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On Counterexamples for Rates of Convergence concerning Numerical Solutions of Initial Value Problems

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Es ist das Anliegen der vorliegenden Note, die Annäherung von Differenzenverfahren an die exakte Lösung sachgemäß gestellter Anfangswertaufgaben zu untersuchen. Genauer wird gezeigt, daß die Konvergenzgeschwindigkeiten, die unlängst über abstrakte Approximationssätze vom Lax-Typ hergeleitet werden konnten, tatsächlich scharf sind. Diese Aussagen folgen aus einem allgemeinen Beschränktheitsprinzip mit Ordnung, das hier im Rahmen diskreter Approximationen von Banach-Räumen hergeleitet wird. Die dabei benutzte Beweismethode ist die klassische Methode des gleitenden Höckers, die jedoch nun mit Ordnungen versehen ist. Die aufgeführten Anwendungen betreffen hyperbolische wie parabolische Aufgaben und dienen in erster Linie dem Zweck, den einheitlichen Zugang zu Fragen der Schärfe von Fehlerabschätzungen aufzuzeigen.

В работе исследуется приближение методов сеток к точным решениям корректно поставленных задач Коши. Волее подробно: показывается, что оценки скорости сходимости, недавно полученные из общих аппроксимационных теорем типа Lax, являются точными. Результаты эти получаются из некоторого общего принципа ограниченности с порядком, который в работе устанавливается в рамках дискретной аппроксимации банаховых пространств. Метод доказательства совпадает по существу с классическим методом скользящего горба, снабженный однако порядком. Данные применения относятся и к гиперболическим и к параболическим задачам и служат в первой очереди демонстрацией общего подхода к вопросам точности оценок погрешности.

The present note is concerned with the approximation of the exact solution of a properly posed initial value problem by finite difference methods. It is shown that those rates of convergence obtained in previous papers on abstract Lax-type theorems with rates are indeed sharp. This is achieved as a consequence of a general uniform boundedness principle with rates, given in the setting of discrete approximations of Banach spaces. The method of proof is the familiar gliding hump method but now equipped with rates. The applications presented emphasize the unifying approach to various concrete results scattered in the literature.

1. Introduction

This note is concerned with the approximation of the exact solution of an initial value problem by difference methods. It will be shown that those rates of convergence given in previous papers on abstract Lax-type theorems with rates (e.g. [4, 6, 7]) are indeed sharp. This will be a consequence of a general gliding hump method equipped with rates. Corresponding concrete results given e.g. by HEDSTROM [14] and BRENNER-THOMÉE [1] (see also [2, pp. 76, 111]) use rather specific arguments, for example, in the framework of Fourier analysis.

Given a properly posed initial value problem

$$d/dt\, u(t) = Au(t), \qquad u(0) = f \in X,$$

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on a Banach space X (with norm $\|\cdot\|_X$), the solution may be represented in the form $u(t) = E(t) f$, $t \geqq 0$, where

$$\begin{aligned} &E(t) \in [X] \quad \text{and} \quad \|E(t)\|_{[X]} \leqq C && (0 \leqq t \leqq T < \infty),\\ &E(t)\, E(s) = E(t+s) && (s, t \geqq 0),\\ &\lim_{t\to 0+} E(t) f = E(0) f = f && (f \in X). \end{aligned} \tag{1.1}$$

Here $[X] := [X, X]$ denotes the space of bounded linear operators from X into itself.

A difference method for the approximation of the continuous solution semigroup (1.1) is described by operators $\{E_\tau \in [X_h]; 0 \leqq \tau \leqq 1\}$ where for each $h = h(\tau)$ the linear space X_h is endowed with norm $\|\cdot\|_h$. The spaces X_h are related to X in terms of operators $P_h \in [X, X_h]$ satisfying

$$\begin{aligned} &\text{(i)}\ \|P_h f\|_h \leqq C\, \|f\|_X && (h > 0, f \in X),\\ &\text{(ii)}\ \lim_{h\to 0} \|P_h f\|_h = \|f\|_X && (f \in X), \end{aligned} \tag{1.2}$$

i.e., the operators P_h and spaces X_h define a discrete approximation of X (cf. [17] and the literature cited).

The difference method is assumed to be consistent of order α with the solution semigroup (1.1) on the linear subspace $U \subset X$ with seminorm $|\cdot|_U$, i.e.,

$$\|[E_\tau P_h - P_h E(\tau)]\, E(t) f\|_h \leqq C\tau^{1+\alpha}\, |f|_U \tag{1.3}$$

uniformly for all $f \in U$, $0 \leqq \tau \leqq 1$, $0 \leqq t \leqq T$. If the difference method is stable, i.e.,

$$\|E_\tau^j\|_{[X_h]} \leqq C \qquad (0 \leqq \tau \leqq 1, 0 \leqq j\tau \leqq T, j \in \mathbf{N}) \tag{1.4}$$

($\mathbf{N}:=$ natural numbers), then the approximation error satisfies

$$\|E_\tau^n P_h f - P_h E(n\tau) f\|_h \leqq C\mathfrak{K}(n\tau^{1+\alpha}, f; X, U) \tag{1.5}$$

uniformly for all $f \in X$, $0 \leqq \tau \leqq 1$, $0 \leqq n\tau \leqq T$, $n \in \mathbf{N}$ (see [4, 6, 7], also [15], and for related material [12]). Here the $\mathfrak{K}$-functional

$$\mathfrak{K}(t, f; X, U) := \inf_{g \in U} \{\|f - g\|_X + t\, |g|_U\} \qquad (f \in X, t \geqq 0) \tag{1.6}$$

serves as an abstract measure of the "smoothness" of $f \in X$. Note that the constants C in (1.1—5) and in the following may have different values at each occurrence.

