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On Entropy-Like Invariants for Dynamical Systems

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In der Arbeit wird eine Methode aus der Theorie der Ordnungsstruktur der Zustände auf die der Kolmogorov-Sinai-Entropie übertragen. Die Funktion $h(x) = -x \log x$ wird in allen Definitionen der Entropietheorie durch beliebige beschränkte und konkave Funktionen ersetzt. Dieses Verfahren führt zu einer Klasse von Isomorphie-Invarianten, wobei die dynamische Entropie verallgemeinert wird. Ihre allgemeinen Eigenschaften werden untersucht, und in einigen einfachen Fällen ist die explizite Berechnung der neuen Invarianten ausgeführt.

Основной метод теории частичного упорядочения состояний переносится в теорию энтропии Колмогорова-Синай. При этом во всех соотношениях теории функция $h(x) = -x \log x$ заменяется произвольной ограниченной выпуклой функцией. Эта процедура приводит к возникновению класса автоморфизмов пространств Лебега, обобщая, таким образом, понятие динамической энтропии. В работе исследуются общие свойства этих обобщенных энтропий; в некоторых простых случаях эти инварианты вычислены в явном виде.

This paper transfers the theory of the Kolmogorov-Sinai-entropy a method which is a basic tool in the theory of the order-structure of states. The function $h(x) = -x \log x$ is replaced by arbitrary bounded, concave functions in all definitions of the entropy-theory. This procedure leads to a class of isomorphy invariants, thus generalizing the notion of dynamical entropy. The general properties of the generalized dynamical entropies are investigated and an explicit calculation of the new invariants is accomplished on some simple cases.

0. Introduction

In 1958 Kolmogorov introduced the notion of the dynamical entropy into the ergodic theory of dynamical systems [3]. He showed, with the aid of the entropy, that there are nonisomorphic Bernoulli shifts. In 1969 Ornstein solved the isomorphy problem for the class of all Bernoulli shifts by showing that all Bernoulli shifts with the same entropy are isomorphic [8, 9]. The isomorphy question for K-systems, however, is unsolved at present [10]. Kouchnirenko constructed generalizations of the K-entropy, called "sequence entropies". It was shown by Newton that they give new information about the isomorphy of transformations only in the zero-entropy case [7]. Versik [15] introduced the notion of the scale of a transformation, and Juzvinskij [2] proved that for any positive entropy there are countably many subclasses of K-systems with pairwise distinct scales.

All the isomorphy invariants listed above, and a large number of others, are successfully used in the ergodic theory, but all the invariants we know cannot completely solve the isomorphy question for the class of systems with positive entropy, especially for K-systems. Therefore new invariants are needed.

1. Content of this paper

We construct simple generalizations of entropy using the following idea. The K-entropy $H(T)$ is defined as a supremum over all finite partitions $\underline{C}$ of the relative entropies $H(\underline{C}/T)$ of the transformation T with respect to $\underline{C}$. ($H(\underline{C}/T)$ is called in this paper "entropy of the process $(\underline{C}/T)$".) The entropy of a process is well defined, because it is the limit of the entropy of the partitions $\underline{C} \vee T^{-1}\underline{C} \vee \cdots \vee T^{-1}\underline{C}$ divided by n, which always exists. The reason for the existence of this limit lies in the subadditivity of the entropy of partitions, i.e. $H(\underline{C} \vee \underline{D}) \leqq H(\underline{C}) + H(\underline{D})$. The subadditivity is a consequence of special properties of the function $h(x) = -x \log x$ used in the definition of the entropy [1]. It is well known [1, 12] that

$$H(\underline{C}/T) := \lim_n \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\underline{C}\right) = \lim_n H\left(\underline{C} \Big/ \bigvee_{i=1}^{n} T^{-i}\underline{C}\right). \tag{1.1}$$

The existence of the right-hand side of (1.1), however, is guaranteed only by the concavity and boundedness of the function $h(x)$ (and $h(0) = 0$). Consequently, if we replace $h(x)$ by any arbitry concave, bounded function g of the closed unit interval with $g(0) = 0$ and then repeat all constructions of the entropy-theory starting with $H(\underline{C}/T) = \lim H\left(\underline{C} \Big/ \bigvee_{i=1}^{n} T^{-i}\underline{C}\right)$, we get a large class of new isomorphy invariants.

