Zeitschrift für Analysis
und ihre Anwendungen
1 (1982), S. 71–73

On some class of holomorphic functions of several complex variables

R. MAZUR

In der Arbeit werden gewisse bekannte Abschätzungen über holomorphe Funktionen einer komplexen Variablen auf den Fall mehrerer komplexer Variabler verallgemeinert.

В работе обобщаются некоторые известные оценки для голоморфных функций одного комплексного переменного на случай многих комплексных переменных.

The aim of the paper is to generalize some well-known estimations for holomorphic functions of one complex variable to the case of several complex variables.

Introduction

Let $D \subset \mathbf{C}^n$ be an arbitrarily fixed full Reinhardt domain containing the origin $z = 0$. By $H(D)$ we denote a class of holomorphic functions $h: D \to \mathbf{C}$. We define in $H(D)$ an operator of the form

$$L(h(z)) = h(z) + \sum_{k=1}^{n} z_k h'_{z_k}(z) \qquad \text{for } z = (z_1, z_2, \ldots, z_n). \tag{1}$$

Let $Q(D) \subset H(D)$ denote a family of functions q holomorphic in the domain D and normed by the conditions $q(0) = 0$, $|q(z)| < 1$ for $z \in D$. The author of paper [3] has introduced the family $\mathscr{P}(a, b; D)$ composed of the functions p such that

$$p(z) = \frac{1 + a \cdot q(z)}{1 + b \cdot q(z)}, \qquad z \in D, \tag{2}$$

where $q \in Q(D)$ and a, b — arbitrary fixed numbers satisfying the inequalities $-1 \leqq b < a \leqq 1$.

The class $\mathscr{V}(a, b; D)$

We now introduce the following d e f i n i t i o n.

Let L denote the operator defined in (1). The subclass of $H(D)$ of functions f satisfying the conditions:

$$f(0) = 1 \tag{3}$$

$$L(f(z)) = p(z), \qquad z \in D, \qquad p \in \mathscr{P}(a, b; D), \tag{4}$$

is denoted by $\mathscr{V}(a, b; D)$.

Let us assume next for a fixed $r \in (0, 1)$, D_r is the set of points $z \in D$ for which

$\frac{z}{r} \in D$. By combining Theorem 1 [3] with the definition of the class $\mathscr{V}(a, b; D)$, we obtain the following result.

Theorem 1. *If $f \in \mathscr{V}(a, b; D)$, then for $z \in D_r$ we have*

$$\frac{1-ar}{1-br} \leqq \left|L(f(z))\right| \leqq \frac{1+ar}{1+br}, \tag{5}$$

$$\frac{1-ar}{1-br} \leqq \operatorname{re} L(f(z)) \leqq \frac{1+ar}{1+br}, \tag{6}$$

$$\left|\operatorname{im} L(f(z))\right| \leqq \frac{a-br}{1-b^2r^2}, \tag{7}$$

$$\left|\arg L(f(z))\right| \leqq \arcsin \frac{(a-b)\,r}{1-abr^2}. \tag{8}$$

These inequalities are sharp.

Theorem 2. *If $f \in \mathscr{V}(a, b; D)$, then for $z \in D_r$ we have*

$$|f(z)| \leqq \begin{cases} 1 + \frac{1}{2}\,ar & \text{for } b = 0, \\ \frac{a}{b} - \frac{a-b}{b^2 r} \log(1+br) & \text{for } b \neq 0, \end{cases} \tag{9}$$

$$|f(z)| \geqq \begin{cases} 1 - \frac{1}{2}\,ar & \text{for } b = 0, \\ \frac{a}{b} + \frac{a-b}{b^2 r} \log(1-br) & \text{for } b \neq 0. \end{cases} \tag{10}$$

These inequalities are sharp.

