{dN}} \mu \log f\, d\mathbf{x} = \frac{1}{\eta}\int_{\mathbb{T}^{dN}} \mu\Phi\, d\mathbf{x} + \frac{1}{N}\int_{\mathbb{T}^{dN}} \mu \log \nu\, d\mathbf{x} - \frac{\log\theta}{N}. \quad ∎$$

The next two statements are crucial theorems of [18].

Theorem 3.3 ([18, Theorem 3]). *Consider any probability measure μ on $\mathbb{T}^d$ and a scalar function $\psi \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ with $\|\psi\|_{L^\infty} < \frac{1}{2e}$ and such that for all $z \in \mathbb{T}^d$, $\int_{\mathbb{T}^d} \psi(z,x)\,\mu(dx) = 0$. Then*

$$\int_{\mathbb{T}^{dN}} \exp\left(\frac{1}{N}\sum_{j_1,j_2=1}^{N} \psi(x_1,x_{j_1})\psi(x_1,x_{j_2})\right)\mu^{\otimes N}\,d\mathbf{x} \le C := 2\left(1+\frac{10\alpha}{(1-\alpha)^3}+\frac{\beta}{1-\beta}\right), \tag{3.2}$$

where

$$\alpha = (e\|\psi\|_{L^\infty})^4 < 1, \quad \beta = (\sqrt{2e}\,\|\psi\|_{L^\infty})^4 < 1.$$

The second one is a nice improvement of the usual level 2 large deviations bound for i.i.d. random variables.

Theorem 3.4 ([18, Theorem 4]). *Consider any probability measure μ on $\mathbb{T}^d$ and $\phi \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ with*

$$\gamma := (1600^2 + 36e^4)\left(\sup_{p\ge 1} \frac{\left\|\sup_z |\phi(\cdot,z)|\right\|_{L^p(\mu)}}{p}\right)^2 < 1. \tag{3.3}$$

Assume that ϕ satisfies the following cancellations:

$$\forall z \in \mathbb{T}^d, \quad \int_{\mathbb{T}^d} \phi(x,z)\,\mu(dx) = 0 = \int_{\mathbb{T}^d} \phi(z,x)\,\mu(dx)\,.$$

Then, for all $N \in \mathbb{N}$,

$$\int_{\mathbb{T}^{dN}} \exp\left(\frac{1}{N}\sum_{i,j=1}^{N} \phi(x_i,x_j)\right)\mu^{\otimes N}\,d\mathbf{x} \le \frac{2}{1-\gamma} < \infty. \tag{3.4}$$

3.3. Bounding the error terms

Lemma 3.5. *The terms A_N and B_N introduced in Lemma 3.1 satisfy*

$$A_N(t) + \frac{1}{2}B_N(t) \le C\left(\mathcal{H}_N(t) + \frac{1}{N}\right)$$

with

$$C = \hat{C}_1 \lambda d \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} + \hat{C}_2 \lambda^2 d^2 \|V\|_{L^\infty}^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2,$$

where $\hat{C}_1, \hat{C}_2$ are universal constants.

Proof. Recall from Theorem 2.1 that $\bar{\rho}_t \in \mathcal{C}_\lambda^\infty(\mathbb{T}^d)$ for all $t \ge 0$. We first bound B_N. For $(X_t^i)_i$ given in (1.3), we have

$$B_N = \frac{1}{N}\sum_i \int_{\mathbb{T}^{dN}} \rho_N \frac{|\nabla \bar{\rho}_t|^2}{\bar{\rho}_t^2}(x_i) \left|\frac{1}{N}\sum_j V(x_i - x_j) - V * \bar{\rho}_t(x_i)\right|_f^2 d\mathbf{x}$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_i \mathbb{E}\left(\left|\frac{\nabla\bar\rho_t}{\bar\rho_t}(X_t^i)\right|^2\left|\frac{1}{N}\sum_j V(X_t^i - X_t^j) - V*\bar\rho_t(X_t^i)\right|_f^2\right)\\
&= \frac{1}{N}\sum_i\sum_{\alpha,\beta=1}^d \mathbb{E}\left(\left|\frac{\nabla\bar\rho_t}{\bar\rho_t}(X_t^i)\right|^2\left(\frac{1}{N}\sum_j V_{\alpha,\beta}(X_t^i - X_t^j) - V_{\alpha,\beta}*\bar\rho_t(X_t^i)\right)^2\right)\\
&\leq \frac{\lambda^2\|\nabla\bar\rho_t\|_{L^\infty}^2}{N}\sum_i\sum_{\alpha,\beta=1}^d \mathbb{E}\left(\left(\frac{1}{N}\sum_j V_{\alpha,\beta}(X_t^i - X_t^j) - V_{\alpha,\beta}*\bar\rho_t(X_t^i)\right)^2\right).
\end{aligned}
$$

We apply Lemma 3.2 to each

$$
\Phi_{\alpha,\beta} = \left(\frac{1}{N}\sum_j V_{\alpha,\beta}(x_i - x_j) - V_{\alpha,\beta}*\bar\rho_t(x_i)\right)^2
$$

to get, for all $C_B > 0$,

$$
\begin{aligned}
&\mathbb{E}\left(\left(\frac{1}{N}\sum_j V_{\alpha,\beta}(X_t^i - X_t^j) - V_{\alpha,\beta}*\bar\rho_t(X_t^i)\right)^2\right)\\
&\quad\leq C_B\mathcal{H}_N(t) + \frac{C_B}{N}\log\mathbb{E}\left(\exp\left(\frac{1}{C_B}\left(\frac{1}{\sqrt{N}}\sum_j V_{\alpha,\beta}(\bar X_t^i - \bar X_t^j) - V_{\alpha,\beta}*\bar\rho_t(\bar X_t^i)\right)^2\right)\right).
\end{aligned}
$$

