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On the existence of dense ideals in LMC*-algebras

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In der Arbeit wird folgender Satz bewiesen: Besitzt eine LMC*-Algebra (mit Einselement) ein unbeschränktes Element, so gibt es in ihr ein dichtes Ideal.

В работе доказывается следующее предложение: Если LMC*-алгебра (с единичным элементом) содержит неограниченный элемент, то она содержит плотный идеал.

In this paper we prove the following proposition: The existence of an unbounded element in an LMC*-algebra (with unity) implies the existence of a dense ideal in this algebra.

The concept of LMC*-algebras is a natural generalization of the concept of C*-algebras. LMC*-algebras were investigated in [2, 3, 5–7]. Many of the results on C*-algebras can be extended to the larger class of LMC*-algebras, nevertheless there are also essential differences between these classes of algebras. One of these is the existence of dense ideals in LMC*-algebras. In a C*-algebra with unity every maximal left (right, two-sided) ideal is automatically closed. This follows from the well known result that the closure of a proper regular ideal in a Banach algebra is again a proper ideal. Želasko proved that in commutative lmc-algebras (locally multiplicatively-convex algebras) the existence of an unbounded element implies the existence of a dense ideal (of infinite codimension) [8].

In [2] we conjectured that the following theorem holds.

Theorem 1: *The existence of an unbounded element in an* LMC*-*algebra* (*with unity*) *implies the existence of a dense ideal in this algebra.*

For commutative LMC*-algebras this proposition is obviously a special case of the result of Želasko; thanks to the isomorphy of such algebras to algebras $C(X)$ of all continuous complex-valued functions on a topological space X (see Theorem 3) the structure of maximal (closed and dense) ideals is known [4]. In this paper we will prove the conjectured theorem. First of all we recall the definition and some basic properties of LMC*-algebras.

Definition 2 [6]: An LMC*-*algebra* is a complete locally convex $*$-algebra $\mathscr{A}[\tau]$, whose topology τ can be given by a system of seminorms p with the following properties:

(i) $p(xy) \leqq p(x)\, p(y)$ and

(ii) $p(x^*x) = p(x)^2 \ \forall\, x, y \in \mathscr{A}$.

Such seminorms are called C*-*seminorms*.

We will always assume in this paper the existence of a unity e in an LMC*-algebra $\mathscr{A}[\tau]$. For C*-seminorms p we have $p(x^*) = p(x)$ and $p(e) = 1$. $\Gamma_{\max}^{\tau}$ denotes the set of all τ-continuous C*-seminorms on $\mathscr{A}[\tau]$, it is an upwards directed system under the order relation $p \leqq q$ iff $p(x) \leqq q(x)\ \forall\, x \in \mathscr{A}$. By Γ^{τ} we denote a directed subsystem of $\Gamma_{\max}^{\tau}$ which is generating yet the topology τ.

The following result parallel to the Gelfand-Neumark-Theorem for C*-algebras is very useful for our considerations.

Theorem 3 [6]: *For a commutative* LMC*-*algebra* $\mathscr{A}[\tau]$ *there exists a completely regular topological space* X *such that*

(i) $\mathscr{A}[\tau]$ *is algebraically and topologically isomorph to the Algebra* $C(X)$ *of all continuous complex-valued functions on* X *equipped with a topology* τ_0 *weaker than the compact-open topology.*
(We will write $\mathscr{A}[\tau] \stackrel{I}{\simeq} C(X)\,[\tau]_0$.)

(ii) *Under this isomorphism* I *the seminorms* p *were converted into suprema on compact subsets of* X, *that means for* $p \in \Gamma^\tau$ *there is a compact subset* K_p *of* X *such that*

$$p(x) = p_{K_p}(I(x)) = \sup_{t \in K_p} |x(t)| \quad \text{and} \quad \bigcup_{p \in \Gamma^\tau} K_p = X.$$

Remark: The image of an element of the algebra under I we always denote by the same letter joining the argument t.

