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On a Conjecture of P. Jorgensen and W. Klink

By

Bernard HELFFER*

In [J. K], the authors conjecture that if $\lambda_n(\alpha)$ is the series of the eigenvalues of the operator:

$$H_\alpha = -\Delta_x + (\alpha + x_1 x_2)^2 \quad \text{in } \mathbb{R}^2 (\alpha \in \mathbb{R})$$

then each of the $\lambda_n(\alpha)$ is continuous in α.

It is relatively easy to prove this conjecture using the technics of [HE-NO]$_3$ combined with the more classical technics of Kato [KA]. Using technics of Helffer-Sjöstrand [HE-SJ]$_{1,2}$, we study also the asymptotic behavior of $\lambda_n(\alpha)$ for $|\alpha| \to \infty$. Let us first recall briefly some results which are used in the proof of the hypoellipticity of invariant homogeneous operators on stratified nilpotent groups [HE-NO]$_3$. Let us consider a stratified nilpotent Lie algebra $\mathfrak{G} = \mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_r$ of rank r and $P = \sum_i Y_i^2 \in \mathfrak{U}_2(\mathfrak{G})$ where Y_i is a basis of $\mathfrak{G}_1$. For each irreducible $\pi \in \hat{G}$, $\pi(P)$ is an essentially self-adjoint operator starting from C_π^∞ and the self-adjoint extension has as domain the Sobolev space $H_\pi^2 = \{u \in H_\pi^0, \pi(Y^\beta) u \in H_\pi^0 \text{ for } |\beta| \le 2\}$. H_π^0 is the space of the representation and we put the norm:

$$u \to \Big(\sum_{|\beta| \le 2} \|\pi(Y^\beta) u\|^2\Big)^{1/2} \quad \text{on } H_\pi^2.$$

Moreover $\pi(P)$ has compact resolvent (cf. chapter II of [HE-NO]$_3$). The §1. 6 of the same chapter in this book "How to recognize an induced representation" gives a useful criterion to recognize that:

$$H_\alpha = \pi_{l_\alpha}\Big(\sum_{i=1}^{3} Y_i^2\Big) \quad \text{for some } l_\alpha \in \mathfrak{G}^*$$

where $\mathfrak{G}$ is a stratified Lie algebra with 3 generators of rank 3

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* Centre de Mathématiques de l'Ecole Polytechnique, Plateau de Palaiseau-91128 Palaiseau Cedex, France.
Current address: Departement de Mathématiques et D'Informatique, Université de Nantes, Chemin de la Houssinière, 44072 Nantes Cedex 03, France.

satisfying to

$$\dim \mathfrak{G}_1 = 3, \quad \mathfrak{G}_2 = \boldsymbol{R}[Y_1, Y_3] + \boldsymbol{R}[Y_2, Y_3]$$
$$[Y_1, Y_2] = 0, \quad \dim \mathfrak{G}_3 = 1$$

and π_{l_α} is the irreducible representation attached by the Kirillov map to $l_\alpha = (\underbrace{0, 0, \alpha}_{\mathfrak{G}_1}\, ;\, \underbrace{0, 0}_{\mathfrak{G}_2}\, ;\, \underbrace{1}_{\mathfrak{G}_3}) \in \mathfrak{G}^*$

Observe now that P is hypoelliptic and therefore we have (because $l_\alpha \restriction_{\mathfrak{G}_3} = 1$) the existence of a constant C such that: $\forall \alpha \in \boldsymbol{R}$:

$$\sum_{|\beta| \leq 2} ||\pi_{l_\alpha}(Y^\beta) u||^2 \leq C ||\pi_{l_\alpha}(P) u||^2, \quad \forall u \in H^2_{\pi_{l_\alpha}}. \tag{1}$$

That means in particular:

$$\exists C > 0,\ \forall \alpha \in \boldsymbol{R},\ ||u||^2_{H^2(\boldsymbol{R}^2)} + \sum_{0 \leq k \leq 2} ||(\alpha + x_1 x_2)^k u||^2_{L^2(\boldsymbol{R}^2)} \leq C ||H_\alpha u||^2_{L^2(\boldsymbol{R}^2)}. \tag{2}$$

Using interpolation and the inequality

$$||\alpha u||_{H^0_{\pi_{l_\alpha}}} \leq C ||u||_{H^4_{\pi_{l_\alpha}}} \tag{3}$$

we can complete (2) in proving the existence of $C > 0$ s. t $\forall \alpha$

$$(1 + |\alpha|) ||u||^2_{L^2(\boldsymbol{R}^2)} + \sum_{0 \leq k \leq 2} ||(\alpha + x_1 x_2)^k u||^2_{L^2(\boldsymbol{R}^2)} \leq C ||H_\alpha u||^2_{L^2(\boldsymbol{R}^2)}. \tag{4}$$

