Zeitschrift für Analysis
und ihre Anwendungen
2 (1982), S. 23–39

Toeplitz determinants with piecewise continuous generating function

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Durch konsequente Benutzung der Theorie der Operatordeterminanten und einer speziellen Technik der Störung durch Spuroperatoren wird das asymptotische Verhalten der Toeplitzdeterminanten $D_n(a) = \det\{a_{j-k}\}_{j,k=0}^n$ $(n \to \infty)$ bestimmt, falls die Erzeugerfunktion $a(t) = \sum_{k=-\infty}^{\infty} a_k t^k$ $(|t| = 1)$ stückweise stetig ist und gewissen natürlichen Regularitätsbedingungen genügt. Es gilt $D_n(a) \sim G^{n+1} \cdot E \cdot n^{-\sum_{r=1}^{R} \beta_r^2}$ $(n \to \infty)$, wobei $\beta_r = \frac{1}{2\pi i} \log a(t_r - 0)/a(t_r + 0)$, $|\mathrm{Re}\,\beta_r| < \frac{1}{2}$ ist und $t_1, \ldots, t_R$ die Unstetigkeitsstellen von $a(t)$ sind; die Konstanten G und E werden explizit berechnet.

Применение теории операторных определителей и одна особая техника возмущения ядерными операторами позвольяют определить асимптотическое поведение теплицевых определителей $D_n(a) = \det\{a_{j-k}\}_{j,k=0}^n$ $(n \to \infty)$, если символ (производящая функция) $a(t) = \sum_{k=-\infty}^{\infty} a_k t^k$ $(|t| = 1)$ является кусочно-непрерывной функцией, удовлетворяющей некоторым естественным условиям регулярности. Имеет место соотношение $D_n(a) \sim G^{n+1} \cdot E \cdot n^{-\sum_{r=1}^{R} \beta_r^2}$ $(n \to \infty)$, где $\beta_r = \frac{1}{2\pi i} \log a(t_r - 0)/a(t_r + 0)$, $|\mathrm{Re}\,\beta_r| < \frac{1}{2}$ и где $t_1, \ldots, t_R$ — точки разрыва символа $a(t)$. При этом константы G и E даны в явном виде.

Consequent application of the theory of operator determinants and a special technique of perturbing by trace class operators allow to determine the asymptotic behavior of the Toeplitz determinants $D_n(a) = \det\{a_{j-k}\}_{j,k=0}^n$ $(n \to \infty)$, if the generating function $a(t) = \sum_{k=-\infty}^{\infty} a_k t^k$ $(|t| = 1)$ is piecewise continuous and satisfies some natural conditions of regularity. There holds $D_n(a) \sim G^{n+1} \cdot E \cdot n^{-\sum_{r=1}^{R} \beta_r^2}$ $(n \to \infty)$, where $\beta_r = \frac{1}{2\pi i} \log a(t_r - 0)/a(t_r + 0)$, $|\mathrm{Re}\,\beta_r| < \frac{1}{2}$ with $t_1, \ldots, t_R$ being the points of discontinuity of $a(t)$; thereby the constants G und E are explicitely given.

§ 1 Introduction

Let $a(t)$ be a piecewise continuous function given on the complex unit circle $\Gamma = \{t \in \mathbf{C}: |t| = 1\}$, i.e. a is continuous on Γ with exception of finitely many points $t_1, \ldots, t_R$ where, however, a possesses finite limits $a(t_r - 0)$ and $a(t_r + 0)$ $(r = 1, \ldots, R)$. By $a^\#$ we denote the continuous curve obtained from the range of a by filling in the line segments joining $a(t_r - 0)$ to $a(t_r + 0)$ for each discontinuity. We suppose that $a^\#$ does not contain the origin and that the winding number $\mathrm{ind}\, a^\#$ of $a^\#$ is zero.

Then there exist complex numbers $\beta_1, \ldots, \beta_R$ satisfying $e^{2\pi i \beta_r} = a(t_r - 0)/a(t_r + 0)$, $-1/2 < \operatorname{Re} \beta_r < 1/2$ and a continuous function $b(t)$ on Γ with $b(t) \neq 0$ $(|t| = 1)$, $\operatorname{ind} b = 0$ such that

$$a(t) = (-t)_{t_1}^{\beta_1} \ldots (-t)_{t_R}^{\beta_R} b(t) \qquad (|t| = 1) \tag{1}$$

holds. Here $(-t)_{t_r}^{\beta_r}$ is defined by $\exp \left\{ i\beta_r \arg \left(-\frac{t}{t_r} \right) \right\}$, $\left| \arg \left(-\frac{t}{t_r} \right) \right| < \pi$; so $(-t)_{t_r}^{\beta_r}$ has a discontinuity at $t = t_r$. We remark that we could take a branch of t^{β_r} instead a branch of $(-t)^{\beta_r}$ but we choose the latter, since we have a more "symmetrical" factorization

$$(-t)_{t_r}^{\beta_r} = \left(1 - \frac{t_r}{t}\right)^{-\beta_r} \left(1 - \frac{t}{t_r}\right)^{\beta_r} \qquad (|t| = 1)$$

in this case; here $(1 - t/t_r)^{\beta_r}$ denotes the limit on the unit circle of that branch of the function which is analytic in $|t| < 1$ and takes the value 1 at $t = 0$ and where $(1 - t_r/t)^{-\beta_r}$ is defined similarly.

The asymptotic behavior of the Toeplitz determinants

$$D_n(a) = \det \{a_{j-k}\}_{j,k=0}^n, \qquad a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(e^{i\varphi}) \, e^{-ik\varphi} \, d\varphi \qquad (k \in \mathbf{Z})$$

as $n \to \infty$ has been the object of study by many people for some time (cf. [1, 2, 6, 12, 13]). If the so-called generating function $a(t)$ $(|t| = 1)$ is continuous, sufficiently smooth, satisfies $a(t) \neq 0$ $(|t| = 1)$ and $\operatorname{ind} a = 0$, then one has the well-known asymptotic formula

$$D_n(a) \sim G(a)^{n+1} E(a) \qquad (n \to \infty),$$

where $G(a) = \exp (\log a)_0$, $E(a) = \exp \sum_{k=1}^{\infty} k(\log a)_k (\log a)_{-k}$, ($(\log a)_k$ being the Fourier coefficients of $\log a$). There exist asymptotic formulas in the case where $\operatorname{ind} a \neq 0$ holds or where a has zeros on the unit circle, too. Much less is known, if a has discontinuities. This problem was probably for the first time considered by HARTWIG and FISHER in [2]. Using heuristic arguments they arrived at the conjecture that

$$D_n(a) \sim n^{-\sum_{r=1}^{R} \beta_r^2} G(b)^{n+1} E(t_1, \ldots, t_R; \beta_1, \ldots, \beta_R; b) \tag{2}$$

as $n \to \infty$ holds, if a is given by (1). Here E denotes some constant. They were able to verify (2) in the case $R = 1$, $b(t) = 1$ and they proved

$$D_n\left((-t)_{t_0}^{\beta}\right) \sim n^{-\beta^2} \mathfrak{G}(1 + \beta) \, \mathfrak{G}(1 - \beta) \qquad (n \to \infty).$$

Here $\mathfrak{G}(z)$ is the *Barnes $\mathfrak{G}$-function*; this is an entire function defined by

$$\mathfrak{G}(1 + z) = (2\pi)^{\frac{1}{2} z} \, e^{-\frac{1}{2} z(z+1) - \frac{1}{2} \gamma_E z^2} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^n e^{-z + z^2/2n} \right\}$$

($\gamma_E = 0.577\ldots$ Euler's constant) and its role in analysis will be clear if one takes into consideration the relation $\mathfrak{G}(1 + z) = \Gamma(z) \, \mathfrak{G}(z)$ (cf. [14], p. 264).