The set $\mathscr{A}_b = \left\{x \in \mathscr{A} \mid \sup_{p \in \Gamma^\tau} p(x) < \infty\right\}$ is called the *bounded part of* $\mathscr{A}$. Hence, unbounded elements are the elements of $\mathscr{A} \setminus \mathscr{A}_b$. $\mathscr{A}_b$ is τ-dense in $\mathscr{A}$ and a C*-algebra under the norm $\|x\| = \sup p(x)$ [6].

The set $\mathscr{P}(\mathscr{A}) = \left\{\sum_{\text{finite}} x_i{}^* x_i \mid x_i \in \mathscr{A}\right\}$ is called the positive cone of $\mathscr{A}$, it organize the hermitian part $\mathscr{A}_h = \{x \in \mathscr{A} \mid x = x^*\}$ of $\mathscr{A}$ to a partially ordered topological space. We have $\mathscr{P}(\mathscr{A}_b) = \mathscr{A}_b \cap \mathscr{P}(\mathscr{A})$ where $\mathscr{P}(\mathscr{A}_b) = \left\{\sum_{\text{finite}} y_i{}^* y_i \mid y_i \in \mathscr{A}_b\right\}$ [6] and $\mathscr{P}(\mathscr{A}_b)$ is τ-dense in $\mathscr{P}(\mathscr{A})$, even one can approximate elements of $\mathscr{P}(\mathscr{A})$ by increasing sequences of elements of $\mathscr{P}(\mathscr{A}_b)$. The simple proof of this fact is contained in the proof of our theorem.

A further essential result is that every LMC*-algebra is the projective limit of C*-algebras. For $p \in \Gamma^\tau$ the set $\mathscr{N}_p = \{x \in \mathscr{A} \mid p(x) = 0\}$ is a τ-closed two-sided $*$-Ideal in $\mathscr{A}$. Let π_p be the natural homomorphism of $\mathscr{A}$ on $\mathscr{A}_p = \mathscr{A}/\mathscr{N}_p$. $\mathscr{A}_p$ is a C*-algebra under the norm $\|\pi_p(x)\|_p = p(x)$ and $\mathscr{A}[\tau] = \lim\operatorname{proj}_{p \in \Gamma^\tau} (\mathscr{A}_p, \|\cdot\|_p)$ [6].

The following facts about continuous linear functionals are immediately clear. For $f \in \mathscr{A}[\tau]'$ there exists $p \in \Gamma^\tau$ such that $|f(x)| \leq cp(x)\ \forall x \in \mathscr{A}$ (c is a positive constant). Then

$$f_p(\pi_p(x)) = f(x) \tag{$*$}$$

defines $f_p \in \mathscr{A}_p[\|\cdot\|_p]'$ and converse, for $f_p \in \mathscr{A}_p[\|\cdot\|_p]'$ we get by ($*$) an element f of $\mathscr{A}[\tau]'$, continuous with respect to p. f is positive iff f_p is positive. Further we have: f is a continuous state iff f_p is a state; f is an extremal continuous state iff f_p is an extremal state [5]. We denote by S (resp. S_p) the set of all continuous states of $\mathscr{A}[\tau]$ (resp. $\mathscr{A}_p[\|\cdot\|_p]$), by ex S (resp. ex S_p) the subsets of extremal states.

We will make use of the following result on the ideal structure of LMC*-algebras.

Proposition 4 [2]:

(i) *Every maximal closed left ideal* $\mathscr{J}$ *in an* LMC*-*algebra* $\mathscr{A}[\tau]$ *is the left kernel of an extremal continuous state, i.e.* $\exists\, \omega \in \text{ex}\, S$ *such that* $\mathscr{J} = \{x \in \mathscr{A} \mid \omega(x^*x) = 0\}$.

(ii) *Every closed left ideal* $\mathscr{J}$ *in an* LMC*-*algebra is the intersection of all maximal closed left ideals containing* $\mathscr{J}$.

Now we prove a lemma on the possibility of extension of continuous states. This result is well known for C*-algebras (see for instance [1], 2.10.1.).