A consequence of (4) and of the trivial fact that H_α is positive is that the first eigenvalue of H_α satisfies the minoration:

$$\lambda_1(\alpha) \geq \frac{1}{\sqrt{C}} [1 + |\alpha|]^{1/2}. \tag{5}$$

In particular, H_α is invertible for all α and his spectrum is contained in $\left[\frac{1}{\sqrt{C}} [|\alpha| + 1]^{1/2}, +\infty\right]$.

Continuity of $\lambda_n(\alpha)$ The continuity follows from theorem 3. 16 chapter IV p. 212 and remarks in § 3. 5 p. 213–214 in the book of Kato [KA]. It is sufficient to observe that the H_α are uniformly semi-bounded and that for each $\alpha_0 \in \boldsymbol{R}$: $H_\alpha \to H_{\alpha_0}$ as $\alpha \to \alpha_0$ in the generalized sense i. e.

$$||H_\alpha^{-1} - H_{\alpha_0}^{-1}||_{\mathscr{L}(L^2, L^2)} \xrightarrow[\alpha \to \alpha_0]{} 0.$$

For $u \in \mathcal{S}$, we have:

$$(H_\alpha^{-1} - H_{\alpha_0}^{-1}) u = -H_\alpha^{-1} (H_\alpha - H_{\alpha_0}) H_{\alpha_0}^{-1} u.$$

Note that $H_\alpha^{-1}\in\mathfrak{L}(\mathfrak{H},\mathfrak{H})$; this is proved in the rank 3 case in [HE-NO]$_1$, in the rank r case in [HE-NO]$_2$ or in Melin [ME]). Observe now that $H_\alpha-H_{\alpha_0}=(\alpha-\alpha_0)^2+2(\alpha-\alpha_0)(\alpha_0+x_1x_2)$ and from the relations

$$\begin{cases}(H_\alpha-H_{\alpha_0})H_{\alpha_0}^{-1}=2(\alpha-\alpha_0)[(\alpha_0+x_1x_2)H_{\alpha_0}^{-1}]+(\alpha-\alpha_0)^2H_{\alpha_0}^{-1}\\ \|(H_\alpha-H_{\alpha_0})H_{\alpha_0}^{-1}\|_{\mathfrak{L}(L^2,L^2)}\leq\tilde{C}|\alpha-\alpha_0| \quad (\text{using}(4)) \text{ for } |\alpha-\alpha_0|<1/\tilde{C}\\ \|H_\alpha^{-1}\|\leq\tilde{C}\end{cases}$$

we get

(6) $$\|H_\alpha^{-1}-H_{\alpha_0}^{-1}\|\leq C|\alpha-\alpha_0| \qquad \#$$

Let us prove now that H_α is an analytic family in the sense of Kato. We have to look to the limit of

$$\frac{H_\alpha^{-1}-H_{\alpha_0}^{-1}}{\alpha-\alpha_0}$$

for $\alpha\to\alpha_0$ (α and $\alpha_0\in\boldsymbol{C}$) and to prove that this limit is: $-2H_{\alpha_0}^{-1}(\alpha_0+x_1x_2)H_{\alpha_0}^{-1}$. This is easy to verify if we observe that:

$$\frac{(H_\alpha-H_{\alpha_0})H_{\alpha_0}^{-1}}{(\alpha-\alpha_0)}-2(\alpha_0+x_1x_2)H_{\alpha_0}^{-1}=3(\alpha-\alpha_0)H_{\alpha_0}^{-1}$$

(Here we have used that the maximal estimates (2) and (4) are true for $\alpha\in\boldsymbol{C}$ with $|\mathrm{Im}\,\alpha|<\varepsilon$ for $\varepsilon>0$ small enough).
As a corollary we get by the Kato-Rellich's theorem (Th XII 8) in [RE-SI]) and the Rellich's theorem (Th XII. 3 in [RE-SI]) that:

(7) The eigenvalues are analytic in α (that means that is the nhd of some α_0 the eigenvalues of H_α which are near $\lambda_n(\alpha_0)$ cas be described by analytic functions of α.
This answers to a question of P. Jorgensen [J]. In particular:

(8) The first eigenvalue (which is simple) $\lambda_1(\alpha)$ is analytic.