This way,

$$
\begin{aligned}
B_N \leq{}& \frac{C_B\lambda^2\|\nabla\bar\rho_t\|_{L^\infty}^2}{N^2}\\
&\times\sum_{i,\alpha,\beta}\log\int\bar\rho_N\exp\left(\frac{1}{C_B}\left(\frac{1}{\sqrt{N}}\sum_j V_{\alpha,\beta}(x_i - x_j) - V_{\alpha,\beta}*\bar\rho_t(x_i)\right)^2\right)\\
&+ C_B d^2\lambda^2\|\nabla\bar\rho_t\|_{L^\infty}^2\mathcal{H}_N(t).
\end{aligned}
$$

In the following we choose $C_B = 64e^2\|V\|_{L^\infty}^2$. Applying Theorem 3.3 to

$$
\psi(z,x) = \frac{1}{8e\|V\|_{L^\infty}}(V(z-x) - V*\bar\rho_t(z)),
$$

which satisfies $\|\psi\|_{L^\infty} \leq \frac{1}{4e}$ and is such that

$$
\begin{aligned}
&\int_{\mathbb{T}^d}\psi(z,x)\bar\rho_t(x)\,dx\\
&\quad= \frac{1}{8e\|V\|_{L^\infty}}\int_{\mathbb{T}^d}V(z-x)\bar\rho_t(x)\,dx - \frac{1}{8e\|V\|_{L^\infty}}\int_{\mathbb{T}^d}V*\bar\rho_t(z)\bar\rho_t(x)\,dx = 0,
\end{aligned}
$$

we get

$$
B_N \leq \hat C_B\|V\|_{L^\infty}^2\lambda^2 d^2\|\nabla\bar\rho_t\|_{L^\infty}^2\left(\mathcal{H}_N(t) + \frac{\tilde C_B}{N}\right), \tag{3.5}
$$

where $\hat C_B$ and $\tilde C_B$ are universal constants.

We now proceed with the bound on A_N. Applying Lemma 3.2 to

$$\Phi = \frac{1}{N^2}\sum_{i,j}(V(x_i - x_j) - V * \bar\rho_t(x_i)) : \frac{\nabla^2_{x_i}\bar\rho_N}{\bar\rho_N},$$

we obtain, for all $C_A > 0$,

$$A_N \le \frac{C_A}{N}\log\int_{\mathbb{T}^{dN}}\bar\rho_N \exp\Bigg(\frac{1}{C_A N}\sum_{i,j}(V(x_i - x_j) - V * \bar\rho_t(x_i)) : \frac{\nabla^2_{x_i}\bar\rho_N}{\bar\rho_N}\Bigg)\,d\mathbf{x} \\ + C_A\mathcal{H}_N(t).$$

In the following we choose

$$C_A = 4\sqrt{1600^2 + 36e^4}\,\|\nabla^2\bar\rho_t\|_{L^\infty}\|V\|_{L^\infty}\lambda d =: \hat C_A\lambda d\,\|\nabla^2\bar\rho_t\|_{L^\infty}\|V\|_{L^\infty}.$$

Then, we apply Theorem 3.4 to

$$\phi(z,x) = \frac{1}{C_A}\Bigg((V(z-x) - V * \bar\rho_t(z)) : \frac{\nabla^2\bar\rho_t}{\bar\rho_t}(z)\Bigg),$$

which satisfies, thanks to Assumption 1.3,

$$\begin{aligned}\int_{\mathbb{T}^d}\phi(z,x)\bar\rho_t(z)\,dz &= \frac{1}{C_A}\int_{\mathbb{T}^d}\Bigg((V(z-x) - V * \bar\rho_t(z)) : \frac{\nabla^2\bar\rho_t}{\bar\rho_t}(z)\Bigg)\bar\rho_t(z)\,dz \\ &= \frac{1}{C_A}\int_{\mathbb{T}^d}\big(\operatorname{div}K(z-x) - \operatorname{div}K * \bar\rho_t(z)\big)\bar\rho_t(z)\,dz = 0,\end{aligned}$$

and, thanks to $\int_{\mathbb{T}^d}(V(z-x) - V * \bar\rho_t(z))\bar\rho_t(x)\,dx = 0$,

$$\int_{\mathbb{T}^d}\phi(z,x)\bar\rho_t(x)\,dx = 0.$$

Through our choice of C_A, (3.3) is satisfied, because

$$\gamma \le (1600^2 + 36e^4)\left(\frac{2d\,\|V\|_{L^\infty}\|\nabla^2\bar\rho_t\|_{L^\infty}\lambda}{C_A}\right)^2 = \frac{1}{4} < 1.$$

Hence

$$A_N \le \hat C_A\|\nabla^2\bar\rho_t\|_{L^\infty}\|V\|_{L^\infty}\lambda d\left(\mathcal{H}_N(t) + \frac{\tilde C_A}{N}\right), \tag{3.6}$$

where $\hat C_A$ and $\tilde C_A$ are universal constants. The conclusion easily follows. ∎

