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The Poisson formula for Euclidean space groups and some of its applications I

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Die klassische Poissonsche Summenformel bezieht sich auf eine von n linear unabhängigen Translationen erzeugte Gruppe von Translationen des n-dimensionalen affinen oder euklidischen Raumes. In dieser Arbeit wird eine Verallgemeinerung der Poissonformel gegeben, die sich auf eine allgemeine, eigentlich diskontinuierliche Gruppe affiner Transformationen mit kompaktem Fundamentalbereich bezieht.

Классическая формула суммирования Пуассона относится к группе сдвигов аффинного и эвклидового пространств размерности n, порождённой n линейнонезависимыми сдвигами. В настоящей работе даётся обобщение формулы Пуассона, которое относится к общим собственно дискретным группам аффинных преобразований с компактной фундаментальной областью.

The classical Poisson summation formula refers to a translation group with n linearly independent generators in the n-dimensional affine or euclidean space. In this paper a generalization of the Poisson formula is given which belongs to a general properly discontinuous group of affine transformations with compact fundamental domain.

The classical Poisson formula can be written as an equality of distributions in $\mathbf{R}^1$:

$$\sum_{l\in\mathbf{Z}} e^{il\cdot} = 2\pi \sum_{l\in\mathbf{Z}} \delta_{2\pi l}. \tag{0.1}$$

During the last years this formula was generalized by several authors in various directions mainly in connection with the celebrated Selberg trace formula. ([1, 3, 4, 5, 6, 9, 10, 11, 12, 13].) Special interest has been shown for the study of the distribution

$$\sum_{l=0}^{\infty} \cos\left(\sqrt{\lambda_l}\,\cdot\right), \tag{0.2}$$

where $\{\lambda_l\}_{l\geq 0}$ is the sequence of the eigenvalues of a compact Riemannian manifold M. Already in 1959 H. Huber [7] showed the equivalence of the eigenvalue spectrum and the length spectrum for hyperbolic space forms; recently beautiful relations between (0.2) and the closed geodesics of a general M were discovered. (In this interpretation (0.1) corresponds to the case $M = S^1$.)

The classical multidimensional variant of (0.1) is as follows. Let $\mathfrak{V}$ be an n-dimensional vector space over $\mathbf{R}$ and Γ a lattice in $\mathfrak{V}$ with n linearly independent generators and a fundamental domain $\mathscr{F}(\Gamma)$. Let $\mathfrak{V}^*$, Γ^* be the dual of $\mathfrak{V}$, Γ respectively. If $\mathfrak{V}$ is equipped with a Lebesgue measure μ then we have the following equality of distributions of $\mathfrak{V}$:

$$\sum_{\mathfrak{u}\in\Gamma^*} e^{2\pi i\langle\mathfrak{u},\cdot\rangle} = \mu\big(\mathscr{F}(\Gamma)\big) \sum_{\mathfrak{t}\in\Gamma} \delta_{\mathfrak{t}}. \tag{0.3}$$

In this paper we shall give the following generalization of (0.3). Regarding $\mathfrak{V}$ as an affine space we consider a properly discontinuous group $\mathfrak{G}$ of affine transformations of $\mathfrak{V}$ with compact fundamental domain $\mathscr{F}(\mathfrak{G})$. For the elements $S \in \mathfrak{G}$ fixed points are allowed. The translations contained in $\mathfrak{G}$ form a normal subgroup $\mathfrak{T}$ of $\mathfrak{G}$; their translation vectors form a lattice Γ spanned by n linearly independent vectors. $\mathfrak{G}/\mathfrak{T}$ is finite. ([2, 8, 15, 14].) Let $f\colon \mathfrak{V} \to \mathbf{C}$ be an element of the Schwartz space $\mathfrak{S}(\mathfrak{V})$. We consider the series

$$\sum_{S\in\mathfrak{G}} \int_{\mathscr{F}(\mathfrak{G})} f\big(S(\mathfrak{x}) - \mathfrak{x}\big)\, d\mu(\mathfrak{x}). \tag{0.4}$$

We shall give two quite different expressions for it (a) and b)), their equality constitutes a formula which we call the *Poisson formula for* $\mathfrak{G}$. (Prop. 3.2 and the theorem of § 3.) (In the case $\mathfrak{G} = \mathfrak{T}$ the right-hand side of (0.3) applied to f gives (0.4).)

a) The homogeneous transformations σ occuring in the transformations $S \in \mathfrak{G}$ form a finite group $\mathfrak{L} \cong \mathfrak{G}/\mathfrak{T}$. Each σ^T gives a one-to-one mapping of Γ^* onto itself. Two vectors $\mathfrak{u}, \mathfrak{u}' \in \Gamma^*$ are said to be equivalent, if there is a $\sigma \in \mathfrak{L}$ with $\sigma^T(\mathfrak{u}) = \mathfrak{u}'$. This way Γ^* is decomposed into equivalence classes $\mathfrak{k}$; some of these classes are characterized as principal classes. Let $\mathfrak{H}$ be the set of these principal classes. Then (0.4) equals

$$\sum_{\mathfrak{k}\in\mathfrak{H}} (1/\mathrm{Card}\,\mathfrak{k}) \sum_{\mathfrak{u}\in\mathfrak{k}} \tilde{f}(2\pi\mathfrak{u}). \tag{a}$$

Here $\tilde{f}$ is the Fourier transform of f. We remark: To each principal class $\mathfrak{k} \in \mathfrak{H}$ there corresponds exactly one (over $\mathbf{C}$) linearly independent $\mathfrak{G}$-automorphic function $\psi_{\mathfrak{k}}\colon \mathfrak{V} \to \mathbf{C}$ such that $\{\psi_{\mathfrak{k}} \mid \mathfrak{k} \in \mathfrak{H}\}$ is a complete orthogonal system in the Hilbert space $L_2(\mathfrak{G})$ of $\mathfrak{G}$-automorphic functions.

b) The group $\mathfrak{G}$ is decomposed into $\mathfrak{T}$-conjugacy classes; let $\mathscr{T}$ be the set of these classes. To each $\tau \in \mathscr{T}$ we assign a distribution

$$\mathfrak{S}(\mathfrak{V}) \ni f \to I_\tau(f) \in \mathbf{C}.$$

Essentially $I_\tau(f)$ is the integral of f over a lower dimensional plane in the affine space $\mathfrak{V}$. This plane has dimension zero if and only if τ contains a translation; it contains the origin if and only if the elements of τ have fixed points. Now, (0.4) equals

$$\frac{1}{r} \sum_{\tau\in\mathscr{T}} I_\tau(f); \qquad r = \mathrm{Ord}\,\mathfrak{L}. \tag{b}$$