In this paper we give the construction and some properties of these new invariants. We compute them explicitly for the cases of zero and infinite entropy ($H(T) = 0 \Rightarrow G(T) = g(1)$ for all g, $H(T) = \infty \Rightarrow G(T) = \lim_{x \downarrow 0} g'(x)$ for all continuous g, if the limit exists). These extreme cases cannot give new information, but it is shown that or the case of finite, positive entropy

$$g(1) < G(T) < \lim_{x \downarrow 0} g'(x). \tag{1.2}$$

We show, for a special class of concave functions, that $G(T) < \lim_{0 \downarrow x} g'(x)$ provided the entropy is not too large. All these statements depend on the entropy, but the results available at present lead us to hope that new information for the solution of the isomorphy problem of the dynamical systems will be obtained from the invariants constructed in this paper.

2. Basic notations and definitions (see [1, 11, 16])

Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space, where $\mathfrak{B}$ is the σ-algebra of all measurable sets, and μ is a probability measure on $(X, \mathfrak{B})$. A mapping $T: X \mapsto X$, measure-preserving and one-to-one a.e. is called an *automorphism* of $(X, \mathfrak{B}, \mu)$. An aggregate $(X, \mathfrak{B}, \mu, T)$ with T being an automorphism of the Lebesgue space $(X, \mathfrak{B}, \mu)$ will be referred to as a *dynamical system*.

Definition 1:

1. Dynamical systems $(X', \mathfrak{B}', \mu', T')$ and $(X, \mathfrak{B}, \mu, T)$ are *isomorphic* iff there is a mapping $I: X' \mapsto X$ measure-preserving and one-to-one a.e. (i.e. I is a measure space isomorphism) such that $IT' = TI$ a.e.

2. If I is not a measure space isomorphism but a measure space homomorphism such that $IT' = TI$ a.e., then $(X, \mathfrak{B}, \mu, T)$ is said to be a *factor* of $(X', \mathfrak{B}', \mu', T')$.

3. A *dynamical invariant* is a property of dynamical systems which is invariant under isomorphisms of dynamical systems.

The following definitions are concerned with partitions of $(X, \mathfrak{B}, \mu)$ and sub-σ-algebras of $\mathfrak{B}$. All relations between partitions and σ-algebras are understood to hold only up to measure zero. We write $\underline{C} \vee \underline{D}$ for the common refinement of the partitions $\underline{C}$ and $\underline{D}$. For a finite family $\{\underline{C}^i\}_{i=n}^m$ the symbol $\bigvee_{i=n}^m \underline{C}^i$ denotes the partition which is the common refinement of all the partitions $\underline{C}^i$. If $n = -\infty$ or $m = \infty$, then $\bigvee_{i=n}^m \underline{C}^i$ is the smallest σ-algebra containing all the listed partitions. If $\underline{C} \vee \underline{D} = \underline{D}$, we write $\underline{C} \leqq \underline{D}$. Analogously, if $\{\mathfrak{A}_i\}_{i=n}^m$ is a family of sub-σ-algebras, we denote with $\bigvee_{i=1}^m \mathfrak{A}_i$ the smallest sub-σ-algebra of $\mathfrak{B}$ containing all $\mathfrak{A}_i$.

Let $\mathfrak{A}$ be a sub-σ-algebra of $\mathfrak{B}$ and $\underline{C}$ a partition. We write $\sigma(\underline{C}) = \mathfrak{A}$ iff $\underline{C}$ generates $\mathfrak{A}$. The one-to-one correspondence between the set of finite partitions and the set of finite subalgebras contained in the relation $\sigma(\underline{C}) = \mathfrak{A}$ is freely used throughout this paper.

Definition 2: Let g: $[0, 1] \to R$ be a real, bounded, continuous, concave function of the closed unit interval, and let $g(0) = 0$. Let further $(X, \mathfrak{B}, \mu)$ and $\underline{C}$ be a Lebesgue space and a finite partition, respectively. We define

1. $$G(\underline{C}) := \sum_i g\big(\mu(C_i)\big), \qquad \underline{C}(C_1, \ldots, C_n). \tag{2.1}$$

2. For any measurable set $A \in \mathfrak{B}$, $\underline{C}/A := (C_1 \cap A, \ldots, C_n \cap A)$ is a partition of A induced by $\underline{C}$. The measure $\mu(.)$ induces a probability measure $\mu(\cdot/A)$ on A (A being a set of positive measure) by

$$\mu(B/A) := \frac{\mu(B \cap A)}{\mu(A)} \quad \forall B \in \mathfrak{B}. \tag{2.2}$$

We define

$$G(\underline{C}/A) := \sum_i g\big(\mu(C_i/A)\big) \tag{2.3}$$

and

$$G(\underline{C}/\underline{D}) := \sum_j \mu(D_j)\, G(\underline{C}/D_j) \tag{2.4}$$

where $\underline{D} = (D_1, \ldots, D_m)$ is a second finite partition of X. (See Remark 1.)