Let ω be a modulus of continuity, thus a function defined on $[0, \infty)$, continuous and monotonely increasing there such that (cf. [16, p. 96ff.])

$$\begin{aligned} &\omega(0) = 0, \qquad \omega(t) > 0 \quad \text{for} \quad t > 0,\\ &\omega(t_1 + t_2) \leqq \omega(t_1) + \omega(t_2). \end{aligned} \tag{1.7}$$

Then one may introduce intermediate spaces $U \subset X_\omega \subset X$ by

$$X_\omega := \{f \in X;\ \mathfrak{K}(t, f; X, U) = \mathcal{O}(\omega(t)),\ t \to 0+\} \tag{1.8}$$

with seminorm

$$|f|_\omega := \sup_{t>0} \mathfrak{K}(t, f; X, U)/\omega(t). \tag{1.9}$$

For $f \in X_\omega$ and $n\tau = t_0 = 1$ (fixed), the estimate (1.5) now implies the rate of convergence

$$\|E^n_{1/n} P_h f - P_h E(1) f\|_h = \mathcal{O}\big(\omega(n^{-\alpha})\big) \qquad (n \to \infty). \tag{1.10}$$

In fact, concerning the operator norms of the approximation error

$$\|E^n_{1/n} P_h - P_h E(1)\|_{[X_\omega, X_h]} := \sup_{|f|_\omega \neq 0} \frac{\|E^n_{1/n} P_h f - P_h E(1) f\|_h}{|f|_\omega}, \tag{1.11}$$

the estimate (1.5) even delivers

$$\|E^n_{1/n} P_h - P_h E(1)\|_{[X_\omega, X_h]} \leqq C\omega(n^{-\alpha}). \tag{1.12}$$

It is the purpose of this note to show that convergence assertions of the previous type are indeed sharp in the sense that there exist elements $f_\omega \in X_\omega$ for which e.g. the right side of (1.10) cannot be improved to $o\big(\omega(n^{-\alpha})\big)$.

To this end, Sec. 2 exhibits a general approach (Thm. 1) concerning the existence of counterexamples dealing with rates of convergence for X_ω. Sec. 3 presents a theorem on Voronovskaja-type relations for iterates of operators and gives a more detailed description of the difference methods employed. Sec. 4 is concerned with a hyperbolic initial value problem, the solution of which is represented by the translation semigroup. An application of Thm. 1 then improves results of HEDSTROM [14] and BRENNER-THOMÉE [1] (see also [2, p. 111]) in as much as a sequence $\{f_{n,\omega}\} \subset X_\omega$ may be replaced by a single element $f_\omega \in X_\omega$ to achieve certain lower bounds (cf. (2.7), (4.4)). In Sec. 5 the solution semigroups are certain holomorphic semigroups. Here an application of Thm. 1 mainly reproduces results given by HEDSTROM [14] (see also [2, p. 76]) in the special case of the Gauss-Weierstrass semigroup. In any case, the choice of applications presented here is of course not complete but rather exemplary, emphasizing the unifying approach via Thm. 1 to various concrete results known so far in the literature but derived there by different and specific methods.

2. Counterexamples via the Gliding Hump Method

Let us commence with an observation concerning operator norms on X_ω (cf. (1.11)).

Proposition 1: *Let X be a Banach space, $\{Y_n\}$ a sequence of normed linear spaces, and $U \subset X$ a seminormed linear subspace. Let $\{\varphi_n\}$ denote a sequence of positive numbers with*

$$\lim_{n\to\infty} \varphi_n = 0 \quad \textit{monotonely decreasing.} \tag{2.1}$$

If for each $R_n \in [X, Y_n]$ there exists $g_n \in U$ such that for all $n \in \mathbf{N}$

$$\|g_n\|_X \leqq C_1, \tag{2.2}$$

$$|g_n|_U \leqq C_2/\varphi_n, \tag{2.3}$$

$$\|R_n g_n\|_n \geqq C_3 > 0, \tag{2.4}$$

then there exists a constant $C_4 > 0$ such that for any modulus of continuity ω and for each $n \in \mathbf{N}$

$$\|R_n\|_{[X_\omega, Y_n]} \geqq C_4 \omega(\varphi_n). \tag{2.5}$$

Proof: It is a well-known fact that for each modulus of continuity one has (cf. [16, p. 99])

$$\omega(s)/s \leqq 2\omega(t)/t \qquad (s \geqq t > 0). \tag{2.6}$$

Consider now the elements $g_{n,\omega} := \omega(\varphi_n)\, g_n \in U \subset X_\omega$. Since (cf. (1.7), (2.2/3), (2.6))

$$\mathfrak{K}(t, g_{n,\omega}; X, U) \leqq \begin{cases} \|g_{n,\omega}\|_X \leqq C_1\omega(\varphi_n) \leqq C_1\omega(t) & \text{for } t \geqq \varphi_n, \\ t\,|g_{n,\omega}|_U \leqq C_2 t\omega(\varphi_n)/\varphi_n \leqq 2C_2\omega(t) & \text{for } t \leqq \varphi_n, \end{cases}$$

one has that $|g_{n,\omega}|_\omega \leqq C^* := \max\{C_1, 2C_2\}$. Thus by (2.4)

$$\|R_n\|_{[X_\omega, Y_n]} \geqq \|R_n g_{n,\omega}\|_n / C^* \geqq C_3\omega(\varphi_n)/C^*,$$

i.e., one has (2.5) with $C_4 := C_3/C^* > 0$ independent of ω and $n \in \mathbf{N}$ ∎

