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Solution of a degenerated elliptic equation of second order in an unbounded domain

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Es wird die Gleichung $\operatorname{div}(\varrho(x)(\nabla u + f(x))) = 0$ im $\mathbf{R}^N$ untersucht, wobei $\varrho \in L_1(\mathbf{R}^N)$ und $f \in L_2(\mathbf{R}^N, \varrho)$ vorgegebene glatte Funktionen sind. Die Gleichung entartet auf der glatten Fläche $\Gamma = \{\varrho(x) = 0\}$, an der sich $\varrho(x)$ wie eine Potenz von $\operatorname{dist}(x, \Gamma)$ verhält. Folgende Resultate werden bewiesen: 1. Existenz und Eindeutigkeit (bis auf additive Konstanten) einer Lösung mit $\int |\nabla u|^2 \varrho \, dx < \infty$; der Beweis benutzt eine Variationsmethode in einem Sobolewraum mit Gewicht; 2. Regularität der Lösung in der Nähe von Γ; 3. Konvergenz und Korrektheit eines numerischen Verfahrens (Differenzenverfahren); 4. Konvergenz einer Iterationsmethode zur Lösung des diskreten Problems.

Рассматривается уравнение $\operatorname{div}(\varrho(x)(\nabla u + f(x))) = 0$ в $\mathbf{R}^N$, где $\varrho \in L_1(\mathbf{R}^N)$ и $f \in L_2(\mathbf{R}^N, \varrho)$ — заданные гладкие функции. Уравнение вырождается на гладкой поверхности $\Gamma = \{\varrho(x) = 0\}$, у которой $\varrho(x)$ ведёт себя как некоторая степень $\operatorname{dist}(x, \Gamma)$. Получены следующие результаты: 1. Существование и единственность (до аддитивной постоянной) решения из класса $\int |\nabla u|^2 \varrho \, dx < \infty$; доказательство опирается на вариационный метод в некотором пространстве Соболева с весом; 2. Регулярность решения вблизи Γ; 3. Сходимость и корректность некоторого вычислительного метода (метод разностей); 4. Сходимость итерационного метода решения дискретной задачи.

The paper deals with the equation $\operatorname{div}(\varrho(x)(\nabla u + f(x))) = 0$ in $\mathbf{R}^N$, where $\varrho \in L_1(\mathbf{R}^N)$ and $f \in L_2(\mathbf{R}^N, \varrho)$ are given smooth functions. The equation degenerates on the smooth surface $\Gamma = \{\varrho(x) = 0\}$ where $\varrho(x)$ behaves like a power of $\operatorname{dist}(x, \Gamma)$. The following results are proved: 1. Existence and uniqueness (up to additive constants) of a solution with $\int |\nabla u|^2 \varrho \, dx < \infty$; the proof uses a variational method in a weighted Sobolev space; 2. Regularity of the solution near Γ; 3. Convergence and correctness of a numerical (difference) method; 4. Convergence of an iteration method to solve the discrete problem.

1. Introduction. Formulation of the problem

The subject matter of the present paper is the linear partial differential equation

$$\operatorname{div}\left[\varrho(x)\left(\nabla u + f(x)\right)\right] = 0 \tag{1}$$

in the whole Euclidean space $\mathbf{R}^N$ of indepedent variables[1]). The data ϱ and $f = (f_1, \ldots, f_N)$ are C^∞ functions[2]) in $\mathbf{R}^N$ with the following properties:

$$0 \leqq \varrho(x) \leqq \varrho_0, \qquad \int_{\mathbf{R}^N} \varrho(x) \, dx < \infty, \tag{2}$$

$$\int_{\mathbf{R}^N} |f|^2 \varrho(x) \, dx = \int_{\mathbf{R}^N} (f_1^2 + \cdots f_N^2) \varrho(x) \, dx < \infty. \tag{3}$$

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[1]) Scalar product and norm denoted by $x \cdot x$ and $|y|$.

[2]) In the sequel functions and function spaces are real.

Equation (1) arises if we consider a system of electrons (imaginating as a stationary continuous cloud around the nucleus) under the influence of an external magnetic field. In this case, ϱ is the charge density of the electrons and the vector function $\boldsymbol{f}$ is determedined by the magnetic field. Induction processes lead to a stationary motion of the electronic cloud with the velocity $\boldsymbol{v} = \nabla u + \boldsymbol{f}$ and the current density $\boldsymbol{j} = \varrho(\nabla u + \boldsymbol{f})$. This motion may be used for the calculation of some effects in NMR (*nuclear magnetic resonance*). For details see SCHMIEDEL [9], SALZER [8].

Note some specialities of Equation (1).

Remark 1.1: The density function ϱ may have zeros. In this points the equation degenerates: it is elliptic only for $\varrho > 0$. We suppose that

$$\Gamma = \{x \in \mathbf{R}^N \mid \varrho(x) = 0\} \qquad \text{is an } (N-1)\text{-dimensional manifold of the class } C^\infty, \tag{4}$$

the so called *knode surface*. Γ may be unbounded or disconnected, e.g. two parallel planes. By the continuity of ϱ, Γ is a closed set of Lebesgue measure zero and divides $\mathbf{R}^N$ in a finite or denumerable number of disjoint domains:

$$\Omega = \mathbf{R}^N \setminus \Gamma = \Omega \cup \Omega_2 \cup \cdots \cup \Omega_k \cup \cdots.$$

Remark 1.2: As the physical interpretation shows we shall not look to the properties of u but to those of the *gradient*. This concerns for example the behavior of the solution near the knode surface or the question of uniqueness.

Remark 1.3: We have no boundary conditions or conditions in the infinity and shall make the only restriction of *finite energy*:

$$\int_{\mathbf{R}^N} |\nabla u|^2 \varrho \, dx < \infty.$$

After these remarks we can formulate the

Primary problem: *Find a function $u \in C^2(\Omega)$ with finite energy and satisfying Eq. (1) in Ω.* (5)

Here and later we assume tacitly that the data are C^∞-smooth and satisfy (2), (3), (4). Note that Eq. (1) must be satisfied only in the points of positive density. We shall see that even in this large class we have a unique solution.