3. Let $\mathfrak{A} \subset \mathfrak{B}$ be a sub-σ-algebra of $\mathfrak{B}$.

$$G(\underline{C}/\mathfrak{A}) := \inf_{\underline{D}} G(\underline{C}/\underline{D}), \tag{2.5}$$

$\underline{D}$ runs over all finite partitions with elements in $\mathfrak{A}$.

Remarks:

1. We use the following convention. If $A \in \mathfrak{B}$ is a set of measure zero, then we set $\mu(A)\, g\big(\mu(B/A)\big) = 0 \ \forall B \in \mathfrak{B}$. This is no further restriction on g because $\mu(B \cap A) \leqq \mu(A)$, and g is bounded on $[0, 1]$. With this convention we have a correct definition in equ. 2.4.

2. If we take the function $h(x) = \begin{cases} -x \log x, & \partial \in (0, 1] \\ 0, & x = 0 \end{cases}$ which is continuous, bounded, and concave we get the definitions of the entropy theory. $H(\underline{C}) = \sum h\big(\mu(C_i)\big)$ is the entropy of the partition $\underline{C}$. $H(\underline{C}/\underline{D})$ and $H(\underline{C}/\mathfrak{A})$ are the relative entropies of the partition $\underline{C}$ with respect to the partition $\underline{D}$ and the sub-σ-algebra $\mathfrak{A}$, respectively.

3. Generalizations of the K-entropy

We want to construct new dynamical invariants along the lines of the entropy theory. To this end we need the following statements on generalized relative entropies.

Proposition 1: *Let* $\underline{C}$, $\underline{C}^1$, $\underline{C}^2$, $\underline{D}$, $\underline{D}^1$, $\underline{D}^2$ *be finite partitions and let* $G(\cdot/\cdot)$ *be the functionals of Definition* 2.2. *Then*

1. $G(\underline{C}/\underline{D}) \geqq g(1)$; (3.1)
2. $\underline{C}^1 \leqq \underline{C}^2 \Rightarrow G(\underline{C}^1/\underline{D}) \leqq G(\underline{C}^2/\underline{D})$; (3.2)
3. $\underline{D}^1 \leqq \underline{D}^2 \Rightarrow G(\underline{C}/\underline{D}^1) \geqq G(\underline{C}/\underline{D}^2)$; (3.3)
4. $\underline{C} \leqq \underline{D} \Rightarrow G(\underline{C}/\underline{D}) = g(1)$; (3.4)
5. *If* g *is strongly concave,* $\underline{C} \leqq \underline{D} \Leftrightarrow G(\underline{C}/\underline{D}) = g(1)$; (3.5)
6. $G(\underline{C}/\underline{D}) \leqq G(\underline{C})$; (3.6)
7. $G(\underline{C}/\underline{D}) \leqq \lim_{x \downarrow 0} g'(x)$ (*if the limit exists*). (3.7)

Proof: We use the results of Lemma A. 1 (Appendix).

1. $$G(\underline{C}/D_j) = \sum_i g\big(\mu(C_i/D_j)\big) \geqq g\left(\sum_i \mu(C_i/D_j)\right) = g(1). \tag{3.8}$$

2. For each i, $C_i^1 = \bigcup_{l_i} C_{l_i}^2$, and therefore

$$g\big(\mu(C_i^1/D_j)\big) = g\left(\sum_{l_i} \mu(C_{l_i}^2 D_j)\right) \leqq \sum_{l_i} g\big(\mu(C_{l_i}^2 D_j)\big). \tag{3.9}$$

3. For each j, $D_j^1 = \bigcup_{l_j} D_{l_j}^2$, and therefore

$$\mu(D_j^1)\, g\left(\frac{\mu(C_i \cap D_j^1)}{\mu(D_j^1)}\right) = \left(\sum_{l_j} \mu(D_{l_j}^2)\right) g\left(\sum_{l_j} \mu\,(C_i \cap D_{l_j}^2) \Big/ \sum_{l_j} \mu(D_{l_j}^2)\right)$$
$$\geqq \sum_{l_j} \mu(D_{l_j}^2)\, g\big(\mu(C_i \cap D_{l_j}^2)/\mu(D_{l_j}^2)\big). \tag{3.10}$$

4. For any j, there is one and only one i such that $D_j \subset C_i$. Therefore $\mu(C_i/D_j) = 0$ for all but one $i = \hat{\imath}$, and $\mu(C_{\hat{\imath}}/D_j) = 1$. This leads to

$$G(\underline{C}/D_j) = g(1) \qquad \forall\, j. \tag{3.11}$$

5. Because of A.1.2, we have $G(\underline{C}/D_j) = g(1)$ if and only if $\mu(C_i/D_j) = 0$ for all but one $i = \hat{\imath}$ and $\mu(C/D_j) = 1$. This is equivalent to $\underline{C} \leqq \underline{D}$ (up to measure zero).
6., 7. are obvious ∎

Proposition 2: *Let* $\underline{C}$ *be a finite partition and* $\{\mathfrak{A}_n\}_1^\infty$ *be an increasing sequence of sub-*σ*-algebras of* $\mathfrak{B}$ (*i.e.* $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}\ \forall\, n$). *If* $\sigma(\underline{C}) \subseteq \bigvee_{n=1}^{\infty} \mathfrak{A}_n$ *then* $C(\underline{C}/\mathfrak{A}_n) \xrightarrow{n\to\infty} g(1)$.

