The Poisson formula for Euclidean space groups and some of its applications I

P. GÜNTHER

Die klassische Poissonsche, Summenformel bezieht sich auf eine von *n* linear unabhängigen Translationen erzeugte Gruppe von Translationen des n-thmensionalen affinen oder euklidiachen Raumes. In dieser Arbeit wird eine Veraligemeinerung der Poissonformel gegeben, die sich auf eine ailgemeine, eigentlich diskontinuierliche Gruppe affiner Transformationen mit kompaktem Fundamentalbereich bezieht.

Нлассическая формула суммирования Пуассона относится к группе сдвигов аффинного **эвклидового пространств размерности** *n***, порождённой** *n* **линейнонезависимыми** сдвигами. В настоящей работе даётся обобщение формулы Пуассона, которое относится н общим собственно дискретным группам аффинных преобразований с компактной фундаментальной областью. aumes. In dieser Arbeit wird ein
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histem Fundamentalbereich bezieh
cкая формула суммирования Пу
ового пространств размерност
. В настоящей работе даётся обо
обоственно дискретным групп

The classical Poisson summation formula refers to a translation group with *n* linearly independent generators in the n-dimensional affine or euclidean space. In this paper a generalization of the Poisson formula is given which belongs to a general properly discontinuous group of affine transformations with compact fundamental domain. **TATLERON** Objects

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The classical Poisson formula can be written as an equality of distributions in \mathbb{R}^1 :

$$
\sum_{l\in\mathbb{Z}}e^{il\cdot}=2\pi\sum_{l\in\mathbb{Z}}\delta_{2nl}.\tag{0.1}
$$

During the last years this formula was generalized by several authors in various directions mainly in connection with the celebrated Selberg trace formula. ([1, 3, 4, 5, 6, 9, 10, 11, 12, 13].) Special interest has been shown for the study of the distribution Poisson summation formula refers to a translation group with *n* linearly inde-
ators in the *n*-dimensional affine or euclidean space. In this paper a generalization
in formula is given which belongs to a general properl

$$
\sum_{l=0}^{\infty} \cos (\sqrt{l} \lambda_l \cdot), \qquad (0.2)
$$

where $\{\lambda_i\}_{i\geq 0}$ is the sequence of the eigenvalues of a compact Riemannian manifold M. Already in 1959 H. HUBER $[7]$ showed the equivalence of the eigenvalue spectrum and the length spectrum for hyperbolic space forms; recently beautiful relations between (0.2) and the closed geodesics of a general *M* were discovered. (In this interpretation (0.1) corresponds to the case $M = S¹$.) sequence of the eigenvalues of a compact Riemannian manifold *M*.

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The classical multidimensional variant of (0.1) is as follows. Let \mathfrak{B} be an n-dimensional vector space over R and Γ a lattice in \mathfrak{B} with n linearly independent generators and a fundamental domain $\mathcal{F}(\Gamma)$. Let \mathfrak{B}^* , Γ^* be the dual of \mathfrak{B} , Γ respectively. If \mathfrak{B} is equipped with a Lebesgue measure μ then we have the following equality of distributions of \mathfrak{B} : (0.2) and the closed g
tation (0.1) corresponds
lassical multidimensional
ector space over **R** and
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butions of \mathfrak{B} :
 $\sum_{u \in \Gamma^*} e^{2\pi i \langle u, \cdot \rangle} = \mu(\mathscr{F}(\Gamma)) \sum_{u \in \Gamma^*} e^{2$

$$
\sum_{u \in \Gamma^*} e^{2\pi i \langle u, \cdot \rangle} = \mu \big(\mathscr{F}(\Gamma)\big) \sum_{t \in \Gamma} \delta_t.
$$
 (0.3)

In this paper we shall give the following generalization of (0.3) . Regarding \mathfrak{B} as an affine space we consider a properly discontinuous group $\mathfrak G$ of affine transfor-

mations of \mathfrak{B} with compact fundamental domain $\mathscr{F}(\mathfrak{G})$. For the elements $S \in \mathfrak{G}$ fixed points are allowed. The translations contained in $\mathfrak G$ form a normal subgroup $\mathfrak T$ of \mathcal{G} ; their translation vectors form a lattice *I*' spanned by *n* linearly independent
vectors. \mathcal{G}/\mathfrak{X} is finite. ([2, 8, 15, 14].) Let $f: \mathfrak{B} \to \mathbb{C}$ be an element of the Schwartz
space $\mathcal{G}(\mathfrak{$ vectors. $\mathfrak{G}/\mathfrak{T}$ is finite. ([2, 8, 15, 14].) Let $f: \mathfrak{B} \to \mathbb{C}$ be an element of the Schwartz space $\mathfrak{S}(\mathfrak{B})$. We consider the series **EP**. GÜNTHER
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$$
\sum_{\xi \in \mathfrak{B}} \int f(S(\xi)-\xi) d\mu(\xi). \tag{0.4}
$$

We shall give two quite different expressions for it (a) and b)), their equality constitutes a formula which we call the *Poisson formula for (IS.* (Prop. 3.2 and the theorem of § 3.) (In the case $\mathfrak{G} = \mathfrak{X}$ the right-hand side of (0.3) applied to *f* gives (0.4).)

a) The homogeneous transformations σ occuring in the transformations $S \in \mathbb{G}$ form a finite group $\Omega \cong \mathcal{B}/\mathbb{Z}$. Each σ^T gives a one-to-one mapping of Γ^* onto itself. Two vectors u, $u' \in \Gamma^*$ are said to be equivalent, if there is a $\sigma \in \bar{\mathfrak{L}}$ with $\sigma^T(u) = u'$. This way Γ^* is decomposed into equivalence classes f ; some of these classes are characterized as principal classes. Let \tilde{p} be the set of these principal classes. Then (0.4) equals Then ℓ is similar (ℓ^2 , o, 13, 14;). Let $f: \mathcal{Z}_0 \to \mathcal{U}_0$ and element of the schwartz
 $\int_{\mathcal{F}(\mathbb{S})} f(|S(\xi) - \xi)| d\mu(\xi)$. (0.4)
 $\int_{\mathcal{F}(\mathbb{S})} f(|S(\xi) - \xi|) d\mu(\xi)$. (0.4)
 $\int_{\mathcal{F}(\mathbb{S})} f(|S(\xi) - \xi|) d\mu(\xi)$.

$$
\sum_{\mathbf{t}\in\mathbf{\hat{v}}} (1/\mathrm{Card} \; \mathbf{f}) \sum_{\mathbf{u}\in\mathbf{t}} \tilde{f}(2\pi \mathbf{u}). \tag{a}
$$

Here \tilde{f} is the Fourier transform of *f*. We remark: To each principal class $\tilde{f} \in \tilde{g}$ there corresponds exactly one (over C) linearly independent G-automorphic function $\psi_1: \mathfrak{B} \to \mathbb{C}$ such that $\{\psi_1 \mid \mathfrak{k} \in \mathfrak{D}\}\$ is a complete orthogonal system in the Hilbert space $L_2(\mathfrak{G})$ of $\mathfrak{G}\text{-}$ automorphic functions.

b) The group $\mathfrak G$ is decomposed into $\mathfrak X$ -conjugacy classes; let $\mathscr F$ be the set of these classes. To each $\tau \in \mathscr{T}$ we assign a distribution

$$
\mathfrak{S}(\mathfrak{B})\ni f\to I_{\tau}(f)\in\mathbb{C}.
$$

Essentially $I_{\tau}(f)$ is the integral of *f* over a lower dimensional plane in the affine space \mathcal{R} . This plane has dimension zero if and only if τ contains a translation; it contains the origin if and only if the elements of τ have fixed points. Now, (0.4) equals Fourier transform of *J*. We remark: 10 each principal class $f \in \mathfrak{H}$ there

xactly one (over C) linearly independent $\mathfrak{G}\text{-automorphic function}$

h that $\{\psi_i | f \in \mathfrak{H}\}$ is a complete orthogonal system in the Hilbert
 $\mathfrak{G}\$

$$
\frac{1}{r}\sum_{\tau\in\mathscr{F}}I_{\tau}(f); \qquad r=\text{Ord }\mathfrak{L}.
$$
 (b)

If we make the additional assumption \forall $\xi \in \mathcal{B}$, \forall $\sigma \in \mathcal{C}$, $f(\sigma(\xi)) = f(\xi)$ then we have $\tilde{f}(2\pi u_1) = \cdots = \tilde{f}(2\pi u_k)$ für alle $u_1, \ldots, u_k \in \Gamma^*$ contained in the same principal class $f \in \mathfrak{H}$; we denote this common value by $\tilde{f}(2\pi f)$. Further: *I*, depends only on the G-conjugacy class θ containing the *X*-conjugacy class τ ; we write I_{θ} instead of I_{τ} and we denote the set of \mathfrak{G} -conjugacy classes by Ω . Each $\theta \in \Omega$ contains a finite number $m(\theta)$ of $\mathfrak T$ -conjugacy classes. Our Poisson formula thus reads *n* is plane has dimension zeto it and
origin if and only if the elements
 $\sum_{f \in \mathcal{F}} I_i(f)$; $r = \text{Ord } \Omega$.
 \vdots $r = \cdots = \tilde{f}(2\pi u_k)$ fur alle $u_1, \ldots, u_k \in$

we denote this common value by $\tilde{f}(2\pi)$

class θ conta

$$
\sum_{\mathbf{f}\in\mathfrak{D}}\tilde{f}(2\pi\mathbf{t})=\frac{1}{r}\sum_{\theta\in\Omega}m(\theta)\,I_{\theta}(f). \qquad (0.5)
$$

The paper is self-contained and the proofs are quite elementary except for the proof of proposition 2.2 where a trace formula is used which occurs in the representation theory of finite groups. The considerations are independent of the theory of Lie groups and Lie algebras. The relations between our formula (0.5) and closed geodesics as well as some applications shall be given in a subsequent paper.

