Zeitschrift für Analysis
und ihre Anwendungen
Bd. 1 (3) (1982), S. 85–90

Remarks on the dual least action principle

M. WILLEM

Es sei L ein selbstadjungierter Operator mit abgeschlossenem Wertevorrat in einem Hilbertraum H, und es sei $\psi: H \to \mathbf{R}$ eine konvexe Funktion. Unter der Voraussetzung, daß keine Resonanz vorhanden ist, wird die Frage behandelt, ob der Wertevorrat von $L + \partial\psi$ mit ganz H zusammenfällt.

Пусть L — самосопряженный оператор с замкнутой областью значений в гильбертовом пространстве H, $\psi: H \to \mathbf{R}$ — выпуклая функция. В одном нерезонансном случае даются достаточные условия для того, чтобы область значений оператора $L + \partial\psi$ совпадала с пространством H.

Let L be a self-adjoint operator with a closed range in a Hilbert space H and let ψ be a convex function on H. Under a non resonance assumption the surjectivity of $L + \partial\psi$ is studied.

Introduction

Let H be a real Hilbert space, let $L: D(L) \subset H \to H$ be a self-adjoint operator with a closed range and let $\psi: H \to \mathbf{R}$ be a continuous convex function. The surjectivity of $L + \partial\psi$ is studied under a non resonance condition due to DOLPH [8]. The basic tool is the dual least action principle of Clarke and Ekeland. In contrast to the previous applications of this principle, we consider the case when the right inverse of L is not necessarily a compact operator.

1. The dual least action principle

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and corresponding norm $|\cdot|$. Let $L: D(L) \subset H \to H$ be a self-adjoint operator with a closed range and let $\psi: H \to \mathbf{R}$ be a continuous convex function. Let α, β, γ and c be real numbers such that $0 < \beta \leqq \gamma < \alpha$ and

$$(A_1)\ \sigma(L) \cap]-\alpha, 0[= \phi,$$

where $\sigma(L)$ denotes the spectrum of L, (A_2) for every $u \in H$,

$$\beta \frac{|u|^2}{2} - c \leqq \psi(u) \leqq \gamma \frac{|u|^2}{2} + c.$$

Let us write

$$K = \big(L \mid D(L) \cap R(L)\big)^{-1},$$

$$\psi^*(v) = \sup_{u \in H} [(v, u) - \psi(u)], \qquad v \in H,$$

and

$$\varphi(v) = \frac{1}{2} (Kv, v) + \psi^*(v), \qquad v \in R(L).$$

The function ψ^* is the Fenchel transform of ψ. The present formulation of the "dual action" φ was introduced in [5] for hyperbolic problems and in [9] for hamiltonian systems. See [7] and [10] for other abstract formulations.

Lemma 1: *Under assumption* A_2, *if* φ *has a local minimum on* $R(L)$, *then equation*

$$-Lu \in \partial\psi(u) \tag{1}$$

is solvable.

Proof: If φ has a local minimum at $v \in R(L)$, then for every $h \in R(L)$ sufficiently small and for every $t \in]0, 1[$, we have

$$\psi^*(v) \leqq \psi^*(v + th) + t(Kv, h) + \frac{t^2}{2}(Kh, h).$$

Thus

$$-(Kv, h) \leqq \frac{\psi^*(v + th) - \psi^*(v)}{t} + \frac{t}{2}(Kh, h).$$

If $t \downarrow 0$, we obtain denoting by $\delta^+\psi^*(v, \cdot)$ the right Gateaux variation at v

$$-(Kv, h) \leqq \delta^+\psi^*(v, h).$$

Since $\delta^+\psi^*(v, \cdot)$ is positively homogeneous and subadditive, the Hahn-Banach theorem insures the existence of $w \in \operatorname{Ker} L$ such that, for every $h \in H$,

$$(w, h) - (Kv, h) \leqq \delta^+\psi^*(v, h).$$

But then

$$(w - Kv, h) \leqq \psi^*(v + h) - \psi^*(v),$$

i.e. $w - Kv \in \partial\psi^*(v)$. It follows that $v \in \partial\psi(w - Kv)$. If $u = w - Kv$, $-Lu = v$ and u is a solution of (1) ∎

The following lemma has been widely used in the study of hamiltonian systems (see [9]).

Lemma 2: *Under assumptions* A_1 *and* A_2, φ *is coercive on* $R(L)$, *i.e.* $\varphi(v) \to \infty$, *as* $|v| \to \infty$.

Proof: It suffices to observe that A_1 and A_2 imply that

$$\forall v \in R(L) \qquad -\frac{1}{\alpha}|v|^2 \leqq (Kv, v)$$

and

$$\forall v \in H \qquad \frac{1}{\gamma}\frac{|v|^2}{2} - c \leqq \psi^*(v) \text{ ∎}$$

2. Surjectivity theorems

The following result generalizes Theorem 2.1 of [8] and Theorem 3.2 of [6]. It extends Theorem 1 of [7].

