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Remarks on quadratic optimal control problems in Hilbert spaces

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Es werden quadratische Probleme der optimalen Steuerung im Hilbertraum betrachtet. Für diese Aufgaben werden Eigenschaften der optimalen Steuerung und Schranken hergeleitet.

В статье рассматриваются квадратичные проблемы оптимального управления в гильбертовом пространстве. Для этих проблем выводятся свойства оптимального управления и границы.

In this paper quadratic optimal control problems in Hilbert spaces are considered. For this problem properties of the optimal control and bounds are given.

1. Introduction

In this paper we shall consider the following quadratic optimal control problem (optimal regulator problem)

$$j(Q, u) = \|Q - R\|_1^2 + K \|u\|_2^2 \doteq \underset{u \in U \subset H_2}{\text{Infimum}}, \tag{1}$$

$$AQ + Bu + f = 0 \tag{2}$$

with

H_i — Hilbert spaces with the norm $\|\cdot\| = (\cdot, \cdot)_i^{1/2}$ $(i = 1, 2, 3)$,
K — constant $\geqq 0$,
R, f — given elements of H_1 and H_3, respectively,
$U \subset H_2$ — convex, closed set $(\neq \emptyset)$,
$A \in \mathcal{L}(H_1 \to H_3)$, $B \in \mathcal{L}(H_2 \to H_3)$ — linear operators (additionally we assume that A^{-1} exists, A^{-1} and B are bounded and $D(A)$ is dense in H_1).

If the inverse operator A^{-1} is known, the problem (1), (2) may be written in the well-known form

$$J(u) = \|Lu - r\|_1^2 + K \|u\|_2^2 \doteq \underset{u \in U}{\text{Infimum}} \tag{3}$$

with $L = -A^{-1}B \in \mathcal{L}(H_2 \to H_1)$ bounded and $r = R + A^{-1}f \in H_1$.
Quadratic problems of the form (3) have been extensively studied by many authors (see [1, 2, 3, 7, 11, 14, 15, 19]).

Performance indexes of the form (1) arise practically if we want to approximate a given state R and additionally (for $K > 0$) to minimize the costs of control. Furthermore, the problem (1), (2) is considered in order to obtain the least squares solution of minimal norm of the linear operator equation (2) (by solving problem (3) the least squares solution of the operator equation $Lu = r$ is obtained).

The aim of this paper is to give a short survey of known results in the field of quadratic optimal control problems described by operator equations, and to derive some new results. The survey is given in chapter 2. In chapter 3 properties for the problem (3) are proved for the case where the operator L is only closed. Applying the spectral theory for closed linear mappings, developed by HESTENES [10] and exactly proved in [12], we obtain further properties of the optimal control and upper and lower bounds for the optimal control and the cost functional. The given bounds are better than the bounds published in the literature. Such bounds are very useful for estimating how close a sub-optimal control is to the optimal one, without actually calculating the latter. Furthermore, bounds for the optimal control can be successfully applied in the proof of *bang-bang-ness* of the optimal control. Sufficient conditions for the *bang-bang-ness* of an optimal control are derived in chapter 4, making use of the bounds given in chapter 3.

2. Known results

It is well-known that under the above assumptions the problems (1), (2) and (3) for $K > 0$ have exactly one solution. For $K = 0$ we must additionally assume, for the existence of an optimal control, that the control set U is bounded. In the following we denote an optimal control for our problem, belonging to a fixed constant K, by $u(K)$.