3.4. Proof of Theorem 1.8 in the smooth case

It only remains to gather the previous results. Inequalities (3.1), (3.5) and (3.6) yield

$$\frac{d}{dt}\mathcal{H}_N(t) \le \left(\hat C_A\lambda d\,\|\nabla^2\bar\rho_t\|_{L^\infty}\|V\|_{L^\infty} + \frac{\hat C_B\|V\|^2_{L^\infty}\lambda^2\|\nabla\bar\rho_t\|^2_{L^\infty}d^2}{2}\right)\mathcal{H}_N(t) \\ + \frac{C_2}{N} - \frac{1}{2}\mathcal{I}_N(t),$$

and using Corollary 2.6 and

$$\hat{C}_A \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} \lambda d \le \frac{1}{2}\left(\frac{2\pi}{\lambda}\right)^2 + \frac{1}{2}\left(\frac{\lambda}{2\pi}\right)^2 \hat{C}_A^2 \|\nabla^2 \bar{\rho}_t\|_{L^\infty}^2 \|V\|_{L^\infty}^2 \lambda^2 d^2,$$

we get

$$\begin{aligned}
&\frac{d}{dt}\mathcal{H}_N(t) \\
&\le -\left(\left(\frac{2\pi}{\lambda}\right)^2 - \hat{C}_A \lambda d \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} - \frac{\hat{C}_B \|V\|_{L^\infty}^2 \lambda^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2 d^2}{2}\right)\mathcal{H}_N(t) \\
&\quad + C_2/N \\
&\le -\frac{1}{2}\left(\left(\frac{2\pi}{\lambda}\right)^2 - \hat{C}_A^2 \frac{\lambda^4}{4\pi^2} d^2 \|\nabla^2 \bar{\rho}_t\|_{L^\infty}^2 \|V\|_{L^\infty}^2 - \hat{C}_B \|V\|_{L^\infty}^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2 \lambda^2 d^2\right)\mathcal{H}_N(t) \\
&\quad + C_2/N.
\end{aligned}$$

In a more concise way, using Lemma 2.2, this means there are constants $C_1, C_2^\infty, C_3 > 0$ and a function $t \mapsto C_2(t) > 0$ with $\int_0^t C_2(s)\,ds \le C_2^\infty$ for all $t \ge 0$ such that for all $t \ge 0$,

$$\frac{d}{dt}\mathcal{H}_N(t) \le -(C_1 - C_2(t))\mathcal{H}_N(t) + C_3/N.$$

Multiplying by $\exp(C_1 t - \int_0^t C_2(s)\,ds)$ and integrating in time we get

$$\begin{aligned}
\mathcal{H}_N(t) &\le e^{-C_1 t + \int_0^t C_2(s)\,ds}\mathcal{H}_N(0) + \frac{C_3}{N}\int_0^t e^{C_1(s-t) + \int_s^t C_2(u)\,du}\,ds \\
&\le e^{C_2^\infty - C_1 t}\mathcal{H}_N(0) + \frac{C_3}{C_1 N} e^{C_2^\infty},
\end{aligned}$$

which concludes the proof.

3.5. *Dealing with the regularity of ρ_N*

As mentioned at the beginning of Section 3, up to now we have proven the result under the additional assumption that there exists a smooth solution ρ_N to (1.4). Let us now remove this assumption. Let $(\zeta_\varepsilon)_{\varepsilon \ge 0}$ be a sequence of mollifiers such that $\|\zeta_\varepsilon\|_{L^1} = 1$ with support strictly contained in $[-1/2, 1/2]^d$. Set $K^\varepsilon = K * \zeta_\varepsilon$. We have $K^\varepsilon \in \mathcal{C}^\infty(\mathbb{T}^d)$ and $\mathrm{div}(K^\varepsilon) = 0$.

Let ρ_N^ε be the unique smooth solution (see Lemma 8 below) of the parabolic equation with smooth coefficients

$$\partial_t \rho_N^\varepsilon + \frac{1}{N}\sum_{i,j=1}^N K^\varepsilon(x_i - x_j)\cdot \nabla_{x_i}\rho_N^\varepsilon = \sum_{i=1}^N \Delta_{x_i}\rho_N^\varepsilon \tag{3.7}$$

with initial condition $\rho_N^\varepsilon(0,\cdot) = \rho_N(0,\cdot)$.

Lemma 3.6. *Let $\gamma > 1$ be such that $\rho_N(0) \in \mathcal{C}^\infty_\gamma(\mathbb{T}^{dN})$. Then, for all $t \geq 0$ and all $\varepsilon > 0$, we have $\rho^\varepsilon_N(t) \in \mathcal{C}^\infty_\gamma(\mathbb{T}^{dN})$.*

Proof. Let $\mathbf{x} \in \mathbb{T}^{dN}$. Consider the particle system

$$dX^\varepsilon_i(t) = -\frac{1}{N}\sum_{j=1}^N K^\varepsilon(X^\varepsilon_i(t) - X^\varepsilon_j(t))dt + \sqrt{2}\,dB^i_t,$$

with initial condition $\mathbf{X}^\varepsilon_0 = \mathbf{x}$, where we denote $\mathbf{X}^\varepsilon_t = (X^\varepsilon_1(t), \ldots, X^\varepsilon_N(t))$. We have strong existence and uniqueness for this SDE. Then

$$\rho^\varepsilon_N(t, \mathbf{x}) = \mathbb{E}(\rho^\varepsilon_N(0, \mathbf{X}^\varepsilon_t)),$$

and the bounds on ρ^ε_N follow. ∎

Using Lemma 3.6, we see that $(\rho^\varepsilon_N)_\varepsilon$ is a sequence of smooth functions uniformly bounded in $L^\infty(\mathbb{R}^+ \times \mathbb{T}^{Nd})$. This yields two results.