Proposition 2 is a modified version of Corollary 4.8 of [16]. A sketch of the proof will be given in the Appendix (for details see [11]).

Definition 3: Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system. If $\underline{C}$ is a partition of X we call the pair $(\underline{C}/T)$ a *process* in $(X, \mathfrak{B}, \mu, T)$. Let $(\underline{C}/T)$ be a process with $\underline{C}$ being

a finite partition. We define

$$G(\underline{C}/T) := \lim_n G\left(\underline{C}\Big/\bigvee_{i=1}^{n} T^{-i}\underline{C}\right) \tag{3.12}$$

for all functionals defined as in Def. 2.2.

Proposition 3: *The limit in* Equ. 3.12 *exists for all processes* $(\underline{C}/T)$.

Proof:

$$\bigvee_{i=1}^{n} T^{-i}\underline{C} = T^{-n}\underline{C} \vee \left[\bigvee_{i=1}^{n-1} T^{-i}\underline{C}\right] \geqq \bigvee_{i=1}^{n-1} T^{-i}\underline{C}.$$

Therefore we get $G\left(\underline{C}\Big/\bigvee_{i=1}^{n} T^{-i}\underline{C}\right) \leqq G\left(\underline{C}\Big/\bigvee_{i=1}^{n-1} T^{-i}\underline{C}\right)$. The sequence of the relative entropies is monotone decreasing and bounded from below by $g(1)$ ∎

Definition 4: Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system. We define for all functionals, according to Def. 3

$$G(T) := \sup_{\underline{C}\in\mathfrak{A}} G(\underline{C}/T) \tag{3.13}$$

where $\mathfrak{A}$ denotes the set of partitions $\underline{C}$ measurable $\mathfrak{B}$.

Theorem 4: *All* $G(T)$ *in Def.* 4 *are dynamical invariants.*

Proof: The theorem is clear from the definitions, because all mappings involved are measure preserving and $1-1$ a.e., and sets of measure zero can be neglected because of $g(0) = 0$ ∎

Remarks:

1. Actually, the theorem holds if we use arbitrary real functions in the Definitions 2, 3, 4. The point is, however, that for a bounded concave function g the definition of $G(T)$ makes sense. Only in this case can we be sure to have a supremum over well-defined objects (Proposition 3).

2. Dynamical entropy (K-entropy) is a special case of Def. 4. Therefore the $G(T)$ are called *generalized* dynamical entropies.

In the remainder of the paper we derive some properties of the new dynamical invariants. The general properties if the invariants are the content of this section, but in the next section we deal with a special class of concave functions.

Theorem 5: *Let* $(X', \mathfrak{B}', \mu', T')$ *be a factor of* $(X, \mathfrak{B}, \mu, T)$, *then we have for all invariants* G

$$G(T') \leqq G(T). \tag{3.14}$$

Proof: The transformation T' is isomorphic to T restricted to a T-invariant sub-σ-algebra $\mathfrak{A}_T \subset \mathfrak{B}$. Therefore

$$G(T') = G(T|_{\mathfrak{A}_T}) = \sup_{\underline{D}\subset\mathfrak{A}_T} G(\underline{D}/T|_{\mathfrak{A}_T}) = \sup_{\underline{D}\subset\mathfrak{A}_T} G(\underline{D}/T) \leqq \sup_{\underline{C}\subset\mathfrak{B}} G(\underline{C}/T)$$ ∎

Corollary 6: *If* $(X', \mathfrak{B}', \mu', T')$ *and* $(X, \mathfrak{B}, \mu, T)$ *are weakly isomorphic (i.e. each system is a factor of the other) then* $G(T') = G(T)$ *for all invariants* G.