Later we shall need some assumptions about the behavior of ϱ near the knode surface. The most restrictive but in always all concrete examples fulfilled is the following one:

The density function has the form $\varrho(x) = \psi^2(x)$ with an arbitrary smooth function ψ and $\nabla\psi(x) \neq 0$ on Γ. (6)

This includes the behavior of ϱ like $d^2(x) = dist^2(x, \Gamma)$. More general we may suppose that

there is a $\varkappa \geqq 1$ such that

$$c_1 d^\varkappa(x) \leqq \varrho(x) \leqq c_2 d^\varkappa(x) \tag{7}$$

holds locally, in some neighbourhood of any point of Γ,

or that

$$\varrho(x) \leqq c \cdot d(x) \quad \textit{locally near } \Gamma. \tag{8}$$

Summary of results: It will be shown that the primary problem always has a solution and that this solution is unique up to additive constants. The proof use a

variational method in a Sobolev space with weight. The behavior of the solution near Γ will be described. The main subject will be a numerical (difference) method in a sufficiently large domain for solving the variational equation. We will show the correctness of replacing $\mathbf{R}^N$ by the domain and the convergence of the discrete solutions in the energetic norm.

The present paper concludes essentially the results of the dissertation of the author [1]. The most proofs have been simplified, some facts are new. We shall give only a brief review of the proofs and of the important ideas.

2. Statement of main results and ideas

2.1. The basic function space

A function u is said to have *finite energy* if it is locally square integrable in Ω (not necessarily in $\mathbf{R}^N$) and its distribunal gradient (in Ω!) belongs to $L_2(\Omega, \varrho)$:

$$\int_\Omega |\nabla u|^2 \varrho \, dx < \infty. \tag{9}$$

Contrary to the usual weighted Sobolev spaces, in our definition we have essentially only a restriction on the gradient, not on the function (cf. the spaces in DENY/LIONS [2], Théorème 2.1. or MAZ'JA [5], Satz 1.1.). Clearly, two finite energy functions have the same gradient in Ω if and only if they differ (a.e.) by an additive constant in each component of Ω.

Definition: $H = H(\mathbf{R}^N)$ denotes the space of function classes $\dot{u}$ arising by identification of finite energy functions with the same gradient.

The following properties of H are fundamental for all later considerations.

Proposition 2.1: *H is an Hilbert space with respect to the correctly defined scalar product*

$$(\dot{u}, \dot{v}) = \int_\Omega \nabla u \cdot \nabla v \, \varrho \, dx.$$

Proposition 2.2: *If* (8) *is satisfied then $C_0^\infty(\Omega)$, the set of arbitrary smooth functions with support in Ω, is dense in H (in a natural sense).*

Proposition 2.3 (*Imbedding property*): *Suppose* (7) *satisfied and let G be any bounded domain, $G_+ = \{x \in G \mid d(x) > \delta\} \neq \emptyset$ with $\delta > 0$ sufficiently small. Then we have for all $\dot{u} \in H$*

$$\int_G |u|^2 \sigma \, dx \leqq C \left(\int_{G_+} |u|^2 \, dx + \int_G |\nabla u|^2 \varrho \, dx \right)$$

with a constant depending only on G, G_+, ϱ. The weight $\sigma(x)$ equals $d^{\varkappa-2}(x)$ for $\varkappa > 1$ and $d^{-1}(x) \cdot \log^{-2}(\text{const}/d(x))$ for $\varkappa = 1$. Especially we have $u \in L_2(\mathbf{R}^N, \varrho)$ locally because $\varrho(x) \leqq C \cdot \sigma(x)$ holds locally.

2.2. Existence and uniqueness of the solution

Clearly, every solution of the primary problem is also a solution in the sense of distributions over Ω or, equivalent, a solution of the following

Variational equation: *Find* $\dot{u} \in H$ *so that*

$$\int_{\mathbf{R}^N} (\nabla u + \boldsymbol{f}) \cdot \nabla\varphi \, \varrho \, dx = 0 \tag{10}$$

holds for all $\varphi \in C_0^\infty(\Omega)$.

If $C_0^\infty(\Omega)$ is dense in H then the variational equation may be also formulated as a minimisation problem for the full kinetic energy:

$$\int_{\mathbf{R}^N} |\nabla u + \boldsymbol{f}|^2 \, \varrho \, dx \to \min! \text{ in } H.$$

The proof is a well known Hilbert space technique.

Conversely, by usual regularity arguments in any subdomain $\Omega' \Subset \Omega$ we find that every weak solution is a solution of the primary problem. Consequently, the problems (5) and (10) are equivalent. It remains to solve the variational equation in a standard way: Representing the bounded linear functional

$$l(\dot{\varphi}) = \int_{\mathbf{R}^N} (-\boldsymbol{f}) \cdot \nabla\varphi \, \varrho \, dx, \qquad \dot{\varphi} \in H,$$

as a scalar product in H we establish the existence of some weak solution. This will be the only one if $C_0^\infty(\Omega)$ is dense in H, e.g. if (8) is satisfied. We have proved the following theorem:

Theorem 2.4 (*Existence and uniqueness*): *Assume* (8) *fullfilled. Then the primary problem* (5) *is solvable. Every two solutions differ by an additive constant in each component of* Ω. *For the solutions we have*

$$\int_{\mathbf{R}^N} |\nabla u|^2 \, \varrho \, dx \leqq C \int_{\mathbf{R}^N} |\boldsymbol{f}|^2 \, \varrho \, dx \tag{11}$$

with a constant indepedent of u *and* f.

In particular, all solutions have the same gradient and therefore determine the same current density $\boldsymbol{j} = \varrho(\nabla u + \boldsymbol{f})$. Note that we have found a unique solution without any boundary or contact conditions on Γ and without special conditions in the infinity. All restrictions on the solution are concluded in the finiteness of the energy (9).

2.3. Regularity

Under more special restrictions on ϱ we may describe the current density near Γ.

Theorem 2.5 (*Regularity*): *If* ϱ *satisfies* (6) *and* $\dot{u}$ *is a solution of the primary problem then* $\boldsymbol{j} = \varrho(\nabla u + \boldsymbol{f})$ *is in* H^1 *locally in* $\mathbf{R}^N$ *with a zero trace on* Γ.[3]) *Equation* (1) *holds in the sense of distributions over* $\mathbf{R}^N$.