Let $\mathfrak B$ be an *n*-dimensional vector space over the real field **R**. We consider $\mathfrak B$ also as an affine space, taking the elements of \mathfrak{B} in the usual way both as vectors and as points. Let $\mathfrak G$ be a properly discontinuous group of affine transformations of $\mathfrak R$ with compact fundamental domain. The translations $T \in \mathcal{G}$ form an invariant subgroup \mathcal{Z} of (, which contains *n* linearly independent generators and has a finite factor group $\mathfrak{B}/\mathfrak{X}$ (*Bieberbach's theorem* [2, 8, 15]). The translation vectors of the elements $T \in$ Let \Re be an *n*-dimensional vector space over the real field **R**. We consider \Re also
as an affine space, taking the elements of \Re in the usual way both as vectors and as
points. Let \Im be a properly discontinuo form a lattice $\Gamma \subset \mathfrak{B}$ over \mathbb{Z} ; k vectors of Γ are \mathbb{Z} -linearly independent if and only if they are R-linearly independent. Each $S \in \mathfrak{B}$ has the form: $\forall x \in \mathfrak{B} : S(x) = \sigma(x) + \alpha$; here σ is a linear transformation of \mathfrak{B} and $\mathfrak{a} \in \mathfrak{B}$. We use the wellknown symbol: $S = (\sigma, \mathfrak{a})$ with the multiplication rule $(\sigma', \mathfrak{a}') (\sigma'', \mathfrak{a}'') = (\sigma' \sigma'', \sigma'(\mathfrak{a}'') \tilde{\tau} + \mathfrak{a}')$. The set $\Omega := \{\sigma \mid \exists S \in \mathcal{G} \text{ with } S = (\sigma, \mathfrak{a})\}$ is a finite group isomorphic to \mathcal{G}/\mathcal{X} ; for the sake of simplicity we call $\mathfrak L$ the *homogeneous group*. Each $\sigma \in \overline{\mathfrak L}$ maps Γ onto itself. Set $r = \text{Ord } 2$. If $\sigma \in \mathcal{Q}$ and $(\sigma, \mathfrak{a}) \in \mathcal{G}$ then we call \mathfrak{a} a vector belonging to σ . For (σ', α') , (σ'', α'') , $(\sigma'\sigma'', \alpha) \in \mathfrak{B}$ the so-called *Frobenius congruences* are satisfied: an *n*-dimensional vector space ov
ne space, taking the elements of \mathfrak{B}
et \mathfrak{G} be a properly discontinuous gr
fundamental domain. The translatii
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berback's theorem [2, 8, At contains *n* linearly independent generators and has a finite factor group erback's theorem [2, 8, 15]). The translation vectors of the elements $T \in \mathcal{X}$ tice $\Gamma \subset \mathcal{X}$ over \mathcal{Z} ; k vectors of Γ are $\mathbf{Z$

$$
\sigma'(\mathfrak{a}'') + \mathfrak{a}' \equiv \mathfrak{a} \bmod \varGamma. \tag{1.1}
$$

Definition 1.1: For $\sigma \in \mathcal{Q}$ *we set:*

Ord Ω. If
$$
\sigma \in \mathcal{X}
$$
 and $(\sigma, a) \in \mathcal{Y}$ then we call a a vector belonging to σ . For r' , a''), $(\sigma' \sigma'$, a) ∈ \mathcal{Y} the so-called *Frobenius congruences* are satisfied:

\n $\sigma'(\mathfrak{a'}) + \mathfrak{a}' \equiv \mathfrak{a} \mod \Gamma$.

\nition 1.1: For $\sigma \in \mathcal{Q}$ we set:

\n $\mathfrak{B}(\sigma) = \ker (\sigma - \text{Id}), \quad \mathfrak{B}^{\perp}(\sigma) = \text{im } (\sigma - \text{Id});$ (1.2)

\n $n(\sigma) = \dim \mathfrak{B}(\sigma)$.

\nna 1.1: For $\sigma \in \mathcal{Q}$ we have:

\n $\mathfrak{B} = \mathfrak{B}(\sigma) \oplus \mathfrak{B}^{\perp}(\sigma)$.

\nFrom the dimension theorem for linear mappings there follows:

\ndim $\mathfrak{B} = \dim \mathfrak{B}(\sigma) + \dim \mathfrak{B}^{\perp}(\sigma)$.

\n(1.4)

\n $\sigma \in \mathfrak{B}(\sigma) \cap \mathfrak{B}^{\perp}(\sigma)$; then there is a vector $\mathfrak{z} \in \mathfrak{B}$ with $\sigma(\mathfrak{z}) = \mathfrak{z} + \mathfrak{z}$. Applying equation we obtain

Lemma 1.1: For $\sigma \in \mathfrak{L}$ we have:

$$
\mathfrak{B} = \mathfrak{B}(\sigma) \oplus \mathfrak{B}^1(\sigma). \tag{1.3}
$$

Proof: From the dimension theorem for linear mappings there follows:

$$
\dim \mathfrak{B} = \dim \mathfrak{B}(\sigma) + \dim \mathfrak{B}^{\perp}(\sigma). \tag{1.4}
$$

Assume $g \in \mathfrak{B}(\sigma) \cap \mathfrak{B}^{\perp}(\sigma)$; then there is a vector $g \in \mathfrak{B}$ with $\sigma(g) = g + g$. Applying *a'* to this equation we obtain *a'(b)=+v, v=1,2,...,* because a() = . 2 is a finite group and we have with *r* = Ord 2: *a' =* Id. It $n(\sigma) = \text{dim } \mathfrak{B}(\sigma).$

a 1.1: $For \sigma \in \mathfrak{L} \text{ we have:}$
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: From the dimension theorem if

dim $\mathfrak{B} = \dim \mathfrak{B}(\sigma) + \dim \mathfrak{B}^1(\sigma)$
 $\in \mathfrak{B}(\sigma) \cap \mathfrak{B}^1(\sigma)$; then there is a

equatio

$$
\sigma'(\lambda) = \lambda + \nu \zeta, \qquad \nu = 1, 2, \ldots,
$$

follows $r\mathfrak{x} = 0$ and

$$
\mathfrak{B}(\sigma) \cap \mathfrak{B}^{\perp}(\sigma) = \{0\}.
$$
 (1.5)

The equations (1.4) , (1.5) imply the assertion \blacksquare

Definition 1.2: For $\sigma \in \mathcal{Q}$ we set:

$$
g \in \mathfrak{B}(\sigma) \cap \mathfrak{B}^1(\sigma);
$$
 then there is a vector $\mathfrak{z} \in \mathfrak{B}$ with $\sigma(\mathfrak{z}) = \mathfrak{z} + \mathfrak{z}$. Applying
equation we obtain

$$
\sigma'(\mathfrak{z}) = \mathfrak{z} + \nu \mathfrak{z}, \qquad \nu = 1, 2, ...
$$

$$
\sigma(\mathfrak{z}) = \mathfrak{z}. \quad \mathfrak{z} \text{ is a finite group and we have with } r = \text{Ord } \mathfrak{Z}: \sigma^r = \text{Id}. \quad \text{It}
$$

$$
\mathfrak{x} = 0 \text{ and}
$$

$$
\mathfrak{B}(\sigma) \cap \mathfrak{B}^1(\sigma) = \{0\}.
$$

$$
\text{ations (1.4), (1.5) imply the assertion } \blacksquare
$$

$$
\text{ation 1.2: For } \sigma \in \mathfrak{L} \text{ we set:}
$$

$$
\Gamma(\sigma) = \Gamma \cap \mathfrak{B}(\sigma), \qquad \Gamma^{\perp}(\sigma) = \Gamma \cap \mathfrak{B}^{\perp}(\sigma),
$$

$$
\Gamma_{\epsilon}^{\perp}(\sigma) = (\sigma - \text{Id}) \cdot (\Gamma) \subseteq \Gamma^{\perp}(\sigma).
$$

$$
\text{In a 1.2. The Z-modules } \Gamma(\sigma), \Gamma^{\perp}(\sigma) \text{ contain exactly } n(\sigma), n - n(\sigma) \text{ linearly}
$$

$$
\text{then generators. The difference module } \Gamma^{\perp}(\sigma) - \Gamma_{\epsilon}^{\perp}(\sigma) \text{ is finite.}
$$

Lemma 1.2. The Z-modules $\Gamma(\sigma)$, $\Gamma^{\perp}(\sigma)$ contain exactly $n(\sigma)$, $n - n(\sigma)$ linearly *independent generators. The difference module* $\Gamma^{\perp}(\sigma) - \Gamma_{e^{\perp}}(\sigma)$ *is finite.*

Proof: We choose a basis of Γ ; it is also a basis of \mathfrak{B} . With respect to such a basis the mapping σ - Id is described by an integer matrix \mathfrak{A} with rank $n - n(\sigma)$. Taking $n - n(\sigma)$ linearly independent rows of $\mathfrak V$ we get the coordinates of $n - n(\sigma)$ Lemma 1.2. The Z-modules $\Gamma(\sigma)$, $\Gamma^{\perp}(\sigma)$ containal
independent generators. The difference module $\Gamma^{\perp}(\sigma)$
Proof: We choose a basis of Γ ; it is also a bas
basis the mapping σ – Id is described by an integ
T cannot have more than $n - n(\sigma)$

linearly independent vectors, because $\Gamma_e^{-1}(\sigma) \subset \mathfrak{B}^{\perp}(\sigma)$ and dim $\mathfrak{B}^{\perp}(\sigma) = n - n(\sigma)$. Now it is clear, that $\Gamma_e^L(\sigma)$ has a basis with $n - n(\sigma)$ vectors. The same must be true for $\Gamma^{\perp}(\sigma)$. From this it follows, that $\Gamma^{\perp}(\sigma) - \Gamma_{e^{\perp}}(\sigma)$ has finitely many elements. In order to find the elements of $\Gamma(\sigma)$, one has to find the integer solutions of the system of linear homogeneous equations belonging to the matrix W. This system has $n(\sigma)$ linearly independent solutions. \blacksquare

Definition 1.3: For $\sigma \in \mathfrak{L}$ *we set:* $e(\sigma) = \text{Card } \{ \Gamma^{\perp}(\sigma) - \Gamma_{e^{\perp}}(\sigma) \}.$

Remark 1.1: Using the elementary divisor theorem one can find a basis $t_{n-n(\sigma)}$ of $\Gamma^{\perp}(\sigma)$ such that $\varepsilon_1 t_1, \ldots, \varepsilon_{n-n(\sigma)} t_{n-n(\sigma)}$ form a basis of $\Gamma_{\varepsilon}^{\perp}(\sigma)$. The $\varepsilon_1, \ldots, \varepsilon_{n-n(\sigma)}$ are the non-zero elementary divisors of the matrix $\mathfrak A$ used in the proof 10 F. GUNTHER

Ilnearly independent vectors, because $\Gamma_e^L(\sigma) \subset \mathfrak{B}^L(\sigma)$ and dim $\mathfrak{B}^L(\sigma) = n - n(\sigma)$.