A partition $\underline{C}$ is said to be a *generator* for T iff $\bigvee_{-\infty}^{\infty} T^{-i}\underline{C} = \mathfrak{B}$ (up to measure zero). Kolmogorov's theorem which says that for any generator $\mathbf{C}$ $H(\mathbf{C}/T) = H(T)$ does not generally hold for the G's. We have instead:

Corollary 7: *Suppose T has finite generators. Then $G(T) = \sup_{\tilde{\underline{C}}} G(\tilde{\underline{C}}/T)$, where $\tilde{\underline{C}}$ runs over all finite generators of T.*

Proof: Let $\underline{C}$ be not a generator. Then $\bigvee_{-\infty}^{\infty} T^{-i}\underline{C} = \mathfrak{A} \subset \mathfrak{B}$ and $T\mathfrak{A} = \mathfrak{A}$. Therefore $T|_{\mathfrak{A}}$ is a factor of T.

$$G(\underline{C}/T) \leqq \sup_{\underline{D}\subset\mathfrak{A}} G(\underline{D}/T|_{\mathfrak{A}}) = G(T|_{\mathfrak{A}}) \leqq G(T) \quad \blacksquare$$

Remark: The existence of finite generators is guaranteed for ergodic automorphisms of finite entropy [5].

The aim of the following statements is to compute the new invariants explicitly or to give estimations of them. To do this we need the following lemmas.

Lemma 8: *Let $\underline{C}$ be a partition such that $\bigvee_{i=0}^{\infty} T^{-i}\underline{C} = \mathfrak{B}$. Then $C(\underline{C}/T) = g(1)$ for all functionals defined in* Def. 3.

Proof: We use Proposition 2. Let $\mathfrak{A}_n = \sigma\left(\bigvee_{i=1}^{n} T^{-i}\underline{C}\right)$. $\{\mathfrak{A}_n\}$ is an increasing sequence of sub-σ-algebras.

$$\bigvee_{i=1}^{\infty} T^{-i}\underline{C} = T^{-1}\bigvee_{i=0}^{\infty} T^{-i}\underline{C} = T^{-1}\mathfrak{B} = \mathfrak{B}.$$

Consequently, because of $\sigma(\underline{C}) \subset \mathfrak{B}$ we have $\sigma(\underline{C}) \subset \mathfrak{B} = \bigvee_{i=1}^{\infty} T^{-i}\underline{C} = \bigvee_{i=1}^{\infty} \mathfrak{A}_n$. (up to measure zero). Therefore Prop. 2 holds $\blacksquare$

Definition 5: We say that a functional $G(\underline{C}/\mathfrak{A})$ (cf. Def. 2.3) has the *martingale-property* iff for all increasing sequences $\{\mathfrak{A}_n\}$ of sub-σ-algebras of $\mathfrak{B}$ and for all finite partitions $\underline{C}$

$$\lim_n G(\underline{C}/\mathfrak{A}_n) = G\left(\underline{C}\Big/\bigvee_n \mathfrak{A}_n\right). \tag{3.15}$$

Lemma 9: *Let g be strongly concave, and assume the functional $G(\underline{C}/\mathfrak{A})$ constructed with the function g has the martingale-property. Then*

$$G(\underline{C}/T) = g(1) \Leftrightarrow \sigma(\underline{C}) \subset \bigvee_{i=1}^{\infty} T^{-i}\underline{C}. \tag{3.16}$$

Proof: We have $G(\underline{C}/T) = G\left(\underline{C}\Big/\bigvee_{i=1}^{\infty} T^{-i}\underline{C}\right)$ from the martingale-property. Prop. 1.5 leads to

$$G(\underline{C}/T) = g(1) \Rightarrow \underline{C} \subset \bigvee_{i=1}^{\infty} T^{-i}\underline{C}.$$

The inverse conclusion follows from Prop. 2 $\blacksquare$

Corollary 10: *For all dynamical systems $(X, \mathfrak{B}, \mu, T)$ with entropy $H(T) = 0$ and for all generalized dynamical entropies $G(T) = g(1)$ holds.*

Proof: The relative entropy has the martingale-property (see e.g. [13], Th. 4.28). The remainder of the proof follows from Prop. 2 $\blacksquare$

The suppositions of Lemma 8 can be shown to hold for any finite partition in the case of transformations with discrete spectrum. So we find $G(T) = g(1)$ without explicit use of the entropy for these systems.

Corollary 10 says that all the new invariants give the same information in the case of zero entropy, i.e. they are trivial if the entropy is given to be zero.

From the point of view of the entropy, the opposites of the dynamical systems with $H(T) = 0$ are the Bernoulli shifts. The following proposition gives an estimation for the generalized entropies for Bernoulli shifts. A *Bernoulli shift* is defined as a shift on $X = \{1, \ldots, n\}^Z$ (Z being the set of all integers) with a measure μ given by a product measure on the cylinder sets (see e.g. [16]).