Theorem 1: *Under assumptions* A_1 *and* A_2, *if* $\sigma(L) \cap]-\infty, 0[$ *consists of isolated eigenvalues with finite multiplicity, then* $L + \partial\psi$ *is onto.*

Proof: Since, for any $f \in H$, the function $\psi(u) - (f, u)$ has the same properties as $\psi(u)$, it suffices to prove that (1) is solvable. By assumption $\frac{1}{2}(Kv, v)$ is weakly lower semi-continuous (w.l.s.c.). Therefore φ itself is w.l.s.c. By Lemma 2, φ has a minimum on $R(L)$ and, by Lemma 1, (1) is solvable ∎

The following result extends Theorem 3.7 of [1].

Theorem 2: *Under assumptions* A_1 *and* A_2, *if*
(a) ψ *is differentiable and* $\partial\psi$ *is Lipschitzian with constant* k
(b) $\sigma(L) \cap]-k, 0[$ *consists of isolated eigenvalues with finite multiplicity, then* $L + \partial\psi$ *is onto.*

Proof: As in Theorem 1, it suffices to prove that (1) is solvable. If $f_i \in \partial\psi^*(v_i)$ we have $v_i = \partial\psi(f_i)$ $(i = 1, 2)$. By Corollary 10 of [3], assumption (a) implies that

$$(v_1 - v_2, f_1 - f_2) \geqq \frac{1}{k} |v_1 - v_2|^2.$$

Thus $\varphi_1(v) = \psi^*(v) - \frac{1}{k}\frac{|v|^2}{2}$ is convex.

Let $\{P_\lambda : \lambda \in \mathbf{R}\}$ be the spectral resolution of L and let us write

$$Q_1 = \int_{]-\infty,-k]} dP_\lambda, \qquad Q_2 = \int_{]-k,\infty[} dP_\lambda, \qquad u_i = Q_i u \qquad (i = 1, 2;\, u \in H).$$

It follows from assumption (b) that $\varphi_2(v) = \frac{1}{2}(Kv_2, v_2)$ is w.l.s.c. on $R(L)$. Moreover, for any $v \in R(L)$,

$$(Kv_1, v_1) \geqq -\frac{1}{k}|v_1|^2 \geqq -\frac{1}{k}|v|^2.$$

Thus $\varphi_3(v) = \frac{1}{2}(Kv_1, v_1) + \frac{1}{k}\frac{|v|^2}{2}$ is convex. Finally $\varphi = \varphi_1 + \varphi_2 + \varphi_3$ is w.l.s.c. and, by Lemmas 1 and 2, (1) is solvable ∎

As an obvious consequence of Theorem 2 we obtain:

Corollary 1: *Under assumptions* A_1 *and* A_2, *if* ψ *is differentiable and* $\partial\psi$ *is Lipschitzian with constant* α, *then* $L + \partial\psi$ *is onto.*

Remark: Corollary 1 generalizes Theorem 1.2 of [8], Theorem 1 of [11] and extends Theorem I.12 of [4].

3. Periodic solutions of a nonlinear hyperbolic equation

This section is devoted to the existence of 2π-periodic solutions in t and x of the nonlinear hyperbolic equation

$$u_{tt} - u_{xx} + \lambda u + \partial j(u) = f(t, x)$$

where $j: \mathbf{R} \to \mathbf{R}$ is convex and $f \in H = L^2([0, 2\pi]^2)$. We shall only consider the case when $\lambda = 1$. The other cases are left to the reader. The case when $\lambda = 0$ is treated in [7].

Let A be the linear operator defined by

$$D(A) = \{u \in C^2([0, 2\pi]^2) : u(0, \cdot) - u(2\pi, \cdot) = u_t(0, \cdot) - u_t(2\pi, \cdot)$$
$$= u(\cdot\,, 0) - u(\cdot\,, 2\pi) = u_x(\cdot\,, 0) - u_x(\cdot\,, 2\pi) = 0\}$$
$$Au = u_{tt} - u_{xx}.$$

Let us write $A = A^*$. Then A is self-adjoint and $\sigma(A)$, which is the set of odd integers and of multiples of 4, consists of eigenvalues which are of finite multiplicity except 0 (see [12]).

Let us define $\psi : H \to \overline{\mathbf{R}}$ by

$$\psi(u) = \int_0^{2\pi} \int_0^{2\pi} j\big(u(t, x)\big)\, dt\, dx.$$

The following theorem extends the results of [13].

Theorem 3: *Assume that there exists β, γ and $c \in \mathbf{R}$ such that $0 < \beta \leqq \gamma < 2$ and, for every $u \in \mathbf{R}$,*

$$\beta \frac{u^2}{2} - c \leqq j(u) \leqq \gamma \frac{u^2}{2} + c, \tag{2}$$

then equation

$$Au + u + \partial\psi(u) = f$$

is solvable for every $f \in H$.

Proof: It suffices to apply Theorem 1 with $L = A + I$ and $\alpha = 2$ ∎

4. The growth of $\partial\psi$

A sharp estimate of the growth of $\partial\psi$ under assumption A_2 is given.