On the other hand, note that (2.5) is equivalent to

$$\text{for each } n \in \mathbf{N} \text{ there exists } f_{n,\omega} \in X_\omega \text{ with } |f_{n,\omega}|_\omega \leqq C^* \text{ and } \|R_n f_{n,\omega}\|_n \geqq C_4'^*\omega(\varphi_n), \tag{2.7}$$

in other words, a first glance at conditions (2.2—4) yields the existence of a sequence $\{f_{n,\omega}\}$. For moduli of continuity satisfying

$$\lim_{t\to 0+} \omega(t)/t = \infty, \tag{2.8}$$

however, one may even formulate the following *resonance principle.*

Theorem 1: *Let X, U, $\{Y_n\}$, $\{\varphi_n\}$, $\{R_n\}$, and $\{g_n\}$ satisfy the conditions of Prop.* 1. *Then for any modulus of continuity satisfying* (2.8) *there exists an element $f_\omega \in X_\omega$ such that*

$$\|R_n f_\omega\|_n \neq o\big(\omega(\varphi_n)\big) \qquad (n \to \infty). \tag{2.9}$$

Proof: Assume that (2.9) does not hold, i.e., for each $f \in X_\omega$ one has

$$\|R_n f\|_n = o\big(\omega(\varphi_n)\big) \qquad (n \to \infty). \tag{2.10}$$

Starting with an arbitrary $n_1 \in \mathbf{N}$, one may successively construct an increasing subsequence $\{n_k\} \subset \mathbf{N}$ such that the following conditions are satisfied simultaneously ($k \geqq 2$):

$$\omega(\varphi_{n_k}) \leqq (1/2)\,\omega(\varphi_{n_{k-1}}), \tag{2.11}$$

$$\sum_{j=1}^{k-1} \omega(\varphi_{n_j})/\varphi_{n_j} \leqq \omega(\varphi_{n_k})/\varphi_{n_k}, \tag{2.12}$$

$$\|R_{n_{k-1}}\|_{[X, Y_{n_{k-1}}]} \leqq (C_3/6C_1)\,\omega(\varphi_{n_{k-1}})/\omega(\varphi_{n_k}), \tag{2.13}$$

$$\|R_{n_k} h_{k-1}\|_{n_k} \leqq (C_3/3)\,\omega(\varphi_{n_k}), \tag{2.14}$$

$$h_{k-1} := \sum_{j=1}^{k-1} \omega(\varphi_{n_j})\, g_{n_j} \in U.$$

Indeed, (2.11—13) may be satisfied in view of (1.7), (2.1/8), whereas (2.14) is a consequence of (2.10). By (2.2/11) it follows that

$$\sum_{j=1}^{\infty} \|\omega(\varphi_{n_j})\, g_{n_j}\|_X \leqq C_1 \sum_{j=1}^{\infty} \omega(\varphi_{n_j}) \leqq C_1\omega(\varphi_{n_1}) \sum_{j=1}^{\infty} 2^{-j+1} < \infty.$$

Therefore $h_\omega := \sum_{j=1}^{\infty} \omega(\varphi_{n_j})\, g_{n_j}$ is well-defined as an element of X since X is complete. Moreover, $h_\omega \in X_\omega$. Indeed, for each $t \in (0, \varphi_{n_1})$ there exists $k \in \mathbb{N}$ such that $\varphi_{n_{k+1}} \leqq t < \varphi_{n_k}$. Using the corresponding $h_k \in U$ and conditions (2.2/3), (2.11/12), and finally (2.6), one has in view of definition (1.6)

$$\begin{aligned}\mathfrak{K}(t, h_\omega; X, U) &\leqq \|h_\omega - h_k\|_X + t\, |h_k|_U \\ &= \left\| \sum_{j=k+1}^{\infty} \omega(\varphi_{n_j})\, g_{n_j} \right\|_X + t \left| \sum_{j=1}^{k} \omega(\varphi_{n_j})\, g_{n_j} \right|_U \\ &\leqq 2C_1 \omega(\varphi_{n_{k+1}}) + 2C_2 t \omega(\varphi_{n_k})/\varphi_{n_k} \\ &\leqq (2C_1 + 4C_2)\, \omega(t).\end{aligned}$$

This proves that $h_\omega \in X_\omega$. Applying R_{n_k} to

$$h_\omega = \omega(\varphi_{n_k})\, g_{n_k} + h_{k-1} + (h_\omega - h_k),$$

one obtains by (2.4) and (2.13/14) that

$$\begin{aligned}\|R_{n_k} h_\omega\|_{n_k} &\geqq \|R_{n_k} \omega(\varphi_{n_k})\, g_{n_k}\|_{n_k} - \|R_{n_k} h_{k-1}\|_{n_k} - \|R_{n_k}\|_{[X, Y_{n_k}]}\, \|h_\omega - h_k\|_X \\ &\geqq C_3 \omega(\varphi_{n_k}) \left[1 - \frac{1}{3} - \frac{1}{3}\right].\end{aligned}$$