Now we give some known results for the problem

$$J(u) = \|Lu - r\|_1^2 + K\,\|u\|_2^2 \doteq \underset{u \in H_2}{\mathrm{Minimum}}. \tag{3'}$$

Theorem 1: *The following properties are valid for a solution $u(K)$ of problem (3′):*

a) $(L^T L + KI)\, u(K) = L^T r$ [1]; (4)

and for $K > 0$ $u(K) = (L^T L + KI)^{-1} L^T r$; (5)

$$\left.\begin{array}{ll}\text{b)} & \|u(K)\|_2 \\ \text{c)} & J(K) := J(u(K)) \\ \text{d)} & \|Lu(K) - r\|_1\end{array}\right\} \textit{are for } L^T r \neq 0 \textit{ monotone strictly} \left\{\begin{array}{l}\textit{decreasing}\\ \textit{increasing}\\ \textit{increasing}\end{array}\right\} \textit{with resp. to } K;$$

e) $K_2(\|u(K_2)\|_2^2 - \|u(K_1)\|_2^2) \leqq \|Lu(K_1) - r\|_1^2 - \|Lu(K_2) - r\|_1^2$

$$\leqq K_1(\|u(K_2)\|_2^2 - \|u(K_1)\|_2^2) \quad \textit{for} \quad K_1 \leqq K_2; \tag{6}$$

f) $\|L^T r\|_2/(K + \|L\|^2) \leqq \|u(K)\|_2 \leqq \mathrm{Minimum}\,\{\|L^T r\|_2/K,\, \|r\|_2/K^{1/2}\}$ (*for* $K > 0$); (7)

$$\left.\begin{array}{ll}\text{g)} & J(K)\\ \text{h)} & u(K)\end{array}\right\} \textit{are continuous for } K \in (0, +\infty);$$

i) $\lim_{K \to \infty} u(K) = L^+ r$ *if* $r \in D(L^+)$ [2];

$\lim_{K \to \infty} \|u(K)\|_2 = \infty$ *if* $r \notin D(L^+)$;

j) *Problem (3′) has a solution for $K = 0$ if and only if $r \in D(L^+)$ holds and $u(0) = L^+ r$ is then the solution with minimal norm;*

k) $\lim_{K \to 0} \|Lu(K) - r\|_1 = \underset{u \in H_2}{\mathrm{Infimum}}\, \|Lu - r\|_1$;

l) $\|u - u(K)\|_2 \leqq \|(L^T L + KI)\, u - L^T r\|_2/K \qquad \forall\, u \in H_2, \quad K > 0$; (8)

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[1] L^+ denotes the pseudoinverse of L (see [13] and the appendix).

[2] L^T denotes the adjoint of the operator L.

m) $\|r\|_1^2 - \|L^T r\|_2^2/(K + \|L^T r\|_2^2/\|r\|_1^2) \leq J(K)$
$\leq \|r\|_1^2 - \|L^T r\|_2^2/(K + \|LL^T r\|_1^2/\|L^T r\|_2^2)$ (*for* $K > 0$); (9)

n) $J(u) - J(K) \leq \|L^T L + KI) u - L^T r\|_2^2/K \qquad \forall u \in H_2, \quad K > 0;$ (10)

o) $u(K)$ *is also a solution of the problem*

$$\|Lu - r\|_1 \doteq \underset{\|u\|_2 \leq \|u(K)\|_2}{\text{Minimum}}.$$

Proof: a), b), c), d), g), h) were proved in [2, 16, 18], e), f), l), m), n), o) in [3—6], i), j), k) in [7] ∎

Remarks:

1. Expansions into series for the optimal control $u(K)$ are given by WEIGAND/D'SOUZA [19] and VIDYASAGAR [17].

2. Obviously, for $L^T r = 0$ (i.e. $r \in \operatorname{Ker} L^T$) it follows that the optimal control $u(K) = 0$. In this case the functionals $\|Lu - r\|_1$ and $\|u\|_2$ are simultaneously minimized by $u = 0$ because $\operatorname{Ker} L^T = R(L)^\perp$ holds. For $r \notin \operatorname{Ker} L^T$ it follows $\|u(K)\|_2 > 0$ and

$$r - Lu(K) = K(KI + LL^T)^{-1} r \notin \operatorname{Ker} L^T = R(L)^\perp \qquad \forall K > 0.$$

In this case neither $\|Lu - r\|_1$ nor $\|u\|_2$ are minimized by $u(K)$ ($\forall K > 0$).

3. Further properties

In the following we assume only that the operator L from (3') is closed (with $D(L)$ dense in H_2).