Lemma 5: Let $\mathscr{A}[\tau]$ be an LMC*-algebra, $\mathscr{B}$ a closed subalgebra of $\mathscr{A}$ and $e \in \mathscr{B}$. If g is a continuous state of $\mathscr{B}$, then
(i) g can be extended to a continuous state of $\mathscr{A}$ and
(ii) the extension can be chosen extremal for extremal g.

Proof: *ad* (i): There is a seminorm $p \in \Gamma^\tau$ such that $|g(b)| \leqq p(b) \;\forall\, b \in \mathscr{B}$. Regard the algebras $\mathscr{A}_p = \mathscr{A}/\mathscr{N}_p$ and $\mathscr{B}_p = \mathscr{B}/\mathscr{B} \cap \mathscr{N}_p$, π_p': $\mathscr{B} \to \mathscr{B}_p$ the natural homomorphism. $\mathscr{B}_p$ is a C*-algebra under the norm $\|\pi_p'(b)\| = p(b)$ and $\pi_p'(b) \to \pi_p(b)$ is an imbedding of $\mathscr{B}_p$ in $\mathscr{A}_p$ preserving the norm, thus we can regard $\mathscr{B}_p$ as a C*-subalgebra of $\mathscr{A}_p$. g_p is a state of $\mathscr{B}_p$ and so it can be extended to a state f_p of $\mathscr{A}_p$. Define f by (*). Then f is a continuous state of $\mathscr{A}$ and for $b \in \mathscr{B}$ we have $f(b) = f_p(\pi_p(b)) = g_p(\pi_p(b)) = g(b)$.
ad (ii): For extremal g g_p is extremal too (Prop. 4). Then one can choose f_p extremal [1] and so f is extremal ∎

Remark: We cannot directly use extension theorems, because in general e is not an inner point of the positive cone.

We are able now to prove our theorem:

Proof: Let $a \in \mathscr{A}$ be an unbounded element. Without loss of generality we can assume $a \in \mathscr{P}(\mathscr{A})$, since for unbounded a a^*a is unbounded too. Let us regard the commutative closed subalgebra $\mathscr{A}_0[\tau]$ of $\mathscr{A}[\tau]$ generated by a and e. We have $\mathscr{A}_0[\tau] \overset{I}{\simeq} C(X)\,[\tau_0]$ (Th. 3). Then $a(t) \geqq 0 \;\forall\, t \in X$ and $a(t)$ is an unbounded function. Set $a_n(t) = \min\big(a(t), n\big) \in C(X) \;\;\forall\, n \in \mathrm{N}$ (N is the set of natural numbers); $a_n = I^{-1}\big(a_n(t)\big) \in \mathscr{A}_0$. By Theorem 3 we get $\forall\, n \in \mathrm{N}$

$$0 \leqq a_n < a, \qquad a_n \in \mathscr{A}_b \quad \text{with} \quad \|a_n\| = n, \qquad a_n \leqq a_{n+1}$$

and $a = \tau\text{-}\lim\limits_{n\to\infty} a_n$. Therefore $0 < a_n \leqq ne$ and there is a number $n_0 \in \mathrm{N}$ such that $a_n < ne \;\forall\, n \geqq n_0$.
In the following we consider only indices $n \geqq n_0$. Put $b_n = ne - a_n$. Regarding the functions $b_n(t)$ one finds: $0 < b_n \leqq b_{n+1}$. For $F_n = \{t \in X \mid b_n(t) = 0\}$ we obtain

$$F_n \neq \emptyset, \qquad F_n \neq X \quad \text{and} \quad F_n \supseteq F_{n+1}. \tag{**}$$

Further, the extremal continuous states of $\mathscr{A}_0$ are the "point functionals" of $C(X)$, i.e. the states $\omega_{t_0}(a) = a(t_0)$ $(t_0 \in X)$. These states can be extended to elements of ex S by Lemma 5. From this considerations and (**) it follows for the sets

$$R_n = \{\omega \in \text{ex}\, S \mid \omega(b_n) = 0\}:$$

$$R_n \neq \emptyset \;\forall\, n \in \mathrm{N} \quad \text{and} \quad R_n \neq \text{ex}\, S; \qquad R_{n+1} \subseteq R_n \quad \text{since} \quad b_n \leqq b_{n+1}.$$