Proposition 11: *Let T be the $(q_1 \ldots q_n)$-Bernoulli shift with the entropy $H(T) = \sum_{i=1}^{n} h(q_i) = s$. For all invariants G*

$$X_g(s) := \sup_{\{p_i\} \in \mathscr{S}} \sum g(p_i) \leqq G(T) \leqq \lim_{x \downarrow 0} g'(x) \tag{3.17}$$

holds. $\mathscr{S}$ denotes the set of all probability vectors $\{p_i\}$ with $\sum h(p_i) = s$.

Proof: The upper bound is clear from Prop. 1.7. The partition $\underline{C} = (A_0^1, \ldots, A_0^n)$ (A_0^i is the cylinder set of all elements of X which have the value i on the 0-th place. $\mu(A_0^i) = q_i$ by def.) is a generator for T, and $\underline{C}$ is independent of $\underline{D}^n = \bigvee_{i=1}^{n} T^{-i}\underline{C}$ for all n. Therefore

$$G(\underline{C}/T) = \lim_n G(\underline{C}/\underline{D}^n) = \sum_j \mu(D_j^n)\, G(\underline{C}/D_j^n) = \sum_j \mu(D_j^n)\, G(\underline{C}) = \sum g(q_i).$$

According to Ornstein's theorem [8], we find for any probability vector $\{p_j\} \in \mathscr{S}$ an independent generator $\tilde{\underline{C}}$ for T consisting of sets $\tilde{C}_j$ with $\mu(\tilde{C}_j) = p_j$. The proposition now follows from the definition of $G(T)$ ∎

Corollary 12: *Let T be an ergodic automorphism of positive entropy $H(T) > 0$. Then the Inequalities* 3.17 *hold. Moreover, if g is not identically zero then $X_g(H(T)) > g(1)$.*

Proof: The first assertion is a simple consequence of Sinai's weak-isomorphism theorem [12] and Theorem 5. The second is obvious ∎

We have computed an explicit (but trivial) result for the zero entropy case and an estimation (depending on the entropy) for the case of positive entropy. Now we are going to calculate the new invariants for ergodic automorphisms of infinite entropy. This again leads to a trivial result.

Corollary 13: *Suppose T is ergodic and let $H(T) = \infty$. Then for all generalized dynamical entropies $G(T) = \lim_{x \downarrow 0} g'(x)$ holds provided the limit exists.*

Proof: We consider the following sequence of Bernoulli shifts. T_n is the $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$-Bernoulli shift. Then, T_n is a factor of T_m iff $n \leqq m$, and all the T_n ($n = 1, 2, \ldots$) are factors of T. Therefore we have

$$\sup_{\{p_i\} \in \mathscr{S}_n} \sum g(p_i) \leqq G(T_n) \leqq G(T) \leqq \lim_{x \downarrow 0} g'(x). \tag{3.18}$$

Here $\mathscr{S}_n$ denotes the set of all probability vectors with $\sum h(p_i) = \log n$. But $q = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \in \mathscr{S}_n$ and therefore $\sum g(q_i) = ng\left(\frac{1}{n}\right) \leqq G(T)$. Now $\lim_n n\left[g\left(\frac{1}{n}\right) - g(0)\right] = g'(0)$ and $g(0) = 0$ complete the proof ∎

At the end of this section we formulate the obvious

Proposition 14: *All generalized dynamical entropies G are monotone increasing functions of the entropy on the class of all Bernoulli shifts.*

Proof: We get the desired result if we combine Ornstein's [8, 9] and Sinai's [12] isomorphism theorems and Theorem 5 ∎

This result is not surprising, because the entropy completely determines the isomorphy classes of Bernoulli shifts and is itself contained in the new family of isomorphy invariants. Proposition 14 reflects the fact that not more information on the isomorphy of Bernoulli shifts is to be expected if we know the entropy. The monotonicity results from the fact that all G-invariants are defined in the same manner as a supremum over partitions.

4. A special class of invariants

The new dynamical invariants constructed in the previous section are very hard to compute in nontrivial cases because there is no analogue of Kolmogorov's theorem. To find further nontrivial properties we consider a special family of concave functions having a simple structure. We define

$$g_r(x) = \begin{cases} x & \text{if } 0 \leqq x \leqq r \\ r & \text{if } x > r \end{cases} \qquad (0 < r < 1) \tag{4.1}$$

and denote all functionals associated with g_r by G_r. In a forthcoming paper [11] we will use the G_r to construct an example showing that in 3.17 equality doesn't hold in general even for the case of Bernoulli shifts. Here we only want to answer the question of whether there are dynamical systems with $G_r(T) < 1 = g'(0)$ or not, thus proving the existence of systems with $G_r(T)$ between the trivial values $g(1) = r$ and $g'(0) = 1$.