First we investigate the connections between problem (3') and the problems

$$\|Lu - r\|_1 \doteq \underset{u \in D(L), \|u\|_2 \leq \beta}{\text{Infimum}}, \tag{11}$$

$$\|u\|_2 \doteq \underset{u \in D(L), \|Lu - r\|_1 \leq \alpha}{\text{Infimum}} \tag{12}$$

which also arise in control theory.

Lemma 1: *Problem* (11) *has a solution for every* $\beta > 0$. *If* $r \in D(L^+)$ *and* $\|L^+ r\|_2 \leq \beta$, *then* $u^0 = L^+ r$ *is a solution. Otherwise* (i.e. $r \notin D(L^+)$ *or* $\|L^+ r\|_2 > \beta$) *exactly one* $K_0 > 0$ *exists, such that* $\|u(K_0)\|_2 = \beta$ *holds* ($u(K)$ — *solution of problem* (3')). $u(K_0)$ *is then the unique solution of* (11).

Proof: The first part of the lemma follows from the definition of the pseudoinverse L^+. For $r \notin D(L^+)$ we have $r \notin \operatorname{Ker} L^+ = \operatorname{Ker} L^T$ and therefore $\|u(K)\|_2$ is continuous and monotone strictly decreasing (see Theorem 1) with $\lim_{K \to \infty} \|u(K)\|_2 = 0$ and $\lim_{K \to 0} \|u(K)\|_2 = \infty$. From these properties of $\|u(K)\|_2$ it follows that exactly one $K_0 > 0$ exists with $\|u(K_0)\|_2 = \beta$. The inequality

$$\|Lu - r\|_1 \geq \|Lu(K_0) - r\|_1 \qquad \forall \|u\|_2 \leq \beta$$

which we obtain from

$$\|Lu - r\|_1^2 + K \|u\|_2^2 \geq \|Lu(K_0) - r\|_1^2 + K \|u(K_0)\|_2^2 \qquad \forall u \in D(L),$$

concludes the proof ∎

Lemma 2: *Problem* (12) *has exactly one solution for every*

$$\alpha > I := \underset{u \in D(L)}{\text{Infimum}} \|Lu - r\|_1 .$$

For $\alpha \in (I, \|r\|_1)$ *exactly one* $K_0 > 0$ *exists, such that* $\|Lu(K_0) - r\|_1 = \alpha$ *holds* ($u(K)$ *— solution of problem* (3′)). *This* $u(K_0)$ *is then the unique solution of* (12). *If* $\alpha \geqq \|r\|_1$ *then* $u^0 = 0$ *is the unique solution of* (12).

Proof: Let be $\alpha \in (I, \|r\|_1)$. Then it follows that $\eta < \|r\|_1$, i.e. $r \notin \operatorname{Ker} L^T = R(L)^\perp$. With $\lim_{K\to\infty} \|Lu(K) - r\|_1 = \|r\|_1$, $\lim_{K \to 0} \|Lu(K) - r\|_1 = I$ and the continuity and strict monotonicity of $\|Lu(K) - r\|_1$ we find that exactly one $K > 0$ exists such that $\|Lu(K) - r\|_1 = \alpha$ holds. The inequality

$$\|Lu - r\|_1^2 + K\,\|u\|_2^2 \geqq \|Lu(K) - r\|_1^2 + K\,\|u(K)\|_2^2 \quad \forall\, u \in D(L)$$

yields $\|u\|_2 \geqq \|u(K)\|_2\ \forall\, u$ with $\|Lu - r\|_1 \leqq \alpha$. If $\alpha \geqq \|r\|_1$ then $u^0 = 0$ is admissible and therefore the unique solution of (12) ▮

Remark: Applying Lemma 1 and 2 we see that the sets

$$\{u(K)/K \in (0, \infty)\} \cup \{0\} \quad \text{for } r \notin D(L^+)$$

and

$$\{u(K)/K \in (0, \infty)\} \cup \{0, L^+ r\} \quad \text{for } r \in D(L^+)$$

are the sets of coefficient points of the vector minimum problem

$$\begin{pmatrix} \|u\|_2 \\ \|Lu - r\|_1 \end{pmatrix} \doteq \underset{u \in D(L)}{\text{Infimum}} .$$