This is a contradiction to (2.10), proving the assertion ∎

The proof is based on a gliding hump method as known since the last century, but now equipped with rates. This building in of rates was initiated in relevant work of Teljakovskii and Mertens-Nessel concerning multipliers of strong convergence for the one-dimensional trigonometric system and for regular biorthogonal systems in Banach spaces, respectively. For further comments, however, one may consult [9, 10], in particular for an interpretation of Thm. 1 as a uniform boundedness principle with rates as well as for applications to approximation and interpolation theory, numerical quadrature formulae, and for its connections to Banach-Steinhaus theorems with rates. In the following we will give applications to the numerical solution of initial value problems.

3. A Voronovskaja-Type Relation for Iterates

Apart from some interest in itself, the following (first) asymptotic expansions for iterates of operators are used to verify condition (2.4) in the applications. In fact, only very special cases of the following, more general treatment are needed (cf. Thm. 2).

Lemma 1: *Let $\{E(t)\} \subset [X]$ satisfy (1.1) and $\{E_\tau \in [X_h]\}$ be a stable difference method. Let B be a closed linear operator from $D(B) \subset X$ into X such that $D(B)$ is dense in X, $E(t)\big(D(B)\big) \subset D(B)$, and*

$$\|BE(t)\, f\|_X \leqq C\, \|Bf\|_X \qquad \big(f \in D(B), 0 \leqq t \leqq T\big). \tag{3.1}$$

Then the Voronovskaja-type condition ($\alpha > 0$, $\tau \to 0+$)

$$\|[E_\tau P_h - P_h E(\tau)]\, E(t)\, f - \tau^{1+\alpha} P_h E(t)\, Bf\|_h = o(\tau^{1+\alpha}), \tag{3.2}$$

for each $f \in D(B)$ *uniformly for* $t \in [0, T]$, *implies correspondingly for the iterates that*

$$\|E_\tau^n P_h f - P_h E(n\tau) f - n\tau^{1+\alpha} P_h E(n\tau) Bf\|_h = n\mathit{o}(\tau^{1+\alpha}) \tag{3.3}$$

for each $f \in D(B)$ *uniformly for* $n\tau \in [0, T]$, $n \in \mathbf{N}$.

Proof: First observe that $D(B) =: U$ is a Banach space with respect to the norm $|f|_U := \|f\|_X + \|Bf\|_X$. Therefore by the uniform boundedness principle condition (3.2) implies consistency of order α on U, and thus convergence (1.5). For each $f \in D(B)$ one has

$$\begin{aligned}&\|E_\tau^n P_h f - P_h E(n\tau) f - n\tau^{1+\alpha} P_h E(n\tau) Bf\|_h \\ &\le \left\| \sum_{j=0}^{n-1} E_\tau^{n-j-1}[[E_\tau P_h - P_h E(\tau)] E(j\tau) f - \tau^{1+\alpha} P_h E(j\tau) Bf] \right\|_h \\ &+ \tau^{1+\alpha} \left\| \sum_{j=0}^{n-1} \left[E_\tau^{n-j-1} P_h - P_h E\big((n-j-1)\tau\big)\right] E(j\tau) Bf \right\|_h \\ &+ n\tau^{1+\alpha} \|P_h E\big((n-1)\tau\big) [E(0) - E(\tau)] Bf\|_h.\end{aligned}$$

Stability (1.4), convergence (1.5) as well as conditions (1.1/2) and (3.2) yield

$$\begin{aligned}&\|E_\tau^n P_h f - P_h E(n\tau) f - n\tau^{1+\alpha} P_h E(n\tau) Bf\|_h \\ &\le nC\mathit{o}(\tau^{1+\alpha}) + \tau^{1+\alpha} \sum_{j=0}^{n-1} C\mathfrak{K}\big((n-j-1)\tau^{1+\alpha}, E(j\tau) Bf; X, U\big) + n\tau^{1+\alpha}\mathit{o}(1).\end{aligned}$$

But in view of (1.1) and (3.1) one has $|E(j\tau) g|_U \le C |g|_U$ for each $g \in U$, and hence uniformly for $0 \le j \le n-1$ $(\tau \to 0+)$

$$\mathfrak{K}\big((n-j-1)\tau^{1+\alpha}, E(j\tau) Bf; X, U\big) \le C\mathfrak{K}(\tau^\alpha, Bf; X, U) = \mathit{o}(1).$$