Consider now the sets

$$\mathscr{I}_n = \bigcap_{\omega \in R_n} \mathscr{I}_\omega \quad \text{where} \quad \mathscr{I}_\omega = \{x \in \mathscr{A} \mid \omega(x^*x) = 0\}.$$

Then, $\mathscr{I}_n$ is a closed left ideal in $\mathscr{A}[\tau]$, $b_n \in \mathscr{I}_n$ (and hence $\mathscr{I}_n \neq \{0\}$) and $\mathscr{I}_n \subseteq \mathscr{I}_{n+1}$.
Now, let us regard $\mathscr{I} = \bigcup\limits_{n \in \mathrm{N}} \mathscr{I}_n$. Obviously, $\mathscr{I}$ is a proper left ideal in $\mathscr{A}$. Now we show that $\mathscr{I}$ is dense. Assuming the converse then by Prop. 4 there is an element $\sigma \in \text{ex}\, S$ such that $\mathscr{I} \subseteq \mathscr{I}_\sigma = \{x \in \mathscr{A} \mid \sigma(x^*x) = 0\}$. But $\sigma(b_n) = \sigma(ne - a_n) = n - \sigma(a_n) \geqq n - \sigma(a) > 0$ for sufficently large n, hence $b_n \notin \mathscr{I}_\sigma$ for such n, and so we have a contradiction. Thus, our proof is complete ∎

Remarks: 1. The converse of our theorem is not true, i.e. there are LMC*-algebras without unbounded elements containing dense ideals. Such algebras one can find already in the commutative case. To see this take a pseudocompact and locally compact, but not compact, completely regular space X. (An example of such a space one finds in [4], it is a space of ordinals with suitable chosen topology.) We take the algebra $\mathscr{A} = C(X)$ with the topology τ given by the seminorms $p_K(x) = \sup_{t \in K} |x(t)|$ where K runs over all compact subsets of X. $\mathscr{A}[\tau]$ is a LMC*-algebra, the completeness is given by the locally compactness of X. Since X is pseudocompact, every continuous function on X is bounded, hence $\mathscr{A}_b = \mathscr{A}$. But there is at least one dense ideal in $\mathscr{A}[\tau]$. To see that take the one-point-compactification X^* of X and the ideal of all functions vanishing in a neighbourhood of the adjoint point.

2. In the commutative case, the dense maximal ideals are in one-to-one correspondence to the extremal states of $\mathscr{A}_b[\|\cdot\|]$, not extendable to continuous states of $\mathscr{A}[\tau]$. The question, wether it is so in the general (noncommutative) case, is yet open. The structure of dense maximal ideals was described only in the case that the LMC*-algebra is a direct product of C*-algebras [2].

REFERENCES:

[1] DIXMIER, J.: Les C^*-algebres et leurs representations. Gauthier-Villars: Paris 1969.
[2] FRITZSCHE, M.: Zur Idealtheorie topologischer *-Algebren. Diss. A: Leipzig 1978.
[3] FRITZSCHE, M.: Über Ideale in LMC*-Algebren. Wiss. Zeitschr. der KMU Leipzig 3 (1978), 253–259.
[4] GILLMAN, L., and N. JERISON: Rings of Continuous Functions. Van Nostrand: Princeton 1960.
[5] INOUE, A.: Locally C*-algebras. Mem. Fac. Sci. Kyushu Univ. Ser. A 25 (1971), 197–235.
[6] SCHMÜDGEN, K.: Beiträge zur Theorie topologischer *-Algebren und ihrer Realisierungen. Diss. A: Leipzig 1973.
[7] SCHMÜDGEN, K.: Über LMC*-Algebren. Math. Nachrichten 68 (1975), 167–181.
[8] ŽELASKO, W.: On maximal ideals in commutative m-convex algebras. Studia mathematica 58 (1976), 291–298.

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