In [12] WIDOM proved (2) if $R = 1$, $\operatorname{Im} \beta_1 = 0$, $-1/2 < \beta_1 < 1/2$, and if b has a derivative satisfying a Lipschitz condition with exponent greater than $|2\beta_1|$. In this case a has only one point of discontinuity $t = t_1$ and there holds $\arg a(t_1 + 0) = \arg a(t_1 - 0)$. Furthermore, (2) was proved by ESTELLE BASOR in [1] in the case $\operatorname{Re} \beta_r = 0$ ($r = 1, \ldots, R$) and under the assumption that b has a derivative satisfying a Lipschitz condition. Moreover, she was able to determine the constant E under her conditions. This result corresponds to the case where at the points of discontinuity $t_1, \ldots, t_R$ of a the relations $|a(t_r + 0)| = |a(t_r - 0)|$ hold.

It is the aim of this paper to prove formula (2) and to determine the constant E if a is an arbitrary piecewise continuous function with $0 \notin a^{\#}$, $\operatorname{ind} a^{\#} = 0$, and if b in (1) satisfies some smoothness conditions.

§ 2 Preliminaries

Here some facts from the theory of Toeplitz operators, the theory of trace class operators and operator determinants are presented.

For $a \in L^\infty(\Gamma)$ with Fourier coefficients a_k ($k \in \mathbf{Z}$) we denote by $T(a)$ both the semi-infinite matrix $\{a_{j-k}\}_{j,k=0}^{\infty}$ and the bounded operator induced by this matrix in a natural way on l^2, the so-called *Toeplitz operator*. It is well-known that the use of *Hankel operators* in the study of Toeplitz operators is of great importance. The Hankel operator $H(a)$ generated by $a \in L^\infty(\Gamma)$ is given by the semi-infinite matrix $\{a_{j+k+1}\}_{j,k=0}^{\infty}$ on l^2. For $a \in L^\infty(\Gamma)$ we write $\tilde{a}(t) = a(1/t)$. Then one has the following simple identities (cf. [13])

$$T(fg) = T(f)\,T(g) + H(f)\,H(\tilde{g}), \tag{1}$$

$$H(fg) = T(f)\,H(g) + H(f)\,T(\tilde{g}). \tag{2}$$

By $T_n(a)$ we denote the finite matrix $\{a_{j-k}\}_{j,k=0}^{n}$. If we define the operators P_n and W_n on l^2 by

$$P_n\colon \{\xi_0, \xi_1, \ldots\} \to \{\xi_0, \ldots, \xi_n, 0, \ldots\},$$

$$W_n\colon \{\xi_0, \xi_1, \ldots\} \to \{\xi_n, \ldots, \xi_0, 0, \ldots\}$$

then $T_n(a)$ may be identified with $P_n T(a) P_n / \operatorname{Im} P_n$. An analogue of (1) is the identity

$$T_n(fg) = T_n(f)\,T_n(g) + P_n H(f)\,H(\tilde{g})\,P_n + W_n H(\tilde{f})\,H(g)\,W_n. \tag{3}$$

If A is an invertible operator on l^2, if the operators $A_n = P_n A P_n \mid \operatorname{Im} P_n$ are invertible for n large enough and if A_n^{-1} converges strongly to A^{-1} as $n \to \infty$ then we say that the *reduction method for A converges* and we write $A \in \Pi\{P_n\}$. For a piecewise continuous function a we have the well-known fact

$$T(a) \in \Pi\{P_n\} \Leftrightarrow T(a) \text{ invertible} \Leftrightarrow 0 \notin a^{\#}, \operatorname{ind} a^{\#} = 0 \tag{4}$$

(cf. [3]). We remark also that from $A \in \Pi\{P_n\}$, T a compact operator, $A + T$ invertible, it follows that $A + T \in \Pi\{P_n\}$ (cf. [3]).

An important class of operators on l^2 are the so-called trace operators. An operator $A \in \mathscr{L}(l^2)$ is called a *trace operator* (we write $A \in \mathscr{C}_1$), if $\sum_{n=1}^{\infty} s_n(A) < \infty$ where $s_n(A)$ is defined by

$$s_n(A) = \inf \{\|A - K\|_\infty \colon \dim \operatorname{Im} K \leq n\}$$

($\|\cdot\|_\infty$ denoting the usual norm in the space $\mathscr{L}(l^2)$ of bounded operators on l^2).

Under the norm

$$\|A\|_1 = \sum_{n=1}^{\infty} s_n(A)$$

$\mathscr{C}_1$ is a closed ideal in $\mathscr{L}(l^2)$. We remark that every trace operator is compact.

If $\{\lambda_n(A)\}$ denotes the sequence of eigenvalues of $A \in \mathscr{C}_1$, then we have $\sum |\lambda_n(A)| < \infty$ and the operator determinant $\det(I + A)$ is defined by

$$\det(I + A) = \prod_n \big(1 + \lambda_n(A)\big).$$

For equivalent definitions and properties of these determinants we refer to the relevant literature (cf. [4, 11]). We should notice here only the following facts:

$$\begin{aligned} &\det(I + \cdot) \text{ is continuous on } \mathscr{C}_1; \\ &\det(I + A) \cdot \det(I + B) = \det(I + A)(I + B) \qquad (A, B \in \mathscr{C}_1); \end{aligned} \tag{5}$$

$$\begin{aligned} &\det C^{-1}(I + A)\, C = \det(I + C^{-1}AC) = \det(I + A) \\ &(C^{\pm 1} \in \mathscr{L}(l^2), A \in \mathscr{C}_1); \end{aligned} \tag{6}$$

$$\det P(I + A)\, P = \det(I + PAP), \tag{7}$$

where P is a finite-dimensional projection and where the det on the left refers to the ordinary finite-dimensional determinant for operators defined on Im P.

Finally, we will often apply the following proposition (cf. [13]):

Suppose $\{B_n\}$ and $\{C_n\}$ are two sequences of bounded operators satisfying

$B_n \to B$ strongly, $C_n^* \to C^*$ strongly.

Then if $A \in \mathscr{C}_1$

$$\lim_{n\to\infty} \|B_n A C_n - BAC\|_1 = 0.$$

§ 3 Perturbations by operators of the trace class

The following simple proposition is the key of our investigations.

Proposition 1: *If $A \in \Pi\{P_n\}$* and *$K \in \mathscr{C}_1$ then*

$$\lim_{n\to\infty} \frac{\det P_n(A + K)\, P_n}{\det P_n A P_n} = \det(I + A^{-1}K).$$

Proof: Putting $P_n A P_n = A_n$, $P_n K P_n = K_n$ we have for n large enough

$$\frac{\det(A_n + K_n)}{\det A_n} = \frac{\det A_n \cdot \det(I_n + A_n^{-1}K_n)}{\det A_n}$$

$$= \det(I_n + A_n^{-1}K_n) = \det(I + A_n^{-1}P_n \cdot K \cdot P_n)$$

and since $A_n^{-1}P_n \to A^{-1}$ strongly, $P_n^* \to I^*$ strongly it follows by the proposition stated at the end of § 2 that

$$\|A_n^{-1}P_n K P_n - A^{-1}K\|_1 \to 0 \qquad (n \to \infty),$$

thus

$$\det(I + A_n^{-1}P_n K P_n) \to \det(I + A^{-1}K) \qquad (n + \infty) \blacksquare$$

Our next concern is the question under which conditions Hankel operators or products of them belong to the trace class $\mathscr{C}_1$. More about this will be said at the end of this paper; here we remark only that $H(b) \in \mathscr{C}_1$ if for instance b has a derivative satisfying a Lipschitz condition (we write $b \in C^{1+\epsilon}$). This can be proved as follows: if p_n is the trigonometric polynomial of best uniform approximation of degree n for b, then $\|b - p_n\|_\infty \leqq C \cdot n^{-1-\epsilon}$ and this implies

$$s_n(H(b)) \leqq \|H(b) - H(p_n)\|_\infty \leqq \|b - p_n\|_\infty \leqq C \cdot n^{-1-\epsilon}$$

thus $\sum_{n=1}^{\infty} s_n(H(b)) < \infty$. Later we will need the following fact.