Asymptotic for $|\alpha|\to\infty$. In a personnal letter [J] P. Jorgensen asks if the $\lambda_n(\alpha)$ are simple, monotone. We just write the type of results it is reasonable to prove and give the principal steps of "a proof" (we have not verified all the details).

Conjecture. For $\alpha\to+\infty$, we have

(9) (i) $\lambda_{2k+1}(\alpha) \equiv \lambda_{2k+2}(\alpha) \bmod (0(|\alpha|^{-\infty})$ $\forall k \in \boldsymbol{N}$

(10) (ii) $\lambda_{2k+2}(\alpha) \sim E_2\alpha^{1/2} + E_3(2k+1)\alpha^{-1/4} + \sum_{j\geq 4} E_j^k \alpha^{2-3j/4}$ $\forall k \in \boldsymbol{N}.$

$\forall k, \forall \varepsilon > 0, \exists C_\varepsilon^k > 0$

(11) (iii) $|\lambda_{2k+2}(\alpha) - \lambda_{2k+1}(\alpha)| \geq C_k^\varepsilon e^{-(S_0+\varepsilon)|\alpha|^{3/2}}$

with $S_0 = 4\sqrt{2}/3$.

Remark 1. Using the symmetry $(x_1, x_2) \to (-x_1, x_2)$, we see that:

(12) $$\lambda_n(\alpha) = \lambda_n(-\alpha) \qquad \forall n \in \boldsymbol{N}^*, \ \forall \alpha \in \boldsymbol{R}$$

so we hope similar results (9), (10), (11) for $\alpha \to -\infty$. #

"Proof of the conjecture"

We make the scaling: $x_1 = \sqrt{\alpha}\ y_1$, $x_2 = -\sqrt{\alpha}\ y_2$ and we get the unitary equivalent operator:

(13) $$\alpha^2 \tilde{H}_\alpha(y, D_y) \qquad \text{with}$$

(14) $$\tilde{H}_\alpha(y, D_y) = \frac{1}{\alpha^3}(D_{y_1}^2 + D_{y_2}^2) + (1 - y_1 y_2)^2.$$

We recognize the Schrödinger operator:

(15) $$\tilde{H}_\alpha = P(h) = -h^2\Delta + V$$

with $h = \alpha^{-3/2}$ and $V = (1 - y_1 y_2)^2$

and we are interested in studying the asymptotic behavior as $h \to 0$ of the first eigenvalues of $P(h)$.

We then follow the strategy of [HE-SJ]$_{1,2}$. We see that

(16) $$V^{-1}(0) = \{y_1 y_2 = 1\} = U_1 \cup U_2$$

where
$$U_1 = \{(y_1 y_2 = 1\} \cap y_1 < 0\}$$
$$U_2 = \{(y_1 y_2 = 1\}) \cap y_1 > 0\}$$

U_1 and U_2 are disjoint wells and the symmetry $(x_1, x_2) \to (-x_1, -x_2)$ sends U_1 on U_2 and U_2 and U_1. This explains why we have (i), and in fact we can hope that the eigenvalues $\mu_k(h)$ verify

(17) $$|\mu_{2k+2}(h) - \mu_{2k+1}(h)| \leq C_k^\varepsilon e^{-S_0/h} e^{\varepsilon/h}$$

where S_0 is the Agmon distance attached to the metric $V\ dy^2$ between the two wells U_1 and U_2. Between U_1 and U_2 there is a unique minimal geodesic which is the straight line between $(-1, -1)$ and $(+1, +1)$ and we get:

$$S_0=\sqrt{2}\int_{-1}^{+1}(1-y_1^2)\,dy_1=4\sqrt{2}/3\,. \tag{18}$$

The link between $\mu_k(h)$ and $\lambda_k(\alpha)$ is given according to (13)-(15) by

$$\lambda_k(\alpha)=\alpha^2\mu_k(\alpha^{-3/2}) \qquad \text{for } \alpha>0\,. \tag{19}$$

According to $[\text{HE-SJ}]_2$ and to be "almost" in the case treated in this paper (in $[\text{HE-SJ}]_2$ the U_i are assumed to be compact but I think that $[\text{HE-SJ}]_2$ can be applied in the case treated here), we have a problem with two miniwells inside two wells.

Let us look to the V''_m at a point m of U_2; the eigenvalues are 0 and $2(y_1^2+y_2^2)$ *where* $m=(y_1,y_2)$. The non zero eigenvalue admits a minimum at $m=(1,1)$ in U_2, and it is proved in $[\text{HE-SJ}]_2$ that the eigenfunctions are localized in the miniwells $(-1,-1)$ and $(+1,+1)$.