Now we investigate the mapping $L^T(LL^T + KI)^{-1}$. This mapping is continuous and defined on the whole space H_1 (for $K > 0$). Furthermore, $(L^TL + KI)^{-1} L^T$ is the restriction of the mapping $L^T(LL^T + KI)^{-1}$ to $D(L^T)$. Representing the functional (3′) as

$$\begin{aligned} J(u) &= \|Lu - r\|_1^2 + K\,\|u\|_2^2 \\ &= \|L(u - L^T(LL^T + KI)^{-1} r\|_1^2 \\ &\quad + K\,\|u - L^T(LL^T + KI)^{-1} r\|_2^2 + K\big((LL^T + KI)^{-1} r, r\big)_1 , \end{aligned} \tag{13}$$

we see that the optimal control $u(K)$ can be expressed as

$$u(K) = L^T(LL^T + KI)^{-1} r. \tag{5′}$$

Using the equalities

$$L^T(LL^T + KI)^{-1} = (L + KL^{+T})^+ \qquad (\forall\, K > 0) \tag{14}$$

and

$$(L + KL^{+T})^+ = (KL^{+U} + K(KL^{+T})^{+T})^+ , \tag{15}$$

we find that $u(K)$ from (5′) is also a solution of the problem

$$\|KL^{+T}u - r\|_1^2 + K\,\|u\|_2^2 \doteq \underset{u \in D(L^{+T})}{\text{Minimum}} . \tag{16}$$

Employing the spectral representation

$$L = \int_0^\infty \beta \, dR_\beta \tag{17}$$

for the operator L (see appendix), it can be shown by using the expressions (5'), (14), (15) that the optimal control for problem (3') can be expressed as

$$u(K) = \int_0^\infty \beta/(K + \beta^2)\, dR_\beta{}^T r. \tag{18}$$

This leads to

$$\|u(K)\|_2{}^2 = \int_0^\infty \beta^2/(K + \beta^2)^2\, d\, \|R_\beta{}^T r\|_2{}^2 \tag{19}$$

and

$$J(K) = \|r\|_1{}^2 - \int_0^\infty \beta^2/(K + \beta^2)\, d\, \|R_\beta{}^T r\|_2{}^2. \tag{20}$$

The expressions (18), (19) and (20) for the optimal control and the cost functional, now give the possibility of deriving properties of the optimal control.

Lemma 3: *The functions $J(K)$ and $\|u(K)\|_2{}^2$ are analytical for $K \in (0, \infty)$, i.e. there is an expansion into a power series for every $K > 0$ in a neighbourhood of K. Furthermore, $\frac{\mathrm{d}}{\mathrm{d}K} J(K) = \|u(K)\|_2{}^2$ holds, and $J(K)$ and $\|u(K)\|_2{}^2$ are concave and convex, respectively.*

Proof: Making use of a theorem of Dieudonné [8] the property of being analytic follows from the fact that $\beta^2/(K + \beta^2)$ is analytic with respect to K. Therefore

$$\frac{\mathrm{d}^n}{\mathrm{d}K^n} \int_0^\infty \beta^2/(K + \beta^2)\, d\, \|R_\beta{}^T r\|_2{}^2 = \int_0^\infty \frac{\mathrm{d}^n}{\mathrm{d}K^n} \beta^2/(K + \beta^2)\, d\, \|R_\beta{}^T r\|_2{}^2$$

also holds, and particulary $\frac{\mathrm{d}}{\mathrm{d}K} J(K) = \|u(K)\|_2{}^2$. The remaining results now follow immediately from the inequalities

$$\frac{\mathrm{d}^2}{\mathrm{d}K^2} J(K) \leqq 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}K^2} \|u(K)\|_2{}^2 \geqq 0 \; \blacksquare$$

Making use of the spectral representation (19), (20) and the equations

$$\|R^T r\|_2{}^2 = \int_0^\infty d\, \|R_\beta{}^T r\|_2{}^2, \quad \|L^T r\|_2{}^2 = \int_0^\infty \beta^2 d\, \|R_\beta{}^T r\|_2{}^2, \quad \|LL^T r\|_1{}^2 = \int_0^\infty \beta^4 d\, \|R_\beta{}^T r\|_2{}^2 \tag{21}$$

we are able to derive bounds for $\|u(K)\|_2$ and $J(K)$.