Lemma 1: *If b has a continuous second derivative b'', then* $\|H(b)\|_1 \leqq \frac{\pi^4}{12} \|b''\|_\infty$.

Proof: If p_n denotes the trigonometric polynomial of best uniform approximation of degree n for b, then (cf. [7], eq. (9.5))

$$\begin{aligned} s_n(H(b)) &\leqq \|H(b) - H(p_n)\|_\infty \leqq \|b - p_n\|_\infty \\ &\leqq \frac{\pi}{2n}\, \omega\left(b', \frac{\pi}{n}\right) \leqq \frac{\pi}{2n} \|b''\|_\infty \frac{\pi}{n}, \end{aligned}$$

thus

$$\|H(b)\|_1 = \sum_{n=1}^{\infty} s_n(H(b)) \leqq \sum_{n=1}^{\infty} \frac{\pi^2}{2n^2} \|b''\|_\infty = \frac{\pi^4}{12} \|b''\|_\infty \quad \blacksquare$$

Lemma 2: *If $H(b) \in \mathscr{C}_1$ and $H(\tilde{b}) \in \mathscr{C}_1$, then there holds $b \in W$. Here W denotes the Wiener algebra of all functions on Γ with absolutely convergent Fourier series.*

Proof: $\{e_n\}_{n=0}^{\infty}$, $e_n = \{\delta_{nj}\}_{j=0}^{\infty} \in l^2$ is an orthonormal basis in l^2 and from $H(b) \in \mathscr{C}_1$, $H(\tilde{b}) \in \mathscr{C}_1$ it follows (cf. [11])

$$\begin{aligned} &\sum |(H(b)\, e_n, e_n)| = |b_1| + |b_3| + \cdots < \infty, \\ &\sum |(H(\tilde{b})\, e_n, e_n)| = |b_{-1}| + |b_{-3}| + \cdots < \infty. \end{aligned}$$

From (2.2) we get

$$\begin{aligned} &H(tb) = T(t)\, H(b) + H(t)\, T(\tilde{b}) \in \mathscr{C}_1, \\ &H(\tilde{t}\tilde{b}) = T(t^{-1})\, H(\tilde{b}) + H(t^{-1})\, T(b) \in \mathscr{C}_1, \end{aligned}$$

and therefore

$$\begin{aligned} &\sum |(H(tb)\, e_n, e_n)| = |b_0| + |b_2| + \cdots < \infty, \\ &\sum |(H(\tilde{t}\tilde{b})\, e_n, e_n)| = |b_{-2}| + |b_{-4}| + \cdots < \infty \quad \blacksquare \end{aligned}$$

Every $b \in W$, $b(t) \neq 0$ ($|t| = 1$), ind $b = 0$ has a canonical factorization $b = b_- b_+$ where

$$b_-(t) = \exp\left\{\sum_{k=-\infty}^{-1} (\log b)_k\, t^k\right\}, \qquad b_+(t) = \exp\left\{\sum_{k=0}^{\infty} (\log b)_k\, t^k\right\} \tag{1}$$

and there holds $b_-^{\pm 1} \in W \cap \overline{H^\infty}$, $b_+^{\pm 1} \in W \cap H^\infty$ (cf. [3]). If $H(b) \in \mathscr{C}_1$, $H(\tilde{b}) \in \mathscr{C}_1$ then we have $H(b_+) \in \mathscr{C}_1$ and $H(\tilde{b}_-) \in \mathscr{C}_1$; this follows immediately from

$$H(b_+) = H(bb_-^{-1}) = T(b)\, H(b_-^{-1}) + H(b)\, T(\tilde{b}_-^{-1}) = H(b)\, T(\tilde{b}_-^{-1}).$$

It is a well-known fact (cf. [5]) that $H(f)\, H(\tilde{g})$ is a compact operator, if f and g are piecewise continuous functions having no common point of discontinuity. Even much more is true.

Proposition 2: *Suppose that f and g are piecewise continuous functions on Γ without common points of discontinuity. If each of these functions belongs to $C^{1+\varepsilon}$ on each closed subset of Γ which contains no of its points of discontinuity, then $H(f)\,H(\tilde{g}) \in \mathscr{C}_1$.*

Proof: Since f and g have only a finite number of points of discontinuity, in virtue of the representations

$$f(t) = \sum_{k=1}^{n} f_k(t), \qquad g(t) = \sum_{k=1}^{n} g_k(t)$$

where $f_k(t)$ and $g_k(t)$ have only one point of discontinuity and belong to $C^{1+\varepsilon}$ on each closed arc of Γ which does not contain this point of discontinuity, we may reduce the proof to the case when both f and g have only one point of discontinuity t_0 and t_1, respectively, and $t_0 \neq t_1$. In this case we can choose two open arcs γ and Δ of Γ such that $t_0 \in \gamma$, $\bar{\gamma} \subset \Delta$, $t_1 \notin \bar{\Delta}$ holds. From $g \in C^{1+\varepsilon}(\bar{\Delta})$ it follows the existence of a function $b \in C^{1+\varepsilon}(\Gamma)$ satisfying

$$b(t) = g(t) \qquad (t \in \gamma), \qquad b(t) \neq g(t) \qquad (t \in \partial\Delta = \bar{\Delta} \setminus \Delta).$$

Furthermore, there exists a function $c \in C^{1+\varepsilon}(\Gamma)$ with

$$c(t) = g(t) - b(t) \qquad (t \in \Delta), \qquad c(t) \neq 0 \qquad (t \in \Gamma \setminus \Delta).$$

In particular there holds $c(t) = 0$ for $t \in \gamma$. Finally, put

$$d(t) = \begin{cases} 1, & t \in \Delta \\ [g(t) - b(t)]/c(t), & t \in \Gamma \setminus \Delta. \end{cases}$$

We have $g = b + cd$ by construction and applying (2.1) some times, we obtain

$$\begin{aligned} T(fcd) &= T(fc)\,T(d) + H(fc)\,H(\tilde{d}) \\ &= T(f)\,T(c)\,T(d) + H(f)\,H(\tilde{c})\,T(d) + H(fc)\,H(\tilde{d}) \\ &= T(f)\,T(cd) - T(f)\,H(c)\,H(\tilde{d}) + H(f)\,H(\tilde{c})\,T(d) + H(fc)\,H(\tilde{d}) \end{aligned}$$

and

$$T(fb) = T(f)\,T(b) + H(f)\,H(\tilde{b}).$$

This implies

$$\begin{aligned} H(f)\,H(\tilde{g}) &= T(fg) - T(f)\,T(g) = T(fcd + fb) - T(f)\,T(b + cd) \\ &= T(fcd) - T(f)\,T(cd) + T(fb) - T(f)\,T(b) \\ &= -T(f)\,H(c)\,H(\tilde{d}) + H(f)\,H(\tilde{c})\,T(d) + H(fc)\,H(\tilde{d}) + H(f)\,H(\tilde{b}). \end{aligned} \tag{2}$$

From b, c, $fc \in C^{1+\varepsilon}(\Gamma)$ (we remark once more that $c = 0$ in a neighbourhood of the point of discontinuity of f) we get $H(\tilde{b})$, $H(c)$, $H(\tilde{c})$, $H(fc) \in \mathscr{C}_1$, and since all the other operators on the right of (2) are bounded, the assertion follows.