Proposition 15: *Let $(X, \mathfrak{B}, \mu, T)$ be an ergodic dynamical system with entropy $0 < H(T) = s < \infty$ and assume $r < \mathrm{e}^{-s}$. Then $G_r(T) < 1$.*

Remark: The logarithm in the function $h(x) = -x \log x$ is to the base e. If another basis b is used then Proposition 15 holds if $r < b^{-s}$.

Proof of Prop. 15: $G_r(T) = \sup_{\underline{C}} G_r(\underline{C}/T)$. We can restrict ourselves to processes $(\underline{C}/T)$ with $H(\underline{C}/T) = s$. For $r < \mathrm{e}^{-s}$ there exist real numbers $\varepsilon, \delta, \gamma > 0$ such that $r = \mathrm{e}^{-(s+\varepsilon)} - \gamma$ and $(s + \varepsilon)(1 - \delta) > s$. We denote $\underline{D}^n = \bigvee_{i=1}^{n} T^{-i}\underline{C}$. Because of $H(\underline{C}/\underline{D}^n) \downarrow s$ there is a $n_0(\underline{C}, \varepsilon, \delta)$ such that $\forall\, n > n_0$

$$s \leqq H(\underline{C}/\underline{D}^n) < (s + \varepsilon)(1 - \delta). \tag{4.2}$$

Assume $n > n_0$. We show that in the partition $\underline{C} \vee \underline{D}^n$ there are some elements $C_i \cap D_j^n$ the union of which has a measure greater than δ and which have the property

$$\mu(C_i \cap D_j^n) > \mathrm{e}^{-(s+\varepsilon)}\mu(D_j^n). \tag{4.3}$$

The elements of $\underline{C} \vee \underline{D}^n$ not fulfilling (4.3) are denoted by $C_k \cap D_l^n$.

Assume $\sum_{i,j} \mu(C_i \cap D_j^n) \leqq \delta$. Then we have for the sets $C_k \cap D_l^n$, because of $\mu(C_k \cap D_l^n) \leqq e^{-(s+\varepsilon)} \mu(D_l^n)$, $-\log \mu(C_k/D_l^n) \geqq s + \varepsilon$. This in turn leads to $-\sum_{\substack{k \neq i \\ k \neq j}} \mu(C_k \cap D_l^n) \log \mu(C_k/D_l^n) \geqq (s + \varepsilon)(1 - \delta)$. By the positivity of $h(x)$ and the definition of the relative entropy we get

$$H(\underline{C}/\underline{D}^n) \geqq (s + \varepsilon)(1 - \delta).$$

This contradicts (4.2). Therefore $\sum_{i,j} \mu(C_i \cap D_j^n) > \delta$. From $r < e^{-(s+\varepsilon)}$ we get $g_r(\mu(C_i/D_j^n)) = r$ and $g_r(\mu(C_k/D_l^n)) \leqq \mu(C_k/D_l^n)$. Now $r = e^{-(s+\varepsilon)} - \gamma$ leads to

$$\begin{aligned} G_r(\underline{C}/\underline{D}^n) &\leqq \sum_{k,l} \mu(C_k \cap D_l) + \sum_{i,j} \mu(D_j^n)\,[\mu(C_i/D_j^n) - \gamma] \\ &= 1 - \gamma \sum_{i,j} \mu(D_j^n) < 1 - \gamma\delta. \end{aligned}$$

Therefore $G_r(\underline{C}/T) < 1 - \gamma\delta < 1$ for all finite partitions $\underline{C}$, and the upper bound is independent of $\underline{C}$. So it also holds for the supremum ∎

5. Discussion

We have constructed a large class of dynamical invariants by generalizing the notion of K-entropy. The construction is based on the idea of replacing the function $h(x) = -x \log x$ by an arbitrary, concave, bounded function with $g(0) = 0$. This method stems from the theory of the order-structure of states [14], which has been used successfully in the analysis of the irreversible behaviour of physical systems.

The dynamical entropy can be computed for many systems. The theorem of Kolmogorov which is a basic tool for the computation of the entropy does not hold in general for the new invariants. Therefore the explicit calculation of the $G(T)$ seems to be a very difficult problem, and results are known only in some cases which are trivial from the point of view of the entropy theory. One sees, however, that any invariant constructed with a function g which is not identically zero is non-trivial, i.e. one can find dynamical systems $(X, \mathfrak{B}, \mu, T)$, $(X', \mathfrak{B}', \mu', T')$ such that $G(T) \neq G(T)'$.