If we make the additional assumption $\forall\, \mathfrak{x} \in \mathfrak{V},\ \forall\, \sigma \in \mathfrak{L},\ f\big(\sigma(\mathfrak{x})\big) = f(\mathfrak{x})$ then we have $\tilde{f}(2\pi\mathfrak{u}_1) = \cdots = \tilde{f}(2\pi\mathfrak{u}_k)$ für alle $\mathfrak{u}_1, \ldots, \mathfrak{u}_k \in \Gamma^*$ contained in the same principal class $\mathfrak{k} \in \mathfrak{H}$; we denote this common value by $\tilde{f}(2\pi\mathfrak{k})$. Further: I_τ depends only on the $\mathfrak{G}$-conjugacy class θ containing the $\mathfrak{T}$-conjugacy class τ; we write I_θ instead of I_τ and we denote the set of $\mathfrak{G}$-conjugacy classes by Ω. Each $\theta \in \Omega$ contains a finite number $m(\theta)$ of $\mathfrak{T}$-conjugacy classes. Our Poisson formula thus reads

$$\sum_{\mathfrak{k}\in\mathfrak{H}} \tilde{f}(2\pi\mathfrak{k}) = \frac{1}{r} \sum_{\theta\in\Omega} m(\theta)\, I_\theta(f). \tag{0.5}$$

The paper is self-contained and the proofs are quite elementary except for the proof of proposition 2.2 where a trace formula is used which occurs in the representation theory of finite groups. The considerations are independent of the theory of Lie groups and Lie algebras. The relations between our formula (0.5) and closed geodesics as well as some applications shall be given in a subsequent paper.

§ 1

Let $\mathfrak{V}$ be an n-dimensional vector space over the real field $\mathbf{R}$. We consider $\mathfrak{V}$ also as an affine space, taking the elements of $\mathfrak{V}$ in the usual way both as vectors and as points. Let $\mathfrak{G}$ be a properly discontinuous group of affine transformations of $\mathfrak{V}$ with compact fundamental domain. The translations $T \in \mathfrak{G}$ form an invariant subgroup $\mathfrak{T}$ of $\mathfrak{G}$, which contains n linearly independent generators and has a finite factor group $\mathfrak{G}/\mathfrak{T}$ (*Bieberbach's theorem* [2, 8, 15]). The translation vectors of the elements $T \in \mathfrak{T}$ form a lattice $\Gamma \subset \mathfrak{V}$ over $\mathbf{Z}$; k vectors of Γ are $\mathbf{Z}$-linearly independent if and only if they are $\mathbf{R}$-linearly independent. Each $S \in \mathfrak{G}$ has the form: $\forall\, \mathfrak{x} \in \mathfrak{V}: S(\mathfrak{x}) = \sigma(\mathfrak{x}) + \mathfrak{a}$; here σ is a linear transformation of $\mathfrak{V}$ and $\mathfrak{a} \in \mathfrak{V}$. We use the wellknown symbol: $S = (\sigma, \mathfrak{a})$ with the multiplication rule $(\sigma', \mathfrak{a}')\,(\sigma'', \mathfrak{a}'') = (\sigma'\sigma'', \sigma'(\mathfrak{a}'') + \mathfrak{a}')$. The set $\mathfrak{L} := \{\sigma \mid \exists\, S \in \mathfrak{G} \text{ with } S = (\sigma, \mathfrak{a})\}$ is a finite group isomorphic to $\mathfrak{G}/\mathfrak{T}$; for the sake of simplicity we call $\mathfrak{L}$ the *homogeneous group*. Each $\sigma \in \mathfrak{L}$ maps Γ onto itself. Set $r = \operatorname{Ord} \mathfrak{L}$. If $\sigma \in \mathfrak{L}$ and $(\sigma, \mathfrak{a}) \in \mathfrak{G}$ then we call $\mathfrak{a}$ a vector belonging to σ. For $(\sigma', \mathfrak{a}')$, $(\sigma'', \mathfrak{a}'')$, $(\sigma'\sigma'', \mathfrak{a}) \in \mathfrak{G}$ the so-called *Frobenius congruences* are satisfied:

$$\sigma'(\mathfrak{a}'') + \mathfrak{a}' \equiv \mathfrak{a} \bmod \Gamma. \tag{1.1}$$

Definition 1.1: *For* $\sigma \in \mathfrak{L}$ *we set:*

$$\mathfrak{V}(\sigma) = \ker(\sigma - \mathrm{Id}), \qquad \mathfrak{V}^{\perp}(\sigma) = \operatorname{im}(\sigma - \mathrm{Id}); \tag{1.2}$$
$$n(\sigma) = \dim \mathfrak{V}(\sigma).$$

Lemma 1.1: *For* $\sigma \in \mathfrak{L}$ *we have:*

$$\mathfrak{V} = \mathfrak{V}(\sigma) \oplus \mathfrak{V}^{\perp}(\sigma). \tag{1.3}$$

Proof: From the dimension theorem for linear mappings there follows:

$$\dim \mathfrak{V} = \dim \mathfrak{V}(\sigma) + \dim \mathfrak{V}^{\perp}(\sigma). \tag{1.4}$$

Assume $\mathfrak{x} \in \mathfrak{V}(\sigma) \cap \mathfrak{V}^{\perp}(\sigma)$; then there is a vector $\mathfrak{z} \in \mathfrak{V}$ with $\sigma(\mathfrak{z}) = \mathfrak{z} + \mathfrak{x}$. Applying σ^{ν} to this equation we obtain

$$\sigma^{\nu}(\mathfrak{z}) = \mathfrak{z} + \nu\mathfrak{x}, \qquad \nu = 1, 2, \dots,$$

because $\sigma(\mathfrak{x}) = \mathfrak{x}$. $\mathfrak{L}$ is a finite group and we have with $r = \operatorname{Ord} \mathfrak{L}$: $\sigma^{r} = \mathrm{Id}$. It follows $r\mathfrak{x} = 0$ and

$$\mathfrak{V}(\sigma) \cap \mathfrak{V}^{\perp}(\sigma) = \{0\}. \tag{1.5}$$

The equations (1.4), (1.5) imply the assertion ∎

Definition 1.2: *For* $\sigma \in \mathfrak{L}$ *we set:*

$$\Gamma(\sigma) = \Gamma \cap \mathfrak{V}(\sigma), \qquad \Gamma^{\perp}(\sigma) = \Gamma \cap \mathfrak{V}^{\perp}(\sigma),$$
$$\Gamma_e^{\perp}(\sigma) = (\sigma - \mathrm{Id})\,(\Gamma) \subseteq \Gamma^{\perp}(\sigma). \tag{1.6}$$

Lemma 1.2. *The* $\mathbf{Z}$*-modules* $\Gamma(\sigma)$, $\Gamma^{\perp}(\sigma)$ *contain exactly* $n(\sigma)$, $n - n(\sigma)$ *linearly independent generators. The difference module* $\Gamma^{\perp}(\sigma) - \Gamma_e^{\perp}(\sigma)$ *is finite.*

