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On some parabolic boundary control problems with constraints on the control and functional-constraints on the state

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In dieser Arbeit werden parabolische Randsteuerprobleme in einem n-dimensionalen beschränkten Gebiet untersucht, bei denen zusätzlich zu einer punktweisen Steuerbeschränkung noch endlich viele lineare Funktionalbeschränkungen an den Zustand gegeben sind. Als Zielfunktional wird die L_∞- oder L_2-Norm der Differenz zwischen dem Endzustand und einem gegebenen (erwünschten) Zustand verwendet.

Es wird ein verallgemeinertes Bang-Bang-Prinzip bewiesen, welches besagt, daß unter natürlichen Voraussetzungen fast überall (bezüglich der Zeitvariablen) entweder die optimale Steuerung oder der zugehörige Zustand Werte am Rand des entsprechenden zulässigen Bereiches annimmt. Die Voraussetzungen werden anhand von Beispielen diskutiert.

В этой работе рассматриваются проблемы оптимального управления, описываемые параболическими уравнениями с третьим краевым условием в n-мерной ограниченной области. При этом кроме точечного ограничения на управление задано конечное число линейных функциональных ограничений на фазовые координаты. Целевой функцией служит норма разности конечного состояния и заданного (потребуемого) состояния в смысле пространств L_∞ или L_2.

Доказывается обобщение принципа релейности. Принцип состоит в том, что при естественных предположениях почти всюду (относительно переменной времени) либо оптимальное управление либо соответствующее состояние принимает свое значение на границе допустимой области. Предположения обсуждаются на примерах.

In this paper parabolic boundary control problems in a n-dimensional bounded domain are investigated, where in addition to a pointwise constraint on the control finitely many linear functional constraints on the state are given. As performance functional the L_∞- or L_2-norm of the difference between the final state and a given (desired) state is used.

A generalized bang-bang-principle is proved which expresses that under natural assumptions almost everywhere (with respect to the time) either the optimal control or the corresponding state is acting on the boundary of its constraints. The assumptions are discussed by examples.

1. Introduction

The problem to heat a given body on its surface from certain initial temperature to a desired temperature distribution may be regarded as the motivation for an enormous list of papers on "minimum-norm boundary control for parabolic differential equations".

Beginning with the basic work of Yu. Egorov [14] in the majority of papers constraints on the (boundary-) control were imposed. To get some information on this topic the reader is referred for instance to Lions [6] (where the solution of the parabolic initial-boundary value problem is defined in Sobolev spaces), Glashoff and Weck [4] or Schmidt and Weck [9] (who use the definition of generalized solutions by means of a Green's function). Some new aspects of time-optimal control were presented by Schmidt [10] and [11].

A basic result obtained in these and other papers is the so-called "bang-bang-principle" asserting that under natural assumptions the optimal control admits almost everywhere values at its lower or upper bound.

The main topic of this paper is to investigate a class of parabolic boundary control problems in a N-dimensional domain, defined in [4], with additional general state-constraints. The result is the "generalized bang-bang-principle" which expresses the fact that almost everywhere (with respect to the time) either the optimal control admits its lower (upper) bound or the corresponding state is acting on its boundary.

In this paper a large amount of results and notation will be adopted from the basic work [4]. The corresponding statements are listed in section 3.

As the presentation of the results requires a large amount of pages other aspects of state-constrained problems are not investigated. The reader is referred for instance to recent papers of ŁASIECKA [5], MACKENROTH [8] or SOKOŁOWSKI and SOSNOWSKI [12].

2. Notation

Sets: $\Omega \subseteq R^N (N \in \mathbf{N}, N \geqq 2)$ is a bounded domain with C_∞-boundary $\partial\Omega$; $T :=$ fixed positive final time; $\Gamma := [0, T] \times \partial\Omega$.

Spaces and pairings: If X is a Banach space its adjoint space is denoted by X'. The adjoint operator to a linear operator A is written A'. For a suitable compact set σ the pairing between $C'(\sigma)$ and $C(\sigma)$ is denoted with $\langle \cdot, \cdot \rangle (\sigma)$, the corresponding one between $L_q(\sigma)$ and $L_p(\sigma)$ with $[\cdot, \cdot] (\sigma)$ $(1 \leqq p < \infty,\ 1/q + 1/p = 1)$.

$$\langle \alpha, y \rangle (0, T) := \langle \alpha, y \rangle ([0, T]),$$

$$\langle \alpha, y \rangle := \langle \alpha, y \rangle (\bar{\Omega}).$$

Norms: $|\cdot|_N :=$ Euclidean norm of R^N, $N \in \mathbf{N}$, $|\cdot| := |\cdot|_1$, $\|\cdot\|_p (\sigma) :=$ norm of $L_p(\sigma)$, $1 < p \leqq \infty$.