First, we can extract a weakly-* converging subsequence in $L^\infty(\mathbb{R}^+ \times \mathbb{T}^{Nd})$, i.e. there exists $\rho_N \in L^\infty(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ such that for all $f \in L^1(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ we have

$$\int_{\mathbb{T}^{Nd}} \rho^\varepsilon_N f \xrightarrow[\varepsilon \to 0^+]{} \int_{\mathbb{T}^{Nd}} \rho_N f.$$

We finally check that ρ_N is indeed a weak solution of (1.4). For all $T \geq 0$ and all smooth test functions f on $[0, T] \times \mathbb{T}^{Nd}$, the following hold:

- Since $\partial_t f$ is smooth and therefore in $L^1([0, T] \times \mathbb{T}^{Nd})$, we have

$$\int_{\mathbb{T}^{Nd}} \rho^\varepsilon_N \partial_t f \to \int_{\mathbb{T}^{Nd}} \rho_N \partial_t f.$$

- Likewise, since $\Delta_{x_i} f$ is smooth and therefore in $L^1([0, T] \times \mathbb{T}^{Nd})$, we have

$$\int_{\mathbb{T}^{Nd}} \rho^\varepsilon_N \Delta_{x_i} f \to \int_{\mathbb{T}^{Nd}} \rho_N \Delta_{x_i} f.$$

- Finally,

$$\begin{aligned}
&\int_{\mathbb{T}^{Nd}} \rho^\varepsilon_N K^\varepsilon(x_i - x_j) \cdot \nabla_{x_i} f - \int_{\mathbb{T}^{Nd}} \rho_N K(x_i - x_j) \cdot \nabla_{x_i} f \\
&= \int_{\mathbb{T}^{Nd}} \rho^\varepsilon_N \big(K^\varepsilon(x_i - x_j) - K(x_i - x_j)\big) \cdot \nabla_{x_i} f + \int_{\mathbb{T}^{Nd}} (\rho^\varepsilon_N - \rho_N) K(x_i - x_j) \cdot \nabla_{x_i} f \\
&\leq \|\rho^\varepsilon_N\|_{L^\infty} \|\nabla_{x_i} f\|_{L^\infty} \|K^\varepsilon - K\|_{L^1} + \int_{\mathbb{T}^{Nd}} (\rho^\varepsilon_N - \rho_N) K(x_i - x_j) \cdot \nabla_{x_i} f \\
&\to 0
\end{aligned}$$

  since $\|K^\varepsilon - K\|_{L^1} \to 0$ and $K(x_i - x_j) \cdot \nabla_{x_i} f \in L^1([0, T] \times \mathbb{T}^{Nd})$.

We have thus proven that ρ_N is a weak solution of (1.4).

Likewise, we may consider $(\bar{\rho}^\varepsilon)_\varepsilon$, which weakly-* converges to a solution which, by uniqueness, is $\bar{\rho}$.

Second, ρ_N^ε satisfies the assumption made at the beginning of Section 3, i.e. $\rho_N^\varepsilon \in \mathcal{C}^\infty_{>0}(\mathbb{R}^+ \times \mathbb{T}^d)$. Since by considering $V^\varepsilon = V * \zeta_\varepsilon$ we have $K^\varepsilon = \text{div}(V^\varepsilon)$, we find that K^ε satisfies Assumption 1.3 and the calculations in Section 3 are valid for this specific kernel, i.e.

$$\mathcal{H}_N(\rho_N^\varepsilon(t), \bar{\rho}_N^\varepsilon(t)) \leq \mathcal{H}_N(\rho_N(0), \bar{\rho}_N(0)) e^{-C_1^\varepsilon t} e^{C^{\infty,\varepsilon}} + \frac{C_3^\varepsilon e^{C^{\infty,\varepsilon}}}{C_1^\varepsilon} \frac{1}{N}. \tag{3.8}$$

Notice that, in the proof of Lemma 2.2, the constants bounding the various derivatives of $\bar{\rho}$ only depend on the initial conditions, on $\|K\|_{L^1}$ and on $\|V\|_{L^\infty}$. Since $(\zeta_\varepsilon)_{\varepsilon \geq 0}$ is a sequence of mollifiers, we have $\|K^\varepsilon\|_{L^1} \to \|K\|_{L^1}$ as $\varepsilon \to 0$, and $\|V^\varepsilon\|_{L^\infty} \leq \|V\|_{L^\infty}$. The right-hand side of (3.8) can thus be chosen independent of ε.

We now use the fact that for $u \geq 0$ and $v \in \mathbb{R}$ we have $uv \leq u \log u - u + e^v$ to obtain the variational formulation of the entropy,

$$N \mathcal{H}_N(\rho_N^\varepsilon(t), \bar{\rho}_N^\varepsilon(t)) = \sup \{\mathbb{E}_{\rho_N^\varepsilon(t)}(g) - \mathbb{E}_{\bar{\rho}_N^\varepsilon(t)}(e^g) + 1 : g \in L^\infty\}, \tag{3.9}$$

the equality being attained for $g = \log(\rho_N^\varepsilon / \bar{\rho}_N^\varepsilon)$. Thus, for $g \in L^\infty$,

$$\frac{1}{N}\left(\mathbb{E}_{\rho_N^\varepsilon(t)}(g) - \mathbb{E}_{\bar{\rho}_N^\varepsilon(t)}(e^g) + 1\right) \leq \mathcal{H}_N(\rho_N(0), \bar{\rho}_N(0)) e^{-C_1 t} e^{C^\infty} + \frac{C_3 e^{C^\infty}}{1 + C_1} \frac{1}{N}.$$

By definition of weak-* convergence in L^∞ (since both g and e^g are in L^1), we have

$$\mathbb{E}_{\rho_N^\varepsilon(t)}(g) \to \mathbb{E}_{\rho_N(t)}(g) \quad \text{and} \quad \mathbb{E}_{\bar{\rho}_N^\varepsilon(t)}(e^g) \to \mathbb{E}_{\bar{\rho}_N(t)}(e^g)$$

as $\varepsilon \to 0$. Therefore, for all $g \in L^\infty$,

$$\frac{1}{N}\left(\mathbb{E}_{\rho_N(t)}(g) - \mathbb{E}_{\bar{\rho}_N(t)}(e^g) + 1\right) \leq \mathcal{H}_N(\rho_N(0), \bar{\rho}_N(0)) e^{-C_1 t} e^{C^\infty} + \frac{C_3 e^{C^\infty}}{1 + C_1} \frac{1}{N},$$

which yields Theorem 1.8, using (3.9) for $\mathcal{H}_N(\rho_N(t), \bar{\rho}_N(t))$.