§ 4 The existence of the limit $\lim\limits_{n\to\infty} D_n(\varphi b)/G(b)^{n+1} n^{-\Sigma\beta_r^2}$

We write $\varphi(t) = (-t)_{t_1}^{\beta_1} \dots (-t)_{t_R}^{\beta_r}$ and $\varphi_r(t) = (-t)_{t_r}^{\beta_r}$. Suppose that $H(b)$, $H(\tilde{b}) \in \mathscr{C}_1$ holds (according to Lemma 2 it follows $b \in W$) and that $b(t) \neq 0$ ($|t| = 1$), ind $b = 0$ is fulfilled. Let b_- and b_+ be given by (3.1) from the canonical factorization $b = b_- b_+$.

In order to apply Proposition 1 we put[1])

$$A = T(b_+)\, T(\varphi)\, T(b_-)$$

and

$$A + K = T(b_-)\, T(\varphi)\, T(b_+) = T(\varphi b).$$

We have $K \in \mathscr{C}_1$, since

$$K = T(\varphi b) - T(b_+)\, T(\varphi)\, T(b_-)$$
$$= H(b_+)\, H(\tilde{\varphi}\tilde{b}_-) + T(b_+)\, H(\varphi)\, H(\tilde{b}_-)$$

and $H(b_+)$, $H(\tilde{b}_-) \in \mathscr{C}_1$. From $A + K = T(\varphi b) \in \Pi\{P_n\}$ (cf. (2.4)), the compactness of K and the invertibility of A we may deduce that $A \in \Pi\{P_n\}$. Now, applying Proposition 1 and the identity

$$P_n A P_n = P_n T(b_+)\, T(\varphi)\, T(b_-)\, P_n = T_n(b_+)\, T_n(\varphi)\, T_n(b_-)$$

following from (2.3), we get

$$\lim_{n\to\infty} D_n(\varphi b)/D_n(b_+)\, D_n(\varphi)\, D_n(b_-)$$
$$= \det \{I + T^{-1}(b_-)\, T^{-1}(\varphi)\, T^{-1}(b_+)\, [T(\varphi b) - T(b_+)\, T(\varphi)\, T(b_-)]\}$$
$$= \det T^{-1}(b_-)\, T^{-1}(\varphi)\, T^{-1}(b_+)\, T(\varphi b).$$

$T_n(b_\pm)$ are triangular matrices and therefore we have $D_n(b_+)\, D_n(b_-) = G(b_+)^{n+1}\, G(b_-)^{n+1} = G(b)^{n+1}$ and because of $T^{-1}(f) = T(f^{-1})$ for $f \in \overline{H^\infty}$ or $f \in H^\infty$ we arrive at

$$\lim_{n\to\infty} D_n(\varphi b)/G(b)^{n+1}\, D_n(\varphi) = \det T(b_-^{-1})\, T^{-1}(\varphi)\, T(b_+^{-1})\, T(\varphi b). \quad (1)$$

Thus, we have eliminated the "regular" factor $b(t)$ from the generating function. Now we are going to delete successively one factor φ_r $(r = 1, \ldots, R - 1)$ from $D_n(\varphi)$.

We write $\varphi = \varphi_1\psi$ and put

$$A = T(\varphi_1)\, T(\psi), \qquad K = H(\varphi_1)\, H(\tilde{\psi}).$$

Then we have $T(\varphi_1\psi) \in \Pi\{P_n\}$, $K \in \mathscr{C}_1$ by Proposition 2, A is invertible and $A = T(\varphi_1\psi) - K$ yields $A \in \Pi\{P_n\}$. Thus, we may apply Proposition 1 and we obtain

$$\lim_{n\to\infty} D_n(\varphi_1\psi)/\det P_n T(\varphi_1)\, T(\psi)\, P_n$$
$$= \det \{I + T^{-1}(\psi)\, T^{-1}(\varphi_1)\, H(\varphi_1)\, H(\tilde{\psi})\}$$
$$= \det T^{-1}(\psi)\, T^{-1}(\varphi_1)\, T(\varphi_1\psi). \quad (2)$$

Now, the following identity may easily be verified:

$$P_n T(f)\, T(g)\, P_n = T_n(f)\, T_n(g) + W_n H(\tilde{f})\, H(g)\, W_n \qquad \big(f, g \in L^\infty(\Gamma)\big).$$

It follows

$$P_n T(\varphi_1)\, T(\psi)\, P_n = T_n(\varphi_1)\, T_n(\psi)\, \{I_n + T_n^{-1}(\psi)\, T_n^{-1}(\varphi_1)\, W_n H(\tilde{\varphi}_1)\, H(\psi)\, W_n\}$$

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[1]) This idea is due to Prof. B. Silbermann.

hence

$$\det P_nT(\varphi_1)\,T(\psi)\,P_n/D_n(\varphi_1)\,D_n(\psi)$$
$$= \det\{I_n + T_n^{-1}(\psi)\,T_n^{-1}(\varphi_1)\,W_nH(\tilde{\varphi}_1)\,H(\psi)\,W_n\}$$
$$= \det W_n\{I_n + W_nT_n^{-1}(\psi)\,W_nW_nT_n^{-1}(\varphi_1)\,W_nH(\tilde{\varphi}_1)\,H(\psi)\,P_n\}\,W_n$$
$$= \det\{I + T_n^{-1}(\tilde{\psi})\,T_n^{-1}(\tilde{\varphi}_1)\,H(\tilde{\varphi}_1)\,H(\psi)\,P_n\}.$$

Here we have used $W_n^2 = P_n = I_n$, $W_nT_n^{-1}(f)\,W_n = T_n^{-1}(\tilde{f})$, $\det W_nAW_n = \det P_nAP_n$, $\det\{I_n + P_nAP_n\} = \det\{I + P_nAP_n\}$. Proposition 2 applied to $\tilde{\varphi}_1$ and $\tilde{\psi}$ implies $H(\tilde{\varphi}_1)\,H(\psi) \in \mathscr{C}_1$ and since $T_n^{-1}(\tilde{\psi})\,T_n^{-1}(\tilde{\varphi}_1) \to T^{-1}(\tilde{\psi})\,T^{-1}(\tilde{\varphi}_1)$ strongly, the proposition stated at the end of § 2 leads to

$$\lim_{n\to\infty} \det P_nT(\varphi_1)\,T(\psi)\,P_n/D_n(\varphi_1)\,D_n(\psi)$$
$$= \det\{I + T^{-1}(\tilde{\psi})\,T^{-1}(\tilde{\varphi}_1)\,H(\tilde{\varphi}_1)\,H(\psi)\}$$
$$= \det T^{-1}(\tilde{\psi})\,T^{-1}(\tilde{\varphi}_1)\,T(\tilde{\varphi}_1\tilde{\psi}). \tag{3}$$

Analogously as this was done in (2) and (3) we may now delete φ_2 from $D_n(\psi)$ and so on. Continuing this process, we get from (1), (2), (3)

$$\lim_{n\to\infty} D_n(\varphi_1 \ldots \varphi_R b)/G(b)^{n+1}\,D_n(\varphi_1) \ldots D_n(\varphi_R)$$
$$= \det T(b_-^{-1})\,T^{-1}(\varphi)\,T(b_+^{-1})\,T(b_+\varphi b_-)$$
$$\times \prod_{r=1}^{R-1} \det T^{-1}(\psi_r)\,T^{-1}(\varphi_r)\,T(\varphi_r\psi_r)$$
$$\times \prod_{r=1}^{R-1} \det T^{-1}(\tilde{\psi}_r)\,T^{-1}(\tilde{\varphi}_r)\,T(\tilde{\varphi}_r\tilde{\psi}_r), \tag{4}$$

where

$$\psi_r = \varphi_{r+1} \ldots \varphi_R \qquad (r = 1, \ldots, R-1).$$

From the computations of Hartwig and Fisher [2] follows

$$D_n(\varphi_r) = D_n\big((-t)_{t_r}^{\beta_r}\big) \sim n^{-\beta_r^2}\,\mathfrak{G}(1+\beta_r)\,\mathfrak{G}(1-\beta_r) \tag{5}$$

and therefore the existence of the limit

$$\lim_{n\to\infty} D_n(\varphi b)/G(b)^{n+1}\,n^{-\Sigma\beta_r^2} = E.$$

By (4) and (5) we have a preliminary information about the constant E, it remains to evaluate the occuring operator determinants. This is straightforward but somewhat extensiv, since we are constrained to use some approximation arguments.