Proof: We choose a basis of Γ; it is also a basis of $\mathfrak{V}$. With respect to such a basis the mapping $\sigma - \mathrm{Id}$ is described by an integer matrix $\mathfrak{A}$ with rank $n - n(\sigma)$. Taking $n - n(\sigma)$ linearly independent rows of $\mathfrak{A}$ we get the coordinates of $n - n(\sigma)$ linearly independent vectors of $\Gamma_e^{\perp}(\sigma)$; but $\Gamma_e^{\perp}(\sigma)$ cannot have more than $n - n(\sigma)$

linearly independent vectors, because $\Gamma_e^{\perp}(\sigma) \subset \mathfrak{V}^{\perp}(\sigma)$ and $\dim \mathfrak{V}^{\perp}(\sigma) = n - n(\sigma)$. Now it is clear, that $\Gamma_e^{\perp}(\sigma)$ has a basis with $n - n(\sigma)$ vectors. The same must be true for $\Gamma^{\perp}(\sigma)$. From this it follows, that $\Gamma^{\perp}(\sigma) - \Gamma_e^{\perp}(\sigma)$ has finitely many elements. In order to find the elements of $\Gamma(\sigma)$, one has to find the integer solutions of the system of linear homogeneous equations belonging to the matrix $\mathfrak{A}$. This system has $n(\sigma)$ linearly independent solutions. ∎

Definition 1.3: For $\sigma \in \mathfrak{L}$ *we set*: $\mathrm{e}(\sigma) = \mathrm{Card}\,\{\Gamma^{\perp}(\sigma) - \Gamma_e^{\perp}(\sigma)\}$.

Remark 1.1: Using the elementary divisor theorem one can find a basis $\mathfrak{y}_1, \ldots, \mathfrak{y}_{n-n(\sigma)}$ of $\Gamma^{\perp}(\sigma)$ such that $\varepsilon_1\mathfrak{y}_1, \ldots, \varepsilon_{n-n(\sigma)}\mathfrak{y}_{n-n(\sigma)}$ form a basis of $\Gamma_e^{\perp}(\sigma)$. The $\varepsilon_1, \ldots, \varepsilon_{n-n(\sigma)}$ are the non-zero elementary divisors of the matrix $\mathfrak{A}$ used in the proof of Lemma 1.2 and their product $\varepsilon_1, \cdots, \varepsilon_{n-n(\sigma)} = d_{n-n(\sigma)}$ is the determinant divisor of $\mathfrak{A}$ of order $n - n(\sigma)$. On the other hand one has $\mathrm{e}(\sigma) = |\varepsilon_1, \cdots, \varepsilon_{n-n(\sigma)}|$ This yields $\mathrm{e}(\sigma) = |d_{n-n(\sigma)}|$.

Remark 1.2: In the following we shall consider the difference $\mathbf{Z}$-module $\Gamma - \Gamma^{\perp}(\sigma)$ as a lattice in the difference $\mathbf{R}$-module $\mathfrak{V} - \mathfrak{V}^{\perp}(\sigma)$. This can be done by identifying a coset $\mathfrak{x} + \Gamma^{\perp}(\sigma)$ of $\Gamma - \Gamma^{\perp}(\sigma)$ with the coset $\mathfrak{x} + \mathfrak{V}^{\perp}(\sigma)$ of $\mathfrak{V} - \mathfrak{V}^{\perp}(\sigma)$.

Definition 1.4: Let $\mathfrak{V}^*$ be the dual vector space of $\mathfrak{V}$ and σ^T the transposed transformation of $\sigma \in \mathfrak{L}$. Further: let Γ^* be the dual lattice of Γ. *We set:*

$$\mathfrak{V}^*(\sigma) = \ker(\sigma^T - \mathrm{Id}), \qquad \mathfrak{V}^{*\perp}(\sigma) = \mathrm{im}(\sigma^T - \mathrm{Id}),$$

$$\Gamma^*(\sigma) = \Gamma \cap \mathfrak{V}^*(\sigma), \qquad \Gamma^{*\perp}(\sigma) = \Gamma \cap \mathfrak{V}^{*\perp}(\sigma).$$

Remark 1.3: $\mathfrak{V}^*(\sigma)$ and $\mathfrak{V}^{*\perp}(\sigma)$ have the dimension $n(\sigma)$ and $n - n(\sigma)$ respectively. Further: $\mathfrak{V}^* = \mathfrak{V}^*(\sigma) \oplus \mathfrak{V}^{*\perp}(\sigma)$. The $\mathbf{Z}$-modules $\Gamma^*(\sigma)$ and $\Gamma^{*\perp}(\sigma)$ have $n(\sigma)$ and $n - n(\sigma)$ linearly independent generators respectively. The proof of these facts follows the lines of the proofs for the Lemmata 1.1 and 1.2.

Remark 1.4: In a natural manner the pairs of vector spaces $\mathfrak{V} - \mathfrak{V}^{\perp}(\sigma)$, $\mathfrak{V}^*(\sigma)$ and $\mathfrak{V}^{\perp}(\sigma)$, $\mathfrak{V}^* - \mathfrak{V}^*(\sigma)$ are pairs of dual vector spaces. The dual lattice of $\Gamma - \Gamma^{\perp}(\sigma)$ considered as a lattice in $\mathfrak{V} - \mathfrak{V}^{\perp}(\sigma)$ (see Remark 1.2) is $\Gamma^*(\sigma)$. In the same way $\Gamma^* - \Gamma^*(\sigma)$ is the dual lattice of $\Gamma^{\perp}(\sigma)$.

§ 2

Let μ be the Lebesgue measure of $\mathfrak{V}$ normed such that a fundamental domain $\mathcal{F}(\mathfrak{T})$ of $\mathfrak{T}$ has measure 1. Let $L_2(\mathfrak{T})$ be the Hilbert space over $\mathbf{C}$ of locally quadratically integrable $\mathfrak{T}$-automorphic functions with scalar product:

$$(\varphi, \psi) = \int\limits_{\mathcal{F}(\mathfrak{T})} \varphi(\mathfrak{x})\,\overline{\psi(\mathfrak{x})}\,d\mu(\mathfrak{x}). \tag{2.1}$$

The functions $\{\varphi_{\mathfrak{u}} \mid \mathfrak{u} \in \Gamma^*\}$ with

$$\forall\, \mathfrak{x} \in \mathfrak{V} : \varphi_{\mathfrak{u}}(\mathfrak{x}) = \exp\{2\pi i\langle \mathfrak{u}, \mathfrak{x}\rangle\} \tag{2.2}$$

form a complete orthonormal system in $L_2(\mathfrak{T})$. Our aim is to find such a system for the subspace $L_2(\mathfrak{G}) \subseteqq L_2(\mathfrak{T})$ of $\mathfrak{G}$-automorphic functions.

Lemma 2.1: *Let* $S = (\sigma, \mathfrak{a}) \in \mathfrak{G}$ *be an element of* $\mathfrak{G}$. *Then we have for* $\mathfrak{x} \in \mathfrak{V}$:

$$\varphi_{\mathfrak{u}}(S(\mathfrak{x})) = \exp\{2\pi i\langle \mathfrak{u}, \mathfrak{a}\rangle\}\, \varphi_{\sigma^T(\mathfrak{u})}(\mathfrak{x}). \tag{2.3}$$

Definition 2.1: Two vectors $\mathfrak{u}, \mathfrak{u}' \in \Gamma^*$ are called *equivalent*, if a $\sigma \in \mathfrak{L}$ exists with $\mathfrak{u}' = \sigma^T(\mathfrak{u})$. Let $\mathfrak{K}$ be the *set of equivalence classes* belonging to this relation. If $\mathfrak{k} = \{\mathfrak{u}_1, \ldots, \mathfrak{u}_l\} \in \mathfrak{K}$, the linear subspace of $L_2(\mathfrak{T})$ spanned by $\varphi_{\mathfrak{u}_1}, \ldots, \varphi_{\mathfrak{u}_l}$ is denoted by $L(\mathfrak{k})$.