To get (10), we have to construct W-K-B approximate eigenfunctions of the type $a(x,h)e^{-\varphi/h-\phi/\sqrt{h}}$ (see $[\text{HE-SJ}]_2$) near $(1,1)$ (for example). As in $[\text{HE-SJ}]_2$, we get the expansion

$$\mu_{2k}(h)\sim E_2h+E_3h^{3/2}+\sum_{j\geq 4}E_j^kh^{j/2} \tag{20}$$

with $E_2=\min\limits_{(y_1,y_2)\in U_2}\sqrt{y_1^2+y_2^2}$, $E_3>0$ (In $[\text{HE-SJ}]_2$, the proof is given for the first eigenvalue). The estimate (10) corresponds to

$$|\mu_{2k+2}(h)-\mu_{2k+1}(h)|\geq \tilde{C}_k^{\varepsilon}e^{-(S_0+\varepsilon)/h} \tag{21}$$

and corresponds to a minoration of the splitting between the eigenvalues due to the tunneling effect.

(21) is relatively clear in the case $k=0$. The case $k>0$ is probably true also adapting technics of A. Martinez [MA].

Remark 2. If the conjecture is true, for each N_0, $\exists C_0$ s.t all the eigenvalues $\lambda_k(\alpha)$ are simple for $k\leq N_0$ and $|\alpha|>C_0$.

If you observe the analyticity in α, for $k\in N$ fixed then $\lambda_k(\alpha)$ is simple and analytic outside a finite number of points. #

Remark 3. Let us give now a slightly different approach to localize the $\lambda_k(\alpha)$ as $\alpha\to\pm\infty$. The results are weaker but the proof is relatively general and uses group theoretical methods. Let us consider the operator $\frac{1}{\sqrt{\alpha}}H_\alpha$ for $\alpha>0$. We have $\frac{1}{\sqrt{\alpha}}\circ H_\alpha=\pi_{l_\alpha}(\sum_{i=1}^{3}Y_i^2)$ with

$\tilde{l}_\alpha = (0, 0, \alpha^{3/4}, 0, 0, \alpha^{-1/2})$. Associated to $\tilde{l}_\alpha$, let us consider the orbit of $\tilde{l}_\alpha$:

$$G.\tilde{l}_\alpha = \{l \in \mathfrak{G}^* \text{ s. t } \exists (\eta_1, \eta_2, y_1, y_2) \in \mathbf{R}^q \text{ s. t } (\eta_1, \eta_2, \alpha^{3/4}(1+y_1 y_2), y_1, y_2, \alpha^{-3/4}).$$

Let us introduced the set

$$\mathfrak{L}(\tilde{l}_\alpha) = \{l \in \mathfrak{G}^* \text{ s. t } \exists \alpha_n \text{ with } \alpha_n \to \infty \text{ and } l^{(n)} \in (Gl_{\alpha_n}) \text{ s. t } l^{(n)} \to l\}.$$

Then using the technics of Helffer-Nourrigat [HE-NO]$_3$ (see Theorem 4.9 in [HE]$_2$) it is possible to prove that

$$\lim_{|\alpha| \to \infty} \frac{\lambda_k(\alpha)}{\sqrt{\alpha}} \in \overline{\bigcup_{l \in \mathfrak{L}(\tilde{l}_\alpha)} Sp\ \pi_l(\textstyle\sum Y_i^2)}\ .$$

This type of argument appears in similar contexts in [HE]$_1$ (after Helffer-Métivier-Nourrigat) and in [HE-MO].

Let us just describe $\mathfrak{L}(\tilde{l}_\alpha)$ and the family of the $Sp(\pi_l(\sum Y_i^2))$

$$\mathfrak{L}(\tilde{l}_\alpha) = \{l \in \mathfrak{G}^* \text{ s. t } \exists (\eta_1, \eta_2, \eta_3, y_1, y_2) \in \mathbf{R}^4$$

with $y_1 y_2 = -1$ and

$$l = (\eta_1, \eta_2, \eta_3, y_1, y_2, 0)\}$$

Then it is not to difficult to verify that $Sp(\Pi_l(\sum_{i=1}^n Y_i^2)) = [\sqrt{y_1^2 + y_2^2}, +\infty]$.

As a corollary we get that for $|\alpha| > \alpha_0$ we have: $\lambda_1(\alpha) \geq \sqrt{2}\,|\alpha|^{1/2}$ which is coherent with (19) and (20).

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