Proof. Let $z_0 = (\mathring{z}_1, \mathring{z}_2, \ldots, \mathring{z}_n)$ be an arbitrary point of the domain D_r. From the properties of the domains D_r and D it follows that the point $\frac{\mathfrak{z} z_0}{r} = \left(\frac{\mathfrak{z}\mathring{z}_1}{r}, \frac{\mathfrak{z}\mathring{z}_2}{r}, \ldots, \frac{\mathfrak{z}\mathring{z}_n}{r}\right)$ also belongs to the domain D for every $\mathfrak{z} \in \mathbb{C}$, $|\mathfrak{z}| < 1$. We define a function $\mathfrak{z} \to g(\mathfrak{z})$, where

$$g(\mathfrak{z}) = f\left(\frac{\mathfrak{z} z_0}{r}\right), \ |\mathfrak{z}| < 1, \quad f \in \mathscr{V}(a, b; D). \tag{11}$$

It is easily to see that the function g is holomorphic in the unit disc, $g(0) = 0$ and $g'(\mathfrak{z}) = L\left(f\left(\frac{\mathfrak{z} z_0}{r}\right)\right)$. Hence and from (2)–(4) we obtain

$$g'(\mathfrak{z}) = \frac{1 + a\tilde{q}(\mathfrak{z})}{1 + b\tilde{q}(\mathfrak{z})},$$

where $\tilde{q}(\mathfrak{z}) = q\left(\frac{\mathfrak{z} z_0}{r}\right)$, $q \in Q(D)$. Since $\tilde{q}(0) = 0$ and $|\tilde{q}(\mathfrak{z})| < 1$ for $|\mathfrak{z}| < 1$, then $g \in \mathscr{V}(a, b)$ (c.f. [2]). By applying Theorem 2 [2] to the function g we obtain

$$\mathfrak{G}(-a, -b; |\mathfrak{z}|) \leqq |g(\mathfrak{z})| \leqq \mathfrak{G}(a, b; |\mathfrak{z}|),$$

where

$$\mathfrak{G}(a, b; |\mathfrak{z}|) = \begin{cases} |\mathfrak{z}| + \dfrac{1}{2} a |\mathfrak{z}|^2 & \text{for } b = 0, \\ \dfrac{a}{b} |\mathfrak{z}| - \dfrac{a-b}{b^2} \log(1 + b|\mathfrak{z}|) & \text{for } b \neq 0. \end{cases}$$

Hence and from (11) we obtain (9)—(10) for $\mathfrak{z} = r$. The functions

$$f(z) = \int_0^1 \frac{1 + a\varepsilon/_n(z_1 + z_2 + \cdots + z_n)\, t}{1 + b\varepsilon/_n(z_1 + z_2 + \cdots + z_n)\, t}\, dt,$$

where $\varepsilon = \pm 1$ respectively, at $z = (r, r, \ldots, r)$ indicate the sharpeness of the result ∎

Let be $z^m = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$, where $m = (m_1, m_1, \ldots, m_n)$, m_i-being non-negative integers, $|m| = m_1 + m_2 + \cdots + m_n$. From the definition of the family $\mathscr{V}(a, b; D)$ and from Theorem 2 [3] we obtain the following result.

Theorem 3. *If $f \in \mathscr{V}(a, b; D)$ and*

$$f(z) = 1 + \sum_{k=1}^{\infty} \Big(\sum_{|m|=k} a_m z^m \Big)$$

then

$$\sup_{z \in D} \Big| \sum_{|m|=k} a_m z^m \Big| \leqq \frac{a-b}{k+1} \qquad \text{for } k = 1, 2, \ldots$$

Remark. For $a = 1$, $b = -1$ or $a = 1$, $b = 0$ and $n = 2$ we arrive at the known result obtained in [1].

REFERENCES

[1] Bavrin, I. I.: Klassy golomorfnych funkcij mnogich kompleksnych peremennych i ekstremalnye voprosy dlja etich klassov funkcij. Moskva: MOPI 1976.

[2] Mazur, R.: On some extremal problems in the class of functions with bounded rotation. Scientific Bulletin of Łódź Technical University No. 254, Matematyka z. 9 (1977), 77—88.

[3] Mazur, R.: Distortion properties of holomorphic functions of several complex variables. Ann. Polon. Math. (to appear).

Manuskripteingang: 01. 07. 1980

VERFASSER:

Dr. R. Mazur
Instytut Matematyki, Wyższa Szkoła Pedagogiczna
P - 25 - 406 Kielce, ul. M. Konopnickiej 29, Poland