Remark 2.1: From (2.3) there follows, that an element $\varphi \in L_2(\mathfrak{T})$ is $\mathfrak{G}$-automorphic if and only if its projection on every $L(\mathfrak{k})$, $\mathfrak{k} \in \mathfrak{K}$, is $\mathfrak{G}$-automorphic. Therefore it is sufficient to find only the $\mathfrak{G}$-automorphic functions contained in each of the $L(\mathfrak{k})$.

Remark 2.2: For $S \in \mathfrak{G}$ the mapping $L(\mathfrak{k}) \ni \varphi \to \varphi \circ S$ is a transformation of $L(\mathfrak{k})$. If S varies in $\mathfrak{G}$ these transformations give a representation $\mathfrak{D}(\mathfrak{k})$ of the group $\mathfrak{L} = \mathfrak{G}/\mathfrak{T}$ in $L(\mathfrak{k})$. The space $L(\mathfrak{k})$ contains exactly h linearly independent $\mathfrak{G}$-automorphic functions, if $\mathfrak{D}(\mathfrak{k})$ contains the identical representation exactly h times. It is wellknown that

$$h = \frac{1}{r} \sum_{\sigma \in \mathfrak{L}} \operatorname{tr} \mathfrak{D}(\mathfrak{k})(\sigma). \tag{2.4}$$

Definition 2.2: For $\mathfrak{u} \in \Gamma^*$ let $\mathfrak{R}(\mathfrak{u})$ be the *subgroup of all* $\sigma \in \mathfrak{L}$ *with* $\sigma^T(\mathfrak{u}) = \mathfrak{u}$; *we set* $\varrho(\mathfrak{u}) = \operatorname{Ord} \mathfrak{R}(\mathfrak{u})$. We choose a vector $\mathfrak{a}$ belonging to $\sigma \in \mathfrak{R}(\mathfrak{u})$ *and define*

$$\chi(\mathfrak{u}, \sigma) = \exp\{2\pi i \langle \mathfrak{u}, \mathfrak{a}\rangle\}. \tag{2.5}$$

This definition is correct, because $\mathfrak{a}$ is determined mod Γ.

The following lemma is obvious.

Lemma 2.2: *For equivalent vectors* $\mathfrak{u}, \mathfrak{u}' \in \Gamma^*$ *the subgroups* $\mathfrak{R}(\mathfrak{u})$, $\mathfrak{R}(\mathfrak{u}')$ *are conjugate. The index of* $\mathfrak{R}(\mathfrak{u})$ *equals the number of vectors equivalent to* $\mathfrak{u}$. $\chi(\mathfrak{u}, \cdot)$ *is a character of* $\mathfrak{R}(\mathfrak{u})$.

Definition 2.3: The vector $\mathfrak{u} \in \Gamma^*$ is called *principal vector*, if $\chi(\mathfrak{u}, \cdot)$ is the principal character of $\mathfrak{R}(\mathfrak{u})$, i. e.

$$\forall\, \sigma \in \mathfrak{R}(\mathfrak{u}): \chi(\mathfrak{u}, \sigma) = 1.$$

Lemma 2.3: *A class* $\mathfrak{k} \in \mathfrak{K}$ *of equivalent vectors contains either only principal vectors or only non-principal vectors.*

Proof: Assume $\mathfrak{k} \in \mathfrak{K}$ and $\mathfrak{u}, \mathfrak{u}' \in \mathfrak{k}$. There is a $\sigma' \in \mathfrak{L}$ with $\mathfrak{u}' = \sigma'^T(\mathfrak{u})$. If $\sigma \in \mathfrak{R}(\mathfrak{u})$ then we have $\sigma'^{-1}\sigma\sigma' \in \mathfrak{R}(\mathfrak{u}')$ and

$$\chi(\mathfrak{u}', \sigma'^{-1}\sigma\sigma') = \chi(\mathfrak{u}, \sigma). \tag{2.6}$$

From this the assertion follows ∎

Proposition 2.1: *If* $\mathfrak{k}$ *is a class of principal vectors, then* $\dim_{\mathbf{C}}\{L(\mathfrak{k}) \cap L_2(\mathfrak{G})\} = 1$; *if* $\mathfrak{k}$ *is a class of non-principal vectors, then* $L(\mathfrak{k}) \cap L_2(\mathfrak{G}) = \{0\}$.

Proof: Let $\mathfrak{k} = \{\mathfrak{u}_1, \ldots, \mathfrak{u}_l\}$ be a class of equivalent vectors and let $S_1, \ldots, S_r$ be a complete system of representatives of $\mathfrak{G}$ with respect to $\mathfrak{T}$. Then we have:

$$h = \frac{1}{r} \sum_{\sigma \in \mathfrak{L}} \operatorname{tr} \mathfrak{D}(\mathfrak{k})(\sigma) = \frac{1}{r} \sum_{j=1}^{r} \sum_{i=1}^{l} (\varphi_{\mathfrak{u}_i} \circ S_j, \varphi_{\mathfrak{u}_i}).$$

According to (2.3) the last scalar product vanishes if $\sigma_j^T(\mathfrak{u}_i) \neq \mathfrak{u}_i$, i.e. if $\sigma_j \notin \mathfrak{R}(\mathfrak{u}_i)$.

Therefore we obtain from (2.3) and (2.5):

$$h = \frac{1}{r} \sum_{i=1}^{l} \sum_{\sigma \in \mathfrak{R}(\mathfrak{u}_i)} \chi(\mathfrak{u}_i, \sigma).$$

We use the following wellknown formula.

$$\sum_{\sigma \in \mathfrak{R}(\mathfrak{u})} \chi(\mathfrak{u}, \sigma) = \begin{cases} \varrho(\mathfrak{u}) & \text{if } \chi(\mathfrak{u}, \cdot) \text{ is principal}, \\ 0 & \text{else}. \end{cases} \tag{2.7}$$

Taking into account that $\varrho(\mathfrak{u}_1) = \cdots = \varrho(\mathfrak{u}_l)$, $l = \text{index } \mathfrak{R}(\mathfrak{u}_i)$, we find $h = 1$ if the $\mathfrak{u}_i$ are principal vectors and $h = 0$ in the contrary case ∎

Definition 2.4: The *set of classes* $\mathfrak{k} \in \mathfrak{K}$ *containing principal vectors* is denoted by $\mathfrak{H}$. Let $\psi_\mathfrak{k}$ *be a* $\mathfrak{G}$-automorphic function with $\|\psi_\mathfrak{k}\| = 1$ contained in $\mathfrak{k} \in \mathfrak{H}$.