Theorem 2: *Let be $r \in D(L^T)$ and $L^T r \neq 0$. Then the inequality*

$$J(K) \geqq \|r\|_1{}^2 - \|L^T r\|_2{}^4/(K + \|L^T r\|_2{}^2/\|R^T r\|_2{}^2) \tag{22}$$

is fulfilled.

Proof: Employing *Jensen's inequality* (see [8, 12]) for $f(x) = x/(K + x)$ we obtain

$$f\left(\int_0^\infty \beta^2 d\, \|R_\beta{}^T r\|_2{}^2 \Big/ \int_0^\infty d\, \|R_\beta{}^T r\|_2{}^2\right) \geqq \int_0^\infty f(\beta^2)\, d\, \|R_\beta{}^T r\|_2{}^2 \Big/ \int_0^\infty d\, \|R_\beta{}^T r\|_2{}^2. \tag{23}$$

From (21) and (23) now follows

$$\|L^T r\|_2^2/(K + \|L^T r\|_2^2/\|R^T r\|_2^2) \geqq \int_0^\infty \beta^2/(K + \beta^2)\, d\, \|R_\beta{}^T r\|_2^2 \quad \blacksquare$$

Theorem 3: *For all $r \in H_2$ and $K > 0$ the inequality*

$$\|u(K)\|_2 \leqq \|R^T r\|_2/2K^{1/2} \tag{24}$$

is fulfilled. If $r \in D(L^T)$ and $K \geqq \|L^T r\|_2^2/\|R^T r\|_2^2$ then the better estimation

$$\|u(K)\|_2 \leqq \|L^T r\|_2/(K + \|L^T r\|_2^2/\|R^T r\|_2^2) \tag{25}$$

follows. Furthermore, for $r \in D(LL^T)$, $L^T r \neq 0$ and $K > 0$ the inequality

$$\|u(K)\|_2 \geqq \|L^T r\|_2/(K + \|LL^T r\|_1^2/\|L^T r\|_2^2) \tag{26}$$

is fulfilled.

Proof: Obviously, the inequality $2s^2/(K + s)^3 + (K - s)\, x/(K + s)^3 \geqq x/(K + x)^2$ is valid for $s \in [0, K]$ and $x \in [0, \infty)$. Setting $x = \beta^2$ in this inequality and integrating with respect to $d\, \|R_\beta{}^T r\|_2^2$ we obtain

$$2s^2\, \|R^T r\|_2^2/(K + s)^3 + (K - s)\, \|L^T r\|_2^2/(K + s)^3 \geqq \|u(K)\|_2^2 \quad \forall\, s \in [0, K]. \tag{27}$$

For the minimum s^0 of the left hand side of the inequality (27) follows that

$$s^0 = \begin{cases} K & \text{if} \quad K \leqq \|L^T r\|_2^2/\|R^T r\|_2^2, \\ \|L^T r\|_2^2/\|R^T r\|_2^2 & \text{if} \quad K \geqq \|L^T r\|_2^2/\|R^T r\|_2^2. \end{cases}$$

This yields the estimations (24) and (25). The inequality (26) again follows by using *Jensen's inequality* for $f(x) = (K + x)^{-2}$: with

$$f\left(\int_0^\infty \beta^4 d\, \|R_\beta{}^T r\|_2^2 \Big/ \int_0^\infty \beta^2 d\, \|R_\beta{}^T r\|_2^2\right) \leqq \int_0^\infty f(\beta^2)\, \beta^2 d\, \|R_\beta{}^T r\|_2^2 \Big/ \int_0^\infty \beta^2 d\, \|R_\beta{}^T r\|_2^2$$

we obtain $(K + \|LL^T r\|_1^2/\|L^T r\|_2^2)^{-2} \leqq \|u(K)\|_2^2/\|L^T r\|_2^2$ $\blacksquare$