In particular, this gives a "natural" (= automatically satisfied) contact condition on Γ: $\boldsymbol{j} \cdot \boldsymbol{n} = 0$ ($\boldsymbol{n}$ — normal vector to Γ). In other words: there is no current across the knode surface.

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[3]) $H^k(G) = \{u \in L_2(G) \mid D^\alpha u \in L_2(G), |\alpha| \leqq k\}$, $k = 0, 1, 2, \ldots$

2.4. The Problem in a bounded domain

The next intention is to solve the variational equation (10) numerically "in a sufficiently large bounded domain", say in a parallelepiped $Q = (a_1, b_1) \times \cdots \times (a_N, b_N)$. More exactly: By analogy to the $\mathbf{R}^N$-case 2.1. we define the space $H(Q)$ replacing in the definition of the finite energy functions $\Omega = \mathbf{R}^N \setminus \Gamma$ by $Q \setminus \Gamma$ and formulate the

Q-problem: *Find* $\dot{u}_Q \in H(Q)$ *such that*

$$\int_Q (\nabla u_Q + \boldsymbol{f}) \cdot \nabla \varphi_Q \, dx = 0, \qquad \dot{\varphi} \in H(Q). \tag{12}$$

This problem has exactly one solution the so called *Q-solution*. The proof is the same as in the $\mathbf{R}^N$-case, cf. 3.1. The Q-solution tends to the "true" solution in the energetic norm as "Q tends to the $\mathbf{R}^N$":

Theorem 2.6: *Suppose* (8) *satisfied and let* $\dot{u}$ *and* $\dot{u}_Q$ *be the solutions of the primary and the Q-problem. If the radius of the largest ball* $\subset \bar{Q}$ *with center in* 0 *tends to infinity then*

$$\int_{\mathbf{R}^N} |\nabla u - \nabla u_Q|^2 \varrho \, dx \to 0.^4)$$

2.5. The discrete problem

Fix the parallelepiped in (12) in such a way that

in each point of $\partial Q \cap \Gamma$ *the normal vectors to* Γ *and to the square surfaces through this point are linarly independent.* (13)

This means particulary that the knode surface is not tangential to any side of Q and does not cross any corner of Q. We shall now describe the so called *discrete problem* the solution of which will approximate the Q-solution of our problem. For this reason we shall construct a partition h of the parallelepiped Q in *h-cells* (= special small parallelepipedes) and a space of *h-function-classes*. These will be step functions (constant on every h-cell) assumed identified if they have the same *discrete gradient*. In this space we will find the solution of the discrete problem, the *Q_h-solution*. Roughly speaking, we get the discrete problem if we replace in the Q-problem all functions by its *discretisations*. It follow the precise definitions.

We get a partition h if we divide Q parallel to its surfaces in a finite number of open parallelepipeds. The only restriction is a "regularity" condition: the quotient of the largest and the smallest lenght is majorated by a absolute constant:

$$\text{for example} \quad L_h \leqq 1000 l_h. \tag{14}$$

The number

$$|h| = l_h/2$$

we call the fineness of the partition. All parallelepipeds without common points with Γ are called *h-cells*. Note that an h-cell may have boundary points on Γ. A step function with constant values on the h-cells and zero outside of the h-cells is

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4) We assume $u_Q = 0$ and $\nabla u_Q = 0$ outside of Q.

called an *h-function*. In view of defining discrete gradients we shall say that h-cells Z and Z' are *neighbours if*

they have a common side and belong to the same component of $Q \setminus \Gamma$. (15)

Only in this cases we will write $Z < Z'$ or $Z > Z'$ in dependence of its position (Z or Z' on the left). To any h-cells $Z_1 < Z_2$ neighbouring in the i-th direction we attach an ih-cell Z_{12} "glueing up two halfs" of the h-cells (Fig. 1) and define the i-th *discrete derivative* of an h-function u_h as a step function of differential quotients:

$$\delta_{ih}u_h = \begin{cases} \dfrac{u_h(z_2) - u_h(z_1)}{|z_2 - z_1|} & \text{on any } ih\text{-cell } Z_{12} \\ 0 & \text{outside of the } ih\text{-cells.} \end{cases} \tag{16}$$

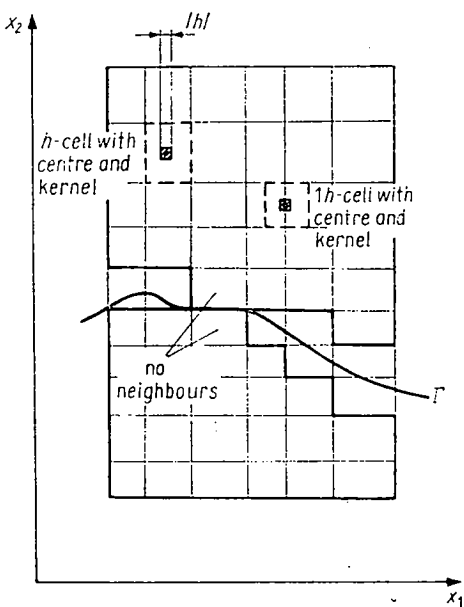

Fig. 1

The vector function $\delta_h u_h = (\delta_{1h}u_h, \ldots, \delta_{Nh}u_h)$ is called the *discrete gradient* of u_h. Clearly (cf. 3.6.(b)), two h-functions have the same discrete gradient if and only of they differ by an additive constant on each component of $Q \setminus \Gamma$ in an obvious sense. Identifying h-functions with the same discrete gradient we get the finite dimensional Hilbert space $H_h(Q)$ with the scalar product

$$(\dot{u}_h, \dot{v}_h) = \int_Q \delta_h u_h \cdot \delta_h v_h \varrho \, dx.$$

Finally, denote by f_{ih} and ϱ_{ih} the discretisations of the data: step functions with constant values $f_i(z)$ and $\varrho(z)$ on an ih-cell with the centre z and vanishing outside of the ih-cells.

Discrete problem: *Find* $\dot{u}_h \in H_h(Q)$ *such that*

$$\sum_{i=1}^{N} \int_Q (\delta_{ih}u_h + f_{ih}) \cdot \delta_{ih}\varphi_h \cdot \varrho_{ih} \, dx = 0 \tag{17}$$

holds for all $\dot{\varphi} \in H_h(Q)$.