Lemma 2.4: $\{\psi_\mathfrak{k} \mid \mathfrak{k} \in \mathfrak{H}\}$ *is a complete orthonormal system in* $\mathfrak{L}_2(\mathfrak{G})$.

Remark 2.3: An explicite expression for $\psi_\mathfrak{k}$ can be found in the following way. Assume $\mathfrak{k} = \{\mathfrak{u}_1, \ldots, \mathfrak{u}_l\} \in \mathfrak{H}$; let $\sigma_1, \ldots, \sigma_l$ be a complete system of representatives of the left cosets of $\mathfrak{L}$ with respect to $\mathfrak{R}(\mathfrak{u}_1)$. For $i = 1, 2, \ldots, l$ we choose a vector $\mathfrak{a}_i$ belonging to σ_i. Then we have

$$\psi_\mathfrak{k}(\mathfrak{x}) = \frac{1}{\sqrt{l}} \sum_{j=1}^{l} \exp\{2\pi i \langle \mathfrak{u}_1, \mathfrak{a}_j \rangle\}\, \varphi_{\mathfrak{u}_j}(\mathfrak{x}). \tag{2.8}$$

It is easy to see that $\psi_\mathfrak{k}$ has the required properties.

Proposition 2.2: *Let* $g\colon \mathfrak{B}^* \to \mathbf{C}$ *be given. Assume that for each* $\sigma \in \mathfrak{L}$ *the series*

$$\sum_{\mathfrak{u} \in \Gamma^*(\sigma)} \chi(\mathfrak{u}, \sigma)\, g(\mathfrak{u}) \tag{2.9}$$

is absolutely convergent. Then we have

$$\sum_{\sigma \in \mathfrak{L}} \sum_{\mathfrak{u} \in \Gamma^*(\sigma)} \chi(\mathfrak{u}, \sigma)\, g(\mathfrak{u}) = r \sum_{\mathfrak{k} \in \mathfrak{H}} (1/\text{Card } \mathfrak{k}) \sum_{\mathfrak{u} \in \mathfrak{k}} g(\mathfrak{u}). \tag{2.10}$$

Proof: On the left-hand side we change the order of summation. Taking into account the Definitions 1.4 and 2.2 we obtain

$$\sum_{\mathfrak{u} \in \Gamma^*} \sum_{\sigma \in \mathfrak{R}(\mathfrak{u})} \chi(\mathfrak{u}, \sigma)\, g(\mathfrak{u}).$$

Finally we apply (2.7) and the equation $\text{Card } \mathfrak{k} = r/\varrho(\mathfrak{u})$. The proof is finished ∎

§ 3

Proposition 3.1: *Let f be an element of the Schwartz space $\mathfrak{S}(\mathfrak{B})$ such that $\forall\, \mathfrak{x} \in \mathfrak{B}$, $\forall\, \sigma \in \mathfrak{L}\colon f(\sigma(\mathfrak{x})) = f(\mathfrak{x})$. Let $\bar{f}$ be the Fourier transform of f; then $\bar{f}(2\pi\mathfrak{u})$ has the same value for every $\mathfrak{u} \in \mathfrak{k}$. Denoting this common value by $\bar{f}(2\pi\mathfrak{k})$ we have*

$$\sum_{S \in \mathfrak{G}} f(S(\mathfrak{y}) - \mathfrak{x}) = r \sum_{\mathfrak{k} \in \mathfrak{H}} \bar{f}(2\pi\mathfrak{k})\, \overline{\psi_\mathfrak{k}(\mathfrak{x})}\, \psi_\mathfrak{k}(\mathfrak{y}). \tag{3.1}$$

Both series in (3.1) *are absolutely convergent.*

Proof: For fixed $\mathfrak{x} \in \mathfrak{B}$ the left-hand side of (3.1) is a $\mathfrak{G}$-automorphic function of $\mathfrak{y}$; its Fourier expansion gives

$$\sum_{S \in \mathfrak{G}} f(S(\mathfrak{y}) - \mathfrak{x}) = \sum_{\mathfrak{k} \in \mathfrak{H}} c_\mathfrak{k}(\mathfrak{x})\, \psi_\mathfrak{k}(\mathfrak{y}). \tag{3.2}$$

The Fourier coefficients $c_{\mathfrak{k}}(\mathfrak{x})$ are given by

$$c_{\mathfrak{k}}(\mathfrak{x}) = \sum_{S\in\mathfrak{G}} \int_{\mathcal{F}(\mathfrak{T})} \overline{\psi_{\mathfrak{k}}(\mathfrak{y})}\, f\big(S(\mathfrak{y}) - \mathfrak{x}\big)\, d\mu(\mathfrak{y}) = r \int_{\mathfrak{B}} \overline{\psi_{\mathfrak{k}}(\mathfrak{z} + \mathfrak{x})}\, f(\mathfrak{z})\, d\mu(\mathfrak{z}). \tag{3.3}$$

In the last equation we have used the fact that almost every point of $\mathfrak{B}$ is contained in exactly r of the sets $S\big(\mathcal{F}(\mathfrak{T})\big)$, $S \in \mathfrak{G}$. Taking into account the expression (2.8) for $\psi_{\mathfrak{k}}$ we obtain

$$c_{\mathfrak{k}}(\mathfrak{x}) = \frac{r}{\sqrt{l}} \sum_{j=1}^{l} \exp\{-2\pi i\langle \mathfrak{u}_1, \mathfrak{a}_j\rangle - 2\pi i\langle \mathfrak{u}_j, \mathfrak{x}\rangle\}\, \Phi_j \tag{3.4}$$

with

$$\Phi_j = \int_{\mathfrak{B}} \exp\{-2\pi i\langle \mathfrak{u}_1, \sigma_j(\mathfrak{z})\rangle\}\, f(\mathfrak{z})\, d\mu(\mathfrak{z}).$$

The function f and the measure μ are invariant under application of σ_j, hence $\Phi_j = \tilde{f}(2\pi\mathfrak{u}_1)$ and

$$c_{\mathfrak{k}}(\mathfrak{x}) = r\tilde{f}(2\pi\mathfrak{u}_1)\, \overline{\psi_{\mathfrak{k}}(\mathfrak{x})}.$$

The absolutely uniform convergence of the series in (3.1) follows from the properties of the function f belonging to $\mathfrak{S}(\mathfrak{B})$. ∎

Corollary 3.1: *If $\mathfrak{G} = \mathfrak{T}$ is a pure translation group then from* (3.1) *with $\mathfrak{y} = \mathfrak{z} + \mathfrak{x}$ the wellknown Poisson formula, valid for every $f \in \mathfrak{S}(\mathfrak{B})$, follows:*

$$\sum_{\mathfrak{t}\in\Gamma} f(\mathfrak{z} + \mathfrak{t}) = \sum_{\mathfrak{u}\in\Gamma^*} \tilde{f}(2\pi\mathfrak{u}) \exp\{2\pi i\langle \mathfrak{u}, \mathfrak{z}\rangle\}. \tag{3.5}$$