Now it is clear, that $\Gamma_e^L(\sigma)$ has a basis with $n - n(\sigma)$ vectors. The same must be

true for $\Gamma^L(\sigma)$. From Detinition 1.3: For $\sigma \in \mathcal{X}$ we set: $e(\sigma) = \text{Card } \{I^{\perp}(\sigma) - \Gamma_e^{-1}(\sigma)\}$.

Remark 1.1: Using the elementary divisor theorem one can find a basis
 $n_1, ..., n_{n-n(\sigma)}$ of $\Gamma^{\perp}(\sigma)$ such that $\varepsilon_1 n_1, ..., \varepsilon_{n-n(\sigma)} n_{n-(\sigma)}$ f of \mathfrak{A} of order $n - n(\sigma)$. On the other hand one has $e(\sigma) = |\varepsilon_1, \dots, \varepsilon_{n-n(\sigma)}|$ This yields $e(\sigma) = |d_{n-n(\sigma)}|$.

Remark 1.2: In the following we shall consider the difference Z-module $\Gamma - \Gamma^{\perp}(\sigma)$ a lattice in the difference R-module $\mathfrak{B} - \mathfrak{B}^{\perp}(\sigma)$. This can be done by identifying coset $\mathfrak{x} + \Gamma^{\perp}(\sigma)$ of $\Gamma - \Gamma^{\per$ as a lattice in the difference R-module $\mathfrak{B} - \mathfrak{B}^1(\sigma)$. This can be done by identifying a coset $\mathfrak{g} + \Gamma^{\perp}(\sigma)$ of $\Gamma - \Gamma^{\perp}(\sigma)$ with the coset $\mathfrak{g} + \mathfrak{B}^{\perp}(\sigma)$ of $\mathfrak{B} - \mathfrak{B}^{\perp}(\sigma)$.

Definition 1.4: Let \mathfrak{B}^* be the dual vector space of \mathfrak{B} and σ^T the transposed transformation of $\sigma \in \mathcal{L}$. Further: let Γ^* be the dual lattice of Γ . We set:

+
$$
I^T(\sigma)
$$
 of $I' - I^T(\sigma)$ with the coset $\zeta + \mathfrak{B}^T(\sigma)$ of
ition 1.4: Let \mathfrak{B}^* be the dual vector space of \mathfrak{B} as
ratio of $\sigma \in \mathfrak{L}$. Further: let Γ^* be the dual lattice of
 $\mathfrak{B}^*(\sigma) = \ker (\sigma^T - \text{Id}), \qquad \mathfrak{B}^{*T}(\sigma) = \text{im } (\sigma^T - \text{Id}),$
 $\Gamma^*(\sigma) = \Gamma \cap \mathfrak{B}^*(\sigma), \qquad \Gamma^{*T}(\sigma) = \Gamma \cap \mathfrak{B}^{*T}(\sigma).$

Remark 1.3: $\mathfrak{B}^*(\sigma)$ and $\mathfrak{B}^{*}(\sigma)$ have the dimension $n(\sigma)$ and $n - n(\sigma)$ respectively. Further: $\mathfrak{B}^* = \mathfrak{B}^*(\sigma) \oplus \mathfrak{B}^{*1}(\sigma)$. The Z-modules $\Gamma^*(\sigma)$ and $\Gamma^{*1}(\sigma)$ have $n(\sigma)$ and $n - n(\sigma)$ linearly independent generators respectively. The proof of these facts follows the lines of the proofs for the Lemmata 1.1 and *1.2.*

Remark 1.4: In a natural manner the pairs of vector spaces $\mathfrak{B} - \mathfrak{B}^1(\sigma)$, $\mathfrak{B}^*(\sigma)$ and $\mathfrak{B}^1(\sigma)$, $\mathfrak{B}^* - \mathfrak{B}^*(\sigma)$ are pairs of dual vector spaces. The dual lattice of $\Gamma - \Gamma^1(\sigma)$ considered as a lattice in $\mathfrak{B} - \mathfrak{B}^1(\sigma)$ (see Remark 1.2) is $\Gamma^*(\sigma)$. In the same way $\Gamma^* - \Gamma^*(\sigma)$ is the dual lattice of $\Gamma^{\perp}(\sigma)$. $n - n(\sigma)$ linearly independent gener

ows the lines of the proofs for the Le

rk 1.4: In a natural manner the pair
 r), $\mathfrak{B}^* - \mathfrak{B}^*(\sigma)$ are pairs of dual vecto

d as a lattice in $\mathfrak{B} - \mathfrak{B}^+(0)$ (see Re

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Let μ be the Lebesgue measure of \mathfrak{B} normed such that a fundamental domain $\mathscr{F}(\mathfrak{D})$ of $\mathfrak X$ has measure 1. Let $L_2(\mathfrak X)$ be the Hilbert space over C of locally quadratically integrable $\mathfrak T$ -automorphic functions with scalar product: of $\mathfrak T$ has measure
integrable $\mathfrak T$ -autor
 $\langle \varphi, \psi \rangle = \mathscr{F}$
The functions $\langle \varphi_u |$
 \forall $\mathbf{r} \in \mathbb{R}$. measure 1. Let $L_2(\mathfrak{X})$ be the Hilbert space over C of locally quadratically
 $(\varphi, \psi) = \int \varphi(\xi) \overline{\psi(\xi)} d\mu(\xi)$.
 $(\varphi, \psi) = \int \varphi(\xi) \overline{\psi(\xi)} d\mu(\xi)$.
 (2.1)

tions $\{\varphi_u \mid u \in \Gamma^* \}$ with
 $\forall \xi \in \mathfrak{B}: \varphi_u(\xi) = \exp \{2\pi$

$$
(\varphi, \psi) = \int_{\mathscr{F}(\mathfrak{T})} \varphi(\mathfrak{x}) \, d\mu(\mathfrak{x}). \tag{2.1}
$$

The functions $\{\varphi_{\mathfrak{u}} \mid \mathfrak{u} \in \Gamma^*\}$ with

$$
\forall \ \xi \in \mathfrak{B} : \varphi_{\mathfrak{u}}(\xi) = \exp \{2\pi i \langle \mathfrak{u}, \xi \rangle \} \tag{2.2}
$$

form a complete orthonormal system in $L_2(\mathfrak{T})$. Our aim is to find such a system for the subspace $L_2(\mathfrak{B}) \subseteq L_2(\mathfrak{D})$ of \mathfrak{B} -automorphic functions.

Lemma 2.1: Let $S = (\sigma, a) \in \mathcal{G}$ *be an element of* \mathcal{G} *. Then we have for* $g \in \mathcal{X}$ *:*

$$
\varphi_{\mathfrak{u}}(S(\mathfrak{x})) = \exp \left\{ 2\pi i \langle \mathfrak{u}, \mathfrak{a} \rangle \right\} \varphi_{\sigma^T(\mathfrak{u})}(\mathfrak{x}). \tag{2.3}
$$

Definition 2.1: Two vectors u, $u' \in \Gamma^*$ are called *equivalent*, if a $\sigma \in \mathfrak{L}$ exists with $u' = \sigma^T(u)$. Let \mathcal{R} be the *set of equivalence classes* belonging to this relation. If $\mathfrak{k} = \{\mathfrak{u}_1, \ldots, \mathfrak{u}_l\} \in \mathbb{R}$, the linear subspace of $L_2(\mathfrak{T})$ spanned by $\varphi_{\mathfrak{u},\mathfrak{v}}$, $\ldots, \varphi_{\mathfrak{u}}$ is denoted by $L(\mathfrak{t})$.

Remark 2.1: From (2.3) there follows, that an element $\varphi \in L_2(\mathfrak{T})$ is $\mathfrak{G}\text{-auto-}$ morphic if and only if its projection on every $L(f)$, $f \in \mathcal{R}$, is \mathcal{C}_1 -automorphic. Therefore it is sufficient to find only the G-automorphic functions contained in each of the $L(f)$.

Remark 2.2: For $S \in \mathcal{G}$ the mapping $L(f) \ni \varphi \rightarrow \varphi \circ S$ is a transformation of $L(f)$. If S varies in $\mathfrak G$ these transformations give a representation $\mathfrak D(f)$ of the group $\Omega = \mathcal{G}/\mathcal{Z}$ in $L(\mathfrak{f})$. The space $L(\mathfrak{f})$ contains exactly *h* linearly independent \mathcal{G} -automorphic functions, if $\mathfrak{D}(t)$ contains the identical representation exactly *h* times. It is wellknown that From (2.3) there follows, that an element $\varphi \in L_2(\mathfrak{X})$ is denoted

From (2.3) there follows, that an element $\varphi \in L_2(\mathfrak{X})$ is denoted

From (2.3) there follows, that an element $\varphi \in L_2(\mathfrak{X})$ is denoted

From *x* and surficient to find only the \mathcal{Y} -action contained in each of
 *x***k** 2.2: For $S \in \mathcal{Y}$ the mapping $L(f) \ni \varphi \rightarrow \varphi \circ S$ is a transformation of

varies in \mathcal{Y} these transformations give a representation

$$
h = \frac{1}{\tau} \sum_{\sigma \in \mathfrak{L}} \text{tr } \mathfrak{D}(\mathfrak{k}) \; (\sigma). \tag{2.4}
$$

Definition 2.2: For $u \in \Gamma^*$ let $\Re(u)$ be the *subgroup of all* $\sigma \in \mathcal{Q}$ with $\sigma^T(u) = u$; *we set* $\rho(u) = \text{Ord } \Re(u)$. We choose a vector a belonging to $\sigma \in \Re(u)$ and define

$$
\chi(\mathfrak{u},\,\sigma)=\exp\left\{2\pi i\langle\mathfrak{u},\,\mathfrak{a}\rangle\right\}.\tag{2.5}
$$

This definition is correct, because a is determined mod Γ .