This completes the proof ∎

As concrete examples of spaces X, U, X_h we consider $X = C_{ub}(\mathbf{R})$, the space of uniformly continuous, bounded (complex-valued) functions on the real axis $\mathbf{R}$ endowed with the usual sup-norm $\|\cdot\|_C$,

$$U := C_{ub}^{(r)}(\mathbf{R}) := \{f \in C_{ub}(\mathbf{R}); f^{(j)} \in C_{ub}(\mathbf{R}), j = 0, 1, \ldots, r\}, \ |f|_U := \|f^{(r)}\|_C,$$

and $X_h = C_h(\mathbf{R})$, the space of bounded functions on the mesh $h\mathbf{Z} := \{\nu h; \nu \in \mathbf{Z}$ ($:=$ integers)$\}$ with sup-norm. The restrictions P_h defined by

$$(P_h f)(\nu h) := f(\nu h) \qquad \big(\nu \in \mathbf{Z}, f \in C_{ub}(\mathbf{R})\big) \tag{3.4}$$

obviously satisfy (1.2). If $T(t)$ denotes the translation operator

$$\big(T(t) f\big)(x) := f(x+t) \qquad \big(x, t \in \mathbf{R}, f \in C_{ub}(\mathbf{R})\big), \tag{3.5}$$

then the difference methods of type

$$E_\tau := \sum_{k=-\infty}^{\infty} a_k(\lambda) T(kh), \qquad \sum_{k=-\infty}^{\infty} |a_k(\lambda)| < \infty, \qquad \lambda := \tau/h \ge 0 \tag{3.6}$$

are well defined in $[C_h(\mathbf{R})]$. The following lemma gives a sufficient condition for (3.2) to be satisfied for the semigroup $\{T(t); t \ge 0\} \subset [C_{ub}(\mathbf{R})]$ in terms of the weights $a_k(\lambda)$.

Lemma 2: *Let* $\{E_\tau\}$ *be given by* (3.6) *such that* $(1 < r \in \mathbf{N})$

$$\text{(i)} \sum_{k=-\infty}^{\infty} k^j a_k(\lambda) = \begin{cases} \lambda^j, & j = 0, 1, \ldots, r-1, \\ \big(1 + b(\lambda)\big)\lambda^r, & j = r, \end{cases}$$

$$\text{(ii)} \sum_{k=-\infty}^{\infty} |k^r a_k(\lambda)| < \infty. \tag{3.7}$$

Then $\{E_\tau\}$ *and* $\{T(t)\}$ *satisfy* (3.1/2) *with* $\alpha = r - 1$, $B = \big(b(\lambda)/r!\big)(d/dx)^r$, *and* $D(B) = C_{ub}^{(r)}(\mathbf{R})$.

Proof: Since (3.7) (i) implies $(\lambda h = \tau)$

$$\sum_{k=-\infty}^{\infty} a_k(\lambda)(kh - \tau)^j = \begin{cases} 1, & j = 0, \\ 0, & j = 1, 2, \ldots, r-1, \\ \tau^r b(\lambda), & j = r, \end{cases}$$

one obtains for $f \in C_{ub}^{(r)}(\mathbf{R})$, $\nu \in \mathbf{Z}$, $t > 0$

$$\begin{aligned}
&|(E_\tau P_h T(t) f)(\nu h) - (P_h T(t+\tau) f)(\nu h) - \tau^r (b(\lambda)/r!)(P_h T(t) f^{(r)})(\nu h)| \\
&= \Bigg| \sum_{k=-\infty}^{\infty} a_k(\lambda) \Bigg\{ \sum_{j=0}^{r-1} f^{(j)}(\nu h + t + \tau)(kh - \tau)^j/j! \\
&\quad + \frac{1}{(r-1)!} \int_\tau^{kh} (kh - u)^{r-1} f^{(r)}(\nu h + t + u)\, du \Bigg\} \\
&\quad - f(\nu h + t + \tau) - \tau^r \big(b(\lambda)/r!\big) f^{(r)}(\nu h + t) \Bigg| \\
&= \Bigg| \sum_{k=-\infty}^{\infty} a_k(\lambda) \frac{1}{(r-1)!} \int_\tau^{kh} (kh - u)^{r-1} [f^{(r)}(\nu h + t + u) - f^{(r)}(\nu h + t + \tau)]\, du \\
&\quad + \tau^r \big(b(\lambda)/r!\big) [f^{(r)}(\nu h + t + \tau) - f^{(r)}(\nu h + t)] \Bigg| \\
&\leqq \sum_{|kh - \tau| \leqq \tau^{1/2}} |a_k(\lambda)| (|kh - \tau|^r/r!) \sup_{0 \leqq u - \tau \leqq \tau^{1/2}} |f^{(r)}(\nu h + t + u) - f^{(r)}(\nu h + t + \tau)| \\
&\quad + \sum_{|kh - \tau| > \tau^{1/2}} |a_k(\lambda)| (|kh - \tau|^r/r!)\, 2\, \|f^{(r)}\|_C \\
&\quad + \tau^r \big(b(\lambda)/r!\big) |f^{(r)}(\nu h + t + \tau) - f^{(r)}(\nu h + t)| \\
&\leqq \tau^r \sum_{k=-\infty}^{\infty} |a_k(\lambda)| (|k/\lambda - 1|^r/r!) \sup_{0 \leqq \delta \leqq \tau^{1/2}} \|f^{(r)}(\cdot) - f^{(r)}(\cdot + \delta)\|_C \\
&\quad + \tau^r \sum_{|k/\lambda - 1| > 1/\tau^{1/2}} |a_k(\lambda)| (|k/\lambda - 1|^r/r!)\, 2\, \|f^{(r)}\|_C \\
&\quad + \tau^r \big(b(\lambda)/r!\big) \|f^{(r)}(\cdot) - f^{(r)}(\cdot + \tau)\|_C.
\end{aligned}$$