This problem also has a unique solution $\dot{u}_h$ (3.6., H 4). There are two ways to describe the derivation of $\dot{u}_h$ from the Q-solution $\dot{u}$: we may compare $\dot{u}_h$ with $\dot{u}$ directly or with a *discretisation* $r_h\dot{u}$ defined as follows. In each h-cell let fix a so called *kernel* (= a cube with the same centre and with the lenght $|h|$) and define for all $\dot{v} \in H(Q)$ the h-function-class

$$r_h\dot{v} = \begin{cases} \dfrac{1}{|h|^N} \displaystyle\int_K v(x)\,dx & \text{in any } h\text{-cell (kernel } K), \\ 0 & \text{outside of the } h\text{-cells.} \end{cases} \tag{18}$$

Theorem 2.7 (*Convergence of the discrete solutions*): *Let* (7) *be satisfied. If* $\dot{u}$ *and* $\dot{u}_h$ *denote the* Q- *and* Q_h-*solutions then we have in* $L_2(Q, \varrho)$

$$\|\nabla u - \delta_h u_h\| \to 0 \quad \text{and} \quad \|\delta_h r_h \dot{u} - \delta_h u_h\| \to 0$$

as $|h| \to 0$.

2.6. Iterative solution of the discrete problem

We can formulate the discrete problem as a square *system of linear equations* choosing the test functions in (17) in a special way. Previously we introduce the following notations:

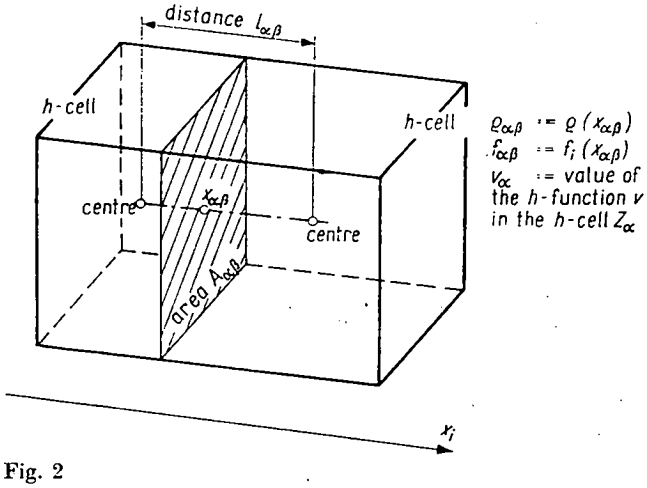

Fig. 2

Then the discrete problem rewrites to

$$\sum_{i=1}^{N} \sum \left(\frac{u_\alpha - u_\beta}{l_{\alpha\beta}} + f_{\alpha\beta} \right) \frac{\varphi_\alpha - \varphi_\beta}{i_{\alpha\beta}} \varrho_{\alpha\beta} l_{\alpha\beta} A_{\alpha\beta} = 0$$

where the inner sum is taken over all h-cells with $Z_\alpha < Z_\beta$ in the i-th direction (cf. (15)). Choosing test functions of the form

$$\varphi(x) = \begin{cases} 1 & \text{in a fixed } h\text{-cell} \\ 0 & \text{elsewhere} \end{cases}$$

we get the following system of equations for the determination of the unknown u_α:

$$\begin{pmatrix} d_1 & & -a_{\alpha\beta} \\ & \ddots & \\ -a_{\alpha\beta} & & d_m \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} q_1 \\ \vdots \\ q_m \end{pmatrix} \tag{19}$$

$$Au = q$$

where

m = number of h-cells,

$$a_{\alpha\beta} = \begin{cases} \dfrac{\varrho_{\alpha\beta} A_{\alpha\beta}}{l_{\alpha\beta}} & \text{if } Z_\alpha \text{ is a neighbour of } Z_\beta \\ 0 & \text{otherwise,} \end{cases}$$

$$d_\alpha = a_{\alpha 1} + a_{\alpha 2} + \cdots + a_{\alpha m},$$

$$q_\alpha = \sum_{Z_\alpha < Z_\beta} f_{\alpha\beta} \varrho_{\alpha\beta} A_{\alpha\beta} - \sum_{Z_\alpha > Z_\beta} f_{\alpha\beta} \varrho_{\alpha\beta} A_{\alpha\beta}.$$

This as a symmetric system with a singular matrix. Since the discrete problem is only another form of the problem

$$(Au, \varphi) = (q, \varphi) \qquad (\varphi \in \mathbf{R}^m)$$

it is equivalent to the system (19). Regardless of the fact $\det A = 0$ we can apply a Gauss-Seidel iteration method or an SOR method with the factor ω for solving the system (19):

$$\begin{cases} \text{start:} & u^0 \in \mathbf{R} \text{ arbitrary,} \\ \text{iteration:} & \left(\dfrac{1}{\omega} D - F\right)(u^{(i)} - u^{(i-1)}) = -(Au^{(i-1)} - q), \end{cases} \tag{20}$$

where

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ a_{\alpha\beta} & & 0 \end{bmatrix}.$$

Theorem 2.8: *For all start vectors and all* $0 < \omega < 2$ *the method* (20) *converges to one of the solutions of* $Au = q$.

Remark 2.1: It is possible to derive the system (19) by a simple heuristic consideration from Equation (1). It is interesting, too, that the iteration method (20) may be interpreted as an "agreement of charges" using the physical explanation of Equation (1). For details cf. BERNDT [1, p. 17].

