

## Modular Interpolation Spaces I

M. KRBEČ

Es wird eine Interpolationsmethode in modularen Räumen definiert, und Grundeigenschaften der Interpolationsräume werden untersucht (Vollständigkeit, Einbettungen usw.). Das wichtigste Resultat ist hier ein Satz über die Stabilität der Methode. Ferner wird gezeigt, wie die Methode in ORLICZ-Räumen arbeitet, und als ein Beispiel von möglichen Anwendungen ist ein Satz über Multiplikatoren vom MICHLIN-Typ bewiesen. Die ausgearbeitete Methode verallgemeinert die  $K$ -Methode von PÉTRÉ.

Определяется интерполяционный метод в модулярных пространствах и изучаются основные свойства полученных интерполяционных пространств (полнота, вложения и т.д.). Самым важным результатом является здесь теорема об устойчивости метода. Показывается, как метод работает в пространствах Орлича и в качестве примера применений доказана теорема типа Михлина о мультипликаторах. Метод обобщает  $K$ -метод ПЕТРЕ.

An interpolation method in modular spaces is introduced and basic properties of obtained interpolation spaces are studied (completeness, imbeddings etc.). The main result here is a reiteration theorem. It is shown how the method works in Orlicz spaces and as an example of applications there is proved a multiplier theorem of the Michlin type. The method generalizes the  $K$ -method of PÉTRÉ.

### 1. Introduction

The aim of this paper is to develop a part of a basic theory of interpolation of modular spaces. We shall use an approach based on a suitable generalization of the  $\bar{L}$ -functional.

There are several papers dealing with the concrete case of Orlicz spaces — they contain an extension of the “basic interpolation property” — i.e. an appropriate version of the Riesz-Thorin theorem. It was shown (see [12, 11]) that the Orlicz space  $L_\Phi$  is an interpolation space with respect to  $L_{\Phi_0}$  and  $L_{\Phi_1}$ , where

$$\Phi^{-1} = (\Phi_0^{-1})^{1-\theta} (\Phi_1^{-1})^\theta \quad (0 < \theta < 1)$$

or, more generally,

$$\Phi^{-1} = \Phi_0^{-1} h \left( \frac{\Phi_1^{-1}}{\Phi_0^{-1}} \right)$$

with some concave function  $h$  (or with some  $h$  equivalent to a concave one). For this case see [2].

Our goal will be to develop a wider theory applicable to Orlicz and Sobolev-Orlicz spaces. The substantial difficulties are, clearly, connected with the structure of spaces of the Orlicz type — they are caused, in part, by the non-homogeneity of Young functions and the rather non-constructive definition of the norm. The last

fact is, above all, the reason why the known interpolation theory (having been, roughly speaking, developed and successfully applied namely to  $L_p$ -type spaces) has not found use in the theory of Orlicz spaces and related ones. There is a more favourable approach which was indicated by J. PEETRE in [10] in a concrete "Orlicz case". We shall make use of this natural idea. For this purpose it will be reasonable to change a little the notion of the interpolation space and interpolation properties — in accordance with the special structure of modular spaces and, especially, of the Orlicz ones. Then we prove assertions concerning basic properties of spaces obtained.

In the prepared paper [6] there will be dealt with another variant of the presented method, with trace spaces and applications to imbedding theorems.

We begin with several definitions.

1.1. Definition. Let  $X$  be a (real) linear space. A function  $\varrho: X \rightarrow \langle 0, \infty \rangle$  is said to be the *modular* (on  $X$ ) if

- (i)  $\varrho(x) = 0 \Leftrightarrow x = 0$ ,
- (ii)  $\varrho(-x) = \varrho(x)$ ,  $x \in X$ ,
- (iii)  $\varrho(\alpha x + \beta y) \leq \alpha \varrho(x) + \beta \varrho(y)$  for all  $x, y \in X$  and  $\alpha, \beta \geq 0$ ,  $\alpha + \beta = 1$ .

Set

$$X(\varrho) = \{x \in X; \lim_{\lambda \rightarrow 0} \varrho(\lambda x) = 0\}.$$

The *modular space* ( $m$ -space) is the couple  $(X(\varrho), \varrho)$ . (See [9].)

For our purposes it will be sufficient to suppose that always  $X = X(\varrho)$  and we shall write only  $(X, \varrho)$  or  $X$ .

In any  $m$ -space  $X = (X, \varrho)$  one can introduce the (Luxemburg) norm

$$\|x\|_X = \inf \left\{ \lambda > 0; \varrho \left( \frac{x}{\lambda} \right) \leq 1 \right\}.$$

The important example of  $m$ -spaces is the Orlicz (Sobolev-Orlicz) space: Let  $\Phi$  be an Young function, i.e.  $\Phi: \mathbb{R} \rightarrow \langle 0, \infty \rangle$ , even, convex and

$$\lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = \lim_{t \rightarrow \infty} \frac{t}{\Phi(t)} = 0.$$

Let  $\Omega \subset \mathbb{R}^N$  be measurable and consider the modular

$$\varrho_\Phi(f) = \int_{\Omega} \Phi(f(x)) dx, \quad f \text{ measurable on } \Omega.$$

The corresponding  $m$ -space is the Orlicz space  $L_\Phi = L_\Phi(\Omega)$ . More generally, if  $\Omega$  is a domain in  $\mathbb{R}^N$  then using the modular

$$\varrho_k(f) = \int_{\Omega} \sum_{|\alpha| \leq k} \Phi(D^\alpha f(x)) dx, \quad k = 0, 1, \dots,$$

( $D^\alpha f$  being regular distribution,  $|\alpha| \leq k$ ) we get the Sobolev-Orlicz space  $W^k L_\Phi(\Omega)$ . It is also reasonable in some connections to define the space  $W^k E_\Phi(\Omega)$  — as the closure in the norm  $W^k L_\Phi(\Omega)$  of  $C^\infty$ -functions in  $\Omega$  with bounded support. (See [7].)

One can extend in a natural way the notion of the so-called  $\Delta_2$ -condition known from the theory of Orlicz spaces (see, e.g. [5, 7]).

1.2. Definition. The modular  $\varrho$  on the space  $X$  is said to satisfy the  $\Delta_2$ -condition if

$$\varrho(2x) \leq c\varrho(x) \tag{1.1}$$

for some  $c = c(\varrho) > 0$  and each  $x \in X$ . We shall also sometimes write  $\varrho \in \Delta_2$ .