The following lemma is obvious.

Lemma 2.2: For equivalent vectors u, $u' \in \Gamma^*$ the subgroups $\Re(u)$, $\Re(u')$ are con*jugate. The index of* $\mathcal{R}(u)$ *equals the number of vectors equivalent to u.* $\chi(u, \cdot)$ *is a character*. *of R(u).* For example tender of $\mathfrak{R}(u)$, and $\mathfrak{R}(u)$ are contained by $\mathfrak{R}(2, 2)$: For equivalent vectors $u, u' \in \Gamma^*$ the subgroups $\mathfrak{R}(u)$, $\mathfrak{R}(u')$ are contained $\mathfrak{R}(u)$ equals the number of vectors equival

Definition 2.3: The vector $u \in \Gamma^*$ is called *principal vector*, if $\chi(u, \cdot)$ is the principal character of $\mathfrak{R}(\mathfrak{u})$, i. e.

$$
\forall \sigma \in \Re(\mathfrak{u}) : \chi(\mathfrak{u}, \sigma) = 1.
$$

Lemma 2.3: A class $f \in \mathbb{R}$ of equivalent vectors contains either only principal *vectors or only non-principal vectors.*

Proof: Assume $f \in \mathbb{R}$ and u, u' \in f. There is a $\sigma' \in \mathcal{Q}$ with $u' = \sigma'^T(u)$. If $\sigma \in \mathbb{R}(u)$ then we have $\sigma'^{-1}\sigma\sigma' \in \Re(\mathfrak{u}')$ and

$$
\chi(\mathfrak{u}',\,\sigma'^{-1}\sigma\sigma')=\chi(\mathfrak{u},\,\sigma). \qquad (2.6)
$$

From this the assertion follows I

Proposition 2.1: *If is a class of principal vectors, then* dim $_{0}(L(\mathbf{t}) \cap L_{2}(\mathcal{G})) = 1$; *if* **f** is a class of non-principal vectors, then $L(f) \cap L_2(\mathbb{G}) = \{0\}.$

Proof: Let $\mathbf{t} = \{u_1, \ldots, u_l\}$ be a class of equivalent vectors and let S_1, \ldots, S_r be a complete system of representatives of $\mathfrak G$ with respect to $\mathfrak T$. Then we have:

only non-principle vectors.
\n: Assume
$$
f \in \mathbb{R}
$$
 and $u, u' \in f$. There is a $\sigma' \in \mathbb{R}$ with a
\nhave $\sigma'^{-1}\sigma\sigma' \in \mathbb{R}(u')$ and
\n $\chi(u', \sigma'^{-1}\sigma\sigma') = \chi(u, \sigma)$.
\nis the assertion follows
\nposition 2.1: If f is a class of principal vectors, then d
\nclass of non-principle vectors, then $L(f) \cap L_2(\mathbb{G}) = \{0\}$.
\n: Let $f = \{u_1, ..., u_l\}$ be a class of equivalent vector
\nthe system of representatives of \mathbb{G} with respect to \mathbb{X}
\n
$$
h = \frac{1}{r} \sum_{\sigma \in \mathbb{R}} tr \mathbb{D}(f) (\sigma) = \frac{1}{r} \sum_{j=1}^r \sum_{i=1}^l (\varphi_{u_i} \circ S_j, \varphi_{u_i}).
$$

\nlog to (2.3) the last scalar product vanishes if $\sigma_j^T(u_i)$
\nand $\sigma_j^T(u_i)$
\nand $\sigma_j^T(u_i)$

According to (2.3) the last scalar product vanishes if $\sigma_i^T(u_i) + u_i$, i.e. if $\sigma_i \notin \Re(u_i)$.

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Therefore we obtain from (2.3) and (2.5) :

e we obtain from (2.3) ar
\n
$$
h = \frac{1}{r} \sum_{i=1}^{l} \sum_{\sigma \in \Re(u_i)} \chi(u_i, \sigma).
$$
\nthe following well-known
\n
$$
\sum_{\sigma \in \Re(u)} \chi(u, \sigma) = \begin{cases} \varrho(u) & \text{if } \beta \\ 0 & \text{else.} \end{cases}
$$
\ninto account that $\varrho(u_1) =$

We use the following wellknown formula.

. GÜNTBER
\nwe obtain from (2.3) and (2.5):
\n
$$
= \frac{1}{r} \sum_{i=1}^{l} \sum_{\sigma \in \Re(u_i)} \chi(u_i, \sigma).
$$
\ne following well-known formula.
\n
$$
\sum_{\tau \in \Re(u)} \chi(u, \sigma) = \begin{cases} \varrho(u) & \text{if } \chi(u, \cdot) \text{ is principal,} \\ 0 & \text{else.} \end{cases}
$$
\n(2.7)
\nto account that $\varrho(u_1) = \cdots = \varrho(u_l), l = \text{index } \Re(u_i), \text{ we find } h = 1 \text{ if the\ncipal vectors and } h = 0 \text{ in the contrary case}$

Taking into account that $\varrho(\mathfrak{u}_1) = \cdots = \varrho(\mathfrak{u}_l)$, $l = \text{index } \Re(\mathfrak{u}_i)$, we find $h = 1$ if the u_i are principal vectors and $h = 0$ in the contrary case

Definition 2.4: The set of classes $f \in \mathbb{R}$ containing principal vectors is denoted by \mathfrak{H} . Let ψ_i *be a* \mathfrak{G} -automorphic function with $\|\psi_i\|=1$ contained in $f \in \mathfrak{H}$.

Lemma 2.4: $\{ \psi_i \mid f \in \mathfrak{H} \}$ *is a complete orthonormal system in* $\mathfrak{L}_2(\mathfrak{G})$.

Remark 2.3: An explicite expression for ψ_l can be found in the following way. Assume $f = \{u_1, ..., u_l\} \in \mathfrak{H}$; let $\sigma_1, ..., \sigma_l$ be a complete system of representatives of the left cosets of \mathfrak{L} with respect to $\mathfrak{R}(\mathfrak{u}_1)$. For $i = 1, 2, ..., l$ we choose a vector \mathfrak{a}_i belonging to σ_i . Then we have $\{2.4: \text{The set of classes } \mathbf{f} \in \mathbb{R} \text{ containing principal vectors is denoted} \in \mathbb{R} \}$
 $\{2.4: \text{The set of classes } \mathbf{f} \in \mathbb{R} \text{ containing principal vectors is denoted} \in \mathbb{R} \}$
 $\{|\psi_1| \in \mathbb{S}\} \text{ is a complete orthonormal system in } \mathbb{Q}_2(\mathbb{S}).$
 $\{2.4: |\psi_1| \in \mathbb{S}\} \text{ is a complete orthonormal system in } \mathbb{Q}_2(\mathbb{S}).$
 $\{2.4: |\psi_1| \in \mathbb{S}\} \text$ 3: An explicite expression for ψ_l can be found in the following way,

1, ..., $u_l \in \mathfrak{F}$; let σ_1 , ..., σ_l be a complete system of representatives

is of 2 with respect to $\Re(u_1)$. For $i = 1, 2, ..., l$ we choose

$$
\psi_t(\xi) = \frac{1}{\sqrt{l}} \sum_{j=1}^l \exp \left\{ 2\pi i \langle u_1, a_j \rangle \right\} \varphi_{u_j}(\xi).
$$
 (2.8)

It is easy to see that ψ_t has the required properties.

$$
\sum_{\mathbf{u}\in\Gamma^{\bullet}(\sigma)}\chi(\mathbf{u},\,\sigma)\,g(\mathbf{u})\tag{2.9}
$$

is absolutely convergent. Then we have

Proposition 2.2: Let
$$
g: \mathfrak{B}^* \to \mathbb{C}
$$
 be given. Assume that for each $\sigma \in \mathbb{Q}$ the series\n
$$
\sum_{u \in \Gamma^*(\sigma)} \chi(u, \sigma) g(u)
$$
\n(2.9)\n\nabsolutely convergent. Then we have\n
$$
\sum_{\sigma \in \mathbb{Q}} \sum_{u \in \Gamma^*(\sigma)} \chi(u, \sigma) g(u) = r \sum_{t \in \mathbb{Q}} (1/\text{Card } t) \sum_{u \in t} g(u).
$$
\n(2.10)\nProof: On the left-hand side we change the order of summation. Taking into

Proof: On the left-hand side we change the order of summation. Taking into account the Definitions 1.4 and 2.2 we obtain

$$
\sum_{\mathfrak{u}\in\Gamma^*}\sum_{\sigma\in\mathfrak{R}(\mathfrak{u})}\chi(\mathfrak{u},\,\sigma)\,g(\mathfrak{u})\,.
$$

Finally we apply (2.7) and the equation Card $f = r/\rho(u)$. The proof is finished **E**

§3

Proposition 3.1: Let f be an element of the Schwartz space $\mathfrak{S}(\mathfrak{B})$ such that $\forall x \in \mathfrak{B}$, $\forall \sigma \in \mathcal{L}: f(\sigma(\mathbf{r})) = f(\mathbf{r}).$ Let \tilde{f} be the Fourier transform of f ; then $\tilde{f}(2\pi\mathfrak{u})$ has the same *value for every* $\mu \in \mathfrak{k}$. Denoting this common value by $\tilde{f}(2\pi\mathfrak{k})$ we have EVE LET $\frac{1}{\epsilon}$ is a left thand side we change the order of summation. Taking into

the Definitions 1.4 and 2.2 we obtain
 $\sum_{\epsilon f \cdot \epsilon} \sum_{\sigma \in \mathbb{N}(1)} \gamma(u, \sigma) g(u)$.