3.6. *Proof of Corollary* 1.9

Let $k \in \mathbb{N}$ and $N \geq k$. The subadditivity of entropy (see for instance [1, Theorem 10.2.3]) implies that the (rescaled) relative entropy of the marginals is bounded by the total (rescaled) relative entropy,

$$k \left\lfloor \frac{N}{k} \right\rfloor \mathcal{H}_k(\rho_N^k(t), \bar{\rho}_k(t)) \leq N \mathcal{H}_N(\rho_N(t), \bar{\rho}_N(t)).$$

The logarithmic Sobolev inequality established in Corollary 2.6 implies a Talagrand transportation inequality (see [26]), so that the L^2-Wasserstein distance is bounded by the relative entropy. Classically, this is also the case of the total variation thanks to Pinsker's inequality, and thus

$$\begin{aligned}\|\rho_N^k(t)-\bar\rho_k(t)\|_{L^1}+\mathcal{W}_2(\rho_N^k(t),\bar\rho_k(t)) &\le C\sqrt{k\mathcal{H}_k(\rho_N^k(t),\bar\rho_k(t))}\\ &\le C\sqrt{\frac{N}{\lfloor N/k\rfloor}\mathcal{H}_N(\rho_N(t),\bar\rho_N(t))}.\end{aligned}$$

With the additional assumption that $\mathcal{H}_N(\rho_N(0),\bar\rho_N(0))=0$, we thus get the result using Theorem 1.8. To obtain the result on the empirical measure, we recall for completeness the arguments of [19, Proposition 8]. Given $x,y\in\mathbb{T}^{dN}$, a coupling of $\pi(x)$ and $\pi(y)$ is obtained by considering (x_J,y_J) where J is uniformly distributed over $[\![1,N]\!]$. From this we get $\mathcal{W}_2(\pi(x),\pi(y))\le|x-y|/\sqrt{N}$. Letting $(\mathbf{X},\mathbf{Y})$ be an optimal coupling of $(\rho_N(t),\bar\rho_N(t))$, we bound

$$\begin{aligned}\mathbb{E}(\mathcal{W}_2(\pi(\mathbf{X},\bar\rho_t)) &\le \mathbb{E}(\mathcal{W}_2(\pi(\mathbf{X}),\pi(\mathbf{Y})))+\mathbb{E}(\mathcal{W}_2(\pi(\mathbf{Y}),\bar\rho_t))\\ &\le \frac{1}{\sqrt{N}}\mathcal{W}_2(\rho_N(t),\bar\rho_N(t))+\mathbb{E}(\mathcal{W}_2(\pi(\mathbf{Y}),\bar\rho_t)).\end{aligned}$$

The last term is tackled with the result for i.i.d. variables established in [14].

Appendix A. Proof of Theorem 2.1

The proof is based on an iterative procedure, and relies heavily on work of Ben-Artzi [3]. Let $\bar\rho^{(-1)}:=0$, and then for $k\in\mathbb{N}$ solve

$$\partial_t\bar\rho^{(k)}=-(u^{(k-1)}\cdot\nabla)\bar\rho^{(k)}+\Delta\bar\rho^{(k)}\quad\text{in }\mathbb{R}^+\times\mathbb{T}^d, \tag{A.1}$$

$$u^{(k)}=K*\bar\rho^{(k)}, \tag{A.2}$$

$$\bar\rho^{(k)}(0,\cdot)=\mu_0. \tag{A.3}$$

Let us recall the following lemma concerning the regularity of a second order parabolic equation. We refer to [13, Chapter 7] for a proof on a bounded domain, which can be extended to the torus.

Lemma A.1. *Let $a(t,x)$ be a $\mathcal{C}^\infty$ function on $\mathbb{R}^+\times\mathbb{T}^d$ and $\psi_0\in\mathcal{C}^\infty(\mathbb{T}^d)$. Then the problem*

$$\begin{aligned}\partial_t\psi &= -a\cdot\nabla\psi+\Delta\psi\quad \textit{in }\mathbb{R}^+\times\mathbb{T}^d,\\ \psi(0,\cdot)&=\psi_0,\end{aligned}$$

has a unique solution, which is $\mathcal{C}^\infty$.

Lemma A.2. *Suppose $\mu_0\in\mathcal{C}^\infty(\mathbb{T}^d)$. Then the system (A.1)–(A.3) defines successively a sequence of $\mathcal{C}^\infty$ solutions $\{\bar\rho^{(k)},u^{(k)}\}_{k\in\mathbb{N}}$. Furthermore, for all $t\ge0$ and all $k\in\mathbb{N}$,*

$$\|\bar\rho^{(k)}(t,\cdot)\|_{L^\infty}\le\|\mu_0\|_{L^\infty}\quad\textit{and}\quad\|u^{(k)}(t,\cdot)\|_{L^\infty}\le\|K\|_{L^1}\|\mu_0\|_{L^\infty}.$$

Finally, given a final time $T\ge0$, $\bar\rho^{(k)}$ (resp. $u^{(k)}$) and all their derivatives, both in time and in space, are bounded on $[0,T]\times\mathbb{T}^d$ uniformly in k.