In the class of modulars satisfying (1.1) there it is possible to get some growth conditions (roughly replacing the role of a homogeneity). If (1.1) holds then the function

$$\lambda \mapsto \sup_{\substack{x \in X \\ x \neq 0}} \frac{\varrho(\lambda x)}{\varrho(x)}, \quad \lambda > 0,$$

is finite and, in addition, submultiplicative and therefore (see [10] and generally [2, Part II, Chapter 7]) there exist  $p_0 = p_0(\varrho) > 0$ ,  $p_1 = p_1(\varrho) > 0$  and  $C = C(\varrho) > 0$  such that

$$\varrho(\lambda x) \leq C \max(\lambda^{p_0}, \lambda^{p_1}) \varrho(x), \quad x \in X, \lambda > 0.$$

1.3. Definition. Let  $(X, \varrho)$  and  $(Y, \bar{\varrho})$  be  $m$ -spaces. A linear mapping  $T$  from  $X$  to  $Y$  will be called  $m$ -continuous and it will be written  $T: X \rightarrow Y$  if there exists  $\gamma > 0$  such that

$$\bar{\varrho}(\gamma Tx) \leq \varrho(x). \tag{1.2}$$

1.4. Remark. It is clear that every  $m$ -continuous mapping is also continuous. The converse does not generally hold. Nevertheless, this is not any large restriction. Many important operators in Orlicz type spaces are  $m$ -continuous or one can choose suitable modulars (in order to reach the  $m$ -continuity — see, e.g. Corollary 4.2). Here, we give an example of an  $m$ -continuous imbedding: Let  $\Omega \subset \mathbb{R}^N$ ,  $\mu(\Omega) = \infty$ , and let  $\Phi_0$  and  $\Phi_1$  be some Young functions. Then (see, e.g. [7, Part II, 3.17])  $L_{\Phi_1}(\Omega)$  is continuously imbedded into  $L_{\Phi_0}(\Omega)$  iff  $\Phi_1(\gamma t) \leq \Phi_0(t)$  for  $t \geq 0$  with some  $\gamma > 0$ . This leads to an inequality of the type (1.2) and it means that the imbedding is  $m$ -continuous. Similar considerations can be used for the case  $\mu(\Omega) < \infty$ , further, for the case of the imbedding of Sobolev-Orlicz spaces into trace spaces and so on.

1.5. Notation. The symbol  $\subset$  will denote an  $m$ -continuous imbedding. If  $\bar{X} = (X_0, X_1)$  and  $\bar{Y} = (Y_0, Y_1)$  are couples of  $m$ -spaces then  $T: \bar{X} \rightarrow \bar{Y}$  means that  $T: X_i \rightarrow Y_i$ ,  $i = 0, 1$ , (in the sense of Definition 1.3).

The results presented in this paper can be generalized; the condition (iii) from Definition 1.1 can be weakened.

## 2. An abstract interpolation method

We shall consider the following situation: Let  $X_0 = (X_0, \varrho_0)$  and  $X_1 = (X_1, \varrho_1)$  be  $m$ -spaces imbedded into some linear Hausdorff space. We define the spaces

$$\Sigma(\bar{X}) = X_0 + X_1 = \{x; x = x_0 + x_1 \text{ for some } x_0 \in X_0, x_1 \in X_1\}$$

and

$$\Delta(\bar{X}) = X_0 \cap X_1.$$

Let us denote by  $\|\cdot\|_i$  the norm in  $X_i$ ,  $i = 0, 1$ . It is easy to prove that  $\Sigma(\bar{X})$  and  $\Delta(\bar{X})$  are  $m$ -spaces, as well; more exactly: The norm in  $\Sigma(\bar{X})$  defined by

$$\|x\|_X = \inf_{\substack{x = x_0 + x_1 \\ x_0 \in X_0, x_1 \in X_1}} (\|x_0\|_0 + \|x_1\|_1) \tag{2.1}$$

is equivalent to the (Luxemburg) norm derived from the modular

$$\varrho_L(x) = \inf (\varrho_0(x_0) + \varrho_1(x_1)) \quad (2.2)$$

(the inf on the right hand side is here and always in analogous situations in the sequel to be taken as in (2.1)). Similarly, the  $\Delta(\bar{X})$ -norm

$$\|x\|_\Delta = \max (\|x\|_0, \|x\|_1)$$

is equivalent to the norm obtained from the modular

$$\varrho_\Delta(x) = \max (\varrho_0(x), \varrho_1(x)).$$

**2.1. Definition.** A couple  $\bar{X} = (X_0, X_1)$  of  $m$ -spaces imbedded into a linear Hausdorff space is said to be the  $m$ -interpolation couple.

An  $m$ -space  $X$  such that

$$\Delta(\bar{X}) \circ X \circ \Sigma(\bar{X})$$

will be called the  $m$ -intermediate space. If  $X$ , in addition, has the property that  $T: X \rightarrow X$  for each  $T: \bar{X} \rightarrow \bar{X}$  then  $X$  is said to be the  $m$ -interpolation space (with respect to  $\bar{X}$ ).

**2.2. Definition.** Let  $\mathfrak{M}$  be some system of  $m$ -spaces such that each  $\bar{X} \in \mathfrak{M} \times \mathfrak{M}$  forms an  $m$ -interpolation couple. A mapping  $F: \mathfrak{M} \times \mathfrak{M} \rightarrow \mathfrak{M}$  is said to be the  $m$ -interpolation functor on  $\mathfrak{M}$  if

- (i)  $\bar{X} \in \mathfrak{M} \times \mathfrak{M} \Rightarrow \Delta(\bar{X}) \circ F\bar{X} \circ \Sigma(\bar{X})$ ,
- (ii)  $\bar{X}, \bar{Y} \in \mathfrak{M} \times \mathfrak{M}, T: \bar{X} \rightarrow \bar{Y} \Rightarrow T: F\bar{X} \rightarrow F\bar{Y}$ .

In the sequel, each couple of  $m$ -spaces in question will be supposed to be an  $m$ -interpolation couple unless we recall it.