Lagrange-function: In this paper the Lagrange-function

$$\mathscr{L}(u, y) := [S'\alpha, u] (\sigma) - \sum_{i=1}^{k} \langle A_i u - c_i^1, y_i^1 \rangle (0, T) + \sum_{i=1}^{k} \langle A_i u - c_i^2, y_i^2 \rangle (0, T) \tag{2.1}$$

is used, where S, α, u, k, A_i, y_i^j, c_i^j become clear from the context and $\sigma := \Gamma$ in section 4, $\sigma := [0, T]$ in section 5.

3. The parabolic initial-boundary value problem

3.1. Formulation

Let L denote the symmetric and uniformly elliptic operator

$$Lw(x) := \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} w(x) \right) + a(x)\, w(x),$$

where $a(\cdot)$, $a_{ij}(\cdot) \in C_\infty(\bar{\Omega})$, $a_{ij}(\cdot) = a_{ji}(\cdot)$, and

$$\sum_{i,j=1}^{N} a_{ij}(x)\, p_i p_j \geqq c_0\, |p|_N{}^2, \qquad c_0 > 0.$$

Boundary conditions are defined as follows: There are introduced $\beta > 0$, $n(\xi) :=$ outer unit normal in $\xi \in \partial\Omega$,

$$\partial w(\xi) := \sum_{i,j=1}^{N} n_i(\xi)\, a_{ij}(\xi)\, \frac{\partial}{\partial \xi_j}\, w(\xi), \qquad \xi \in \partial\Omega$$

(the conormal derivative), and

$$Bw(\xi) := (1/\beta)\, \partial w(\xi) + w(\xi).$$

The basic initial-boundary value problem is

$$\frac{\partial}{\partial t}\, w(t, x) - L_x w(t, x) = 0, \qquad (t, x) \in (0, T] \times \Omega, \tag{3.1}$$

$$w(0, x) = 0, \qquad x \in \bar{\Omega}, \tag{3.2}$$

$$B_\xi w(t, \xi) = u(t, \xi), \qquad (t, \xi) \in (0, T] \times \partial\Omega. \tag{3.3}$$

(The subscripts in L and B indicate that L and B are acting with respect to the space coordinates x and ξ respectively.) The function u will be given in a certain function space.

3.2. The Greens function and its series-representation

For $u(\cdot, \cdot) \in C(\Gamma)$ there exists a solution of (3.1—3) in classical sense. This solution can be explicitely represented by

$$w(u; t, x) = \beta \int_0^t \int_{\partial\Omega} G(x, \xi, t, s)\, u(s, \xi)\, dS_\xi\, ds \tag{3.4}$$

($dS_\xi :=$ surface element on $\partial\Omega$), where G is the *Green's function* which is defined, continuous and non-negative on

$$\{(x, \xi, t, s) \mid (x, \xi) \in \bar{\Omega} \times \bar{\Omega},\ 0 \leqq s \leqq t \leqq T\} \setminus \{(x, \xi, t, s) \mid x = \xi,\ t = s\}.$$

There holds the estimation

$$|G(x, \xi, t, s)| \leqq c(t - s)^{-N/2} \exp\left(-C\, |x - \xi|_N{}^2/(t - s)\right) \tag{3.5}$$

($c, C > 0$). Using a generalization of Lemma 1, chpt. 1.3, in A. FRIEDMAN's book [2] to L_p-spaces one shows similarly to the proof of Lemma 1 in [3] that $w(u; \cdot, \cdot)$ defined by (3.4) is in $C([0, T] \times \bar{\Omega})$, if $u(\cdot, \cdot) \in L_p(\Gamma)$ and $p > N + 1$. Now once and for all $p > N + 1$ is kept fixed, and for $u \in L_p(\Gamma)$ the corresponding function $w(u; t, x)$ is called *generalized solution* of (3.1—3).

Denote with λ_k and v_k the non-negative eigenvalues and corresponding normalized eigenfunctions of the eigenvalue problem

$$Lv(x) + \lambda v(x) = 0, \qquad x \in \Omega, \tag{3.6}$$

$$Bv(\xi) = 0, \qquad \xi \in \partial\Omega. \tag{3.7}$$

It is known that $\{v_k\}$ is a complete orthonormal system in $L_2(\Omega)$, $\lambda_k \sim ck^{2/N}$ if $k \to \infty$, and that $\|v_k\|_\infty(\bar{\Omega}) = \mathcal{O}(k^m)$ with certain $m \in \mathbf{N}$. The Green's function G can be expressed in terms of the eigenfunctions v_k by

$$G(x, \xi, t, s) = \sum_{n=1}^{\infty} v_n(x)\, v_n(\xi) \exp\big(-\lambda_n(t - s)\big). \tag{3.8}$$

(A nice discussion of that relation is contained in [4], Appendix A.) In [4] it was also shown that the linear hull of the system $\{v_n\}$ is dense in $C(\bar{\Omega})$.