Proof. We use induction on k. The assertion is clear for $\bar{\rho}^{(0)}$ from the explicit solution to the heat equation. Suppose $\{\bar{\rho}^{(j)}, u^{(j-1)}\}_{j=0,\dots,k}$ have been shown to be $\mathcal{C}^\infty$ solutions bounded uniformly in time.

Regularity. By definition

$$\begin{aligned} u^{(k)}(t,x) &= K * \bar{\rho}^{(k)}(t,x) \\ &= \int_{\mathbb{T}^d} K(x-y)\bar{\rho}^{(k)}(t,y)\,dy = -\int_{\mathbb{T}^d} K(y)\bar{\rho}^{(k)}(t,x-y)\,dy. \end{aligned}$$

Then

$$u^{(k)}(t,x) = -\int_{\mathbb{T}^d} \mathrm{div}(V(y))\bar{\rho}^{(k)}(t,x-y)\,dy = -\int_{\mathbb{T}^d} V(y)\nabla_y\bar{\rho}^{(k)}(t,x-y)\,dy.$$

Since we are in the compact set $\mathbb{T}^d$, and $V \in L^\infty(\mathbb{T}^d)$ and $\bar{\rho}^{(k)} \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$ by induction hypothesis, we can easily show that $u^{(k)}$, as well as all its derivatives, are Lipschitz continuous. Hence $u^{(k)}$ is $\mathcal{C}^\infty$. Applying Lemma A.1 to (A.1) with k replaced by $k+1$ yields the desired result for $\bar{\rho}^{(k+1)}$.

Boundedness of $\bar{\rho}^{(k+1)}$ and $u^{(k)}$. Let us show that for all $T \geq 0$, $\bar{\rho}^{(k+1)}$ and $u^{(k)}$ are both bounded on $[0,T] \times \mathbb{T}^d$, with a bound independent of T. Using Young's convolution inequality and the induction hypothesis, we have

$$\|u^{(k)}(t,\cdot)\|_{L^\infty} \leq \|K\|_{L^1}\|\bar{\rho}^{(k)}(t,\cdot)\|_{L^\infty} \leq \|K\|_{L^1}\|\mu_0\|_{L^\infty}.$$

Now $\bar{\rho}^{(k+1)}$ is the unique solution of

$$\begin{aligned} \partial_t\bar{\rho}^{(k+1)} &= -(u^{(k)}\cdot\nabla)\bar{\rho}^{(k+1)} + \Delta\bar{\rho}^{(k+1)}, \\ \bar{\rho}^{(k+1)}(0,x) &= \mu_0(x). \end{aligned}$$

For $t \geq 0$, let $Z_s^{(k+1)}$ be the strong solution of the following stochastic differential equation for $s \in [0,t]$:

$$dZ_s^{(k+1)} = \sqrt{2}\,dB_s - u^{(k)}(t-s, Z_s)\,ds,$$

which exists, is unique and non-explosive since $u^{(k)}$ is smooth, bounded and Lipschitz continuous. Then

$$\bar{\rho}^{(k+1)}(t,x) = \mathbb{E}_x(\mu_0(Z_t^{(k+1)})).$$

We thus get

$$\|\bar{\rho}_t^{(k+1)}\|_{L^\infty} \leq \|\mu_0\|_{L^\infty}.$$

Notice that this is simply a probabilistic way of presenting the use of the maximum principle.

Boundedness of the derivatives of $\bar{\rho}^{(k+1)}$ and $u^{(k)}$. The boundedness of the derivatives of $u^{(k)}$ is a direct consequence of the boundedness of the derivatives of $\bar{\rho}^{(k)}$ thanks to Young's convolution inequality. The proof for $\bar{\rho}^{(k+1)}$ is similar to the proof of Lemma 2.2, using the boundedness of the derivatives of $u^{(k)}$. To show that the bounds are in fact independent of k, we follow the proof of Lemma 2.2, i.e. we argue by induction on the order of the derivative, and in each induction step we prove that both the integrated and uniform bounds are independent of k. This comes from the fact that the proof initially only relies on the bounds on $\|\bar{\rho}_t^{(k+1)}\|_{L^\infty}$ and $\|u_t^{(k)}\|_{L^\infty}$ – which, as we have shown, only depend on $\|\mu_0\|_{L^\infty}$ – and then, for each induction step, on the initial condition and on the bounds constructed at the previous step (therefore independent of k). The bounds concerning the derivatives involving time are then obtained thanks to the bounds on the space derivatives using (A.1). ∎

Proof of Theorem 2.1. It is sufficient to prove existence and uniqueness of the solution in $[0,T]\times\mathbb{T}^d$ for all $T\geq 0$, since then the solutions on $[0,T_1]\times\mathbb{T}^d$ and $[0,T_2]\times\mathbb{T}^d$, with $T_1<T_2$, must coincide in $[0,T_1]\times\mathbb{T}^d$, leading to the existence and uniqueness of the global solution in $\mathbb{R}^+\times\mathbb{T}^d$. Let $T\geq 0$.