We apply (2.7) and the equation Card $f = r/\varrho(u)$. The p ⁸ 3

⁸ 3

(o(z) = f(z). Let f be an element of the Schwartz space $\mathfrak{S}(8)$ such that $\forall \chi \in \mathfrak{B}$,
 $\mathfrak{so}(\pi(\chi)) = f(\chi)$. Let f be the Fourier transform of f; then $\mathfrak{f}(2\pi\chi)$ has the same

ery $\mathfrak{u} \in \mathfrak$

$$
\sum_{S \in \mathfrak{S}} f(S(\mathfrak{y}) - \mathfrak{x}) = r \sum_{\mathfrak{k} \in \mathfrak{S}} \tilde{f}(2\pi \mathfrak{k}) \overline{\psi_{\mathfrak{k}}(\mathfrak{x})} \psi_{\mathfrak{k}}(\mathfrak{y}). \tag{3.1}
$$

Both series in (3.1) are absolutely convergent.

Proof: For fixed $g \in \mathcal{X}$ the left-hand side of (3.1) is a \mathcal{Y} -automorphic function of $\mathfrak n$; its Fourier expansion gives

$$
\sum_{S \in \mathfrak{G}} f(S(\mathfrak{y}) - \mathfrak{x}) = r \sum_{t \in \mathfrak{G}} \tilde{f}(2\pi t) \overline{\psi_t(\mathfrak{x})} \psi_t(\mathfrak{y}).
$$
\n(3.1)
\n*i.e.* in (3.1) are absolutely convergent.
\n**f**: For fixed $\mathfrak{x} \in \mathfrak{B}$ the left-hand side of (3.1) is a \mathfrak{B} -automorphic function
\n*i* Fourier expansion gives
\n
$$
\sum_{S \in \mathfrak{G}} f(S(\mathfrak{y}) - \mathfrak{x}) = \sum_{t \in \mathfrak{G}} c_t(\mathfrak{x}) \psi_t(\mathfrak{y}).
$$
\n(3.2)

$$
\text{Poisson formula for Euclidean space groups I} \qquad \text{19}
$$
\nThe Fourier coefficients $c_l(\xi)$ are given by

\n
$$
c_l(\xi) = \sum_{S \in \mathfrak{G}} \int_{\mathfrak{F}(\mathfrak{T})} \overline{\psi_l(\mathfrak{y})} f(S(\mathfrak{y}) - \xi) d\mu(\mathfrak{y}) = r \int_{\mathfrak{G}} \overline{\psi_l(\mathfrak{F} + \xi)} f(\mathfrak{z}) d\mu(\mathfrak{z}). \tag{3.3}
$$
\nIn the last equation we have used the fact that almost every point of \mathfrak{R} is contained.

In the last equation we have used the fact that almost every point of \mathfrak{B} is contained in exactly *r* of the sets $S(\mathscr{F}(\mathfrak{X}))$, $S \in \mathfrak{G}$. Taking into account the expression (2.8) for ψ_1 we obtain Poisson formula for Euclidean space groups I 19
 $S(\tau)(\tau)$ are given by
 $\overline{\psi_t(\tau)} f(S(\tau)) - \tau \int d\mu(\tau) = r \int d\tau \overline{\psi_t(\tau)} f(\tau) d\mu(\tau)$. (3.3)

thave used the fact that almost every point of $\overline{\mathfrak{B}}$ is contained

is $S(\mathscr{F}(\$

$$
c_{\mathfrak{k}}(\mathfrak{x}) = \frac{r}{\sqrt{l}} \sum_{j=1}^{l} \exp \{-2\pi i \langle \mathfrak{u}_1, \mathfrak{a}_j \rangle - 2\pi i \langle \mathfrak{u}_j, \mathfrak{x} \rangle \} \Phi_j \tag{3.4}
$$

with

$$
\Phi_j = \int_{\mathcal{B}} \exp \{-2\pi i \langle u_1, \sigma_j(\mathfrak{z}) \rangle \} f(\mathfrak{z}) d\mu(\mathfrak{z}).
$$

The function f and the measure μ are invariant under application of σ_j , hence $\Phi_i = \tilde{f}(2\pi u_1)$ and

$$
c_1(\mathfrak{x})=r\overline{f}(2\pi\mathfrak{u}_1)\,\psi_1(\mathfrak{x}).
$$

The absolutely uniform convergence of the series in (3.1) follows from the properties of the function *f* belonging to $\mathfrak{S}(\mathfrak{B})$.

Corollary 3.1: If $\mathfrak{G} = \mathfrak{X}$ is a pure translation group then from (3.1) with $\mathfrak{y} = \mathfrak{z} + \mathfrak{x}$ *the wellknown Poisson formula, valid for every* $f \in \mathfrak{S}(\mathfrak{B})$ *, follows:*

$$
\Phi_j = \int_{\mathbb{R}} \exp \{-2\pi i \langle u_1, \sigma_j(\mathfrak{z})\rangle\} f(\mathfrak{z}) d\mu(\mathfrak{z}).
$$

ction f and the measure μ are invariant under application of σ_j , hence
 $2\pi u_1$ and
 $\sigma(\mathfrak{z}) = r\tilde{f}(2\pi u_1) \psi(\mathfrak{z}).$
slutely uniform convergence of the series in (3.1) follows from the properties
action f belonging to $\mathfrak{S}(\mathfrak{B})$.
lary 3.1: If $\mathfrak{B} = \mathfrak{X}$ is a pure translation group then from (3.1) with $\mathfrak{y} = \mathfrak{z} + \mathfrak{x}$
shown Poisson formula, valid for every $f \in \mathfrak{S}(\mathfrak{B})$, follows:

$$
\sum_{t \in \Gamma} f(\mathfrak{z} + t) = \sum_{u \in \Gamma^*} \tilde{f}(2\pi u) \exp \{2\pi i \langle u, \mathfrak{z}\rangle\}.
$$

position 3.2: Let f be any element of the Schwartz space $\mathfrak{S}(\mathfrak{B})$ with Fourier
 \tilde{f} . Then we have

Proposition 3.2: Let *f* be any element of the Schwartz space $\mathfrak{S}(\mathfrak{B})$ with Fourier *transform /. Then we have*

$$
c_l(\mathbf{z}) = r\tilde{f}(2\pi u_1) \psi_l(\mathbf{z}).
$$

olutely uniform convergence of the series in (3.1) follows from the properties
unction *f* belonging to $\tilde{\mathfrak{S}}(8)$.
llary 3.1: If $\mathfrak{G} = \mathfrak{X}$ is a pure translation group then from (3.1) with $\mathfrak{y} = \mathfrak{z} + \mathfrak{z}$
shown Poisson formula, valid for every $\mathfrak{f} \in \mathfrak{S}(8)$, follows:

$$
\sum_{t \in \Gamma} f(\mathfrak{z} + t) = \sum_{u \in \Gamma^*} \tilde{f}(2\pi u) \exp \{2\pi i \langle u, \mathfrak{z} \rangle\}.
$$
 (3.5)
osition 3.2: Let *f* be any element of the Schwartz space $\mathfrak{S}(8)$ with Fourier
 $\mathfrak{m} \tilde{f}$. Then we have

$$
\sum_{s \in \mathfrak{G}} \int_{\mathfrak{F}(\mathfrak{S})} f(S(\mathfrak{z}) - \mathfrak{z}) d\mu(\mathfrak{z}) = r \sum_{t \in \mathfrak{G}} (1/\text{Card } \mathfrak{k}) \sum_{u \in \mathfrak{k}} \tilde{f}(2\pi u).
$$
 (3.6)
 \mathfrak{f} : From equation (3.2) it follows

Proof: From equation (3.2) it follows

function
$$
f
$$
 belonging to $\mathfrak{S}(\mathfrak{B})$.

\nlary 3.1: $If \mathfrak{B} = \mathfrak{X}$ is a pure translation group then from (3.1) with $\mathfrak{y} = \mathfrak{z} + \mathfrak{x}$ *nown Poisson formula, valid for every* $f \in \mathfrak{S}(\mathfrak{B})$, follows:

\n
$$
\sum_{\mathfrak{i} \in \Gamma} f(\mathfrak{z} + \mathfrak{t}) = \sum_{\mathfrak{u} \in \Gamma^*} \tilde{f}(2\pi \mathfrak{u}) \exp\{2\pi i \langle \mathfrak{u}, \mathfrak{z} \rangle\}.
$$
\n(3.5)

\nposition 3.2: Let f be any element of the Schwartz space $\mathfrak{S}(\mathfrak{B})$ with Fourier $n \bar{f}$. Then we have

\n
$$
\sum_{\mathfrak{i} \in \mathfrak{D}} \int f(S(\mathfrak{x}) - \mathfrak{x}) d\mu(\mathfrak{x}) = r \sum_{\mathfrak{i} \in \mathfrak{D}} (1/\text{Card } \mathfrak{k}) \sum_{\mathfrak{u} \in \mathfrak{A}} \tilde{f}(2\pi \mathfrak{u}).
$$
\n(3.6)

\nFrom equation (3.2) it follows

\n
$$
\sum_{\mathfrak{s} \in \mathfrak{B}} \int f(S(\mathfrak{x}) - \mathfrak{x}) d\mu(\mathfrak{x}) = \sum_{\mathfrak{t} \in \mathfrak{D}} \int c_{\mathfrak{t}}(\mathfrak{x}) \psi_{\mathfrak{t}}(\mathfrak{x}) d\mu(\mathfrak{x}).
$$
\n(3.7)

\nsee expressions (3.4) and (2.8) for $c_{\mathfrak{t}}$, $\psi_{\mathfrak{t}}$ respectively, we obtain

Using the expressions (3.4) and (2.8) for c_f , ψ_f respectively we obtain

\n
$$
\text{transform } \tilde{f}.\text{ Then we have}
$$
\n
$$
\sum_{S \in \mathfrak{G}} \int f(S(\mathfrak{x}) - \mathfrak{x}) \, d\mu(\mathfrak{x}) = r \sum_{\mathfrak{t} \in \mathfrak{G}} (1/\text{Card } \mathfrak{t}) \sum_{u \in \mathfrak{t}} \tilde{f}(2\pi u). \tag{3.6}
$$
\n

\n\n
$$
\text{Proof: From equation (3.2) it follows}
$$
\n
$$
\sum_{S \in \mathfrak{G}} \int f(S(\mathfrak{x}) - \mathfrak{x}) \, d\mu(\mathfrak{x}) = \sum_{\mathfrak{t} \in \mathfrak{G}} \int c_{\mathfrak{t}}(\mathfrak{x}) \, \psi_{\mathfrak{t}}(\mathfrak{x}) \, d\mu(\mathfrak{x}). \tag{3.7}
$$
\n