In view of the uniform continuity of $f^{(r)}$ and the absolute summability of the series, all terms are of order $o(\tau^r)$, $\tau \to 0+$. This establishes the assertion ∎

Note that in the concrete situation of these spaces X and U the $\mathfrak{K}$-functional turns out to be equivalent to the classical r-th modulus of continuity

$$\omega_r(t, f) := \sup_{0 \leq s \leq t} \|[T(s) - T(0)]^r f\|_C,$$

i.e., there exist constants $c_1, c_2 > 0$, independent of $f \in C_{ub}(\mathbf{R})$ and $t \geqq 0$, such that

$$c_1 \omega_r(t, f) \leqq \mathfrak{K}(t^r, f; C_{ub}(\mathbf{R}), C_{ub}^{(r)}(\mathbf{R})) \leqq c_2 \omega_r(t, f). \tag{3.8}$$

Thus for $\omega(t) := t^\beta$, $0 < \beta \leqq 1$, the intermediate spaces $X_\omega =: X_\beta$ are the standard r-th Lipschitz spaces

$$X_\beta = \mathrm{Lip}_r (\beta r) := \{f \in C_{ub}(\mathbf{R}); \omega_r(t, f) = \mathcal{O}(t^{\beta r}), t \to 0+\}$$

see e.g. [3, p. 192f.]).

4. Approximation of the Translation Semigroup

In this section we are concerned with the approximation of the solution of the hyperbolic initial value problem

$$d/dt\, u(x, t) = d/dx\, u(x, t), \qquad u(x, 0) = f(x) \in C_{ub}(\mathbf{R}) \qquad (t \geqq 0)$$

by stable difference methods of type (3.6) satisfying (3.7). Since the solution is given by the translation semigroup (3.5), condition (3.7) implies consistency of order $r - 1$ on $C_{ub}^{(r)}(\mathbf{R})$, as was indicated in the proof of La. 1. Thus one has convergence (1.5), i.e.,

$$\|E_{1/n}^n P_h f - P_h T(1) f\|_h = \mathcal{O}(\omega(n^{-r+1})) \qquad (n \to \infty, f \in X_\omega).$$

The following theorem shows that this rate of convergence is indeed sharp.

Theorem 2: *Let $\{E_\tau \in [C_h(\mathbf{R})];\ 0 \leqq \tau \leqq 1\}$ be a difference method of type* (3.6) *satisfying* (3.7). *Then for each modulus of continuity ω and each $\lambda > 0$ with $b(\lambda) \neq 0$ there exists an element $f_{\omega,\lambda} = f_\omega \in X_\omega$ such that*

$$\|E_{1/n}^n P_h f_\omega - P_h T(1) f_\omega\|_h \neq o(\omega(n^{-r+1})) \qquad (n \to \infty).$$

Proof: First of all, consider moduli of continuity satisfying $\omega(t) = \mathcal{O}(t)$. Then any $f_\omega \in D(B)$ with $Bf_\omega = (b(\lambda)/r!)\, f_\omega^{(r)} \neq 0$ (cf. La. 1/2) may serve as a counterexample since the Voronovskaja-type relation (3.3) and (1.2) (ii) deliver

$$\lim_{n\to\infty} n^{r-1} \|E_{1/n}^n P_h f_\omega - P_h T(1) f_\omega\|_h = \lim_{n\to\infty} \|P_h T(1) B f_\omega\|_h$$
$$= \|T(1) B f_\omega\|_C = \|B f_\omega\|_C \neq 0.$$

Thus let ω satisfy (2.8). For an application of Thm. 1 consider the operators ($h = 1/\lambda n^r$)

$$R_{n,\lambda} = R_n := E_{1/n^r}^{n^r} P_h - P_h T(1) \in [C_{ub}(\mathbf{R}), C_h(\mathbf{R})]$$

and the functions θ_c and $g_{n,\lambda} = g_n$ defined by ($c > 0$)

$$\theta_c(x) := \exp\{icx\}, \qquad g_n(x) = \theta_c(\lambda n^{r-1} x) = \exp\{ic\lambda n^{r-1} x\},$$

respectively. Thus (2.2/3) follow for $U = C_{ub}^{(r)}(\mathbf{R})$ and $\varphi_n = n^{-r(r-1)}$. To show (2.4), note that one has $\tau = n^{-r}$, $h = 1/\lambda n^r$ as well as the eigenvalue properties

$$\begin{aligned} E_{1/n^r} P_h g_n &= \sum_k a_k(\lambda) \exp \{ic\lambda n^{r-1} kh\} P_h g_n \\ &= \sum_k a_k(\lambda)\, \theta_c(k/n)\, P_h g_n \\ &= \big([E_{\lambda/n} P_{1/n} \theta_c]\,(0)\big)\, P_h g_n, \end{aligned}$$

$$P_h T(1)\, g_n = P_h \exp \{ic\lambda n^{r-1}\}\, g_n = [\theta_c(\lambda)]^{n^{r-1}}\, P_h g_n,$$

$$\begin{aligned} [E_{\lambda/n} P_{1/n} \theta_c]\,(\nu/n) &= \sum_k a_k(\lambda) \exp \big\{ic\big((\nu + k)/n\big)\big\} \\ &= \big([E_{\lambda/n} P_{1/n} \theta_c]\,(0)\big)\, \theta_c(\nu/n). \end{aligned}$$