Proposition 3.2: *Let f be any element of the Schwartz space $\mathfrak{S}(\mathfrak{B})$ with Fourier transform $\tilde{f}$. Then we have*

$$\sum_{S\in\mathfrak{G}} \int_{\mathcal{F}(\mathfrak{T})} f\big(S(\mathfrak{x}) - \mathfrak{x}\big)\, d\mu(\mathfrak{x}) = r \sum_{\mathfrak{k}\in\mathfrak{H}} (1/\mathrm{Card}\,\mathfrak{k}) \sum_{\mathfrak{u}\in\mathfrak{k}} \tilde{f}(2\pi\mathfrak{u}). \tag{3.6}$$

Proof: From equation (3.2) it follows

$$\sum_{S\in\mathfrak{G}} \int_{\mathcal{F}(\mathfrak{T})} f\big(S(\mathfrak{x}) - \mathfrak{x}\big)\, d\mu(\mathfrak{x}) = \sum_{\mathfrak{k}\in\mathfrak{H}} \int_{\mathcal{F}(\mathfrak{T})} c_{\mathfrak{k}}(\mathfrak{x})\, \psi_{\mathfrak{k}}(\mathfrak{x})\, d\mu(\mathfrak{x}). \tag{3.7}$$

Using the expressions (3.4) and (2.8) for $c_{\mathfrak{k}}$, $\psi_{\mathfrak{k}}$ respectively we obtain

$$c_{\mathfrak{k}}(\mathfrak{x})\, \psi_{\mathfrak{k}}(\mathfrak{x}) = \frac{r}{l} \sum_{j,m=1}^{l} \exp\{2\pi i[\langle \mathfrak{u}_1, \mathfrak{a}_m - \mathfrak{a}_j\rangle + \langle \mathfrak{u}_m - \mathfrak{u}_j, \mathfrak{x}\rangle]\}\, \Phi_j. \tag{3.8}$$

The integration over $\mathcal{F}(\mathfrak{T})$ yields:

$$\int_{\mathcal{F}(\mathfrak{T})} c_{\mathfrak{k}}(\mathfrak{x})\, \psi_{\mathfrak{k}}(\mathfrak{x})\, d\mu(\mathfrak{x}) = \frac{r}{l} \sum_{j=1}^{l} \Phi_j. \tag{3.9}$$

Taking into account that $\sigma_j{}^T(\mathfrak{u}_1) = \mathfrak{u}_j$, $j = 1, 2, \ldots, l = \mathrm{Card}\,\mathfrak{k}$ the formula (3.4) gives

$$\Phi_j = \tilde{f}(2\pi\mathfrak{u}_j). \tag{3.10}$$

From (3.7), (3.9), (3.10) the assertion follows ∎

Corollary 3.2: *Under the additional assumption $\forall\, \mathfrak{x} \in \mathfrak{B}$, $\forall\, \sigma \in \mathfrak{L}$: $f\big(\sigma(\mathfrak{x})\big) = f(\mathfrak{x})$*

the formula (3.6) *reads:*

$$\sum_{S\in\mathfrak{G}}\int_{\mathcal{F}(\mathfrak{G})} f\big(S(\mathfrak{x})-\mathfrak{x}\big)\,d\mu(\mathfrak{x})=\sum_{\mathfrak{k}\in\mathfrak{H}}\tilde{f}(2\pi\mathfrak{k}). \tag{3.11}$$

Of course $\mathcal{F}(\mathfrak{G})$ is a fundamental domain with respect to $\mathfrak{G}$.

Definition 3.1: Let $\mu_\sigma{}^\perp$ and $\mu_{|\sigma}^\perp$ be the Lebesgue measures in $\mathfrak{V}^\perp(\sigma)$ and $\mathfrak{V}-\mathfrak{V}^\perp(\sigma)$ such that a fundamental domain of the lattices $\Gamma^\perp(\sigma)$ and $\Gamma-\Gamma^\perp(\sigma)$ respectively has measure 1.

Lemma 3.1: *If* $\varphi\in L_1(\mathfrak{V})$ *then the following integral formula is valid:*

$$\int_{\mathfrak{V}}\varphi(\mathfrak{x})\,d\mu(\mathfrak{x})=\int_{\mathfrak{V}-\mathfrak{V}^\perp(\sigma)}\int_{\mathfrak{V}^\perp(\sigma)}\varphi(\mathfrak{y}+\mathfrak{z})\,d\mu_\sigma{}^\perp(\mathfrak{y})\,d\mu_{|\sigma}^\perp(\bar{\mathfrak{z}}). \tag{3.12}$$

Here $\bar{\mathfrak{z}}$ *denotes the coset* $\mathfrak{z}+\mathfrak{V}^\perp(\sigma)$.

Proof: It is possible to find a basis $\mathfrak{x}_1,\dots,\mathfrak{x}_n$ of $\mathfrak{V}$ such that $\{\mathfrak{x}_1,\dots,\mathfrak{x}_n\}$, $\{\mathfrak{x}_1,\dots,\mathfrak{x}_{n-n(\sigma)}\}$, $\{\bar{\mathfrak{x}}_{n-n(\sigma)+1},\dots,\bar{\mathfrak{x}}_n\}$ span a fundamental domain of Γ, $\Gamma^\perp(\sigma)$, $\Gamma-\Gamma^\perp(\sigma)$ respectively. Using this basis we obtain the assertion from the Fubini theorem. ∎

Remark 3.1: In the vector spaces $\mathfrak{V}^*$, $\mathfrak{V}^*(\sigma)$ and $\mathfrak{V}^*-\mathfrak{V}^*(\sigma)$ we introduce Lebesgue measures μ^*, $\mu_\sigma{}^*$, $\mu_{|\sigma}^*$ which are normed with the help of the lattices Γ^*, $\Gamma^*(\sigma)$, $\Gamma^*-\Gamma^*(\sigma)$ respectively.

Definition 3.2: Let $\mathcal{T}$ be the *set of* $\mathfrak{T}$*-conjugacy classes of* $\mathfrak{G}$, (S, $S'\in\mathfrak{G}$ are in the same $\mathfrak{T}$-conjugacy class, if a $T\in\mathfrak{T}$ exists with $S'=TST^{-1}$.) Further let Ω be the *set of* $\mathfrak{G}$*-conjugacy classes of* $\mathfrak{G}$. Each $\theta\in\Omega$ is the union of a finite number of $\mathfrak{T}$-conjugacy classes. Let $m(\theta)$ be that number.

Remark 3.2: Assume $\theta\in\Omega$, $S\in\theta$ and let $\mathfrak{N}(S)$ be the normalizer of S in $\mathfrak{G}$. Let $\overline{\mathfrak{N}(S)}$ be the image of $\mathfrak{N}(S)$ under the natural homomorphism of $\mathfrak{G}$ onto $\mathfrak{G}/\mathfrak{T}$. Then we have $m(\theta)=(\text{ord}\,\mathfrak{G}/\mathfrak{T}):\big(\text{ord}\,\overline{\mathfrak{N}(S)}\big)$.