\n\n
$$
\text{Using the expressions (3.4) and (2.8) for } c_{\mathfrak{t}}, \psi_{\mathfrak{t}} \text{ respectively, we obtain}
$$
\n
$$
c_{\mathfrak{t}}(\mathfrak{x}) \, \psi_{\mathfrak{t}}(\mathfrak{x}) = \frac{r}{l} \sum_{j,m=1}^{l} \exp \{2\pi i [\langle u_1, a_m - a_j \rangle + \langle u_m - u_j, \mathfrak{x} \rangle] \} \, \Phi_j.
$$
\n

\n\n
$$
\text{The integration over } \mathcal{F}(\mathfrak{X}) \text{ yields:}
$$
\n
$$
\int c_{\mathfrak{t}}(\mathfrak{t}) \, \psi_{\mathfrak{t}}(\mathfrak{t}) \, d\mu(\mathfrak{t}) = \frac{r}{l} \sum_{j,m=1}^{l} \Phi.
$$
\n

\n\n (3.9)\n

f c) () d/A() = P,. S (3.9) *- JFM = /(2ru). . (3.10)*

Taking into account that $\sigma_j^T(\mathfrak{u}_1) = \mathfrak{u}_j$, $j = 1, 2, ..., l = \text{Card } \mathfrak{k}$ the formula (3.4) gives Using the express $\mathscr{F}(x)$
Using the express $\mathscr{O}(x)$ $\psi_i(x)$
The integration $\int_{\mathscr{F}(x)} \mathscr{O}_i(x)$
Taking into acsociation $\Phi_i = \tilde{f}$
From (3.7), (3.9)

$$
\Phi_j = \tilde{f}(2\pi u_j). \tag{3.10}
$$

From (3.7) , (3.9) , (3.10) the assertion follows **I**

Corollary 3.2: Under the additional assumption \forall $\underline{r} \in \mathcal{B}$, \forall $\sigma \in \mathcal{X}$: $f(\sigma(\underline{r})) = f(\underline{r})$

the /ormula (3.6) reads:

P. GÜNTIEB
\nula (3.6) reads:
\n
$$
\sum_{S \in \mathfrak{G}} \int_{\mathcal{F}(\mathfrak{G})} f(S(\mathfrak{x}) - \mathfrak{x}) d\mu(\mathfrak{x}) = \sum_{t \in \mathfrak{H}} \tilde{f}(2\pi \mathfrak{k}).
$$
\n(a) is a fundamental domain with respect to \mathfrak{G} .

Of course $\mathcal{F}(\mathcal{G})$ is a fundamental domain with respect to \mathcal{G} .

Definition 3.1: Let μ_{σ}^{-1} and μ_{σ}^{-1} be the Lebesgue measures in $\mathfrak{B}^{\perp}(\sigma)$ and $\mathfrak{B} - \mathfrak{B}^{\perp}(\sigma)$ such that a fundamental domain of the lattices $\Gamma^{\perp}(\sigma)$ and $\Gamma = \Gamma^{\perp}(\sigma)$ respectively has measure 1. *f* (*i*) **f** (*i*) **f** (*i*) *f* (*i*) *f* (*i*) *f* (*i*) *f* (*e*) *f* (*i*) *f* (*e*) *f* (*e*) *f* (*e*) *f* (*e*) θ (*e*) (*e*) (*e*) (*f* (3.11)
 $f \in \mathfrak{F}$
 f \mathfrak{g}
 f f

Lemma 3.1: If $\varphi \in L_1(\mathfrak{B})$ then the following integral formula is valid:

$$
\int_{\mathfrak{B}} \varphi(\mathfrak{x}) \, d\mu(\mathfrak{x}) = \int_{\mathfrak{B} - \mathfrak{B}^{\perp}(\sigma)} \int_{\mathfrak{B}^{\perp}(\sigma)} \varphi(\mathfrak{y} + \mathfrak{z}) \, d\mu_{\sigma}^{\perp}(\mathfrak{y}) \, d\mu_{\sigma}^{\perp}(\bar{\mathfrak{z}}).
$$
\n(3.12)

Here $\bar{\delta}$ *denotes the coset* $\bar{\delta}$ + $\mathfrak{B}^{\perp}(\sigma)$ *.*

Proof: It is possible to find a basis ζ_1, \ldots, ζ_n of \Re such that $\{\zeta_1, \ldots, \zeta_n\}$ **Example 19** \leftarrow $\frac{f}{f}(\mathbf{S}(t) - \overline{t}) d\mu(\overline{t}) = \frac{f}{f(\mathbf{S})}(2\pi \mathbf{f}).$ (3.11)

Of course $\mathcal{F}(\mathbf{\Theta})$ is a fundamental domain with respect to $\mathbf{\Theta}$.

Definition 3.1: Let $\mu_g{}^{\perp}$ and $\mu_g{}^{\perp}$ be the Lebesg $\{\xi_1, \ldots, \xi_{n-n(\sigma)}\}, \{\overline{\xi}_{n-n(\sigma)+1}, \ldots, \overline{\xi}_n\}$ span a fundamental domain of $\Gamma, \Gamma^{\perp}(\sigma), \Gamma^{\perp} \Gamma^{\perp}(\sigma)$ respectively. Using this basis we obtain the assertion from the Fubini theorem. $\int \varphi(\mathfrak{x}) d\mu(\mathfrak{x}) = \int \int \varphi(\mathfrak{y}) + \mathfrak{z} d\mu_{\sigma}(\mathfrak{y}) d\mu_{\sigma}(\mathfrak{z}).$ (3.12)
 \mathfrak{g}
 \mathfrak{g} and \mathfrak{g} and \mathfrak{g} and \mathfrak{g} is \mathfrak{g} and \mathfrak{g} and \mathfrak{g} and \mathfrak{g} and \mathfrak{g} and the

Such that a fundamental domain of the lattices $I^+(r)$ and $I - I^+(r)$ respectively has

measure 1.

Lemma 3.1: If $\varphi \in L_1(\mathfrak{B})$ then the following integral formula is valid:
 $\oint \varphi(\xi) d\mu(\xi) = \int \int \varphi(\eta) + \frac{1}{2} d\mu_{\sigma}(\xi$ Lebesgue measures μ^* , μ_o^* , μ_{lo}^* which are normed with the help of the lattices Γ^* , $\Gamma^*(\sigma)$, $\Gamma^* - \Gamma^*(\sigma)$ respectively.

Definition 3.2: Let $\mathcal T$ be the *set of* $\mathfrak X$ -conjugacy classes of $\mathfrak G$, $(S, S' \in \mathfrak G$ are in the same \mathfrak{X} -conjugacy class, if a $T \in \mathfrak{X}$ exists with $S' = TST^{-1}$.) Further let Ω be the *set of* \mathfrak{G} -conjugacy classes of \mathfrak{G} . Each $\theta \in \Omega$ is the union of a finite number of $\mathfrak T$ -conjugacy classes. Let $m(\theta)$ be that number.

Remark 3.2: Assume $\theta \in \Omega$, $S \in \theta$ and let $\mathfrak{N}(S)$ be the normalizer of S in \mathfrak{G} . Let $\overline{\mathfrak{R}(S)}$ be the image of $\mathfrak{R}(S)$ under the natural homomorphism of \mathfrak{G} onto $\mathfrak{G}/\mathfrak{X}$. Then we have $m(\theta) = (ord \mathfrak{B}/\mathfrak{D}) : (ord \mathfrak{R}(S)).$ njugacy class, if a $T \in \mathcal{X}$ exists with $S' = TST^{-1}$.) Further let Ω be
mjugacy classes of Θ . Each $\theta \in \Omega$ is the union of a finite number of
lasses. Let $m(\theta)$ be that number.
2: Assume $\theta \in \Omega$, $S \in \theta$ and let

Proof: The group $\mathfrak{G}/\mathfrak{T}$ acts as a transformation group in \mathscr{T} (via the inner automorphism of \mathcal{G}). Let $\tau \in \mathcal{T}$, $S \in \tau$, $\tau \subseteq \theta$; then $\mathfrak{N}(S)$ is the stable subgroup of τ and the set $\{\tau' \in \mathcal{T} \mid \tau' \subseteq \theta\}$ is its orbit. From these facts the assertion follows. Then we have $m(\theta) = (\text{ord } \mathfrak{B}/\mathfrak{X}) : (\text{ord } \overline{\mathfrak{R}(S)})$.

Proof: The group $\mathfrak{B}/\mathfrak{X}$ acts as a transformation group in \mathcal{T} (via the innerphism of \mathfrak{G}). Let $\tau \in \mathcal{T}, S \in \tau, \tau \subseteq \theta$; then $\overline{\mathfrak{R}(S)}$ i Let $\mathfrak{N}(S)$ be the image of $\mathfrak{N}(S)$ under the natural

Then we have $m(\theta) = (\text{ord } \mathfrak{B}/\mathfrak{D})$: $(\text{ord } \overline{\mathfrak{R}(S)})$.