§ 5 Approximation of Toeplitz and Hankel operators with piecewise continuous generating function

As already stated in the Introduction, the function $\varphi(t) = (-t)_{t_0}^{\beta}$, $-1/2 < \operatorname{Re}\beta < 1/2$ possesses a factorization

$$\varphi(t) = \left(1 - \frac{t_0}{t}\right)^{-\beta}\left(1 - \frac{t}{t_0}\right)^{\beta} \qquad (|t| = 1).$$

We put

$$\varphi_\mu(t) = \left(1 - \frac{t_0}{\mu t}\right)^{-\beta} \left(1 - \frac{t}{\mu t_0}\right)^{\beta} \qquad (|t| = 1),$$

where $\mu > 1$ is a real parameter. Obviously, we have

$$\varphi_\mu \in W, \qquad \varphi_\mu \in C^\infty, \quad \varphi_\mu(t) \neq 0 \qquad (|t| = 1), \qquad \operatorname{ind} \varphi_\mu = 0$$

if μ is sufficiently close to 1. Furthermore, it is easily seen that $T(\varphi_\mu) \to T(\varphi)$ strongly and $H(\varphi_\mu) \to H(\varphi)$ strongly as $\mu \to 1 + 0$. Somewhat less obvious is the following fact.

Lemma 3: *$T^{-1}(\varphi_\mu) \to T^{-1}(\varphi)$ strongly as $\mu \to 1 + 0$.*

Proof: From $\varphi_\mu(t) \neq 0$ $(|t| = 1)$, $\operatorname{ind} \varphi_\mu = 0$ it follows the invertibility of $T(\varphi_\mu)$ for $\mu \in (1, 1 + \varepsilon)$. It is not difficult to show that

$$T^{-1}(\varphi_\mu)\, e_n = T\left[\left(1 - \frac{t}{\mu t_0}\right)^{-\beta}\right] T\left[\left(1 - \frac{t_0}{\mu t}\right)^{\beta}\right] e_n$$

$$\to T\left[\left(1 - \frac{t}{t_0}\right)^{-\beta}\right] T\left[\left(1 - \frac{t_0}{t}\right)^{\beta}\right] e_n = T^{-1}(\varphi)\, e_n$$

in the norm of l^2, where $e_n = \{\delta_{nj}\}_{j=0}^{\infty} \in l^2$. The set of all finite linear combinations of $\{e_0, e_1, \ldots\}$ is dense in l^2 and the assertion will follow if we prove the uniform boundedness of the norms $\|T^{-1}(\varphi_\mu)\|_\infty$ with respect to $\mu \in (1, 1 + \varepsilon)$. It is elementary function theory to show that the range of $\varphi_\mu(t)$ for $t \in \Gamma$, $\mu \in (1, 1 + \varepsilon)$ is contained in a *closed* sector of an annulus spanned by an angle $2\pi \cdot |\operatorname{Re} \beta| < \pi$ and having the radii $\exp\{\pm \pi \cdot |\operatorname{Im} \beta|\}$. Thus, there exists a real number $c > 0$ such that the disk with the centre in c and the radius c contains this sector of an annulus in its interior. Therefore we have

$$|\varphi_\mu(t) - c| < q|c|, \qquad t \in \Gamma, \qquad \mu \in (1, 1 + \varepsilon)$$

with some q satisfying $0 < q < 1$. This implies

$$\|\varphi_\mu - c\|_\infty \leqq q|c|, \qquad \mu \in (1, 1 + \varepsilon),$$

hence

$$\|T(\varphi_\mu) - cI\|_\infty \leqq q|c|, \qquad \mu \in (1, 1 + \varepsilon),$$

and we obtain

$$T(\varphi_\mu) = cI + T(\varphi_\mu) - cI = cI\{I + c^{-1}(T(\varphi_\mu) - cI)\},$$

$$T^{-1}(\varphi_\mu) = \left\{I + \sum_{n=1}^{\infty} (-1)^n [c^{-1}(T(\varphi_\mu) - cI)]^n\right\} \cdot \frac{1}{c}.$$

From $\|c^{-1}(T(\varphi_\mu) - cI)\|_\infty \leqq q < 1$ it follows $\|T^{-1}(\varphi_\mu)\|_\infty \leqq \dfrac{1}{c} \cdot \dfrac{1}{1 - q}$, $\mu \in (1, 1 + \varepsilon)$ ∎

Lemma 4: *Let $\psi(t)$ $(|t| = 1)$ be a piecewise continuous function which is continuous at $t = t_0$ and which belongs to C^2 on each closed subset of Γ containing no point of discontinuity of ψ. Then we have*

$$\|H(\varphi_\mu)\, H(\tilde{\psi}) - H(\varphi)\, H(\tilde{\psi})\|_1 = o(1)$$

as $\mu \to 1$, $\mu \in (1, 1 + \varepsilon)$.

Proof: We construct b, c, d as in the proof of Proposition 2 and by our assumption it is possible to choose $b, c \in C^2(\Gamma)$. Then (3.2) leads to

$$\begin{aligned}&\|H(\varphi_\mu)\,H(\tilde{\psi}) - H(\varphi)\,H(\tilde{\psi})\|_1\\ &\leqq \|\big(T(\varphi_\mu) - T(\varphi)\big)\,H(c)\,H(\tilde{d})\|_1 + \|(H(\varphi_\mu) - H(\varphi))\,H(\tilde{c})\,T(d)\|_1\\ &\quad + \|H[(\varphi_\mu - \varphi)\,c]\,H(\tilde{d})\|_1 + \|(H(\varphi_\mu) - H(\varphi))\,H(\tilde{b})\|_1\end{aligned}$$

and since $T(\varphi_\mu) \to T(\varphi)$, $H(\varphi_\mu) \to H(\varphi)$ strongly, $H(c), H(\tilde{c}), H(\tilde{b}) \in \mathscr{C}_1$, it follows from the proposition stated at the end of § 2 that all items on the right, with exception of the third, are $o(1)$. Applying Lemma 1 we obtain

$$\begin{aligned}\|H[(\varphi_\mu - \varphi)\,c]\|_1 &\leqq \pi^4/12 \cdot \|[(\varphi_\mu - \varphi)\,c]''\|_\infty\\ &= \pi^4/12 \cdot \|[(\varphi_\mu - \varphi)\,c]''\|_{L^\infty(\Gamma\setminus\gamma)} = o(1)\end{aligned}$$

since $c(t) = 0$ $(t \in \gamma)$ and $\|\varphi_\mu^{(k)} - \varphi^{(k)}\|_{L^\infty(\Gamma\setminus\gamma)} = o(1)$, $k = 1, 2, \ldots$). Thus, the third item is $o(1)$, too ∎

§ 6 The determination of the constant $E(t_1, \ldots, t_R;\ \beta_1, \ldots, \beta_R;\ b)$

The evaluation of the operator determinants $\det(I + C)$, $C \in \mathscr{C}_1$ occuring in (4.4) is based on the representation of $I + C$ as a multiplicative commutator $I + C = e^A e^B e^{-A} e^{-B}$ $\big(A, B \in \mathscr{L}(l^2)\big)$ and the formula

$$\det e^A e^B e^{-A} e^{-B} = \exp \operatorname{tr}(AB - BA), \tag{1}$$

being valid if $AB - BA \in \mathscr{C}_1$ (cf. [13]). An immediate application of (1) to our problem, however, is not possible, since the arising operators do not, in general, satisfy $AB - BA \in \mathscr{C}_1$. Thus, we have to use some approximation arguments.