**2.3. Definition.** Let  $(X_0, \varrho_0), (X_1, \varrho_1)$  be  $m$ -spaces and  $\sigma$  a measurable positive function on  $(0, \infty)$ . Let us define the functional

$$\mathcal{L}(t, x, \bar{X}) = \inf (\varrho_0(x_0) + t\varrho_1(x_1)), \quad x \in \Sigma(\bar{X}),$$

and the modular

$$\varrho_\sigma(x) = \int_0^\infty \mathcal{L}(t, x, \bar{X}) \sigma(t) dt.$$

The corresponding  $m$ -space will be denoted by  $\bar{X}_\sigma = (X_0, X_1)_\sigma$ . We denote its norm by  $\|\cdot\|_\sigma$ .

Obviously,  $\varrho_\sigma$  is a modular — the condition  $\varrho_\sigma(x) = 0 \Rightarrow x = 0$  follows from the fact that if  $\varrho_\sigma(x) = 0$  then there is a  $t \geq 1$  such that  $0 = \mathcal{L}(t, x, \bar{X}) \geq \varrho_L(x)$ .

**2.4. Lemma.** Let  $\bar{X} = (X_0, X_1)$  be an  $m$ -interpolation couple and

$$I_\sigma = \int_0^\infty \min(1, t) \sigma(t) dt < \infty. \quad (2.3)$$

Then  $\bar{X}_\sigma$  is an  $m$ -intermediate space with respect to  $\bar{X}$ .

**Proof.** Let be  $x \in \Delta(\bar{X})$ . We have

$$\mathcal{L}(t, x, \bar{X}) \sigma(t) \leq \min(\varrho_0(x), t\varrho_1(x)) \sigma(t) \leq \min(1, t) \sigma(t) \varrho_\Delta(x).$$

This gives  $\|x\|_\sigma \leq I_\sigma \|x\|_A$ . The rest follows from the inequality

$$\min(1, t) \varrho_X(x) \leq \mathcal{L}(t, x, \bar{X}) \quad \blacksquare$$

Before we introduce the class of admissible weights  $\sigma$  it will be always supposed that (2.3) holds. The condition (2.3), however, gives a stronger result:

2.5. Theorem. (i) Let  $((X_0, \varrho_0), (X_1, \varrho_1))$  be an  $m$ -interpolation couple. Then  $\bar{X}_\sigma$  is an  $m$ -interpolation space with respect to  $(X_0, X_1)$ .

(ii) Let  $\mathfrak{M}$  be any class of  $m$ -spaces from Definition 2.2. Then the mapping

$$F: \bar{X} \rightarrow \bar{X}_\sigma, \quad \bar{X} \in \mathfrak{M} \times \mathfrak{M},$$

is an  $m$ -interpolation functor on  $\mathfrak{M}$ . More precisely: If  $\bar{X}, \bar{Y} \in \mathfrak{M} \times \mathfrak{M}$ ,  $T: \bar{X} \rightarrow \bar{Y}$  then  $T: \bar{X}_\sigma \rightarrow \bar{Y}_\sigma$  and

$$\|T\|_{\bar{X}_\sigma \rightarrow \bar{Y}_\sigma} \leq \max(\|T\|_{X_0 \rightarrow Y_0}, \|T\|_{X_1 \rightarrow Y_1}). \tag{2.4}$$

Proof. The assertion (i) follows from (ii) by the choice  $\bar{Y} = \bar{X}$ . As far (ii) is concerned, let us suppose that  $\varrho_i$  is the modular in  $X_i$ ; and  $\bar{\varrho}_i$  in  $Y_i$ ,  $i = 0, 1$ , and that

$$\bar{\varrho}_i \left( \frac{Tx}{\gamma_i} \right) \leq \varrho_i(x), \quad i = 0, 1.$$

Then  $(\bar{\varrho}_\sigma$  denotes the modular in  $\bar{Y}_\sigma$ )

$$\bar{\varrho}_\sigma \left( \frac{Tx}{\max(\gamma_0, \gamma_1)} \right) \leq \varrho_\sigma(x)$$

and (2.4) follows  $\blacksquare$

2.6. Theorem. If  $(X_1, \varrho_1)$  and  $(X, \varrho_1)$  are Banach  $m$ -spaces (i.e. complete with respect to the corresponding norms) then  $\bar{X}_\sigma$  is a Banach  $m$ -space.

Proof. Let  $\{x^n\}$  be a Cauchy sequence in  $\bar{X}_\sigma$ . Then  $(\Sigma(\bar{X}))$  is complete) there exists  $\lim x^n = x$  in  $\Sigma(\bar{X})$ . We estimate  $\|x - x^n\|_\sigma$ . Let  $\varepsilon > 0$ ,  $0 < s < G$ . Then

$$\begin{aligned} & \int_s^G \inf_{x-x^n = x_0^n + x_1^n} \left[ \varrho_0 \left( \frac{x_0^n}{2\varepsilon} \right) + t\varrho_1 \left( \frac{x_1^n}{2\varepsilon} \right) \right] \sigma(t) dt \\ & \leq \int_s^G \inf_{\substack{x-x^m = y_0^m + y_1^m \\ x^m - x^n = z_0^{mn} + z_1^{mn}}} \left[ \varrho_0 \left( \frac{y_0^m + z_0^{mn}}{2\varepsilon} \right) + t\varrho_1 \left( \frac{y_1^m + z_1^{mn}}{2\varepsilon} \right) \right] \sigma(t) dt \\ & \leq \frac{1}{2} \int_s^G \inf_{x-x^m = y_0^m + y_1^m} \left[ \varrho_0 \left( \frac{y_0^m}{\varepsilon} \right) + t\varrho_1 \left( \frac{y_1^m}{\varepsilon} \right) \right] \sigma(t) dt \\ & \quad + \frac{1}{2} \int_0^\infty \inf_{x^m - x^n = z_0^{mn} + z_1^{mn}} \left[ \varrho_0 \left( \frac{z_0^{mn}}{\varepsilon} \right) + t\varrho_1 \left( \frac{z_1^{mn}}{\varepsilon} \right) \right] \sigma(t) dt = I + J. \end{aligned}$$

It is  $I + J \leq 1$  for  $m, n$  large. Let  $m \rightarrow \infty, s \rightarrow 0, G \rightarrow \infty$  in  $I$ . We get so  $\|x - x^n\|_\sigma \leq 2\varepsilon$  for  $n$  sufficiently large  $\blacksquare$

2.7. Theorem. (i) Let  $\sigma_0$  and  $\sigma_1$  be locally bounded and suppose that

$$\sigma(t) = \mathcal{O}(\sigma_1(t)), \quad t \rightarrow 0,$$

$$\sigma(t) = \mathcal{O}(\sigma_0(t)), \quad t \rightarrow \infty.$$

Then  $\bar{X}_{\sigma_0} \cap \bar{X}_{\sigma_1} \subset \bar{X}_\sigma$ .