Remarks:
1. If the norm $\|R^T r\|_2$ is unknown, we can replace $\|R^T r\|_2$ in the inequalities (22), (24) and (25) by $\|r\|_1$ (because of $\|R^T r\|_2 \leqq \|r\|_1$) and obtain the bounds

$$J(K) \geqq \|r\|_1^2 - \|L^T r\|_2^2/(K + \|L^T r\|_2^2/\|r\|_1^2), \tag{22'}$$

$$\|u(K)\|_2 \leqq \begin{cases} \|r\|_1/2K^{1/2} & \text{if} \quad K \leqq \|L^T r\|_2^2/\|r\|_1^2 \quad \text{or} \quad r \notin D(L^T) \quad (24') \\ \|L^T r\|_2/(K + \|L^T r\|_2^2/\|r\|_1^2 & \text{if} \quad r \in D(L^T) \quad \text{and} \quad K \geqq \|L^T r\|_2^2/\|r\|_1^2. \quad (25') \end{cases}$$

We see that the bound (22′) is the same as that given in Theorem 1 and that the bounds (24′) and (25′) are better than the bounds (7) of Theorem 1.

2. If $r \in D(L^T)$ then for $K \to 0$ the estimation

$$\|L^+ r\|_2 \geqq \|L^T r\|_3^2/\|LL^T r\|_1^2 \quad \text{if} \quad L^T r \neq 0, \quad r \in D(LL^T) \cap D(L^+) \tag{28}$$

immediately follows from (26) for the pseudoinverse L^+.

3. Applying the fact that $u(K)$ is also a solution of problem (16), we can replace L by KL^{+T} in the estimations of theorem 3 and obtain

$$\|u(K)\|_2 \leqq \|L^+r\|_2/(1 + K\,\|L^+r\|_2^2/R^Tr\|_2^2 \quad (29)$$

if

$$r \in D(L^+), \quad K \leqq \|R^Tr\|_2^2/\|L^+r\|_2^2$$

and

$$\|L^+r\|_2/(1 + K\,\|L^{+T}L^+r\|_1^2/\|L^+r\|_2^2) \leqq \|u(K)\|_2 \quad (30)$$

if $K^+r \neq 0$, $K > 0$, $r \in D(L^{+T}L^+)$.

These estimations are better than the estimations given in Theorem 3 if K is sufficiently small.

4. Bang-bang control

A more detailed discussion of *bang-bang* controls is given in [6, 9, 11, 14, 15]. In the following we only derive, for the problem

$$J(u) = \|Lu - r\|_1^2 + K\,\|u\|_2^2 \doteq \underset{\|u\|_2 \leqq 1}{\text{Minimum}} \quad (31)$$

($L \in \mathscr{L}(H_2 \to H_1)$ — bounded), a sufficient condition for the *bang-bang-ness* of an optimal control employing the bounds given in Chapter 2 and 3.

Lemma 4: *Let $u^0(K)$ be an optimal control for the problem* (31) (*with $K > 0$ and $L^Tr \neq 0$*). *Then*

$$\|u^0(K)\|_2 = 1 \quad \text{if} \quad 1 \leqq \|L^Tr\|_2/(K + \|LL^Tr\|_1^2/\|L^Tr\|_2^2) \quad (32)$$

and

$$\|u^0(K)\|_2 < 1 \quad \text{if} \quad \text{Minimum}\,\{\|r\|_1/2K^{1/2},\ \|L^Tr\|_2/(K + \|L^Tr\|_2^2/\|r\|_1^2)\} < 1. \quad (33)$$

Proof: It follows immeditaely by applying of Theorem 3 ∎

Remarks:

1. The sufficient conditions of Lemma 4 are better than the conditions

$$\|u^0(K)\|_2 = 1 \quad \text{if} \quad (\|L\|^2 + K) \leqq \|L^Tr\|_2, \quad (34)$$

$$\|u^0(K)\|_2 < 1 \quad \text{if} \quad 2\,\|L^Tr\|_2 < K, \quad (35)$$

given in [6].