First of all, we split up the right side of (4.4) into in a certain sense "more elementary" factors. Thereby we continuously make use of (2.1), (2.5), (2.6) without to mention this each time.

We have

$$\begin{aligned}&\det T(b_-^{-1})\,T^{-1}(\varphi)\,T(b_+^{-1})\,T(\varphi b)\\ &= \det T(b_-^{-1})\,T^{-1}(\varphi)\,T(\varphi b_-) \cdot T^{-1}(\varphi b_-)\,T(b_+^{-1})\,T(\varphi b)\\ &= \det T(b_-^{-1})\,T^{-1}(\varphi)\,T(b_-)\,T(\varphi) \cdot \det T^{-1}(\varphi b_-)\,T(b_+^{-1})\,T(\varphi b)\end{aligned}$$

and the second determinant is equal to

$$\begin{aligned}&\det T^{-1}(\varphi)\,T(b_-^{-1})\,T(b_+^{-1})\,T(b_-)\,T(\varphi)\,T(b_+)\\ &= \det T(b_-^{-1})\,T(b_+^{-1})\,T(b_-)\,T(\varphi)\,T(b_+)\,T^{-1}(\varphi)\\ &= \det T(b_-^{-1})\,T(b_+^{-1})\,T(b_-)\,T(b_+) \cdot \det T(b_+^{-1})\,T(\varphi)\,T(b_+)\,T^{-1}(\varphi).\end{aligned}$$

Let us now write

$$E(f, g) = \det T^{-1}(f)\,T^{-1}(g)\,T(f)\,T(g)$$

whenever this has a sense. We remark that in virtue of

$$T^{-1}(f)\,T^{-1}(g)\,T(f)\,T(g) = I + T^{-1}(f)\,T^{-1}(g)\,[H(g)\,H(\tilde{f}) - H(f)\,H(\tilde{g})]$$

$E(f, g)$ is always defined, if $H(g)\,H(\tilde{f}) \in \mathscr{C}_1$ and $H(f)\,H(\tilde{g}) \in \mathscr{C}_1$. By our assumptions we have $H(b_+), H(\tilde{b}_-) \in \mathscr{C}_1$ and so we may conclude in this way that $E(\varphi, b_+)$, $E(b_-, b_+)$

and $E(b_-, \varphi)$ are defined and from our calculations follows

$$\det T(b_-^{-1}) T^{-1}(\varphi) T(b_+^{-1}) T(\varphi b) = E(\varphi, b_+) E(b_-, b_+) E(b_-, \varphi). \tag{2}$$

It turns out that $E(f, g)$ has a remarkable multiplicative property.

Lemma 5: *If $H(\tilde{f}) H(g) \in \mathscr{C}_1$, $H(\tilde{b}_-) \in \mathscr{C}_1$, $b_- \in \overline{H^\infty}$ then*

$$E(b_-, fg) = E(b_-, f) E(b_-, g),$$

and if $H(f) H(\tilde{g}) \in \mathscr{C}_1$, $H(b_+) \in \mathscr{C}_1$, $b_+ \in H^\infty$ then

$$E(fg, b_+) = E(f, b_+) E(g, b_+).$$

Here we suppose that all occuring in $E(\cdot\,, \cdot)$ inverses $T^{-1}(f), \ldots$ exist.

Proof: From $H(f) H(\tilde{g}) \in \mathscr{C}_1$ it follows that $T(fg) T^{-1}(g) T^{-1}(f) - I \in \mathscr{C}_1$ and we obtain

$$\begin{aligned} &E(f, b_+) \cdot \det T(fg) T^{-1}(g) T^{-1}(f) \cdot E(g, b_+) \\ &= \det T^{-1}(f) T(b_+^{-1}) T(f) T(b_+) \cdot \det T(b_+^{-1}) T^{-1}(f) T(fg) T^{-1}(g) T(b_+) \\ &\quad \times \det T(b_+^{-1}) T(g) T(b_+) T^{-1}(g) \end{aligned}$$

and

$$\begin{aligned} E(fg, b_+) &= \det T^{-1}(fg) T(b_+^{-1}) T(fg) T(b_+) \\ &= \det T(f) T(g) T^{-1}(fg) \cdot T(b_+^{-1}) T(fg) T(b_+) \cdot T^{-1}(g) T^{-1}(f) \\ &= \det T(f) T(g) T^{-1}(fg) \cdot \det T^{-1}(f) T(b_+^{-1}) T(fg) T(b_+) T^{-1}(g). \end{aligned}$$

From

$$\det T(f) T(g) T^{-1}(fg) \cdot \det T(fg) T^{-1}(g) T^{-1}(f) = 1$$

the second assertion follows. The first may be proved analogously ∎

Lemma 6: *If f resp. b_+ are invertible elements with index zero in W resp. $W \cap H^\infty$ and if $H(b_+) \in \mathscr{C}_1$ then*

$$E(f, b_+) = \exp \sum_{k=1}^{\infty} k (\log f)_{-k} (\log b_+)_k.$$

If f resp. b_- are invertible elements with index zero in W resp. $W \cap \overline{H^\infty}$ and if $H(\tilde{b}_-) \in \mathscr{C}_1$ then

$$E(b_-, f) = \exp \sum_{k=1}^{\infty} k (\log b_-)_{-k} (\log f)_k.$$

Proof: From the assumptions it follows that f has a canonical factorization $f = f_- \cdot f_+$. Thus

$$\begin{aligned} E(f, b_+) &= \det T^{-1}(f) T(b_+^{-1}) T(f) T(b_+) \\ &= \det T(f_+^{-1}) T(f_-^{-1}) T(b_+^{-1}) T(f b_+) \\ &= \det T(f_-^{-1}) T(b_+^{-1}) T(f b_+ f_+^{-1}) \\ &= \det T(f_-^{-1}) T(b_+^{-1}) T(f_-) T(b_+). \end{aligned}$$

Now there holds $T(\psi) = \exp T(\log \psi)$, if ψ is an invertible element with index zero in $W \cap H^\infty$ or $W \cap \overline{H^\infty}$ and we get (applying (1))

$$E(f, b_+) = \exp\{-T(\log f_-)\} \exp\{-T(\log b_+)\} \exp T(\log f_-) \exp T(\log b_+)$$
$$= \exp \operatorname{tr} \{T(\log f_-) T(\log b_+) - T(\log b_+) T(\log f_-)\}$$
$$= \exp \operatorname{tr} H(\log b_+) H\big((\log f_-)^\sim\big)$$

if only $H(\log b_+) H\big((\log f_-)^\sim\big) \in \mathscr{C}_1$. But from $H(\psi) \in \mathscr{C}_1$, ψ being invertible with index zero in W it follows $H(\log \psi) \in \mathscr{C}_1$ (cf. [8]) and so by our assumptions we have $H(\log b_+) \in \mathscr{C}_1$. An easy computation shows

$$\operatorname{tr} H(f) H(\tilde{g}) = \sum_{k=1}^{\infty} k f_k g_{-k}$$

for $H(f) H(\tilde{g}) \in \mathscr{C}_1$ and now the assertion follows immediately. The second may be proved in the same way ∎

Looking at (4.4) we still must investigate expressions of the form $\det T^{-1}(g) T^{-1}(f) \times T(fg)$. We write

$$F(f, g) = \det T^{-1}(g) T^{-1}(f) T(fg)$$

whenever this has a sense (e.g. if $H(f) H(g) \in \mathscr{C}_1$).