(ii) Let  $X_1$  be " $m$ -imbedded" into  $X_0$  (i.e. let  $\varrho_0(x) \leq \varrho_1(\gamma x)$  for each  $x \in X_1$  and some  $\gamma > 0$ ). If  $\sigma_0$  and  $\sigma_1$  are locally bounded and

$$\sigma_0(t) = \mathcal{O}(\sigma_1(t)), \quad t \rightarrow 0,$$

then  $\bar{X}_{\sigma_1} \subset \bar{X}_{\sigma_0}$ .

Proof. (i) Let be  $T > 0$  and

$$\sigma(t) \leq m\sigma_1(t), \quad t \leq T,$$

$$\sigma(t) \leq M\sigma_0(t), \quad t \geq T.$$

Further, let be  $x \in \bar{X}_{\sigma_0} \cap \bar{X}_{\sigma_1}$ ,  $\|x\|_{\sigma_i} < 1$ ,  $i = 0, 1$ , and  $C \geq 2 \max(1, m, M)$ . Then

$$\begin{aligned} \varrho_\sigma\left(\frac{x}{C}\right) &= \int_0^T \mathcal{L}\left(t, \frac{x}{C}, \bar{X}\right) \sigma(t) dt + \int_T^\infty \mathcal{L}\left(t, \frac{x}{C}, \bar{X}\right) \sigma(t) dt \\ &\leq \frac{m}{C} \int_0^\infty \mathcal{L}(t, x, \bar{X}) \sigma_1(t) dt + \frac{M}{C} \int_0^\infty \mathcal{L}(t, x, \bar{X}) \sigma_0(t) dt \leq 1. \end{aligned}$$

(ii) Let be  $C \geq \max(1, \gamma)$  and  $t \geq \gamma$ . Then

$$\varrho_0\left(\frac{x}{2C}\right) \leq \frac{1}{2C} \mathcal{L}(t, x, \bar{X}) \leq \frac{1}{2C} \varrho_0(x).$$

The second inequality is obvious; the first follows from

$$\varrho_0\left(\frac{x}{2C}\right) \leq \frac{1}{2} \inf \left[ \varrho_0\left(\frac{x_0}{C}\right) + \frac{t}{2\gamma} \varrho_1\left(\frac{\gamma x_1}{C}\right) \right] \leq \frac{1}{2C} \mathcal{L}(t, x, \bar{X}).$$

Let us still suppose that

$$C \geq \left( \int_\gamma^\infty \sigma_1(t) dt \right)^{-1} \int_\gamma^\infty \sigma_0(t) dt,$$

$$\sigma_0(t) \leq m\sigma_1(t), \quad t \leq \gamma.$$

Now, if  $x \in \bar{X}_{\sigma_1}$ , then

$$\begin{aligned} \varrho_0\left(\frac{x}{2C}\right) &= \int_0^\infty \mathcal{L}\left(t, \frac{x}{2C}, \bar{X}\right) \sigma_0(t) dt = \int_0^\gamma \dots + \int_\gamma^\infty \dots \\ &\leq m \int_0^\infty \mathcal{L}\left(t, \frac{x}{2C}, \bar{X}\right) \sigma_1(t) dt + \varrho_0\left(\frac{x}{2C}\right) \int_\gamma^\infty \sigma_0(t) dt \end{aligned}$$

It holds

$$\varrho_0\left(\frac{x}{2C}\right) \leq \frac{1}{2C} \mathcal{L}(t, x, \bar{X})$$

for  $t \geq \gamma$ . Therefore

$$\varrho_0\left(\frac{x}{2C}\right) \int_{\gamma}^{\infty} \sigma_1(t) dt \leq \frac{1}{2C} \varrho_{\sigma_1}(x)$$

and we get

$$\varrho_{\sigma}\left(\frac{x}{2C}\right) \leq \frac{m}{2C} \varrho_{\sigma_1}(x) + \frac{1}{2C} \varrho_{\sigma_1}(x) \int_{\gamma}^{\infty} \sigma_0(t) dt \cdot \left(\int_{\gamma}^{\infty} \sigma_1(t) dt\right)^{-1} \leq \varrho_{\sigma_1}(x) \blacksquare$$

2.8. Theorem. Let  $T: X_0 \rightarrow Y$  be compact and  $T: X_1 \rightarrow Y$ . Let  $\sigma$  be nonincreasing near  $+\infty$ ,  $\limsup_{t \rightarrow \infty} t^2 \sigma(t) = \infty$  and  $\varrho_1 \in \Delta_2$  or  $\varrho_Y \in \Delta_2$ . Then  $T: \bar{X}_{\sigma} \rightarrow Y$  is compact.

Proof. Let be  $\{x_n\} \subset \bar{X}_{\sigma}$ ,  $\|x_n\|_{\sigma} < 1$ ,  $x_n = x_{n0} + x_{n1} = x_{n0}(t) + x_{n1}(t)$  and

$$\varrho_0(x_{n0}(t)) + t \varrho_1(x_{n1}(t)) \leq 2\mathcal{L}(t, x, \bar{X}).$$

Let be  $M = M(t) \geq \max(1, 2c_1 \max(1, t))$  with

$$c_1 = \left(\int_0^{\infty} \min(1, \tau) \sigma(\tau) d\tau\right)^{-1}.$$

Then

$$\varrho_0\left(\frac{x_{n0}}{M}\right) \leq 2 \max(1, t) \sigma_t\left(\frac{x_n}{M}\right) \leq \frac{2c_1}{M} \max(1, t) \varrho_{\sigma}(x_n)$$

so that  $\{Tx_{n0}\}$  is relatively compact in  $Y$ . Now, it suffices to prove that for any  $\varepsilon > 0$  there is  $\|Tx_{m1}(t) - Tx_{n1}(t)\|_Y < \varepsilon$  for  $m, n$  large enough and for some  $t > 0$ . Let us choose a decomposition  $x_m - x_n = x_{m0} + x_{m1} = x_{m0}(t) + x_{m1}(t)$  such that

$$t \varrho_1\left(\frac{1}{2} x_{m1}(t)\right) \leq 2\mathcal{L}\left(t, \frac{1}{2} (x_m - x_n), \bar{X}\right),$$

$$x_{m1} = x_{m1} - x_{n1} = x_{m1}(t) - x_{n1}(t), \quad i = 0, 1,$$

$$x_{m0} + x_{m1} = x_m, \quad x_{n0} + x_{n1} = x_n, \quad x_{m1} \in X_i, \quad i = 0, 1.$$