3. Proofs

3.1. Hilbert space property (Proposition 2.1)

All but the completeness may be proved easily. Let $\{u_n\}$ be a Cauchy sequence in H. Then by the completeness of $L_2(\Omega, \varrho)$ the gradients converge in $L_2(\Omega, \varrho)$ to a vector function v. It remains to show $v = \nabla u$, $u \in L_{2,loc}(\Omega)$. At first we do that *locally*, in

an arbitrary (smooth) domain $G \Subset \Omega$ with a fixed open ball $B \subset G$: Choose the representators u_n so that they have zero mean value in B and apply the Poincaré-inequality

$$\int_G |v|^2 \, dx \leqq C_G \cdot \int_G |\nabla v|^2 \, dx,$$

see e.g. KUFNER, JOHN, FUČIK [3, Theorem 5.3.11]. It follows that the u_n form a Cauchy sequence in $L_2(G)$ and hence converge to some $u_G \in L_2(G)$. Obviously, u_G has the gradient v and a zero mean value in B. Finally, we get the desired u *globally* (in a fixed component Ω' of Ω) if we construct the u_G for all domains such that

$$B \Subset G_1 \Subset G_2 \Subset \cdots \Subset \Omega', \qquad \bigcup_{k=1}^{\infty} G_k = \Omega'$$

and note that they coincide a.e. in the smaller domain (all u_G have a zero mean value in B and the same gradient!) ∎

Remark 3.1: Proposition 2.1 remains true if we replace H by $H(Q)$ defined in 2.4. The proof is essential the same.

3.2. Density property (Proposition 2.2)

We shall approximate an arbitrary $u \in H$ step by step: first by a bounded function, then by a function vanishing for large $|x|$, then by a function vanishing near Γ and finally by a smooth function.

Step 1: For sufficiently large $k > 0$ set

$$u_k(x) = \begin{cases} -k & \text{if } u(x) < -k, \\ u(x) & \text{if } -k \leqq u(x) \leqq k, \\ k & \text{if } k < u(x). \end{cases}$$

We have $u_k \in H$ since u is absolutely continuous on straight lines in a well known sense, see e.g. KUFNER, JOHN, FUČIK [3, Theorem 5.6.3]. $\|u - u_k\|_H \to 0$ follows easily from the integrability of ϱ.

In the next steps it would be sufficient to approximate u by finite energy functions u_n having the desired property and satisfying

(i) $u_n \to u$ *in* $\mathscr{D}'(\Omega)$,

(ii) $\|u_n\| \leqq$ *const in* H.

Indeed, such a sequence contains a weakly convergent subsequence $\{u_n'\}$ (with the limit u by (i)). Applying Mazur's theorem (cf. YOSIDA [12, Chapter V 1., Theorem 2]) we find some convex combinations of the u_n, tending to u in the norm of H.

Step 2: Suppose u bounded and set $u_n(x) = \varepsilon_n(|x|)\, u(x)$ with a piecewise linear function ε equal 1 for $|x| < n$ and equal 0 for $|x| > n + 1$. Then (i) is obvious and (ii) follows immediately from the boundedness of u and from the integrability of ϱ.

Step 3: Suppose u bounded with a compact support and define $u_n(x) = \eta_n(x)\, u(x)$ with a *cut-off-function* of the following form:

$$\eta_n \in C_0(\mathbf{R}^N),$$

$$\eta_n(x) = 0 \text{ for } d(x) < \frac{1}{n} \text{ and } = 1 \text{ for } d(x) > \frac{2}{n},$$

$$|D^\alpha \eta_n(x)| \leqq c_\alpha n^{|\alpha|} \text{ for any multiindex } \alpha.$$

Clearly, (i) holds again. In deriving (ii) note that $\nabla u_n(x)$ is zero "near Γ" and equals ∇u "away from Γ". This gives

$$\begin{aligned} \|u_n\|^2 &= \int_{F_n} |\nabla \eta_n \cdot u + \eta_n \cdot \nabla u|^2 \varrho \, dx + \int_{d(x) > \frac{2}{n}} |\nabla u|^2 \varrho \, dx \\ &\leqq C \cdot n^2 \int_{F_n} \varrho \, dx + \|u\|^2 \\ &\leqq C' n^2 \int_{1/n}^{2/n} t^2 \, dt + \|u\|^2 \leqq C''. \end{aligned}$$

We have denoted $F_n = \{x \in \operatorname{supp} u \mid 1/n \leqq d(x) \leqq 2/n\}$ and have applied the behavior (8) near $\Gamma \cap \operatorname{supp} u$.

Step 4: Suppose u bounded with the support in $G \Subset \Omega$. Then u belongs to $H^1(G)$ and we can it mollify in the classical way ∎

Remark 3.2: An analogous density theorem holds for the space $H(Q)$ defined in 2.4. In this case we have the dense subset $\{u \in C^\infty(\bar{Q}) \mid \operatorname{supp} u \cap \Gamma = \emptyset\}$.

3.3. Imbedding property (Proposition 2.3)

Using a finite covering of $\bar{G} \cap \Gamma$ by small parallelepipeds we may restrict us to the following situation:

$$\Gamma = \{(\bar{y}, t) = (y_1, \ldots, y_{N-1}, t) \in \mathbf{R}^N \mid t = 0\},$$

$$\varrho(x) = |t|^\varkappa,$$

G is a parallelepiped (Fig. 3).

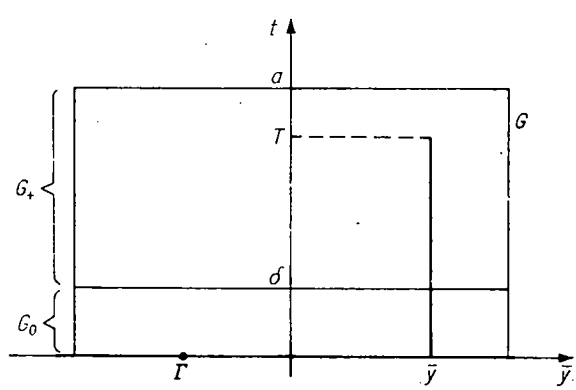

Fig. 3

The main tool is the classical Hardy inequality which states, roughly speaking, an estimate of a function by its derivative in an L_p-norm with weights: If $\varrho(\tau)$ and $\sigma(\tau)$ are nonnegative measurable weights on $(0, T)$ with the property

$$\int_0^t \sigma(\tau)\,\mathrm{d}\tau \cdot \int_t^T \varrho^{-1}(\tau)\,\mathrm{d}\tau \leqq C \qquad \big(t \in (0, t)\big),$$

then we have for all measurable f with $F(t) = \int_t^T f(\tau)\,\mathrm{d}\tau$

$$\int_0^T |F(t)|^2\,\sigma(t)\,\mathrm{d}t \leqq 4C \int_0^T |f(t)|^2\,\varrho(t)\,\mathrm{d}t.$$