Proposition 1: *Under assumption A_2, there exists $c' \in \mathbf{R}$ such that, for every f, $u \in H$,*

$$f \in \partial\psi(u) \Rightarrow |f| \leqq 2\gamma\, |u| + c'. \tag{3}$$

Proof: Let $u \in H$ and $f \in \partial\psi(u)$. Assuming $f \neq 0$, let us write $g = f/|f|$. We have, for every $t \in \mathbf{R}$,

$$(f, tg - u) + \psi(u) \leqq \psi(tg)$$

or

$$0 \leqq \psi(tg) - t\,|f| + (f, u) - \psi(u).$$

By assumption

$$0 \leqq t^2 \frac{\gamma}{2} + c - t\,|f| + |u|\,|f| - c''$$

where $c'' = \inf_{v \in H} \psi(u) > -\infty$. Then

$$|f|^2 \leqq 4 \frac{\gamma}{2} (c - c'' + |u|\,|f|)$$

and (3) follows easly ∎

Remark: The argument of the proof is due to Brézis and Nirenberg [4, p. 312].

The following example shows that estimate (3) is sharp. Let $\gamma \in]1/2, 1[$, $\beta \in]0, 2\gamma - 1[$, $p = \dfrac{\gamma - \beta}{1 - \gamma}$ and $q = \dfrac{p - \beta}{p - 1}$. Let us define inductively $t_0 = \gamma/\beta$, $u_n = pt_n$ and $t_{n+1} = u_n/\beta$. Let us define $r\colon \mathbf{R} \to \mathbf{R}$ by

$$r(u) = \begin{cases} \gamma u, & u \in]-\infty, 1] \\ \gamma, & u \in]1, t_0] \\ qu + (1 - q)\, u_n, & u \in]t_n, u_n] \\ u_n, & u \in]u_n, t_{n+1}]. \end{cases}$$

If $\psi(u) = \int_0^u r(s)\, ds$, for any $u \in \mathbf{R}$,

$$\beta \frac{u^2}{2} \leqq \psi(u) \leqq \gamma \frac{u^2}{2}.$$

But, for any $n \in \mathrm{N}$, $\partial\psi(u_n) = u_n$ and $u_n \to \infty$, $n \to \infty$. Moreover, if $L\colon \mathbf{R} \to \mathbf{R}$ is defined by $Lu = -u$, assumptions A_1 and A_2 are satisfied with $\alpha = 1$, and, for any $n \in \mathrm{N}$, $Lu_n + \partial\psi(u_n) = 0$. Thus conditions A_1 and A_2 doesn't imply any a priori bound for the solutions of (1) (see also [8] and [2]).

REFERENCES

[1] Amman, H.: Saddle points and multiple solutions of differential equations. Math. Z. 169 (1979), 127–166.

[2] Bahri, A., and J. M. Morel: Image de la somme de deux opérateurs. C. R. Acad. Sci. Paris 287 (1978) A, 719–722.

[3] Baillon, J. B., and G. Haddad: Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones. Isr. J. Math. 26 (1977), 137–150.

[4] Brezis, J. H., and L. Nirenberg: Characterizations of the ranges of some nonlinear operators and applications to boundary value problems. Ann. Scuola Norm. Pisa 5 (1978), 225–326.

[5] Brezis, H., Coron, J. M., and L. Nirenberg: Free vibrations of a nonlinear wave equation and an theorem of P. Rabinowitz. Preprint.

[6] Castro, A.: Hammerstein equations with indefinite kernel. Internat. J. Math. & Math. Sci. 1 (1978), 187–201.

[7] Coron, J. M.: Résolution de l'équation $Au + Bu = f$ où A est linéaire auto-adjoint et B est un opérateur potentiel non linéaire. C. R. Acad. Sci. Paris 288 (1979) A, 805–808.

[8] DOLPH, C. L.: Nonlinear integral equations of the Hammerstein type. Trans. Amer. Math. Soc. 66 (1949), 289–307.
[9] EKELAND, I.: Oscillations de systemes hamiltoniens non linéaires III. Preprint.
[10] EKELAND, I., and J. M. LASRY: (to appear).
[11] MAWHIN, J.: Contractive mappings and periodically perturbed conservative systems. Arch. Math. 12 (1976), 67–73.
[12] MAWHIN, J.: Solutions périodiques d'équations aux dérivées partielles hyperboliques non linéaires. In: Mélanges Vogel (Rybak, Janssens et Jessel ed.). Presses Univ. Bruxelles: Bruxelles 1978, 301–319.
[13] MAWHIN, J.: Periodic solutions of nonlinear dispersive wave equations. In: Constructive methods for nonlinear boundary value problems and nonlinear oscillations (Albrecht, Collatz and Kirchgässner ed.). Birkhäuser Verlag: Basel 1979, 102–109.

Manuskripteingang: 08. 05. 1981

VERFASSER:

Dr. MICHEL WILLEM
Universite Catholique de Louvain
Institut de Mathématique Pure et Appliquée, Chemin du Cyclotron 2
B-1348 Louvain-la-Neuve (Belgique)