This yields

$$\begin{aligned} & \mathcal{L}\left(t, \frac{1}{2} (x_m - x_n), \bar{X}\right) \\ & \leq \left(\int_t^{\infty} \sigma(\tau) d\tau\right)^{-1} \int_t^{\infty} \mathcal{L}\left(\tau, \frac{1}{2} (x_m - x_n), \bar{X}\right) \sigma(\tau) d\tau \\ & \leq \left(\int_t^{\infty} \sigma(\tau) d\tau\right)^{-1} \varrho_{\sigma}\left(\frac{1}{2} (x_m - x_n)\right) \end{aligned}$$

and it means that

$$\varrho_1 \left( \frac{1}{2} x_{m+1}(t) \right) = \varrho_1 \left( \frac{1}{2} (x_{m1} - x_{n1}) \right) \leq 2 \left( t \int_t^\infty \sigma(\tau) d\tau \right)^{-1}.$$

The last term is tending to 0 when  $t \rightarrow \infty$ . If  $\varrho_1 \in \Delta_2$  then  $\|x_{m1} - x_{n1}\|_\sigma \rightarrow 0$  for  $m, n \rightarrow \infty$ . If  $\varrho_Y \in \Delta_2$  then we can use the  $m$ -continuity  $T: X_1 \rightarrow Y$  ■

The proof of the following theorem is similar to that of Theorem 2.8 and it is therefore omitted.

2.9. Theorem. Let  $T: Y \rightarrow X_0$  be compact and  $T: Y \rightarrow X_1$ . Let  $\lim_{t \rightarrow \infty} \sigma(t) = 0$  and let  $\varrho_Y \in \Delta_2$  or  $\varrho_i \in \Delta_2$  ( $i = 0, 1$ ). Then  $T: Y \rightarrow \bar{X}_\sigma$  is compact.

### 3. The stability of the method

3.1. Definition. A positive nonincreasing and differentiable function  $\sigma$  on  $(0, \infty)$  will be said *admissible* if (2.3) holds and if there exists an  $\varepsilon > 0$  such that the function

$$t \mapsto t^{1+\varepsilon}\sigma(t)$$

is nonincreasing on  $(0, \infty)$  and the function

$$t \mapsto t^{2-\varepsilon}\sigma(t)$$

is nondecreasing on  $(0, \infty)$ .

Let us notice that an admissible function  $\sigma$  satisfies the conditions

$$\lim_{t \rightarrow 0} t\sigma(t) = \lim_{t \rightarrow \infty} t^2\sigma(t) = \infty, \tag{3.1}$$

$$\lim_{t \rightarrow \infty} t\sigma(t) = \lim_{t \rightarrow \infty} t^2\sigma(t) = 0. \tag{3.2}$$

We shall deal with the following situation: Let us have an  $m$ -interpolation couple  $\bar{X} = (X_0, X_1)$  and positive measurable functions  $\omega_0, \omega_1$ , and  $\lambda$  on  $(0, \infty)$ . Let be  $\bar{E}_i = \bar{X}_{\omega_i}, i = 0, 1$ . These spaces form an  $m$ -interpolation couple, as well. We can define the space  $\bar{E}_\lambda$ .

The so called weak stability of our method is easy to establish:

3.2. Theorem. Let  $T: \bar{X} \rightarrow \bar{X}$ . Then  $T: \bar{E}_\lambda \rightarrow \bar{E}_\lambda$  whenever  $\lambda$  and  $\omega_i, i = 0, 1$ , satisfy (2.3).

The proof is straightforward and easy.

The more interesting question is the problem of the existence of a function  $\theta$  such that  $(\bar{X}_{\omega_0}, \bar{X}_{\omega_1})_\lambda = \bar{E}_\lambda = \bar{X}_\theta$ , i.e. the strong stability of our method. In this section we derive a theorem of this type. In the proof we make use of the natural idea from [8] and [4]: We obtain a formula for  $\mathcal{L}(t, x, \bar{E})$  of the form

$$\mathcal{L}(t, x, \bar{E}) \sim \int_0^{\xi(t)} \mathcal{L}(s, x, \bar{X}) \omega_0(s) ds + t \int_{\xi(t)}^\infty \mathcal{L}(s, x, \bar{X}) \omega_1(s) ds, \tag{3.3}$$

where  $\xi$  is a suitable positive function. (The symbol  $\sim$  in (3.3) and everywhere in the sequel between two terms containing modulars expresses the fact that replacing  $x$  on one side by  $x/C$  with suitable  $C \in \mathbb{R}$  the term obtained can be estimated by the



other side. In other places, the symbol  $\sim$  denotes the usual equivalence). The formula (3.3) will be then applied to the definition of  $\bar{E}_t$  in order to get the function  $\theta$ .

The following auxiliary assertions are devoted to the proof of (3.3).

3.3. Lemma. *Let  $\omega$  be an admissible function. Then*

$$\int_0^t s\omega(s) ds \sim t^2\omega(t) \quad \text{on } (0, \infty), \tag{3.4}$$

$$\int_t^\infty \omega(s) ds \sim t\omega(t) \quad \text{on } (0, \infty). \tag{3.5}$$

Proof. It follows from the admissibility of  $\omega$  that

$$\int_0^1 \omega(s) ds = \infty, \quad \int_1^\infty s\omega(s) ds = \infty, \tag{3.6}$$

and, that

$$[\log (s^{1+\varepsilon}\omega(s))]' \leq 0,$$

$$[\log (s^{2-\varepsilon}\omega(s))]' \geq 0,$$

or some  $\varepsilon > 0$ . The last inequalities yield

$$-2 + \varepsilon < \frac{t\omega'(t)}{\omega(t)} < -1 - \varepsilon. \tag{3.7}$$