Existence in $[0,T]\times\mathbb{T}^d$ *for* T *small enough.* Let us show the existence of the limit solution. We consider here T to be small enough (an explicit bound will be given later). Let

$$G(t,x)=\sum_{k\in\mathbb{Z}^d}\frac{1}{(4\pi t)^{d/2}}\exp\left(-\frac{|x+k|^2}{4t}\right)$$

be the heat kernel on the d-dimensional torus. We have

$$\bar{\rho}^{(k)}(t,x)=G(t,\cdot)*\mu_0(x)-\int_0^t\int_{\mathbb{T}^d}G(t-s,x-y)u^{(k-1)}(s,y)\cdot\nabla_y\bar{\rho}^{(k)}(s,y)\,dy\,ds.$$

Set $N_k(t)=\sup_{0\leq s\leq t}\|\bar{\rho}^{(k+1)}(s,\cdot)-\bar{\rho}^{(k)}(s,\cdot)\|_{L^\infty}$. Using $\nabla_y\cdot u^{(k)}=0$, we have

$$\begin{aligned}
&\bar{\rho}^{(k+1)}(t,x)-\bar{\rho}^{(k)}(t,x)\\
&\quad=-\int_0^t\int_{\mathbb{T}^d}\nabla_y G(t-s,x-y)\big(\bar{\rho}^{(k+1)}(s,y)-\bar{\rho}^{(k)}(s,y)\big)u^{(k)}(s,y)\,dy\,ds\\
&\qquad-\int_0^t\int_{\mathbb{T}^d}\nabla_y G(t-s,x-y)\bar{\rho}^{(k)}(s,y)\big(u^{(k)}(s,y)-u^{(k-1)}(s,y)\big)\,dy\,ds.
\end{aligned}$$

Observe that (using the first moment of the chi distribution), for some constant $\beta>0$ we have

$$\int_{\mathbb{T}^d}|\nabla_x G(t,x)|\,dx\leq\beta t^{-1/2}.$$

We thus get

$$\begin{aligned}
&\|\bar{\rho}^{(k+1)}(t,\cdot)-\bar{\rho}^{(k)}(t,\cdot)\|_{L^\infty}\\
&\qquad\leq\beta\|K\|_{L^1}\|\mu_0\|_{L^\infty}\int_0^t(t-s)^{-1/2}\|\bar{\rho}^{(k+1)}(s,\cdot)-\bar{\rho}^{(k)}(s,\cdot)\|_{L^\infty}\,ds\\
&\qquad\quad+\beta\|\mu_0\|_{L^\infty}\int_0^t(t-s)^{-1/2}\|u^{(k)}(s,\cdot)-u^{(k-1)}(s,\cdot)\|_{L^\infty}\,ds
\end{aligned}$$

and

$$\|u^{(k)}(s,\cdot)-u^{(k-1)}(s,\cdot)\|_{L^\infty}\leq \|K\|_{L^1}\|\bar\rho^{(k)}(s,\cdot)-\bar\rho^{(k-1)}(s,\cdot)\|_{L^\infty}.$$

Therefore

$$\begin{aligned}N_k(t)\leq \beta\|K\|_{L^1}\|\mu_0\|_{L^\infty}&\int_0^t (t-s)^{-1/2}N_k(s)\,ds\\ &+\beta\|K\|_{L^1}\|\mu_0\|_{L^\infty}\int_0^t (t-s)^{-1/2}N_{k-1}(s)\,ds.\end{aligned}$$

Denoting $C=\beta\|K\|_{L^1}\|\mu_0\|_{L^\infty}$ we get

$$N_k(t)\leq C\int_0^t (t-s)^{-1/2}(N_k(s)+N_{k-1}(s))\,ds. \tag{A.4}$$

Since N_k is continuous, there exists $R>0$ such that for all $t\in[0,T]$ we have $N_k(t)\leq R$. We thus have, using this bound in (A.4) and assuming $2C\sqrt{T}\leq 1/2$,

$$\begin{aligned}N_k(t)&\leq RC\int_0^t (t-s)^{-1/2}\,ds+C\int_0^t (t-s)^{-1/2}N_{k-1}(s)\,ds\\ &\leq \frac{R}{2}+C\int_0^t (t-s)^{-1/2}N_{k-1}(s)\,ds.\end{aligned}$$

We use this bound in (A.4) to get

$$\begin{aligned}N_k(t)\leq \frac{R}{2}C\int_0^t (t-s)^{-1/2}\,ds&+C\int_0^t (t-s)^{-1/2}N_{k-1}(s)\,ds\\ &+C^2\int_0^t\int_0^s (t-s)^{-1/2}(s-u)^{-1/2}N_{k-1}(u)\,du\,ds.\end{aligned}$$

We deal with the last term:

$$\begin{aligned}&C^2\int_0^t\int_0^s (t-s)^{-1/2}(s-u)^{-1/2}N_{k-1}(u)\,du\,ds\\ &\quad=C^2\int_0^t N_{k-1}(u)\int_u^t (t-s)^{-1/2}(s-u)^{-1/2}\,ds\,du=C^2\pi\int_0^t N_{k-1}(u)\,du.\end{aligned}$$

Let $\alpha=\sqrt{T}\pi C$ and choose T such that $\alpha\leq 1/2$ (which in turn also yields the previous condition $2C\sqrt{T}\leq 1/2$). We have

$$\begin{aligned}&\alpha C\int_0^t (t-s)^{-1/2}N_{k-1}(s)\,ds-C^2\pi\int_0^t N_{k-1}(s)\,du\\ &\qquad\qquad=C\int_0^t N_{k-1}(s)\big(\alpha(t-s)^{-1/2}-\pi C\big)\,ds,\end{aligned}$$

and since $\alpha=\sqrt{T}\pi C\geq\sqrt{t-s}\,\pi C$ for $0\leq s\leq t\leq T$, we get

$$\alpha C\int_0^t (t-s)^{-1/2}N_{k-1}(s)\,ds\geq C^2\pi\int_0^t N_{k-1}(s)\,du,$$

