Zeitschrift für Analysis
und ihre Anwendungen
2 (1982), S. 7–9

Theorems on polynomials in right invertible operators

D. PRZEWORSKA-ROLEWICZ

Sei D ein rechtsinvertierbarer Operator in einem linearen Raum X und $Q(D)$ ein Polynom von D mit Operatorkoeffizienten. Es wird gezeigt, daß unter gewissen Bedingungen auf X für die Rechtsinverse R und die Koeffizienten des Polynoms $Q(D)$ genau dann $Q(D) = 0$ gilt, falls $Q(D)\, R^k z = 0$ $(k = 0, 1, 2, \ldots)$ für alle z aus ker D.

Предположим, что D — право обратимый оператор, действующий в линейном пространстве X и $Q(D)$ — многочлен с операторными коэффициентами. В этом случае $Q(D) = 0$ тогда и только тогда, когда $Q(D)\, R^k z = 0$ для всех $z \in \ker D$ $(k = 0, 1, 2, \ldots)$ если выполнены некоторые условия на X, правый обратный R и коэффициенты многочлена $Q(D)$.

Suppose that $Q(D)$ is a polynomial in a right invertible operator acting in a linear space X, in general, with operator coefficients. Then $Q(D) = 0$ if and only if $Q(D)\, R^k z = 0$ for all $z \in \ker D$ $(k = 0, 1, 2, \ldots)$ under appropriate assumptions on X, the right inverse R of D and coefficients of $Q(D)$.

Let $P_1(D)$, $P_2(D)$ be polynomial differential operators, i.e. finite sums of monomials of the form

$$A_0(x)\, DA_1(x) \ldots A_{n-1}(x)\, DA_n(x) \quad \text{where} \quad D = \frac{d}{dt}$$

and $A_j(x)$ are given differentiable functions. Using the identity

$$DA_j(x) = A_j(x)\, D + A_j'(x)$$

we can rewrite these polynomials in the form:

$$P_i(D) = \sum_{j=1}^{m} P_j^{(i)}(x)\, D^j \qquad (i = 1, 2). \tag{1}$$

The following theorem holds for polynomial differential operators.

Theorem 1 (cf. CARLITZ [1, 2], INCE [3]): *Suppose that $P_1(D)$ and $P_2(D)$ are two polynomial differential operators. Then*

$$P_1(D) = P_2(D)$$

if and only if

$$P_1(D)\, x^k = P_2(D)\, x^k \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{2}$$

In particular, suppose that $P(D)$ is a polynomial differential operator. Then

$$P(D) = 0$$

if and only if

$$P(D)\, x^k = 0 \qquad (k = 0, 1, 2, \ldots). \tag{3}$$

In order to prove (2) it is enough to prove (3).

This result can be generalized for arbitrary right invertible operators, in particular, for difference operators (cf. [2]). Namely, suppose that X is a linear space over a field $\mathscr{F}$ of scalars, D is a right invertible operator and $\dim \ker D > 0$. If R is an arbitrarily fixed right inverse of D then elements of the form $R^k z_k$, where $z_k \in \ker D$ and $k = 0, 1, 2, \ldots$, play a role of monomials for D and are linearly independent (cf. [4]). The set

$$P(R) = \operatorname{lin}\{R^k z : z \in \ker D \quad (k = 0, 1, 2, \ldots)\}$$

is independent of the choice of a right inverse R.

Now, consider 3 different cases:

Theorem 2: *Let*

$$q(D) = \sum_{j=0}^{n} q_j D^j \tag{4}$$

be an arbitrary polynomial with scalar coefficients. Then $q(D) = 0$ *if and only if*

$$q(D)\, R^k z = 0 \ \textit{ for all } \ z \in \ker D \qquad (k = 0, 1, 2, \ldots) \tag{5}$$

where R *is an arbitrarily fixed right inverse of* D.