Lemma 7: *Put $\varphi(t) = (-t)_{t_0}^{\beta}$ and suppose that ψ, f, g satisfy the conditions of Lemma 4. Then*

$$F(\varphi, \psi) = \lim_{\mu \to 1+0} F(\varphi_\mu, \psi)$$

where φ_μ is defined as in § 5. Furthermore, there holds

$$F(\varphi_\mu, \psi) = E\big(\psi, (\varphi_\mu)_+\big)$$

with $(\varphi_\mu)_+ (t) = \left(1 - \dfrac{t}{\mu t_0}\right)^{\beta}$ *and*

$$F(\varphi, fg) = F(\varphi, f) F(\varphi, g).$$

Here we suppose that all inverses occuring in $F(\cdot, \cdot)$ exist.

Proof: We have

$$F(\varphi, \psi) = \det T^{-1}(\psi) T^{-1}(\varphi) T(\varphi\psi)$$
$$= \det \{I + T^{-1}(\psi) T^{-1}(\varphi) H(\varphi) H(\tilde{\psi})\}$$

and from $T^{-1}(\varphi_\mu) \to T^{-1}(\varphi)$ strongly according to Lemma 3 and $\|H(\varphi_\mu) H(\tilde{\psi}) - H(\varphi) H(\tilde{\psi})\|_1 = o(1)$ according to Lemma 4, we may conclude that

$$F(\varphi, \psi) = \lim_{\mu \to 1+0} \det \{I + T^{-1}(\psi) T^{-1}(\varphi_\mu) H(\varphi_\mu) H(\tilde{\psi})\}$$
$$= \lim_{\mu \to 1+0} \det T^{-1}(\psi) T^{-1}(\varphi_\mu) T(\varphi_\mu \psi)$$
$$= \lim_{\mu \to 1+0} F(\varphi_\mu, \psi).$$

Now there holds

$$F(\varphi_\mu, \psi) = \det T^{-1}(\psi)\, T\big((\varphi_\mu)_+^{-1}\big)\, T\big((\varphi_\mu)_-^{-1}\big)\, T(\varphi_\mu \psi)$$
$$= \det T^{-1}(\psi)\, T\big((\varphi_\mu)_+^{-1}\big)\, T(\psi)\, T\big((\varphi_\mu)_+\big)$$

where $(\varphi_\mu)_- = \left(1 - \frac{t_0}{\mu t}\right)^{-\beta}$, $(\varphi_\mu)_+ = \left(1 - \frac{t}{\mu t_0}\right)^{\beta}$, hence

$$F(\varphi_\mu, \psi) = E\big(\psi, (\varphi_\mu)_+\big).$$

If f and g satisfy the conditions of Lemma 4, then in virtue of $H\big((\varphi_\mu)_+\big) \in \mathscr{C}_1$ Lemma 5 may be applied and what results is

$$F(\varphi_\mu, fg) = F(\varphi_\mu, f) \cdot F(\varphi_\mu, g).$$

Taking the limit $\mu \to 1 + 0$ we get the last assertion. ∎

Now we are ready to evaluate the operator determinants in question.
From Lemma 6 and Lemma 2 immediately follows that

$$E(b_-, b_+) = \exp \sum_{k=1}^{\infty} k (\log b)_{-k} (\log b)_{+k} =: E(b). \tag{3}$$

For $\varphi(t) = (-t)_{t_r}^{\beta_r}$ in virtue of $H(b_+) \in \mathscr{C}_1$ $E(\varphi, b_+)$ is defined and we have

$$E(\varphi, b_+) = \det T^{-1}(\varphi)\, T(b_+^{-1})\, T(\varphi)\, T(b_+)$$
$$= \det \{I + T^{-1}(\varphi)\, T(b_+^{-1})\, H(b_+)\, H(\tilde{\varphi})\}$$
$$= \lim_{\mu \to 1+0} \det \{I + T^{-1}(\varphi_\mu)\, T(b_+^{-1})\, H(b_+)\, H(\tilde{\varphi}_\mu)\},$$

since $T^{-1}(\varphi_\mu) \to T^{-1}(\varphi)$ strongly by Lemma 3, $H(\tilde{\varphi}_\mu) \to H(\tilde{\varphi})$ strongly (obviously) and $H(b_+) \in \mathscr{C}_1$ was supposed. Because of

$$\det \{I + T^{-1}(\varphi_\mu)\, T(b_+^{-1})\, H(b_+)\, H(\tilde{\varphi}_\mu)\}$$
$$= \det T^{-1}(\varphi_\mu)\, T(b_+^{-1})\, T(\varphi_\mu)\, T(b_+) = E(\varphi_\mu, b_+)$$

we get finally $E(\varphi, b_+) = \lim_{\mu \to 1+0} E(\varphi_\mu, b_+)$. Now $E(\varphi_\mu, b_+)$ may be calculated using Lemma 6. What results is

$$E(\varphi_\mu, b_+) = \exp \sum_{k=1}^{\infty} k (\log \varphi_\mu)_{-k} (\log b_+)_k$$
$$= \exp \beta_r \sum_{k=1}^{\infty} (\log b_+)_k \cdot t_r^k / \mu^k$$

(since $(\log \varphi_\mu)_{-k} = \beta_r t_r^k / k \mu^k$) and therefore

$$E(\varphi, b_+) = \lim_{\mu \to 1+0} \exp \beta_r \sum_{k=1}^{\infty} (\log b_+)_k \cdot t_r^k / \mu^k$$
$$= \exp \beta_r \sum_{k=1}^{\infty} (\log b_+)_k \cdot t_r^k = b_+(t_r)^{\beta_r}$$

(we remark that $\log b_+ \in W$ holds!).

Analogously one can show that

$$E(b_-, \varphi) = b_-(t_r)^{-\beta_r}.$$

Lemma 5 now gives

$$E(\varphi_1 \ldots \varphi_R, b_+)\, E(b_-, b_+)\, E(b_-, \varphi_1 \ldots \varphi_R)$$
$$= E(b) \prod_{r=1}^{R} b_+(t_r)^{\beta_r} \prod_{r=1}^{R} b_-(t_r)^{-\beta_r},$$

where $E(b)$ is defined by (3).

Applying Lemma 7 to (4.4) we obtain

$$\prod_{r=1}^{R-1} F(\varphi_r, \psi_r)\, F(\tilde{\varphi}_r, \tilde{\psi}_r) = \prod_{r<s} F(\varphi_r, \varphi_s)\, F(\tilde{\varphi}_r, \tilde{\varphi}_s)$$

and we still must evaluate $F(\varphi_r, \varphi_s)$, $\varphi_r(t) = (-t)_{t_r}^{\beta_r}$, $\varphi_s(t) = (-t)_{t_s}^{\beta_s}$. Also according to Lemma 7 we have

$$F(\varphi_r, \varphi_s) = \lim_{\mu \to 1+0} F[(\varphi_r)_\mu, \varphi_s] = \lim_{\mu \to 1+0} \lim_{\lambda \to 1+0} F[(\varphi_r)_\mu, (\varphi_s)_\lambda]$$

and because of $F[(\varphi_r)_\mu, (\varphi_s)_\lambda] = E[(\varphi_s)_\lambda, (\varphi_r)_{\mu,+}]$ we obtain applying Lemma 6

$$F[(\varphi_r)_\mu, (\varphi_s)_\lambda] = \exp \sum_{k=1}^{\infty} k[\log (\varphi_r)_\mu]_{-k}\, [\log (\varphi_s)_\lambda]_k$$
$$= \exp \sum_{k=1}^{\infty} k \left(\frac{\beta_r t_r^k}{\mu^k k}\right) \left(-\frac{\beta_s}{\lambda^k k t_s^k}\right)$$
$$= \exp \left\{\beta_r \beta_s \log \left(1 - \frac{t_r}{\mu \lambda t_s}\right)\right\} = \left(1 - \frac{t_r}{\mu \lambda t_s}\right)^{\beta_r \beta_s}.$$