For the proof cf. MUCKENHOUPT [6]. Clearly, the pairs

$$\sigma(\tau) = \tau^{\varkappa-2}, \qquad \varrho(\tau) = \tau^{\varkappa} \qquad (\varkappa > 1);$$

$$\sigma(\tau) = \tau^{-1}\log^{-2}(a/\tau), \qquad \varrho(\tau) = \tau \qquad (\varkappa = 1)$$

satisfy the weight condition with $C = (\varkappa - 1)^{-2}$ resp. $= a^{-1}$. Now for fixed $\bar{y}$ and $T \geqq \delta$ let consider u on a strip orthogonal to Γ (Fig. 3) and apply Hardy's inequality to

$$f(t) = \frac{\partial u}{\partial t}(\bar{y}, t), \qquad F(t) = u(\bar{y}, T) - u(\bar{y}, t).^{5})$$

We get

$$\int_0^T |u(\bar{y}, t) - u(\bar{y}, t)|^2\,\sigma(t)\,\mathrm{d}t \leqq C' \int_0^T |\nabla u(\bar{y}, t)|^2\,\varrho(t)\,\mathrm{d}t$$

and after some simple estimate and integration first over $T \in (\delta, a)$ then over $\bar{y}$ the desired estimate is proved ∎

Remark 3.3: The same estimate holds in the space $H(Q)$ defined in 2.4. In this case we may choose $G = Q$ if Q is a parallelepiped with the property (13), cf. BERNDT [1].

3.4. Regularity (Proposition 2.5)

The main idea is the following one: We rewrite Equation (1) in the form

$$-\Delta(\psi^2 u) = \operatorname{div}(\nabla\psi \cdot \psi u - \psi^2 f) \tag{21}$$

and deduce the "good" regularity of $\psi^2 u$ from the "weak" regularity of the right hand side by the aid of a well known regularity result for elliptic equations.

Fix an arbitrary bounded domain $G \subset \mathbf{R}^N$.

Step 1 (*Regularity "a priori"*): By an approximation argument we show that

$$\psi u \text{ and } \psi^2 u \in H^1(G). \tag{22}$$

Indeed, since u has finite energy we can find $u_n \in C^\infty(\mathbf{R}^N)$ such that $\psi\nabla u_n \to \psi\nabla u$ in $L_2(G)$ (Proposition 2.2) and $u_n \to u$ in $L_2(G)$ (Proposition 2.3). This gives $\psi u_n \to \psi u$ in $H^1(G)$ and (22) is proved.

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[5]) $u(\bar{y}, \cdot)$ is absolutely continuous in a well known sense, cf. KUFNER, JOHN, FUČIK [3, Theorem 5.6.3].

Step 2 (*Regularity "a posteriori"*): (22) shows that equation (21) holds in the sense of distributions in G with the right hand side in $L_2(G)$. Thus, by Friedrich's theorem (cf. YOSIDA [12, Chapter VI/9.]) we have

$$\psi^2 u \in H^2(G). \tag{23}$$

Together with the imbedding result of Proposition 2.3 the relations (22) and (23) imply

$$u, \quad \psi \partial_i u, \quad \psi^2 \partial_i \partial_j u \in L_2(G)$$

and therefore the desired result $\psi^2 \nabla u \in H^1(G)$.

Step 3 (*Zero trace*): It remains to approximate $\psi^2 \partial_i u$ in $H^1(G)$ by smooth functions vanishing in some neighbourhood of Γ. We may use the sequence

$$\psi^2 \partial_i u_n \quad \text{with} \quad u_n(x) = \eta_n(x) \cdot u(x)$$

where η_n is the cut-off-function from 3.2., Step 3. For example we have

$$\begin{aligned}
&\int_G |\partial_i \partial_j (u - u_n)|^2 \psi^4 \, dx \\
&\leqq \int_{d(x) > \frac{2}{n}} |\partial_i \partial_j u|^2 \psi^4 \, dx + C \int_{\frac{1}{n} < d(x) < \frac{2}{n}} (|\partial_i \partial_j u|^2 \psi^4 + n^2 |\nabla u|^2 \psi^4 + n^4 |u|^2 \psi^4) \, dx \\
&\leqq C \int_{d(x) > \frac{2}{n}} (|\partial_i \partial_j u|^2 \psi^4 + |\nabla u|^2 \psi^2 + |u|^2) \, dx \to 0
\end{aligned}$$

as $n \to \infty$ because the last three summands are integrable over G (Step 2) ∎

3.5. Replacing of $\mathbf{R}^N$ by a bounded domain (Theorem 2.6)

Let $\{u_n\}$ be an *arbitrary* sequence of Q_n-solutions such that the radius of the largest ball $\subset \overline{Q_n}$ tends to infinity. We shall extract a subsequence with

$$I_n = \int_{Q_n} |\nabla u - \nabla u_n|^2 \varrho \, dx \to 0.$$

This, of course, proves the theorem.

Choosing $\varphi = u$ in the variational equation (10) and $\varphi = u$, $\varphi = u_n$ in the Q_n-problem (12) we get after simple calculations

$$-I_n = \int_{CQ_n} (\nabla u + \boldsymbol{f}) \nabla u \, \varrho \, dx + \int_{Q_n} (\nabla u + \boldsymbol{f}) \nabla u \, \varrho \, dx = A_n + B_n.$$

Clearly, $A_n \to 0$ by the extending property of the Q_n. In the second integral we may suppose $\nabla u_n \to \boldsymbol{w}$ weakly in $L_2(\Omega, \varrho)$ because the ∇u_n are uniformly bounded in this space. If we had $\boldsymbol{w} = \nabla v$ with some finite energy function v than we would have

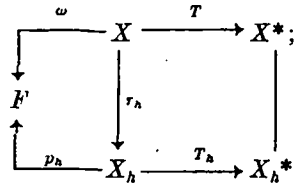
$$B_n \to \int_{\mathbf{R}^N} (\nabla u + \boldsymbol{f}) \nabla v \, \varrho \, dx = 0$$

in view of (10). But we can state $\boldsymbol{w} = \nabla v$ with the same method as in the proof of Proposition 2.1: first locally (u_n are bounded in $L_2(G)$ by Poincaré's inequality, v is the limit of a subsequence of u_n) and then globally (approximate Ω by bounded smooth domains] ∎

Remark 3.4: We did not use hypothesis (8) but only the fact that $u \in H$ satisfies (10) for all $\dot{\varphi} \in H$.