Proof: Necessity is obvious. In order to prove that the condition (4) implies $q(D) = 0$ observe that $R^k z \in \ker D^{k+1}$ $(k = 0, 1, 2, \ldots)$ for all $z \in \ker D$ (cf. also [4]). Then for all $z \in \ker D$ $(k = 0, 1, 2, \ldots)$ we find

$$\begin{aligned} 0 = f(D)\, R^k z &= \sum_{j=0}^{n} q_j D^j R^k z \\ &= \sum_{j=0}^{k} q_j R^{k-j} z + \sum_{j=k+1}^{n} q_j D^{j-k} = \sum_{j=0}^{k} q_j R^{k-j} z \end{aligned}$$

since, by definition, $DR = I$ and $Dz = 0$. This, and the linear independence of elements $z, Rz, R^2 z, \ldots$ together imply that $q_0, \ldots, q_k = 0$ for $k = 0, 1, 2, \ldots, n$, i.e. $q(D) = 0$ ∎

Observe that in Theorem 2 it is sufficient to admit only a finite number of conditions of the form (5). Namely, we obtain the same result if $k = 0, 1, 2, \ldots, m$ where $m = n \cdot \dim \ker D + 1$.

Theorem 3: *Suppose that* X *is a* D*-algebra, i.e. a commutative linear ring (with a non-trivial multiplication) such that*

$$\textit{if } x, y \in \operatorname{dom} D \textit{ then } xy \in \operatorname{dom} D. \tag{6}$$

Suppose, moreover, that $\ker D$ *is not an annihilator in* X, *i.e. if* $x \ker D = \{0\}$ *for an* $x \in X$ *then* $x = 0$. *Let*

$$q(D) = \sum_{k=0}^{n} q_k D^k, \textit{ where } q_0, \ldots, q_n \in X. \tag{7}$$

Then $q(D) = 0$ *if and only if*

$$q(D)\, R^k z = 0 \textit{ for all } z \in \ker D \qquad (k = 0, 1, 2, \ldots) \tag{8}$$

where R *is an arbitrarily fixed right inverse of* D.

Proof: Necessity is obvious. Sufficiency will be proved by induction. Assume that the condition (8) holds. In a similar way, was in the proof of Theorem (8) we shall rewrite (8) in the form

$$\sum_{j=0}^{k} q_j R^{k-j} z = 0 \quad \text{for all } z \ker D \qquad (k = 0, 1, 2, \ldots). \tag{9}$$

Let $k = 0$. Then we have $q_0 R^k z = 0$ and $R^k z \neq 0$ for $z = 0$ (because if $Ru = 0$ then $u = DRu = 0$). The arbitrariness of $z \in \ker D$ and our assumptions together imply that $q_0 = 0$. Suppose that $q_0 = \cdots = q_m = 0$ for an arbitrarily fixed $m \geqq 0$. Then

$$0 = \sum_{j=0}^{m+1} q_j R^{m+1-j} z = q_{m+1} z.$$

The arbitrariness of $z \ker D$ implies $q_{m+1} = 0$, which finishes the proof ∎

Theorem 4: *Suppose that X is a complete linear metric space, R is an arbitrarily fixed right inverse of D and*

$$Q(D) = \sum_{k=0}^{n} Q_k D^k \tag{10}$$

where Q_k: dom $D^n \to X$ are arbitrary linear operators. Suppose, moreover, that the operator $Q(D)$ is closed, $P(R) \subset \ker Q(D)$ and $\overline{P(R)} = X$. Then $Q(D) = 0$ if and only if

$$Q(D)\, R^k z = 0 \quad \textit{for all } z \in \ker D \qquad (k = 0, 1, 2, \ldots). \tag{11}$$

Proof: Necessity is obvious. Sufficiency follows from the fact that by our assumptions we have

$$X = \overline{P(R)} \subset \overline{\ker Q(D)} = \ker Q(D).$$

This implies $Q(D) = 0$ ∎

Note that from proofs of Theorems 2 and 3 follows that

$$P(R) \subset \ker Q(D)$$

which is assumed in Theorem 4.

REFERENCES

[1] Carlitz, L.: Some Operational Formulas. Math. Nachrichten 45 (1970), 379–389.

[2] Carlitz, L.: A Theorem on Differential Operators. Amer. Math. Monthly 83 (1976), 351 bis 354.

[3] Inoe, E. L.: Ordinary Differential Equations, London 1927.

[4] Przeworska-Rolewicz, D.: Algebraic Theory of Right Invertible Operators. Studia Math. 48 (1973), 129–143.

[5] Przeworska-Rolewicz, D.: Shifts and Periodicity for Right Invertible Operators. Research Notes in Mathematics 43, Pitman Advanced Publish. Program. Boston–London–Melbourne 1980.

Manuskripteingang: 20. 2. 1981

VERFASSER:

Prof. Dr. Danuta Przeworska-Rolewicz
Instytut Matematyczny PAN
P-00-950 Warszawa, skr. pocztowa 137