Whether there are systems with equal entropy and $G(T) \neq G(T')$ for some g is unknown at present, although it seems that this should be true. This conjecture is sustained by the non-validity of an analogue of Kolmogorov's theorem on the entropy of a generating process. The new invariants are trivial for systems of zero entropy, but in this case there are invariants such as sequence entropy and support [6] the properties of which are not yet completely investigated and which are trivial for K-systems. So the dynamical invariants presented in this paper could become a useful supplement to the entropy theory, provided the difficulties of their explicit calculation can be overcome.

At the moment there is no hope finding general methods for the computation of all new invariants. Therefore, as a first step the properties of the class of invariants constructed with the functions g_r are considered. The G_r take values which are not trivial (i.e. there are systems T with $g(1) = r < G_r(T) < 1 = g'(0)$). Moreover, from the study of an example, we known that even in the Bernoullian case the supremum of $G_r(\underline{C}/T)$ over the independent generators can be smaller than $G_r(T)$ [11]. This

example, while illustrating the non-triviality of the new invariants, brings out the deep difficulties connected with the explicit computation of the generalized entropies.

We have already remarked that the construction of the G-invariants is based on an idea from the theory of the order-structure of states. This raises the question of whether there is a structure in the set of all automorphisms of a measure space which is induced by the generalized K-entropies. Theorem 5 gives a first hint, but more interesting is a study of the consequences of $G(T) \leqq G(T')$ for all g. It is expected that this will lead to a physical interpretation of the new invariants, which is still an open problem.

Appendix 1: Concave functions on [0, 1]

A function $g: [0, 1] \to R$ is called concave iff $\forall x, y, \lambda \in [0, 1]$

$$g(\lambda x + (1 - \lambda) y) \geqq \lambda g(x) + (1 - \lambda) g(y).$$

If the equality holds only for $x = y$ and (or) ($\lambda = 0$ or $\lambda = 1$), g is called a strongly concave function.

The following properties of concave functions are easily verified [11, 14].

Lemma A.1: *Let $g: [0, 1] \to R$ be concave and continuous, and let $g(0) = 0$.*

1. $g(x) + g(y) \geqq g(x + y) \quad \forall x, y \in [0, 1]$ *with* $x + y \leqq 1$.
2. *If g is strongly concave, $g(x) + g(y) = g(x + y)$ holds if and only if $x = 0$ and* (or) $y = 0$.
3. $g(x) \leqq x \cdot \lim_{y \downarrow 0} g'(y)$ *if the limit exists. g' denotes the first derivative of the function g.*
4. *Let (s_i), (t_i) be sequences of the same length, $s_i, t_i \geqq 0 \; \forall i$, $\sum s_i < \infty$. Then*

$$\left(\sum_i s_i\right) g\left(\sum t_i / \sum s_i\right) \geqq \sum_i s_i g(t_i/s_i).$$

Appendix 2: Proposition 2 (Sketch of the proof)

Proposition 2 is a generalization of Corollary 4.8 of [16]. This corollary holds for the relative entropy of a partition with respect to a sub-σ-algebra of $\mathfrak{B}$ and is widened to the functionals of Def. 2.3. The proof of the generalized version follows exactly the line of the statement cited in [16].

Lemma A.2: *Let $(X, \mathfrak{B}, \mu)$ and $\mathfrak{B}_0 \subset \mathfrak{B}$ be a probability space and an algebra, resp. Assume that $\mathfrak{B}_0$ generates the σ-algebra $\mathfrak{B}$. If $\underline{C}$ is a finite partition measurable $\mathfrak{B}$, then for all $\varepsilon > 0$ and for any continuous, bounded, concave function $g: [0, 1] \to R$ with $g(0) = 0$, there is a finite subalgebra $\mathfrak{D} \subset \mathfrak{B}_0$ such that*

i) $g(1) \leqq G(\underline{C}/\mathfrak{D}) < g(1) + \varepsilon$
ii) $g(1) \leqq G(\underline{D}/\mathfrak{A}) < g(1) + \varepsilon$
where $\mathfrak{A} = \sigma(\underline{C})$ and $\mathfrak{D} = \sigma(\underline{D})$.

For the proof we have only to remark that the lower bounds are trivial and that there is a $0 < \delta_0 < 1$ such that

i) $0 \leqq x < \delta_0 \Rightarrow -\frac{\varepsilon}{r} < g(x) < \frac{\varepsilon}{r}$;

ii) $1 - \delta_0 < x \leqq 1 \Rightarrow g(1) - \frac{\varepsilon}{r} < g(x) < g(1) + \frac{\varepsilon}{r}$.

r is the number of sets which are contained in the partition $\underline{C}$. This simple fact is the only additional argument to the proof of Th. 4.8 in [16], which is the entropy version of the lemma. Lemma A.2 leads directly to Prop. 2 by using the same arguments as in the cited Corollary 4.8.

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