Using (3.4) we get

$$\liminf_{t \rightarrow 0} \frac{t^2\omega(t)}{\int_0^t s\omega(s) ds} \geq \liminf_{t \rightarrow 0} \left[ 2 + \frac{t\omega'(t)}{\omega(t)} \right],$$

$$\limsup_{t \rightarrow 0} \frac{t^2\omega(t)}{\int_0^t s\omega(s) ds} \leq \limsup_{t \rightarrow 0} \left[ 2 + \frac{t\omega'(t)}{\omega(t)} \right]$$

and similarly for  $\liminf$  and  $\limsup$ . It suffices to use (3.7) and (3.4) follows. The equivalence (3.5) is to be treated in the same way ■

3.4. Lemma. *Let be  $C \geq 1, t > 0, \xi(t) > 0$  and  $\omega_i, i = 0, 1$ , be admissible. Then there exists  $\bar{C} > 0$  such that*

$$\mathcal{L} \left( t, \frac{x}{8C}, \bar{E} \right) \leq \bar{C} \left( \int_0^{\xi(t)} \mathcal{L}(s, x, \bar{X}) \omega_0(s) ds + \int_{\xi(t)}^\infty \mathcal{L}(s, x, \bar{X}) \omega_1(s) ds \right)$$

holds for each  $x \in \Sigma(\bar{E})$ , the constant  $\bar{C}$  being independent of  $t$ .

Proof. Let be  $x \in \Sigma(\bar{E}), \xi(t) > 0$ . For each  $t > 0$ , let us consider some decomposition  $x = \bar{x}_0(t) + \bar{x}_1(t), \bar{x}_i(t) = x_i(\xi(t)), i = 0, 1$ , where

$$\varrho_0(\bar{x}_0(t)) + t\varrho_1(\bar{x}_1(t)) \leq 2\mathcal{L}(t, x, \bar{X}). \tag{3.8}$$

Then

$$\begin{aligned}
 \mathcal{L}\left(t, \frac{x}{8C}, \bar{E}\right) &\leq \frac{1}{4C} \int_0^{\xi(t)} \mathcal{L}\left(s, \frac{\bar{x}_0(t)}{2}, \bar{X}\right) \omega_0(s) ds \\
 &\quad + \frac{1}{4C} \int_{\xi(t)}^{\infty} \mathcal{L}\left(s, \frac{\bar{x}_0(t)}{2}, \bar{X}\right) \omega_0(s) ds \\
 &\quad + \frac{t}{4C} \int_0^{\xi(t)} \mathcal{L}\left(s, \frac{\bar{x}_1(t)}{2}, \bar{X}\right) \omega_1(s) ds \\
 &\quad + \frac{t}{4C} \int_{\xi(t)}^{\infty} \mathcal{L}\left(s, \frac{\bar{x}_1(t)}{2}, \bar{X}\right) \omega_1(s) ds \\
 &= I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

Let us still denote

$$P_0 = \int_0^{\xi(t)} \mathcal{L}(s, x, \bar{X}) \omega_0(s) ds,$$

$$P_1 = t \int_{\xi(t)}^{\infty} \mathcal{L}(s, x, \bar{X}) \omega_1(s) ds.$$

We have

$$\begin{aligned}
 I_1 &\leq \frac{1}{8C} P_0 + \frac{1}{8C} \int_0^{\xi(t)} \mathcal{L}(s, \bar{x}_1(t), \bar{X}) \omega_0(s) ds \\
 &\leq \frac{1}{8C} P_0 + \frac{1}{4C} (\xi(t))^{-1} \mathcal{L}(\xi(t), x, \bar{X}) \int_0^{\xi(t)} s \omega_0(s) ds \\
 &\leq \frac{1}{8C} P_0 + \frac{1}{4C} P_0 = \frac{3}{8C} P_0. \tag{3.9}
 \end{aligned}$$

(We have used (3.8) and the fact that the function  $s \mapsto \mathcal{L}(s, x, \bar{X})$  is nondecreasing.) Further, making use of Lemma 3.3 we obtain

$$\begin{aligned}
 I_2 &\leq \frac{1}{8C} \mathcal{L}(\xi(t), x, \bar{X}) \int_{\xi(t)}^{\infty} \omega_0(s) ds \\
 &\leq \frac{1}{8C} \xi(t) \left( \int_0^{\xi(t)} s \omega_0(s) ds \right)^{-1} P_0 \int_{\xi(t)}^{\infty} \omega_0(s) ds \leq \tilde{C} P_0.
 \end{aligned}$$

The proof of the estimates

$$I_3 \leq \tilde{C} P_1, \quad I_4 \leq \tilde{C} P_1$$

is similar and will be therefore omitted ■

3.5. Lemma. Let  $\omega_0$  and  $\omega_1$  be admissible and denote

$$\omega_2(t) = \frac{\omega_0(t)}{\omega_1(t)}, \quad t > 0.$$

Let the function  $t \mapsto t^{-\delta}\omega_2(t)$  be nondecreasing on  $(0, \infty)$  for some  $\delta > 0$ . Then to every  $C \geq 1$  there exists a  $\tilde{C} > 0$  such that

$$\int_0^{\omega_1^{-1}(t)} \mathcal{L}\left(s, \frac{x}{2C}, \bar{X}\right) \omega_0(s) ds + t \int_{\omega_1^{-1}(t)}^{\infty} \mathcal{L}\left(s, \frac{x}{2C}, \bar{X}\right) \omega_1(s) ds \leq \tilde{C} \mathcal{L}(t, x, \bar{E}). \tag{3.10}$$

Proof. Under our assumptions on  $\omega_2$  we have

$$\int_0^1 \frac{\omega_2(s)}{s} ds < \infty, \quad \int_1^{\infty} \frac{ds}{s\omega_2(s)} < \infty, \tag{3.11}$$

$$\int_0^1 \frac{ds}{s\omega_2(s)} = \int_1^{\infty} \frac{\omega_2(s)}{s} ds = \infty.$$

Let be  $x \in \Sigma(\bar{E})$  and  $x = x_0 + x_1$ ,  $x_i \in X_i$ ,  $i = 0, 1$ . Let  $Q = Q(t)$  denote the left hand side of (3.10). Then

$$Q \leq R_1 + R_2 + R_3 + R_4,$$

where

$$R_i = \frac{1}{2C} \int_0^{\omega_1^{-1}(t)} \mathcal{L}(s, x_i, \bar{X}) \omega_0(s) ds, \quad i = 0, 1,$$

$$R_3 = \frac{t}{2C} \int_{\omega_1^{-1}(t)}^{\infty} \mathcal{L}(s, x_0, \bar{X}) \omega_1(s) ds,$$

$$R_4 = \frac{t}{2C} \int_{\omega_1^{-1}(t)}^{\infty} \mathcal{L}(s, x_1, \bar{X}) \omega_1(s) ds.$$