Proof: The group $\mathfrak{G}/\mathfrak{T}$ acts as a transformation group in $\mathcal{T}$ (via the inner automorphism of $\mathfrak{G}$). Let $\tau\in\mathcal{T}$, $S\in\tau$, $\tau\subseteqq\theta$; then $\overline{\mathfrak{N}(S)}$ is the stable subgroup of τ and the set $\{\tau'\in\mathcal{T}\mid\tau'\subseteqq\theta\}$ is its orbit. From these facts the assertion follows. ∎

Theorem: *Assume* $f\in\mathfrak{S}(\mathfrak{V})$. *Let* $\tau\in\mathcal{T}$ *be a* $\mathfrak{T}$*-conjugacy class of* $\mathfrak{G}$ *and* $S=(\sigma,\mathfrak{b})\in\tau$. *Then we set*

$$I_\tau(f)=\frac{1}{\mathrm{e}(\sigma)}\int_{\mathfrak{V}^\perp(\sigma)} f(\mathfrak{y}+\mathfrak{b})\,\mathrm{d}\mu_\sigma{}^\perp(\mathfrak{y}). \tag{3.13}$$

1. $I_\tau(f)$ *depends only on* τ (*i.e.* I_τ *is independent of the choice of* $S\in\tau$). $I_\tau(f)$ *has the alternative expression:*

$$I_\tau(f)=\big(1/(2\pi)^{n(\sigma)}\,\mathrm{e}(\sigma)\big)\int_{\mathfrak{V}^*(\sigma)}\exp\{i\langle\mathfrak{v},\mathfrak{b}\rangle\}\,\tilde{f}(\mathfrak{v})\,d\mu_\sigma{}^*(\mathfrak{v}). \tag{3.14}$$

Here $\tilde{f}$ *is the Fourier transform of* f.

2. *We have*

$$\sum_{\mathfrak{k}\in\mathfrak{H}}(1/\text{Card}\,\mathfrak{k})\sum_{\mathfrak{u}\in\mathfrak{k}}\tilde{f}(2\pi\mathfrak{u})=\frac{1}{r}\sum_{\theta\in\Omega}\sum_{\tau\subseteqq\theta}I_\tau(f). \tag{3.15}$$

3. *Under the additional assumption* $\forall\,\mathfrak{x}\in\mathfrak{V}$, $\forall\,\sigma\in\mathfrak{L}$: $f\big(\sigma(\mathfrak{x})\big)=f(\mathfrak{x})$ *the value of* $I_\tau(f)$ *depends only on* θ *with* $\tau\subseteqq\theta$ *and the value of* $\tilde{f}(2\pi\mathfrak{u})$ *depends only on* $\mathfrak{k}\in\mathfrak{H}$ *with*

$\mathfrak{u} \in \mathfrak{k}$. *Then we can write*

$$\sum_{\mathfrak{k}\in\mathfrak{H}} \bar{f}(2\pi\mathfrak{k}) = \frac{1}{r} \sum_{\theta\in\Omega} m(\theta)\, I_\theta(f). \tag{3.16}$$

Proof: We use Proposition 2.2 with $g(\mathfrak{u})$ replaced by $\bar{f}(2\pi\mathfrak{u})$:

$$\sum_{\mathfrak{k}\in\mathfrak{H}} (1/\mathrm{Card}\,\mathfrak{k}) \sum_{\mathfrak{u}\in\mathfrak{k}} \bar{f}(2\pi\mathfrak{u}) = \frac{1}{r} \sum_{\sigma\in\mathfrak{L}} \sum_{\mathfrak{u}\in\Gamma^*(\sigma)} \chi(\mathfrak{u},\sigma)\, \bar{f}(2\pi\mathfrak{u}). \tag{3.17}$$

For fixed $\sigma \in \mathfrak{L}$ and a vector $\mathfrak{a}$ belonging to σ we consider the series

$$\sum_{\mathfrak{u}\in\Gamma^*(\sigma)} \chi(\mathfrak{u},\sigma)\, \bar{f}(2\pi\mathfrak{u}) = \sum_{\mathfrak{u}\in\Gamma^*(\sigma)} \exp\{2\pi i\langle\mathfrak{u},\mathfrak{a}\rangle\}\, \bar{f}(2\pi\mathfrak{u}). \tag{3.18}$$

Now we apply the Poisson formula for translation groups (eq. (3.5)) together with the Fourier transformation in $\mathfrak{B}^*(\sigma)$:

$$\sum_{\mathfrak{u}\in\Gamma^*(\sigma)} \chi(\mathfrak{u},\sigma)\, \bar{f}(2\pi\mathfrak{u}) = (1/2\pi)^{n(\sigma)} \sum_{\mathfrak{t} \bmod \Gamma^{\perp}(\sigma)} \int_{\mathfrak{B}^*(\sigma)} \exp\{i\langle\mathfrak{a}+\mathfrak{t},\mathfrak{v}\rangle\}\, \bar{f}(\mathfrak{v})\, d\mu_\sigma^*(\mathfrak{v}).$$

The last summation is extended over a complete system of representatives for the cosets forming the difference **Z**-module $\Gamma - \Gamma^{\perp}(\sigma)$. Each of these cosets is the union of exactly $e(\sigma)$ coset elements of the difference **Z**-module $\Gamma - \Gamma_e^{\perp}(\sigma)$. Therefore we can write:

$$\sum_{\mathfrak{u}\in\Gamma^*(\sigma)} \chi(\mathfrak{u},\sigma)\, \bar{f}(2\pi\mathfrak{u}) = \big(1/(2\pi)^{n(\sigma)}\, e(\sigma)\big) \sum_{\mathfrak{A} \bmod \Gamma_e^{\perp}(\sigma)} \int_{\mathfrak{B}^*(\sigma)} \exp\{i\langle\mathfrak{a}+\mathfrak{t},\mathfrak{v}\rangle\}\, \bar{f}(\mathfrak{v})\, d\mu_\sigma^*(\mathfrak{v}). \tag{3.19}$$

Now we turn to the $\mathfrak{T}$-conjugacy classes of $\mathfrak{G}$. Firstly we remark that two $\mathfrak{T}$-conjugate elements of $\mathfrak{G}$ are contained in the same coset of $\mathfrak{G}$ with respect to $\mathfrak{T}$. Let $\mathfrak{C}(\sigma)$, $\sigma \in \mathfrak{L}$ be such a coset; let $S' = (\sigma, \mathfrak{a}+\mathfrak{t}')$, $S'' = (\sigma, \mathfrak{a}+\mathfrak{t}'') \in \mathfrak{C}(\sigma)$ with $\mathfrak{t}', \mathfrak{t}'' \in \Gamma$. It is easy to see that S', S'' are $\mathfrak{T}$-conjugate if and only if $\mathfrak{t}' - \mathfrak{t}'' \in \Gamma_e^{\perp}(\sigma)$. If the vector $\mathfrak{t}$ in $S = (\sigma, \mathfrak{b})$, $\mathfrak{b} = \mathfrak{a} + \mathfrak{t}$ runs through a complete system of representatives of Γ with respect to $\Gamma_e^{\perp}(\sigma)$ then S runs through a complete system of representatives of the $\mathfrak{T}$-conjugacy classes contained in $\mathfrak{C}(\sigma)$. On the other hand we have for $\mathfrak{v} \in \mathfrak{B}^*(\sigma)$:

$$\langle\mathfrak{a}+\mathfrak{t}',\mathfrak{v}\rangle = \langle\mathfrak{a}+\mathfrak{t}'',\mathfrak{v}\rangle \quad \text{if} \quad \mathfrak{t}'-\mathfrak{t}'' \in \Gamma_e^{\perp}(\sigma).$$