3.6. Convergence of discrete solutions (Theorem 2.7)

Theorem 2.7 follows by application of an *external approximation scheme* (TEMAM [11], SCHUMANN [10]):



$$T\dot{u} = f;$$

$$T_h\dot{u}_h = f_h.$$

In our case the abstract spaces and mappings have the following concretisations:

$X = H(Q)$; $X_h = H_h(Q)$; $F = L_2(Q, \varrho)$;

r_h = discretisation operator (18);

p_h = discrete gradient operator δ_h defined in (16);

ω = gradient operator $\omega\dot{u} = \nabla u$.

The two equations stand for the Q- and Q_h-problems:

$$\langle T\dot{u}, \dot{\varphi}\rangle = \int_Q \nabla u \cdot \nabla\varphi\, \varrho \, dx; \quad \langle T_h\dot{u}_h, \dot{\varphi}_h\rangle = \sum_{i=1}^{N} \int_Q \delta_{ih}u_h \cdot \delta_{ih}\varphi_h \cdot \varrho_{ih}\, dx;$$

$$\langle f, \dot{\varphi}\rangle = -\int_Q \boldsymbol{f} \cdot \nabla\varphi\, \varrho \, dx; \quad \langle f_h, \dot{\varphi}_h\rangle = -\sum_{i=1}^{N} \int_Q f_{ih} \cdot \delta_{ih}\varphi_h \cdot \varrho_{ih}\, dx.$$

In the following proof we have to compare elements $\dot{v} \in X$ with families $\{\dot{v}_h\}$, $\dot{v}_h \in X_h$ as $|h| \to 0$. For convenience we shall write

$\dot{v}_h \dashrightarrow \dot{v}$ (*discretely*) if $\|\dot{v}_h - r_h\dot{v}\| \to 0$,

$\dot{v}_h \to \dot{v}$ (*strongly*) if $\|p_h\dot{v}_h - \omega\dot{v}\| \to 0$,

$\dot{v}_h \rightharpoonup \dot{v}$ (*weakly*) if $\|\dot{v}_h\| \leqq$ const. and $p_h\dot{v}_h \rightharpoonup \dot{v}$.

Theorem 2.7 states that the discrete solutions $\dot{u}_n$ converge discretely and strongly to the Q-solution $\dot{u}$. In accordance with the abstract scheme it would be sufficient to verify the following hypothesises (SCHUMANN [10]):

Stability: r_h and p_h are linear operators with uniformly bounded norms.

Convergence I: For any $\dot{v} \in X$: $r_h\dot{v} \to v$ as $|h| \to 0$.

Convergence II: If we have $p_h\dot{v}_h \to \boldsymbol{g}$ for some sequence of partitions $h \to 0$ then $\boldsymbol{g} \in \operatorname{Im}\omega$.

Coercivity: There is a constant $C > 0$ such that

$$\langle T_h\dot{v}, \dot{v}_h\rangle \geqq C\,\|\dot{v}_h\|^2$$

for all sufficiently fine partitions and all $\dot{v}_h \in X_h$.

Approximation property: For any sequence of partitions with $|h| \to 0$ we have

$$\langle T_h r_h\dot{v}, \dot{\varphi}_h\rangle \to \langle T\dot{v}, \dot{\varphi}\rangle$$

and

$$\langle f_h, \dot{\varphi}_h\rangle \to \langle f, \varphi\rangle$$

whenever $\dot{\varphi}_h \to \dot{\varphi}$ and $\dot{v} \in X$.

Remark 3.5: In the following proof we will use some geometrically obvious facts the proof of which may be found in BERNDT [1]. For example: $Q \setminus \Gamma$ has only a finite number of components and two neighbouring h-cells can't have points of Γ on the common side. In a fixed component of $Q \setminus \Gamma$ every two h-cells are connected by a chain of h-cells touching "side by side" (for sufficiently fine partitions). Finally, the values of ϱ in a cell can be uniformly estimated by its values "in the middle of this cell": If Z is the union of two neighbouring h-cells and K the convex hull of the kernels then we have the

fundamental estimate

$$\varrho(x) \leqq C \cdot \varrho(y) \qquad (x \in Z,\ y \in K), \tag{24}$$

with a constant independent of the partition.

Proof of the stability: $\|p_h\| = 1$ is obvious by the definition. $\|r_h\| \leqq$ const. will follow by some simple estimates (apply (14), (24)) from the equality (cf. Fig. 4)

$$\delta_{ih} r_h \dot{v} = \frac{1}{|h|^N} \int_{K^*} \partial_i v(y)\, dy, \qquad \dot{v} \in H(Q). \tag{25}$$

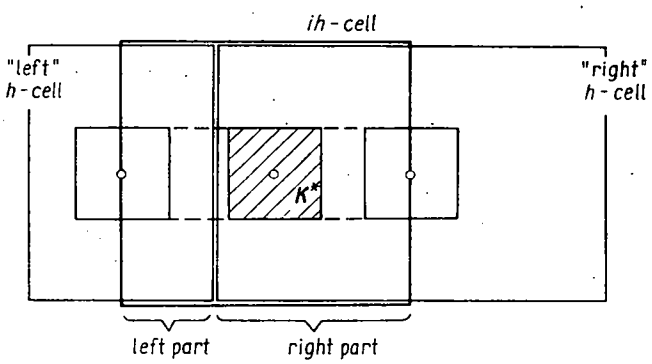


Fig. 4

(25) may be established in every ih-cell if we represent the difference between two values of v by an one-dimensional integral over $\partial_i v$ and apply Fubini's theorem and L_1-continuity ∎

Proof of convergence I: Since the stability is proved it would be sufficient to show $r_h \dot{v} \to \dot{v}$ only in a dense subset of X, e.g. for all $v \in C^\infty(\bar{Q})$ (Remark 3.2). Such v are uniformly continuous and the result follows easily from (25) ∎