Let still

$$L_i = \int_0^{\infty} \mathcal{L}(s, x_i, \bar{X}) \omega_i(s) ds, \quad i = 0, 1, \tag{3.12}$$

then

$$R_1 \leq \frac{1}{2C} L_0, \quad R_4 \leq \frac{t}{2C} L_1.$$

Using Lemma 3.3 and the estimates

$$\mathcal{L}(s, x_i, \bar{X}) \leq \left( \int_s^{\infty} \omega_i(\sigma) d\sigma \right)^{-1} L_i, \quad i = 0, 1;$$

we get

$$R_3 \leq \frac{L_1}{2C} \int_0^{\omega_1^{-1}(t)} \omega_0(s) \left( \int_s^\infty \omega_1(\sigma) d\sigma \right)^{-1} ds \sim L_1 \int_0^{\omega_1^{-1}(t)} \frac{\omega_2(s)}{s} ds. \tag{3.13}$$

For  $\gamma \geq 1 - 2\varepsilon$ , the function  $t \mapsto t^{-\gamma} \omega_2(t)$  is nonincreasing because

$$t^{-\gamma} \omega_2(t) = t^{1-2\varepsilon-\gamma} \frac{t^{1+\varepsilon} \omega_0(t)}{t^{2-\varepsilon} \omega_1(t)}.$$

From this and (3.11) one can show similarly as in the proof of Lemma 3.3 that

$$\begin{aligned} \int_0^t \frac{\omega_2(s)}{s} ds &\sim \omega_2(t) \quad \text{on } (0, \infty), \\ \int_t^\infty \frac{ds}{s\omega_2(s)} &\sim \frac{1}{\omega_2(t)} \quad \text{on } (0, \infty), \end{aligned} \tag{3.14}$$

so that (3.13) yields

$$R_2 \leq \tilde{C}L_1.$$

Quite analogously we get

$$R_3 \leq \tilde{C}L_0 \quad \blacksquare$$

**3.6. Theorem.** *Let  $\omega_0, \omega_1$  and  $\lambda$  be admissible and let  $\omega_2 = \omega_0/\omega_1$  satisfy the condition from Lemma 3.5. Then there exists an admissible function  $\theta$  such that*

$$(\bar{X}_{\omega_0}, \bar{X}_{\omega_1})_1 = \bar{X}_\theta.$$

Moreover, if we define

$$A(t) = \int_t^\infty \lambda(s) ds, \quad t > 0,$$

then

$$\theta(t) \sim \omega_0(t) A(\omega_2(t)).$$

*Proof.* Let  $\varrho_1$  be the modular in  $\bar{E}_1 = (\bar{X}_{\omega_0}, \bar{X}_{\omega_1})_1$ . The foregoing assertions give

$$\begin{aligned} \varrho_1(x) &\sim \int_0^\infty \lambda(t) \int_0^{\omega_1^{-1}(t)} \mathcal{L}(s, x, \bar{X}) \omega_0(s) ds dt + \int_0^\infty \lambda(t) \int_0^{\omega_1^{-1}(t)} \mathcal{L}(s, x, \bar{X}) \omega_1(s) ds dt \\ &= S_0 + S_1. \end{aligned}$$

After change of variables ( $\omega_2$  is increasing and differentiable) we get

$$\begin{aligned} S_0 &= \int_0^\infty \left[ \omega_0(t) \int_t^\infty \lambda(\omega_2(s)) \omega_2'(s) ds \right] \mathcal{L}(t, x, \bar{X}) dt, \\ S_1 &= \int_0^\infty \left[ \omega_1(t) \int_0^t \lambda(\omega_2(s)) \omega_2'(s) \omega_2(s) ds \right] \mathcal{L}(t, x, \bar{X}) dt. \end{aligned}$$

Let us denote

$$\begin{aligned} \theta_0(t) &= \omega_0(t) \int_t^\infty \lambda(\omega_2(s)) \omega_2'(s) ds, \\ \theta_1(t) &= \omega_1(t) \int_0^t \lambda(\omega_2(s)) \omega_2'(s) \omega_2(s) ds. \end{aligned}$$

We show that  $\theta_0(t) \sim \theta_1(t)$  on  $(0, \infty)$ , i.e. that

$$\omega_2(t) \Lambda(\omega_2(t)) \sim \int_0^t \frac{d}{ds} [-\Lambda(\omega_2(s))] \omega_2(s) ds.$$

It holds that

$$\begin{aligned} \lim_{z \rightarrow 0} z \Lambda(z) &= \lim_{z \rightarrow 0} z \int_z^\infty \lambda(s) ds \sim \lim_{z \rightarrow 0} z^2 \lambda(z) = 0, \\ \lim_{z \rightarrow \infty} z \Lambda(z) &= \infty, \end{aligned} \tag{3.15}$$

therefore

$$\begin{aligned} \limsup_{t \rightarrow 0} \omega_2(t) \Lambda(\omega_2(t)) \left[ \int_0^t \frac{d}{ds} [-\Lambda(\omega_2(s))] \omega_2(s) ds \right]^{-1} \\ \leq \limsup_{t \rightarrow 0} \left[ -1 + \frac{\Lambda(\omega_2(t))}{\lambda(\omega_2(t)) \omega_2(t)} \right], \end{aligned} \tag{3.16}$$

which is finite for

$$\lim_{z \rightarrow 0} \frac{\Lambda(z)}{z \lambda(z)} = \lim_{z \rightarrow 0} \frac{1}{z \lambda(z)} \int_z^\infty \lambda(s) ds \sim 1.$$

Further, similarly as above one gets

$$\liminf_{t \rightarrow 0} \omega_2(t) \Lambda(\omega_2(t)) \left[ \int_0^t \frac{d}{ds} [-\Lambda(\omega_2(s))] \omega_2(s) ds \right]^{-1} > 0. \tag{3.17}$$

Using the last inequality in (3.15) we obtain the same relations as in (3.16) and (3.17) for  $\limsup$ ,  $\liminf$ , resp.