This shows that the summand in (3.19) is a function of the $\mathfrak{T}$-conjugacy classes; therefore we can write

$$\sum_{\mathfrak{u}\in\Gamma^*(\sigma)} \chi(\mathfrak{u},\sigma)\, \bar{f}(2\pi\mathfrak{u}) = \sum_{\tau\subseteq\mathfrak{C}(\sigma)} I_\tau(f). \tag{3.20}$$

Here $I_\tau(f)$ is given by (3.14). In (3.20) we sum up over $\sigma \in \mathfrak{L}$; on the left-hand side we use (3.17), on the right-hand side we make a simple change of the order of summation. This gives the assertion (3.15).

In order to obtain formula (3.13) we write down:

$$\bar{f}(\mathfrak{v}) = \int_{\mathfrak{B}} \exp\{-i\langle\mathfrak{v},\mathfrak{x}\rangle\}\, f(\mathfrak{x})\, d\mu(\mathfrak{x});$$

by use of (3.12) we get

$$\bar{f}(\mathfrak{v}) = \int_{\mathfrak{B}-\mathfrak{B}^{\perp}(\sigma)} \int_{\mathfrak{B}^{\perp}(\sigma)} \exp\{-i\langle\mathfrak{v},\mathfrak{y}+\mathfrak{z}\rangle\}\, f(\mathfrak{y}+\mathfrak{z})\, d\mu_\sigma^{\perp}(\mathfrak{y})\, d\mu_{1\sigma}^{\perp}(\mathfrak{z}). \tag{3.21}$$

Inserting (3.21) in (3.14) we obtain an expression for $I_\tau(f)$ with three succesive integrations; two of them cancel by means of the Fourier inversion formula because $\mathfrak{V}^*(\sigma)$ and $\mathfrak{V} - \mathfrak{V}^\perp(\sigma)$ are dual vector spaces. The remaining formula is (3.13). Thus part 1 and 2 of the theorem are proved.

In order to prove part 3 we assume $\forall\, \mathfrak{x} \in \mathfrak{V},\ \forall\, \sigma \in \mathfrak{L}$: $f(\sigma(\mathfrak{x})) = f(\mathfrak{x})$. Then we have $\forall\, \mathfrak{u} \in \mathfrak{V}^*$: $\bar{f}(\sigma^\perp(\mathfrak{u})) = \bar{f}(\mathfrak{u})$. From this it follows that $\bar{f}(2\pi\mathfrak{u})$ has the same value for all $\mathfrak{u} \in \mathfrak{k}$, $\mathfrak{k} \in \mathfrak{H}$; we denote this value by $\bar{f}(2\pi\mathfrak{k})$. Further let S, S' be two $\mathfrak{G}$-conjugate elements of $\mathfrak{G}$ with $S = (\sigma, \mathfrak{b})$, $S' = (\sigma', \mathfrak{b}')$ and $S' = GSG^{-1}$, $G = (\gamma, \mathfrak{c})$. A simple calculation shows:

$$\sigma' = \gamma\sigma\gamma^{-1}, \mathfrak{b}' = \gamma(\mathfrak{b}) + \mathfrak{c} - \gamma\sigma\gamma^{-1}(\mathfrak{c}). \tag{3.22}$$

From this we have

$$\mathfrak{V}^\perp(\sigma') = \gamma(\mathfrak{V}^\perp(\sigma)),\ \Gamma^\perp(\sigma') = \gamma(\Gamma^\perp(\sigma)),$$
$$\Gamma_e{}^\perp(\sigma') = \gamma(\Gamma_e{}^\perp(\sigma)),\ \mathrm{e}(\sigma') = \mathrm{e}(\sigma),\ \mu_{\sigma'}^\perp = \gamma(\mu_\sigma{}^\perp).$$

If the coset of $\mathfrak{b}'$ (resp. $\mathfrak{b}$) modulo $\mathfrak{V}^\perp(\sigma')$ (resp. $\mathfrak{V}^\perp(\sigma)$) is denoted by $\bar{\mathfrak{b}}'$ (resp. $\bar{\mathfrak{b}}$) then we have $\bar{\mathfrak{b}}' = \gamma(\bar{\mathfrak{b}})$. Transforming the integral in (3.13) with the help of the linear transformation γ we obtain

$$\frac{1}{\mathrm{e}(\sigma')} \int\limits_{\mathfrak{V}^\perp(\sigma')} f(\mathfrak{y} + \mathfrak{b}')\, d\mu_{\sigma'}^\perp(\mathfrak{y}) = \frac{1}{\mathrm{e}(\sigma)} \int\limits_{\mathfrak{V}^\perp(\sigma)} f(\mathfrak{y} + \mathfrak{b})\, d\mu_\sigma{}^\perp(\mathfrak{y}).$$

This shows that $I_\tau(f)$ is a function of the $\mathfrak{G}$-conjugacy class $\theta \in \Omega$ containing τ; in this way the notation $I_\theta(f)$ is justified. The proof of the theorem is finished ∎

Proposition 3.3: *Let $\mathfrak{f}(\tau)$, $\tau \in \mathcal{T}$ be the $(n - n(\sigma))$-dimensional plane in the affine space $\mathfrak{V}$ which is the domain of integration occuring in the expression* (3.13) *of* $I_\tau(f)$.

a) $\dim \mathfrak{f}(\tau) = 0$ *if and only if τ contains translations.*

b) $0 \in \mathfrak{f}(\tau)$ *if and only if the elements $S \in \tau$ have fixed points.*

Proof: a) $\dim \mathfrak{f}(\tau) = 0$ means that $\dim \mathfrak{V}^\perp(\sigma) = 0$; owing to Lemma 1.1 this is equivalent to $\dim \mathfrak{V}(\sigma) = n$, i.e. $\sigma = \mathrm{Id}$.

b) From (3.13) it follows that $0 \in \mathfrak{f}(\tau)$ if and only if $\mathfrak{b} \in \mathfrak{V}^\perp(\sigma)$; here $S = (\sigma, \mathfrak{b})$ is any element of the class τ under consideration. $\mathfrak{b} \in \mathfrak{V}^\perp(\sigma)$ is equivalent to the existence of a $\mathfrak{c} \in \mathfrak{V}$ with $\mathfrak{b} = \mathfrak{c} - \sigma(\mathfrak{c})$; in this case we have $S(\mathfrak{c}) = \mathfrak{c}$. Finally we remark: if some element $S \in \tau$ has fixed points, then every element of τ has fixed points.

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