Since $\theta_c(0) = \|P_h g_n\|_h = 1$, this yields

$$\begin{aligned} \|R_n g_n\|_h &= \Big\|[[E_{\lambda/n} P_{1/n} \theta_c]\,(0)]^{n^r}\, P_h g_n - [\theta_c(\lambda)]^{n^{r-1}}\, P_h g_n\Big\|_h \\ &= \Big|[[E_{\lambda/n}^{n} P_{1/n} \theta_c]\,(0)]^{n^{r-1}} - [\theta_c(\lambda)]^{n^{r-1}}\Big|. \end{aligned}$$

Using La. 1/2 one obtains

$$\begin{aligned} \|R_n g_n\|_h &= \Big|[[P_{1/n} T(\lambda)\, \theta_c]\,(0) + n(\lambda/n)^r \big(b(\lambda)/r!\big)\, [P_{1/n} T(\lambda)\, \theta_c^{(r)}]\,(0) \\ &\quad + o(n^{-r+1})]^{n^{r-1}} - [\theta_c(\lambda)]^{n^{r-1}}\Big| \\ &= |\theta_c(\lambda)|^{n^{r-1}}\, |[1 + \lambda(\lambda/n)^{r-1}\, b(\lambda)\, (ic)^r/r! + o(n^{-r+1})]^{n^{r-1}} - 1|. \end{aligned}$$

Since $|\theta_c(\lambda)| = 1$, this implies

$$\lim_{n\to\infty} \|R_n g_n\|_h = |\exp\, (ic\lambda)^r\, b(\lambda)/r!\} - 1|.$$

If r is even, then $b(\lambda) \neq 0$ immediately implies that this limit is different from zero (for each $c > 0$); if r is odd, one may choose $c = c(\lambda)$ to ensure this property. Thus one has (2.4) for all $n \geqq n_0(\lambda)$. Therefore an application of Thm. 1 delivers the existence of an element $f_{\omega,\lambda} = f_\omega \in X_\omega$ such that

$$\limsup_{n\to\infty} \|R_n f_\omega\|_h / \omega\big((n^r)^{-r+1}\big) > 0.$$

This implies, at least for the subsequence $m = n^r$, that

$$\limsup_{m\to\infty} \|E_{1/m}^{m} P_h f_\omega - P_h T(1)\, f_\omega\|_h / \omega(m^{-r+1}) > 0,$$

completing the proof ∎

Examples: The explicit, implicit, semi-discrete difference operators

$$E_\tau := (1 - \lambda)\, T(0) + \lambda T(h), \qquad \lambda \in [0, 1], \tag{4.1}$$

$$E_\tau := \frac{1}{1 + \lambda} \sum_{k=0}^{\infty} \left(\frac{\lambda}{1 + \lambda}\right)^k T(kh), \qquad \lambda \geqq 0, \tag{4.2}$$

$$E_\tau := e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\, T(kh), \qquad \lambda \geqq 0, \tag{4.3}$$

respectively, satisfy (3.6/7) with $r = 2$ and $b(\lambda) = \lambda^{-1}(1 - \lambda)$, $\lambda^{-1}(1 + \lambda)$, λ^{-1}, respectively. Thus in any of these cases one has the rate of convergence

$$\|E_{1/n}^{n} P_h f - P_h T(1)\, f\|_h = \mathcal{O}\big(\omega(1/n)\big) \qquad (n \to \infty)$$

for all $f \in X_\omega$; this is sharp since for each ω and each $\lambda \in (0, 1)$, $\lambda \in (0, \infty)$, respectively, there exists at least one element $f_{\omega,\lambda} = f_\omega \in X_\omega$ such that

$$\|E^n_{1/n} P_h f_\omega - P_h T(1) f_\omega\|_h \neq o(\omega(1/n)) \qquad (n \to \infty).$$

For the difference schemes (4.1—3) the operators $T_n(f; \lambda) := [E^n_{1/n} P_{1/n} f](0)$ occurring in the proof of Thm. 2 turn out to constitute classical approximation processes, namely the Bernstein polynomials, the Baskakov operators and the Szasz-Mirakjan operators, respectively (for details see [8]). Thus La. 1 also allows to reduce the proof of the well-known Voronovskaja relations for these approximation processes to the corresponding simpler ones for the step operators (4.1—3).

Note that Prop. 1 yields a sequence $\{f_{n,\omega}\} \subset X_\omega$ with $|f_{n,\omega}|_\omega \leqq C^*$ and $(n \geqq n_0(\lambda))$

$$\|E^n_{1/n} P_h f_{n,\omega} - P_h T(1) f_{n,\omega}\|_h \geqq C\omega(n^{-r+1}), \tag{4.4}$$

so that in terms of the operator norms (cf. (1.11))

$$\|E^n_{1/n} P_h - P_h T(1)\|_{[X_\omega, C_h(\mathbf{R})]} \geqq C\omega(n^{-r+1}).$$

For the intermediate spaces $X_\beta = \mathrm{Lip}_2\,(2\beta)$, i.e., for $\omega(t) = t^\beta$, $0 < \beta < 1$, this result was obtained by HEDSTROM [14] and BRENNER-THOMÉE [1] (see also [2, p. 111]) for $X_h = C_{ub}(\mathbf{R})$ and $P_h = I(:=$ identity). Thm. 2 (cf. (3.8)) now states that at least for a subsequence $\{n_k\}$ the elements $f_{n_k,\omega}$ may be replaced by a single one $f_\omega \in X_\omega$. In this sense Thm. 2 is an improvement of (4.4) and an extension to all moduli ω; but, on the other hand, due to the general approach via Thm. 1, there is a restriction to an unknown subsequence $\{n_k\}$.