Proof: The group $\mathfrak{B}/\mathfrak{D}$ acts as a transformation

morphism of \mathfrak{B}). Let $\tau \in \math$ *group* \mathcal{B}/\mathcal{Z} acts as a transformation group in \mathcal{T} (via the inner auto-
 \mathcal{B}/\mathcal{L} *t* $\in \mathcal{F}, S \in \tau, \tau \subseteq \theta$; then $\mathcal{R}(S)$ is the stable subgroup of τ and
 $\tau | \tau' \subseteq \theta$ is its orbit. From these f

 $\texttt{Theorem}\colon \textit{Assume}\, f\in \mathfrak{S}(\mathfrak{B}).$ Let $\tau\in \mathscr{T}$ be a \mathfrak{T} -conjugacy class of $\mathfrak{G}% _{n}$ and $S=(\sigma,\mathfrak{b})\in \tau$ *Then we set*

$$
f' \in \mathcal{F} \mid \tau' \subseteq \theta
$$
 is its orbit. From these facts the assertion follows **h**
rem: Assume $f \in \mathfrak{S}(\mathfrak{B})$. Let $\tau \in \mathcal{F}$ be a \mathfrak{X} -conjugacy class of \mathfrak{B} and $S = (\sigma, b) \in \tau$.
set

$$
I_{\tau}(f) = \frac{1}{e(\sigma)} \int_{\mathfrak{B}^{\perp}(\sigma)} f(\eta + b) d\mu_{\sigma}(\eta).
$$
(3.13)
depends only on τ (i.e. I_{τ} is independent of the choice of $S \in \tau$). $I_{\tau}(f)$ has the
expression:

$$
I_{\tau}(f) = (1/(2\pi)^{n(\sigma)} e(\sigma)) \int_{\mathfrak{B}^{\Phi}(\sigma)} \exp \{i \langle \sigma, b \rangle\} \tilde{f}(b) d\mu_{\sigma}(\sigma).
$$
(3.14)
the Fourier transform of f .
have

$$
\sum_{u \in \mathfrak{k}} (1/\text{Card } \hat{\tau}) \sum_{u \in \mathfrak{k}} \tilde{f}(2\pi u) = \frac{1}{\tau} \sum_{\sigma \in \Omega} \sum_{\tau \subseteq \sigma} I_{\tau}(f).
$$
(3.15)
der the additional assumption $\forall \tau \subseteq \mathfrak{B}, \forall \sigma \in \mathfrak{X}: f(\sigma(\tau)) = f(\tau)$ the value of
ends only on θ with $\tau \subseteq \theta$ and the value of $\tilde{f}(2\pi u)$ depends only on $\tilde{\tau} \in \mathfrak{D}$ with

1. $I_{\tau}(f)$ depends only on τ (i.e. I_{τ} is independent of the choice of $S \in \tau$), $I_{\tau}(f)$ has the alternative expression:

$$
I_{\tau}(f) = \left(\frac{1}{2\pi} \right)^{\pi(\sigma)} e(\sigma) \int_{\mathfrak{B}^{\bullet}(\sigma)} \exp \left\{ i \langle \mathfrak{v}, \mathfrak{b} \rangle \right\} \tilde{f}(\mathfrak{v}) d\mu_{\sigma}^{\bullet}(\mathfrak{v}). \tag{3.14}
$$

Here I is the Fourier transform of f. 2. We have

$$
\sum_{\epsilon \in \mathbf{0}} (1/\text{Card } \mathbf{f}) \sum_{u \in \mathbf{f}} \tilde{f}(2\pi u) = \frac{1}{r} \sum_{\theta \in \Omega} \sum_{\tau \subseteq \Theta} I_{\tau}(f).
$$
\n(3.15)

3. Under the additional assumption $\forall x \in \mathcal{X}, \forall \sigma \in \mathcal{X}$: $f(\sigma(x)) = f(x)$ the value of $I_{\mathfrak{r}}(f)$ depends only on θ with $\tau \subseteq \theta$ and the value of $\tilde{f}(2\pi\mathfrak{u})$ depends only on $\mathfrak{k} \in \mathfrak{D}$ with

u € f. Then we can write

$$
\begin{aligned}\n\text{Poisson formula for Euclidean space groups I} & 21 \\
\text{en we can write} \\
\sum_{t \in \Phi} \tilde{f}(2\pi t) &= \frac{1}{r} \sum_{\theta \in \Omega} m(\theta) \, I_{\theta}(f). \\
\text{We use Proposition 2.2 with } g(\mu) \text{ replaced by } \tilde{f}(2\pi\mu):\n\end{aligned}
$$
\n
$$
(3.16)
$$

Proof: We use Proposition 2.2 with $g(\mathfrak{u})$ replaced by $\tilde{f}(2\pi\mathfrak{u})$

Poisson formula for Euclidean space groups I
\nthen we can write
\n
$$
\sum_{t \in \mathfrak{G}} \tilde{f}(2\pi t) = \frac{1}{r} \sum_{\theta \in \mathcal{Q}} m(\theta) I_{\theta}(f).
$$
\n(3.16)
\n
$$
f: \text{We use Proposition 2.2 with } g(\mathfrak{u}) \text{ replaced by } \tilde{f}(2\pi \mathfrak{u}):
$$
\n
$$
\sum_{t \in \mathfrak{G}} (1/\text{Card } t) \sum_{u \in t} \tilde{f}(2\pi \mathfrak{u}) = \frac{1}{r} \sum_{\sigma \in \mathfrak{L}} \sum_{u \in I^*(\sigma)} \chi(\mathfrak{u}, \sigma) \tilde{f}(2\pi \mathfrak{u}).
$$
\n(3.17)
\n
$$
\text{ad } \sigma \in \mathfrak{L} \text{ and a vector } \mathfrak{a} \text{ belonging to } \sigma \text{ we consider the series}
$$
\n
$$
\sum_{u \in I^*(\sigma)} \chi(\mathfrak{u}, \sigma) \tilde{f}(2\pi \mathfrak{u}) = \sum_{u \in I^*(\sigma)} \exp \{2\pi i \langle \mathfrak{u}, \mathfrak{a} \rangle \} \tilde{f}(2\pi \mathfrak{u}).
$$
\n(3.18)
\ne apply the Poisson formula for translation groups (eq. (3.5)) together
\ne Fourier transformation in $\mathfrak{B}^*(\sigma)$:

For fixed $\sigma \in \mathfrak{L}$ and a vector a belonging to σ we consider the series

$$
\sum_{\mathbf{i}\in\Gamma^*(\sigma)}\chi(\mathbf{u},\,\sigma)\,\tilde{f}(2\pi\mathbf{u})=\sum_{\mathbf{u}\in\Gamma^*(\sigma)}\exp\left\{2\pi i\langle\mathbf{u},\,\mathbf{a}\rangle\right\}\tilde{f}(2\pi\mathbf{u}).\tag{3.18}
$$

Now we apply the Poisson formula for translation groups (eq. (3.5)) together with the Fourier transformation in $\mathfrak{B}^*(\sigma)$:

Let
$$
\sigma \in \mathfrak{L}
$$
 and a vector α belonging to σ we consider the series

\n $\sum_{u \in \Gamma^*(\sigma)} \chi(u, \sigma) \tilde{f}(2\pi u) = \sum_{u \in \Gamma^*(\sigma)} \exp\{2\pi i \langle u, \alpha \rangle\} \tilde{f}(2\pi u).$

\n(2.1)

\n(2.2)

\n(3.5)

\n(4)

\n(5)

\n(6)

\n(7)

\n(8)

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\n(15)

The last summation is extended over a complete system of representatives for the cosets forming the difference \mathbb{Z} -module $\Gamma - \Gamma^1(\sigma)$. Each of these cosets is the union of exactly $e(\sigma)$ coset elements of the difference **Z**-module $\Gamma - \Gamma_e^{-1}(\sigma)$. Therefore we can write: x(v) x_0 and x_1 *i* (v) x_2 (v)

mation is extended over a complete system of representatives for the

ig the difference **Z**-module $\Gamma - \Gamma^1(\sigma)$. Each of these cosets is the
 $x(x) = \frac{1}{2\pi x}$ (σ) coset elements $\sum_{u \in \Gamma^*({\sigma})} \chi(u, \sigma) \tilde{f}(2\pi u) = \sum_{u \in \Gamma^*({\sigma})} \exp \{2\pi i \langle u, \alpha \rangle\} \tilde{f}(2\pi u).$

e apply the Poisson formula for translation

e Tourier transformation in $\mathfrak{B}^*({\sigma})$:
 $\sum \chi(u, \sigma) \tilde{f}(2\pi u) = (1/2\pi)^{n({\sigma})} \sum_{\text{mod } \Gamma^*$

$$
\sum_{i\in\Gamma^*(\sigma)} \chi(\mathfrak{u},\sigma) \tilde{f}(2\pi\mathfrak{u}) = (1/(2\pi)^{\mathfrak{u}(\sigma)} e(\sigma)) \sum_{\mathfrak{A}\text{mod}\Gamma_{\sigma}^{-1}(\sigma)} \int \exp \left\{i\langle \mathfrak{a} + \mathfrak{t}, \mathfrak{v} \rangle\right\} \tilde{f}(\mathfrak{v}) d\mu_{\sigma}^*(\mathfrak{v}). \tag{3.19}
$$

Now we turn to the Σ -conjugacy classes of $\mathfrak G$. Firstly we remark that two Σ -conjugate elements of \mathcal{Y} are contained in the same coset of \mathcal{Y} with respect to \mathfrak{X} . Let $\mathfrak{C}(\sigma)$, $\sigma \in \mathcal{L}$ be such a coset; let $S' = (\sigma, \mathfrak{a} + \mathfrak{t}'),$ $S'' = (\sigma, \mathfrak{a} + \mathfrak{t}'') \in \mathcal{L}(\sigma)$ with $\mathfrak{t}', \mathfrak{t}'' \in \Gamma'$. It is easy to see that S', S'' are $\mathfrak X$ -conjugate if and only if $t' - t'' \in \Gamma_{\sigma}^{\perp}(\sigma)$. If the vector t in $S = (\sigma, b)$, $b = a + t$ runs through a complete system of representatives of *F* with respect to $\Gamma_6^{\perp}(\sigma)$ then *S* runs through a complete system of representatives of the X-conjugacy classes contained in $\mathfrak{C}(\sigma)$. On the other hand we have for $\mathfrak{v} \in \mathfrak{B}^{\ast}(\sigma)$: urn to the \mathfrak{X} -conjugacy classes of \mathfrak{G} . Firstly
of \mathfrak{G} are contained in the same coset of
such a coset; let $S' = (\sigma, \mathfrak{a} + \mathfrak{t}'), S'' =$
to see that S', S'' are \mathfrak{X} -conjugate if an
 $1 S = (\sigma, \mathfrak{b}), \mathfr$ in $S = (\sigma, b), b = a + t$ if
the respect to $\Gamma_a^{-1}(\sigma)$ then S if
 \mathfrak{X} -conjugacy classes contain
 σ):
 $\langle a + t', v \rangle = \langle a + t'', v \rangle$
ows that the summand in
re we can write
 $\sum_{u \in \Gamma^*(\sigma)} \chi(u, \sigma) \tilde{f}(2\pi u) = \sum_{v \subseteq \mathfrak{A}(\sigma)}$
 $\langle f \rangle$

$$
\langle \mathfrak{a} + \mathfrak{t}', \mathfrak{v} \rangle = \langle \mathfrak{a} + \mathfrak{t}'', \mathfrak{v} \rangle \quad \text{if} \quad \mathfrak{t}' - \mathfrak{t}'' \in \Gamma_{\epsilon}^{\perp}(\sigma).
$$

This shows that the summand in (3.19) is a function of the \mathfrak{X} -conjugacy classes; therefore we can write

$$
\sum_{\mathfrak{u}\in\Gamma^*(\sigma)} \chi(\mathfrak{u},\sigma) \tilde{f}(2\pi\mathfrak{u}) = \sum_{\mathfrak{r}\subseteq\mathfrak{C}(\sigma)} I_{\mathfrak{r}}(f).
$$
\n(3.20)

Here $I_{\tau}(f)$ is given by (3.14). In (3.20) we sum up over $\sigma \in \mathfrak{L}$; on the left-hand side we use (3.17), on the right-hand side we make a simple change of the order of summation. This gives the assertion (3.15). *f* (2*xu*) = $\sum_{\tau \subseteq G(\sigma)} I_{\tau}(f)$. (3.20)

by (3.14). In (3.20) we sum up over $\sigma \in \mathfrak{L}$; on the left-hand side

the right-hand side we make a simple change of the order of sum-

the assertion (3.15).

in formula (3.