Proof of convergence II: We have to show: For an arbitrary sequence $\dot{v}_h \in X_h$ with $\delta_{ih} v_h \to g_i$ in $L_2(Q, \varrho)$ there is a $v \in L_{2,loc}(Q \setminus \Gamma)$ such that

$$\int_Q g_i \varphi\, dx = -\int_Q v \cdot \partial_i \varphi\, dx, \qquad \varphi \in C_0^\infty(Q \setminus \Gamma).$$

Clearly, it would be sufficient to construct v locally, in an arbitrary domain $G \Subset Q \setminus \Gamma$ and to prove the equality for $\operatorname{supp} \varphi \subset G$. We start from an analogous discrete relation:

$$\int_G \delta_{ih} v_h \cdot \varphi_h{}^i\, dx = -\int_G v_h \cdot \delta_{ih} \varphi_h\, dx.$$

Here φ_h denotes the discretisation of φ defined in (18) and φ_h^i is "almost" equal to φ_h: on the left part of any ih-cell it is to be defined as the mean value of φ over the value of φ over the right kernel and vice versa. The desired result follows by tending to the limit $h \to 0$. Then we have in $L_2(G)$: $\delta_{ih} v_h \rightharpoonup g_i$ by hypothesis, $\delta_{ih}\varphi_h \to \partial_i \varphi$ by convergence I and $\varphi_h^i \to \varphi$ by a simple calculation. Finally, we can extract a subsequence on which v_h converges weakly in $L_2(G)$ to some v, since the $L_2(G)$-norms of the v_h are uniformly bounded. This follows from a discrete Poincaré inequality. ∎

Discrete Poincaré inequality: *For any h-function vanishing in a fixed neighbourhood of* $\partial Q \cap \Gamma$, *we have in the norm of* $L_2(Q)$: $\|v_h\| \leqq \sqrt[N]{\operatorname{mes} Q}\, \|\delta_h v_h\|$ ($|h|$ *sufficiently small*).

For the proof let fix a stripe $S = P_0 \cup P_1 \cup \cdots \cup P_n$ of parallelepipeds "from one side of Q to the other", for convenience in the first direction. Denote

l_k = length of P_k in the first direction,

l_{kj} = distance between the centres in the first direction,

v_k = values of v_h (note that $v_0 = 0$ by hypothesis),

v_{kl} = values of $\delta_{1h} v_h$.

Applying repeatedly the definition (16) of the discrete gradient we get $v_1 = l_{01} v_{01}$, $v_2 = l_{01} v_{01} + l_{12} v_{12}$ etc. That leads to the desired estimate:

$$\sum_k l_k v_k^2 \leqq (b_1 - a_1)^2 \sum_k l_{k,k+1} v_{k,k+1}^2$$

$$\int_S |v_h|^2 \, dx \leqq (b_1 - a_1)^2 \int_S |\delta_{1h} v_h|^2 \, dx \quad ∎$$

The proof of coercivity is obvious in view of (24).

Proof of the approximation property (for T, T_h; for f_h, f analogously): We have to prove (norms and scalar products in $L_2(Q, \varrho)$; $r_h \dot{v}$ denoted by $\dot{v}_h$):

$$\text{if } \|\delta_{ih}\varphi_h\| \leqq C \text{ and } \delta_{in}(\varphi_h \rightharpoonup \partial_i \varphi) \text{ in } L_2(Q, \varrho)$$

$$\text{then } \left(\delta_{ih} v_h \cdot \frac{\varrho_{ih}}{\varrho},\ \delta_{ih}\varphi_h\right) \to (\partial_i v,\ \partial_i \varphi). \tag{26}$$

This is proved if we can show that every subsequence consists another subsequence on which

$$\delta_{ih} v_h \cdot \frac{\varrho_{ih}}{\varrho} \to \partial_i v \quad \text{in } L_2(Q, \varrho). \tag{27}$$

Now, note that the last relation holds if the term ϱ_{ih}/ϱ is absent (Convergence I; the subsequence may be choosed so that we have convergence a.e.). Hence, by the dominated convergence theorem, it holds also in the present form if we can show

$$\frac{\varrho_{ih}(x)}{\varrho(x)} \to 1 \quad (x \in Q \setminus \Gamma) \quad \text{and} \quad \left|\frac{\varrho_{ih}(x)}{\varrho(x)}\right| \leqq C. \tag{28}$$

The first relation follows from the continuity of ϱ. The second relation is satisfied only for x "away from Γ", e.g. for dist $(x, \Gamma) \geqq |h|/2$. Therefore we divide the left integral (26) in an essential part over

A_{ih} = {all ih-cells with a distance to Γ more than $|h|$ and all kernels of the other ih-cells}

and in a rest part over $B_{ih} = Q_{ih} \setminus A_{ih}$. By regularity (16), condition (28) is satisfied for the essential part. Hence, the essential part will tend to the limit (26). The rest part will tend to zero:

$$\left| \int_{B_{ih}} \dots \right| \leqq \sum |\delta_{ih} v_h(x_a)| \cdot |\delta_{ih} \varphi_h(x_a)| \, \varrho(x_a) \cdot \operatorname{mes} Z_a$$

$$\leqq C_1 \sum \dots \operatorname{mes} K_a \leqq C_2 \int_{\frac{|h|}{2} < d(x) < |h|} |\delta_i(v_h| \cdot |\delta_{ih} \varphi_h| \, \varrho \, dx \to 0$$

(summation over all ih-cells Z_a, dist $(Z_a, \Gamma) \leqq |h|$, with centres x_a and kernels K_a) ∎

3.7. Convergence of the iteration method (Theorem 2.8)

By a theorem of MARČUK and KUZNECOV [4, Theorem 3.1.] it remains to show that $(Ax, x) \geqq 0$ for all $x \in \mathbf{R}^m$ or, equivalent, that all eigenvalues of A are nonnegative. This follows from a theorem of Gersgorin (cf. PARODI [7]), which asserts that all eigenvalues are localized in the union of m circles with the centres a_{ii} and the radii $r_i = \sum_{i \neq j} |a_{ji}|$.

In our case we have $a_{ii} = r_i > 0$. Hence, all these circles belong to the half plane $\operatorname{Re} z \geqq 0$ ∎

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