5. Holomorphic Semigroups

In this section we consider the holomorphic semigroups $(0 < \delta \leqq 2)$

$$[W_\delta(t) f](x) := t^{-1/\delta} \int_{-\infty}^{\infty} f(x+u)\, k_\delta(u/t^{1/\delta})\, du \tag{5.1}$$

$(x \in \mathbf{R}, t > 0, f \in C_{ub}(\mathbf{R}))$, where k_δ is given via its Fourier transform

$$k_\delta^\wedge(v) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_\delta(u)\, e^{-ivu}\, du = \exp\{-|v|^\delta\} \qquad (v \in \mathbf{R}).$$

This semigroup solves the initial value problem

$$d/dt\, u(x, t) = -(-(d/dx)^2)^{\delta/2}\, u(x, t), \qquad u(x, 0) = f(x) \in C_{ub}(\mathbf{R}),$$

where the fractional power of the differential operator is to be understood in the sense of BOCHNER and FELLER [13] (see also [5, 18]). For the approximation of these semigroups we consider stable difference methods of type (3.6) with $\lambda = \tau/h^\delta$ that reproduce constant functions, i.e., $\sum a_k(\lambda) = 1$. If the difference method is consistent of order α on $U = C^{(r)}_{ub}(\mathbf{R})$ with $\alpha \leqq r/\delta - 1$, $r > \delta$, then for $f \in \mathrm{Lip}_r\,(\beta)$, $0 < \beta < \beta_0 := r - \delta$, one has the rate of approximation (cf. (3.8))

$$\|E^n_{1/n} P_h f - P_h W_\delta(1) f\|_h = O(n^{-\alpha\beta/(r-\delta)}) \qquad (n \to \infty); \tag{5.2}$$

it is better than that given by (1.5); for a proof see [11, 15]. In case $\alpha = r/\delta - 1$ this rate is again best possible.

Theorem 3: *Let $\{E_\tau\}$ be a difference method of type (3.6) with $\lambda = \tau/h^\delta$ that reproduces constant functions. If the method is stable and consistent of order $r/\delta - 1$ with the semigroup $\{W_\delta(t)\}$, then for each $\beta \in (0, \beta_0)$ there exists an element $f_\beta \in \mathrm{Lip}_r(\beta)$ such that*

$$\|E^n_{1/n} P_h f_\beta - P_h W_\delta(1) f_\beta\|_h \neq o(n^{-\beta/\delta}) \qquad (n \to \infty).$$

Proof: Consider the operators

$$R_n := E^n_{1/n} P_h - P_h W_\delta(1) \in [C_{ub}(\mathbf{R}), C_h(\mathbf{R})]$$

and the elements

$$g_n(x) = \cos(2\pi x/h), \qquad h = (n\lambda)^{-1/\delta}.$$

Obviously these elements satisfy (2.2/3) with $\varphi_n = n^{-r/\delta}$. Since the difference method reproduces constant functions and $P_h g_n(\nu h) = 1$ for all $\nu \in \mathbf{Z}$, one has

$$\|R_n g_n\|_h = \|E^n_{1/n} P_h g_n - P_h W_\delta(1) g_n\|_h = \|1 - k_\delta^\wedge(2\pi/h) P_h g_n\|_h$$

$$= 1 - k_\delta^\wedge(2\pi/h) = 1 + o(1) \qquad (n \to \infty),$$

proving (2.4). Thus the assertion follows by Thm. 1 upon setting $\omega(t) = t^{\beta/r}$, $0 < \beta < \beta_0 < r$ ∎

Example: For $\delta = 2$ the operators (5.1) constitute the Gauss-Weierstrass semigroup. Consider the explicit difference scheme

$$E_\tau = (1 - 2\lambda) T(0) + \lambda[T(h) + T(-h)], \qquad \lambda = \tau/h^2.$$

This scheme is stable for $0 \leqq \lambda \leqq 1/2$ as well as consistent of order 1 on $U = C^{(4)}_{ub}(\mathbf{R})$ for $0 < \lambda \leqq 1/2$, and even consistent of order 2 on $U = C^{(6)}_{ub}(\mathbf{R})$ for $\lambda = 1/6$. Thus one has consistency of order $r/\delta - 1$ and convergence of order $\mathcal{O}(n^{-\beta/2})$ for $f \in \mathrm{Lip}_4(\beta)$, $0 < \beta < 2$, if $\lambda \in (0, 1/2]$, and for $f \in \mathrm{Lip}_6(\beta)$, $0 < \beta < 4$, if $\lambda = 1/6$. Thm. 3 then states that these rates of convergence are indeed best possible.

Let us mention that the result of Thm. 3 (for $\delta = 2$) was also given in [14], [2, p. 76]. The proofs there are constructive (for subsequences of type $n_k = C^k$).

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