In order to obtain formula (3.13) we write down:

$$
\tilde{f}(\mathfrak{v}) = \int_{\mathfrak{B}} \exp \{-i \langle \mathfrak{v}, \mathfrak{x} \rangle f(\mathfrak{x}) d\mu(\mathfrak{x});
$$

by use of (3.12) we get

This gives the assertion (3.15).
\ner to obtain formula (3.13) we write down:
\n
$$
\tilde{j}(b) = \int_{B} \exp \{-i\langle b, \xi | f(\xi) d\mu(\xi);
$$
\n
$$
\tilde{j}(3.12) \text{ we get}
$$
\n
$$
\tilde{j}(b) = \int_{B-B^{\perp}(s)} \exp \{-i\langle b, \eta + \frac{1}{2}\rangle\} f(\eta + \frac{1}{2}) d\mu_{\sigma}(\eta) d\mu_{\sigma}(\tilde{\delta}).
$$
\n(3.21)

Inserting (3.21) in (3.14) we obtain an expression for $I_{\nu}(f)$ with three succesive integrations; two of them cancel by means of the Fourier inversion formula because $\mathfrak{B}^{\ast}(\sigma)$ and $\mathfrak{B} - \mathfrak{B}^{\perp}(\sigma)$ are dual vector spaces. The remaining formula is (3.13). Thus part 1 and 2 of the theorem are proved.

In order to prove part 3 we assume \forall $y \in \mathcal{X}$, \forall $\sigma \in \mathcal{X}$: $f(\sigma(y)) = f(y)$. Then we have $\forall u \in \mathfrak{B}^*$: $\tilde{f}(\sigma^{\perp}(u)) = \tilde{f}(u)$. From this it follows that $\tilde{f}(2\pi u)$ has the same value for all $u \in \mathfrak{k}$, $\mathfrak{k} \in \mathfrak{D}$; we denote this value by $\tilde{f}(2\pi\mathfrak{k})$. Further let S, S' be two \mathfrak{G} -conjugate elements of \emptyset with $S = (\sigma, \mathfrak{b}), S' = (\sigma', \mathfrak{b}')$ and $S' = GSG^{-1}, G = (\gamma, \mathfrak{c}).$ A simple calculation shows: **P.** GÜNTHER
 ag (3.21) in (3.14) we obtain an expression for $I_1(f)$ with three succesive
 ag (3.21) in (3.14) we obtain an expression for $I_1(f)$ with three succesive
 d 2 of the theorem are proved.
 d 2 of the

$$
\sigma' = \gamma \sigma \gamma^{-1}, \mathfrak{b}' = \gamma(\mathfrak{b}) + \mathfrak{c} - \gamma \sigma \gamma^{-1}(\mathfrak{c}). \tag{3.22}
$$

From this we have

$$
\mathfrak{B}^{\perp}(\sigma') = \gamma(\mathfrak{B}^{\perp}(\sigma)), \Gamma^{\perp}(\sigma') = \gamma(\Gamma^{\perp}(\sigma)),
$$

$$
\Gamma_e^{\perp}(\sigma') = \gamma(\Gamma_e^{\perp}(\sigma)), e(\sigma') = e(\sigma), \mu_{\sigma'}^{\perp} = \gamma(\mu_{\sigma}^{\perp}).
$$

If the coset of b' (resp. b) modulo $\mathfrak{B}^{\perp}(\sigma')$ (resp. $\mathfrak{B}^{\perp}(\sigma)$) is denoted by \bar{b}' (resp. \bar{b}) then we have $\bar{b}' = \gamma(\bar{b})$. Transforming the integral in (3.13) with the help of the linear transformation y we obtain

$$
\frac{1}{e(\sigma')} \int\limits_{\mathfrak{B}^{\perp}(\sigma')} f(\mathfrak{y} + \mathfrak{b}') d\mu_{\sigma'}^{\perp}(\mathfrak{y}) = \frac{1}{e(\sigma)} \int\limits_{\mathfrak{B}^{\perp}(\sigma)} f(\mathfrak{y} + \mathfrak{b}) d\mu_{\sigma}^{\perp}(\mathfrak{y}).
$$

This shows that $I_{\tau}(f)$ is a function of the G-conjugacy class $\theta \in \Omega$ containing τ ; in this way the notation $I_{\theta}(f)$ is justified. The proof of the theorem is finished \blacksquare

Proposition 3.3: Let $f(\tau)$, $\tau \in \mathcal{T}$ be the $(n - n(\sigma))$ -dimensional plane in the *affine space* $\mathfrak B$ *which is the domain of integration occuring in the expression* (3.13) *of* $I_{\mathfrak r}(f)$. *I*_c¹ (*x*), $\sqrt{(x - \sqrt{x})^2 + (x - \sqrt{x})^2}$
 If the coset of b' (resp. b) modulo \mathfrak{B}^1 (*co*) (resp. \mathfrak{B}^1 (*d*)) is denoted by \bar{b}' (resp. \bar{b}) if

we have \bar{b}' $\sqrt{(b^2 - \sqrt{b^2})^2 + (c^2 - \sqrt{b^2})^2}$

a) dim $f(\tau) = 0$ *if and only if* τ *contains translations.*

b) $0 \in \mathfrak{f}(\tau)$ if and only *if* τ contains translations.
 b) $0 \in \mathfrak{f}(\tau)$ if and only if the elements $S \in \tau$ have fixed points.

Proof: a) dim $f(\tau) = 0$ means that dim $\mathfrak{B}^{\perp}(\sigma) = 0$; owing to Lemma 1.1 this is equivalent to dim $\mathfrak{B}(\sigma) = n$, i.e. $\sigma = \text{Id}$.

b) From (3.13) it follows that $0 \in \mathfrak{f}(\tau)$ if and only if $\mathfrak{b} \in \mathfrak{B}^{\perp}(\sigma)$; here $S = (\sigma, \mathfrak{b})$ is any element of the class τ under consideration. $\mathfrak{b} \in \mathfrak{B}^{\perp}(\sigma)$ is equivalent to the existence of a $c \in \mathcal{X}$ with $b = c - \sigma(c)$; in this case we have $S(c) = c$. Finally we remark: if some element $S \in \tau$ has fixed points, then every element of τ has fixed points.

REFERENCES

- **[1] BERARD-BERGBY,** L.: Laplacien et geodesiques fermeés sur les formes do espace hyperbolique compactes. Séminaire Bourbaki 24 (1971-72), Exp. 406.
- [2] BURKHARDT, J. J.: Die Bewegungsgruppen der Kristallographie. Birkhäuser Verlag: Basel—Stuttgart 1966.
- [3] CHAZARAIN, J.: Formule de Poisson pour les varietes Riemanniennes. Invent. Math. 24 $(1974), 65-82.$
- [4] DUISTEBMAAT, *J.* J., and V. W. GvILLEMIN: The spectrum of positive elliptic operators and periodic bicharacteristics. Invent. Math. 29 (1975), 39-79.
- [5] GANGOLLI, R., and G. WARNER: On Selberg's trace formula. J. Math. Soc. Japan 27 (1975), 328-343.
- [6] GÜNTHER, P.: Problème de réseaux dans les espaces hyperboliques. C. R. Acad. Sci. Paris, Sér. A 288 (1979), 49-52.
- [7] HUBER, H.: Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen I. Math. Ann. 138 (1959), 1-26.
- [8] KOBAYASBI, S., and K. Nonzu: Foundations of differential geometry. Vol. I. Inter. science Publishers 1963.
- [9] KOLK, J.: Formule de Poisson et distribution asymptotique du spectre simultané d'opérateurs différential. C. R. Acad. Sci. Paris. Sér. A-B 284 (1977), 1045-1048.
- [10] KOLK, J.: The Selberg trace formula and asymptotic behaviour of spectra. Thesis. Rijkauniversiteit Utrecht 1977.
- [11] SELBEBG, A.: Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichiet series. J. Indian Math. Soc. 20 (1956), $47 - 87.$ Paris, Sér. A 288 (1979), 49-52.

Paris, Sér. A 288 (1979), 49-52.

ETOBER, H.: Zur analytischen Theorie hyperbolischer Raumformen und Bewegungs

Fruppen I. Math. Ann. 188 (1959), 1-26.

KOBAYASHI, S., and K. NOMIZU: Found
- [12] SELBERG, A.: Discontinuous groups and harmonic analysis. Proc. Int. Congr. of Math. Stockholm 1962, 177-189.
- [13] TANAKA, S.: Selberg's trace formula and spectrum. Osaka J. Math. 3 (1966), 205-216.
- [14] WnrroEN, G.: Zur Darstellungstheorie der Raumgruppen. Math. Ann. 118 (1941/43),
- [15] WOLF, J. A.: Spaces of constant curvature. McGraw-Hill: New York 1967.

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