and thus

$$N_k(t) \le \frac{R}{4} + C(1+\alpha) \int_0^t (t-s)^{-1/2} N_{k-1}(s)\, ds.$$

Iterating this method we obtain, for all $n \in \mathbb{N}$,

$$N_k(t) \le 2^{-n} R + C(1+\alpha+\cdots+\alpha^{n-1}) \int_0^t (t-s)^{-1/2} N_{k-1}(s)\, ds,$$

and thus

$$N_k(t) \le 2C \int_0^t (t-s)^{-1/2} N_{k-1}(s)\, ds.$$

We now show that this implies that

$$N_k(t) \le N_0(T) \Big(2C\Gamma\Big(\frac{1}{2}\Big)\Big)^k t^{k/2} \Gamma\Big(\frac{k+2}{2}\Big)^{-1}, \tag{A.5}$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$. Indeed, for $k=0$, (A.5) is satisfied and, by induction, we have

$$\int_0^t (t-s)^{-1/2} s^{k/2}\, ds = t^{(k+1)/2} \int_0^1 (1-u)^{-1/2} u^{k/2}\, du = t^{(k+1)/2} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k+3}{2})}.$$

Using the fact that $\Gamma(k+1) = k!$ and $\Gamma(k+\frac{3}{2}) = k!\Gamma(\frac{1}{2})$, we find that $\sum_{k=0}^\infty N_k(t)$ converges uniformly for $t \in [0,T]$ and the limits

$$\bar\rho(t,x) = \lim_{k\to\infty} \bar\rho^{(k)}(t,x) \quad \text{and} \quad u(t,x) = \lim_{k\to\infty} u^{(k)}(t,x)$$

exist in $\mathcal{C}([0,T]\times\mathbb{T}^d)$. Now, since for all $l, n \in \mathbb{N}$ and all $\alpha_1,\dots,\alpha_n$, $\|\partial_t^l \partial_{\alpha_1,\dots,\alpha_n} \bar\rho^{(k)}\|_{L^\infty}$ and $\|\partial_t^l \partial_{\alpha_1,\dots,\alpha_n} u^{(k)}\|_{L^\infty}$ are bounded uniformly in k, using the Arzelà–Ascoli theorem we have uniform convergence of the derivatives, up to extracting a subsequence. Hence the validity of the limits in $\mathcal{C}^\infty([0,T]\times\mathbb{T}^d)$, i.e. there is convergence of the functions along with their derivatives of all orders in $[0,T]\times\mathbb{T}^d$. This implies that the limit $\bar\rho$ satisfies (2.1).

Uniqueness in $[0,T]\times\mathbb{T}^d$. Suppose $\bar\rho^1$ and $\bar\rho^2$ are two bounded solutions of (2.1) on $[0,T]\times\mathbb{T}^d$. Then

$$\partial_t(\bar\rho^1-\bar\rho^2) - \Delta(\bar\rho^1-\bar\rho^2) = -(K*\bar\rho^1)\cdot\nabla(\bar\rho^1-\bar\rho^2) - \nabla\cdot\big((K*\bar\rho^1 - K*\bar\rho^2)\bar\rho^2\big),$$

so that

$$\begin{aligned}
&\bar\rho^1(t,x) - \bar\rho^2(t,x) \\
&\quad= -\int_0^t \int_{\mathbb{T}^d} \nabla_y G(t-s,x-y)\cdot\big(K *_y \bar\rho^1(s,y)\big)(\bar\rho^1(s,y)-\bar\rho^2(s,y))\, dy\, ds \\
&\qquad - \int_0^t \int_{\mathbb{T}^d} \nabla_y G(x-y,t-s)\cdot\big(K *_y \bar\rho^1(s,y) - K *_y \bar\rho^2(s,y)\big)\bar\rho^2(s,y)\, dy\, ds.
\end{aligned}$$

Let $N(t) := \sup_{0\le s\le t} \|\bar\rho^1(s,\cdot) - \bar\rho^2(s,\cdot)\|_{L^\infty}$. Recall

$$\|K * \bar\rho^1(s,\cdot) - K * \bar\rho^2(s,\cdot)\|_{L^\infty} \le \|K\|_{L^1}\|\bar\rho^1(s,\cdot) - \bar\rho^2(s,\cdot)\|_{L^\infty},$$

which implies, as previously, the existence of a constant C such that

$$N(t) \le C\int_0^t (t-s)^{-1/2} N(s)\, ds.$$

We choose $L > 0$ such that $C\int_0^T s^{-1/2} e^{-Ls}\, ds \le 1/2$, and let $Q(t) = e^{-Lt}N(t)$. Then for all t,

$$Q(t) \le C\int_0^t (t-s)^{-1/2} Q(s) e^{-L(t-s)}\, ds.$$

Let $R > 0$ be such that $Q(t) \le R$. Then

$$Q(t) \le RC\int_0^t (t-s)^{-1/2} e^{-L(t-s)}\, ds \le \frac{R}{2}.$$

By induction, we get $N(t) = 0$ for $t \in [0,T]$. This concludes the proof of uniqueness.

Existence in $\mathbb{R}^+ \times \mathbb{T}^d$. For T small enough, there exists a solution in $[0,T]\times\mathbb{T}^d$. Notice that T only depends on constants independent of time (it depends on the L^∞ bound of the initial condition, which we have shown propagates). It is therefore possible to construct the (unique) smooth solution on all intervals $[t_0, T+t_0]\times\mathbb{T}^d$. Uniqueness allows us to iteratively construct the (unique) smooth solution on $\mathbb{R}^+\times\mathbb{T}^d$. This concludes the proof.

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