It remains to show that the function  $\theta(t) = \omega_0(t) \Lambda(\omega_2(t))$  is admissible. The condition (2.3) is satisfied for

$$\begin{aligned} \int_0^1 t \theta(t) dt &\sim \int_0^1 t \omega_1(t) \int_0^t \lambda(\omega_2(s)) \omega_2'(s) \omega_2(s) ds dt < \infty, \\ \int_1^\infty \theta(t) dt &\sim \int_1^\infty \omega_0(t) \int_t^\infty \lambda(\omega_2(s)) \omega_2'(s) ds dt < \infty. \end{aligned}$$

Finally, the function

$$t \mapsto t^{1+\theta}(t) = t^{1+\omega_0(t)} \Lambda(\omega_2(t))$$

is nonincreasing and the function

$$t \mapsto t^{2-\varepsilon}\theta(t)$$

is equivalent to

$$t \mapsto (\omega_2(t))^{\varepsilon'} t^{2-\varepsilon}\omega_1(t) (\omega_2(t))^{2-\varepsilon'} \lambda(\omega_2(t)),$$

which is nondecreasing if we choose  $\varepsilon' > 0$  so that the function  $z \mapsto z^{2-\varepsilon'}\lambda(z)$  is nondecreasing ■

3.7. Remark. Similarly, one can prove reiteration theorems of the form

$$(X_0, \bar{X}_\omega)_\lambda = \bar{X}_\theta \quad \text{or} \quad (\bar{X}_\omega, X_1)_\lambda = \bar{X}_\theta$$

with some suitable  $\theta$ .

#### 4. Miscellaneous

Firstly, we present an important example which can serve as a justification and, roughly speaking, as the motivation of the presented method. (The example can be found in a somewhat different form with the sketch of the proof in the already cited paper [10].)

Let  $\Phi_0$  and  $\Phi_1$  be Young functions and  $\Omega$  a measurable set in  $\mathbb{R}^N$ . Then for any  $\sigma$  satisfying (2.3) we have

$$(L_{\Phi_0}(\Omega), L_{\Phi_1}(\Omega))_\sigma = L_\Phi(\Omega),$$

where

$$\Phi(t) \sim \Phi_0(t) h \left( \frac{\Phi_1(t)}{\Phi_0(t)} \right)$$

with

$$h(t) = \int_0^\infty \min(1, t\tau) \sigma(\tau) d\tau.$$

The function  $h$  is pseudoconcave (see, e.g. [1]) i.e. it is equivalent to some concave function. It can be shown (see, again, e.g. [1]) that

$$h(\lambda t) \leq C \max(1, \lambda) h(t). \quad (4.1)$$

It also holds that any  $h$  satisfying (4.1) is pseudoconcave.

In the same way one can prove the "modular version" of the Stein-Weiss interpolation theorem. (See [13, 1.18.7].)

The rest will be devoted to the proof of a multiplier theorem of Michlin type in Orlicz spaces. This theorem can serve as a basic tool in interpolation of Sobolev-Orlicz spaces. This will be also briefly dealt with in [6].

4.1. Lemma. *Let  $\Phi$  be a Young function satisfying the  $\Delta_2$ -condition. Then there exist  $p_0, p_1 > 0$  and a pseudoconcave function  $h$  so that*

$$\Phi(t) = t^{p_0} h(t^{p_1 - p_0}), \quad t > 0. \quad (4.2)$$

Proof. The function

$$\lambda \mapsto \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)}, \quad \lambda > 0,$$

is submultiplicative and therefore (see Section 1) there exist  $q_0, q_1 > 0$  and  $C > 0$  such that

$$\Phi(\lambda t) \leq C \max(\lambda^{q_0}, \lambda^{q_1}) \Phi(t). \tag{4.3}$$

Let  $Q_0$  and  $Q_1$  be the "best"  $q_0$  and  $q_1$  for which (4.3) holds and  $Q_0 \leq p_0 \leq Q_1 \leq p_1$ . If there is such a function  $h$  that (4.2) holds with so chosen  $p_0$  and  $p_1$  then it is necessary

$$h(s) = s^{-p_0/(p_1-p_0)} \Phi(s^{1/(p_1-p_0)}).$$

If  $\lambda \leq 1$  we have

$$h(\lambda s) \leq \frac{C \lambda^{p_0/(p_1-p_0)} \Phi(s^{1/(p_1-p_0)})}{\lambda^{p_0/(p_1-p_0)} s^{p_0/(p_1-p_0)}} \leq Ch(s)$$

and for  $\lambda > 1$  it is

$$h(\lambda s) \leq \frac{C \lambda^{Q_1/(p_1-p_0)} \Phi(s^{1/(p_1-p_0)})}{\lambda^{p_0/(p_1-p_0)} s^{p_0/(p_1-p_0)}} \leq C \lambda h(s),$$

i.e.  $h$  is pseudoconcave ■

4.2. Corollary. Let  $\Phi$  be a Young function satisfying (4.3) with some  $q_0, q_1 > 1$ . Let  $M$  be a mapping from  $\mathbb{R}^N$  into itself such that

$$|x|^{|\alpha|} |D^\alpha M(x)| \leq C, \quad x \in \mathbb{R}^N,$$

( $\alpha$  is here a multiindex) holds for each  $|\alpha| \leq L$ , where  $L > N/2$ . Then  $M$  is a Fourier multiplier in  $L_\Phi(\mathbb{R}^N)$ , i.e. the mapping

$$f \rightarrow \mathcal{F}^{-1} M \cdot f, \tag{4.4}$$

$\mathcal{F}$  being the Fourier transform and  $\cdot$  denotes the convolution, is continuous from  $L_\Phi(\mathbb{R}^N)$  into itself.

Proof. The mapping (4.4) is (by the Michlin multiplier theorem, see, e.g. [13, 2.2.4]) continuous from  $L_{p_1}(\mathbb{R}^N)$  into itself. It suffices to realize that (4.4) is also  $m$ -continuous with respect to the modularity  $f \mapsto \|f\|_{L_{p_1}}^{p_1}$  in  $L_{p_1}(\mathbb{R}^N)$  and to use the foregoing lemma.

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VERFASSER:

Dr. MIROSLAV KRBEČ

Matematický ústav ČSAV, Žitná 25, 11567 Praha 1, ČSSR