2. For the case $K = 0$ $\|u^0(0)\|_2 = 1$ is valid if $\|L^Tr\|_2 \geqq \|L\|^2$. For the case $L^T = \{0\}$ $\|u^0(0)\|_2 = 1$ is valid if $\|r\|_1 > \|L\|$.

Lemma 5: *Let be* $K_0 := \|L^Tr\|_2 - \dfrac{\|LL^Tr\|_1^2}{\|L^Tr\|_2^2} > 0$. *Then the problem* (31) *has the same solution u^0, with $\|u^0\|_2 = 1$, for all $K \in [0, K_0]$.*

Proof: Applying (32), we see that for the optimal control $u^0(K)$ it follows that $\|u^0(K)\|_2 = 1\ \forall K \in [0, K_0]$. With

$$\underset{\|u\|_2 \leqq 1}{\text{Minimum}}\,(\|Lu - r\|_1 + K\,\|u\|_2^2) = \|Lu^0(K) - r\|_1^2 + K$$

($\forall K \in [0, K_0]$) the lemma is proved ∎

5. Appendix

The spectral theory for closed linear mappings $L \in \mathscr{L}(H_2 \to H_1)$ (with $D(L)$ dense in H_2) used in this paper was developed by HESTENES [10] and exactly proved in [12]. In this theory it is shown that for a closed linear mapping L exactly one partial isometry $R \in \mathscr{L}(H_2 \to H_1)$ exists, such that $L = RL^TR$, L^TR is positive definite and $\operatorname{Ker} L = \operatorname{Ker} R$. Using this partial isometry, we can give for L the *spectral representation*

$$L = \int_0^\infty \beta \, dR_\beta,$$

which possesses the following properties (R_β — decomposition of R):

a) $\int_0^\infty f(\beta) \, dR_\beta$ is a closed linear mapping with a dense domain, if f is a Borel-measurable function;

b) $\left(\int_0^\infty f(\beta) \, dF_\beta\right)^T = \int_0^\infty f(\beta) \, dR_\beta^T$;

c) $\left(\int_0^\infty f(\beta) \, dR_\beta\right)^+ = \int_0^\infty f(\beta)^+ \, dR_\beta^T$ with $f(\beta)^+ = \begin{cases} 1/f(\beta) & \text{if } f(\beta) \neq 0, \\ 0 & \text{if } f(\beta) = 0; \end{cases}$

d) $\left(\int_0^\infty g(\beta) \, dR_\beta^T\right)\left(\int_0^\infty f(\beta) \, dR_\beta\right) \subset \int_0^\infty g(\beta) \, f(\beta) \, dR_\beta^T R_\beta$;

e) $\left(\int_0^\infty f(\beta) \, dR_\beta u, x\right) = \int_0^\infty f(\beta) \, d(R_\beta u, x) \quad \forall u \in D\left(\int_0^\infty f(\beta) \, dR_\beta\right), \ \forall x \in H_2$;

f) $\left\|\int_0^\infty f(\beta) \, dR_\beta u\right\|^2 = \int_0^\infty f(\beta)^2 \, d \, \|R_\beta u\|^2, \ \forall u \in D\left(\int_0^\infty f(\beta) \, dR_\beta\right)$;

g) $\int_0^\infty 1 \, dR_\beta = R$.

The *pseudoinverse* (*generalized inverse*) L^+ of the linear operator L may be defined in many different ways. We have chosen the Penrose definition, which in our notation may be phrased as follows (see [13]):

If $L \in \mathscr{L}(H_1 \to H_1)$ has closed range, then L^+ is the unique operator in $\mathscr{L}(H_1 \to H_2)$ satisfying

$$LL^+ = (LL^+)^T, \quad L^+L = (L^+L)^T, \quad LL^+L = L, \quad L^+LL^+ = L^+.$$

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