The limit $\mu \to 1 + 0$, $\lambda \to 1 + 0$ then gives

$$(F\varphi_r, \varphi_s) = \left(1 - \frac{t_r}{t_s}\right)^{\beta_r \beta_s}.$$

Analogously one can show that

$$F(\tilde{\varphi}_r, \tilde{\varphi}_s) = \left(1 - \frac{t_s}{t_r}\right)^{\beta_r \beta_s}.$$

§ 7 Summary

Let $\varphi_r(t) = (-t)_{t_r}^{\beta_r}$, $-1/2 < \operatorname{Re} \beta_r < 1/2$, be defined as in § 1. Suppose that $b \in L^\infty(\Gamma)$ and $H(b)$, $H(\tilde{b})$ are operators of the trace class. Then necessarily $b \in W$. If $b(t) \neq 0$ ($|t| = 1$), ind $b = 0$ then b has a canonical factorization $b = b_- \cdot b_+$, where the factors $b_\pm$ are defined by (3.1).

We have proved that

$$\lim_{n \to \infty} \frac{D_n(\varphi_1 \ldots \varphi_R b)}{G(b)^{n+1}\, n^{-\sum_{r=1}^{R} \beta_r^2}} = \tilde{E}(t_1, \ldots, t_R; \beta_1, \ldots, \beta_R; b)\, E(b)$$

holds. Here

$$G(b) = \exp(\log b)_0,$$

$$E(b) = \exp\sum_{k=1}^{\infty} k(\log b)_{-k}(\log b)_k,$$

$$\tilde{E}(t_1, \ldots, t_R; \beta_1, \ldots, \beta_R; b)$$
$$= \prod_{r=1}^{R} \mathfrak{G}(1+\beta_r)\,\mathfrak{G}(1-\beta_r) \prod_{r=1}^{R} E(b_-, \varphi_r)\,E(\varphi_r, b_+) \prod_{r<s} F(\varphi_r, \varphi_s)\,F(\tilde{\varphi}_r, \tilde{\varphi}_s),$$

and

$$E(f, g) = \det T^{-1}(f)\,T^{-1}(g)\,T(f)\,T(g),$$

$$F(f, g) = \det T^{-1}(g)\,T^{-1}(f)\,T(fg).$$

An equivalent expression is

$$\tilde{E}(t_1, \ldots, t_R; \beta_1, \ldots, \beta_R; b)$$
$$= \prod_{r=1}^{R} \mathfrak{G}(1+\beta_r)\,\mathfrak{G}(1-\beta_r) \prod_{r=1}^{R} b_+(t_r)^{\beta_r}\,b_-(t_r)^{-\beta_r} \prod_{r\neq s} \left(1 - \frac{t_r}{t_s}\right)^{\beta_r\beta_s}.$$

$\mathfrak{G}(z)$ is the *Barnes* $\mathfrak{G}$*-function* defined in § 1.

Appendix A: Hankel operators of the trace class

We remark that V. V. PELLER in a recent paper [9] announced a necessary and sufficient condition for a Hankel operator to belong to the ideal $\mathscr{C}_p (1 \leqq p < \infty)$.

Given a function $b(t) = \sum_{n=-\infty}^{\infty} b_n t^n$ $(|t| = 1)$, we denote by Pb the function defined by $(Pb)(t) = \sum_{n=0}^{\infty} b_n t^n$, whenever this series converges. By $B_p^{1/p}$ $(1 \leqq p < \infty)$ we denote the Besov class of all measurable functions on Γ satisfying

$$\int_{-\pi}^{\pi} y^{-2} \int_{-\pi}^{\pi} |f(e^{ix+iy}) + f(e^{ix-iy}) - 2f(e^{ix})|^p \,dx\,dy < \infty,$$

which for $p > 1$ is equivalent to

$$\int_{-\pi}^{\pi} y^{-2} \int_{-\pi}^{\pi} |f(e^{ix+iy}) - f(e^{ix})|^p \,dx\,dy < \infty.$$

Further, we put

$$A_p^{1/p} = \left\{ f \in B_p^{1/p} : \int_{-\pi}^{\pi} f(e^{ix})\,e^{ikx}\,dx = 0, \qquad k > 0 \right\},$$

i.e. $A_p^{1/p}$ is the subclass of all analytical functions of $B_p^{1/p}$. Then one has $f \in A_p^{1/p}$ $1 \leqq p < \infty)$ if and only if

$$\int_{\mathbf{D}} |f''|^p\,\chi(1 - |z|)^{2p-2}\,dx\,dy < \infty,$$

which for $p > 1$ is equivalent to

$$\int_{\mathbf{D}} |f'|^p (1 - |z|)^{p-2} \, dx \, dy < \infty .$$

Here a function $f \in A_p^{1/p}$ is identified with its analytical extension into $\mathbf{D} = \{z \in \mathbf{C}: |z| < 1\}$.

The result of Peller then reads:

$$H(b) \in \mathscr{C}_p \Leftrightarrow Pb \in B_p^{1/p} \Leftrightarrow Pb \in A_p^{1/p} \qquad (1 \leq p < \infty).$$

In particular we obtain

$$H(b), H(\tilde{b}) \in \mathscr{C}_1 \Leftrightarrow Pb \in B_1^1, \quad (I - P)\, b \in B_1^1 \Leftrightarrow b_- \in B_1^1, \quad b_+ \in B_1^1,$$

$b_\pm$ being the factors in the canonical factorization $b = b_- b_+$. Furthermore using the boundedness of P on B_1^1 (cf. [9]) we get $H(b), H(\tilde{b}) \in \mathscr{C}_1 \Leftrightarrow b \in B_1^1$.

Appendix B: The block case

We remark that the techniques used here are available in the block case, too. In fact, given a matrix generating function $a(t) = \{a_{ij}(t)\}_{i,j=1}^N$ with elements being piecewise continuous, then we have a factorization $a = b\varphi c$ if only $\det a(t \pm 0) \neq 0$ ($|t| = 1$) holds; here b and c are continuous matrix functions and φ is an upper triangular matrix with piecewise continuous elements (cf. [10], p. 124). Under certain conditions concerning smoothness and invertibility one may eliminate b and c and then, in virtue of the triangular form of φ, the results for the scalar case lead to

$$D_n(a) \sim G^{n+1} \cdot E \cdot n^{-\sum_{k=1}^{N} \sum_{r=1}^{R} \beta_{kr}^2}$$

with some constants G and E; the β_{rk}'s are given by $\beta_{rk} = \frac{1}{2\pi i} \log \lambda_{rk}$, $-1/2 < \operatorname{Re} \beta_{rk} < 1/2$ where λ_{rk} ($k = 1, \ldots, N$) are the N eigenvalues of the matrices $a(t_r + 0)^{-1} \times a(t_r - 0)$, and $t_1, \ldots, t_R$ being the points of discontinuity. More about this will be published elsewhere.

Acknowledgement: I would like to thank Prof. B. Silbermann for suggesting this problem and for numerous stimulating and helpful discussions.

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Added in Proof. Having already submitted the present paper to the Editors, I observed that the same problem was considered in

Basor, E. L.: A Localization Theorem for Toeplitz Determinants. Indiana Univ. Math. J. 28 (1979), 975—983, and for the special case were the generating function has only one point of discontinuity in

Блехер, П. М.: О гипотезе Фишера—Хартвига в теории теплицевых матриц. Функц. анализ и его прилож. 16, вып. 2 (1982), 1—6.

Manuskripteingang: 09. 01. 1981

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