3.3. Some linear operators and their adjoints

Here some important properties of certain linear operators are presented, which will be frequently used in the following sections. Throughout this paper the operator S is defined by

$$(Su)\,(x) := w(u;\, T, x), \qquad x \in \bar{\Omega}, \tag{3.9}$$

where $w(u; t, x)$ is now defined by (3.4). As already mentioned, the representation (3.4) can be continuously extended to L_p-spaces with $p > N + 1$, and hence S is a linear continuous mapping from $L_p(\Gamma)$ into $C(\bar{\Omega})$. For the adjoint operator $S' : C'(\bar{\Omega}) \to L_q(\Gamma)$ $\big(q = p/(p - 1)\big)$ there holds

Lemma 3.1: *Define for fixed $\alpha \in C'(\bar{\Omega})$ and $(t, x) \in [0, T)$*

$$p(t, x) := \beta \sum_{n=1}^{\infty} \langle \alpha, v_n \rangle \exp\big(-\lambda_n(T - t)\big)\, v_n(x). \tag{3.10}$$

Then

(i) $p \in C_\infty([0, T) \times \bar{\Omega})$;

(ii) $-\dfrac{\partial}{\partial t}\, p(t, x) = L_x p(t, x)$ *on* $[0, T) \times \Omega$,

$B_\xi p(t, \xi) = 0$ *on* $[0, T) \times \partial\Omega$;

(iii) $p|_\Gamma \in L_q(\Gamma)$;

(iv) $S'\alpha = p|_\Gamma$.

Proof: Relations (i) to (iv) were proved for the case $p = \infty$, $q = 1$ by GLASHOFF and WECK [4], Lemma 3. The only difference is given here by $q > 1$. Since $S : L_p(\Gamma) \to C(\Omega)$ it must hold $S'\alpha \in L_q(\Gamma)$. Hence (iii) is implied by (iv) of Lemma 3, [4] ∎

It was proved by SCHMIDT and WECK [9] that for non-zero $p(t, x)$ the set

$$\mathcal{N} = \{(t, x) \in \Gamma \mid p(t, x) = \partial p(t, x)/\partial n(x) = 0\}$$

has measure zero. This is an important tool to show bang-bang-theorems.

Now another linear operator will be investigated. Let φ be a continuous linear functional on $C(\bar{\Omega})$ and define A by

$$(Au)\,(t) := \langle \varphi, w(u; t, \cdot) \rangle. \tag{3.11}$$

A is a continuous linear mapping from $L_p(\Gamma)$ into $C[0, T]$, since $w(u; \cdot, \cdot) \in C([0, T] \times \bar{\Omega})$, if $u \in L_p(\Gamma)$. Thus there is a continuous adjoint operator $A' : C'[0, T] = \mathrm{NBV}[0, T] \to L_q(\Gamma)$. The representation of A' is needed only on certain subintervals of $[0, T]$. There holds

Lemma 3.2: *For $y(\cdot) \in \mathrm{NBV}[0, T]$ assume $y(t) \equiv y(a)$ on $[a, b) \subseteq [0, T]$. Then*

$$(A'y)(t, x) = \beta \sum_{n=1}^{\infty} \langle \varphi, v_n \rangle v_n(x) \int_t^T \exp\left(-\lambda_n(s - t)\right) dy(s) \tag{3.12}$$

on $[a, b) \times \partial\Omega$.

Proof: Take $\varepsilon > 0$ sufficiently small, define $z(t, x)$ to be the right side of (3.12) and put

$$H := \{u \in L_p(\Gamma) \mid \operatorname{supp} u \subseteq [a, b - 2\varepsilon] \times \partial\Omega\}.$$

H is a linear subspace of $L_p(\Gamma)$. For $u \in H$

$$\langle y, Au \rangle (0, T) = \int_0^a (Au)(t)\, dy(t) + \int_{b-\varepsilon}^T (Au)(t)\, dy(t),$$

and, because $u(t, x) = 0$ on $[0, a] \times \partial\Omega$

$$\langle y, Au \rangle (0, T) = \int_{b-\varepsilon}^T (Au)(t)\, dy(t)$$

$$= \beta \int_{b-\varepsilon}^T \left\langle \varphi, \sum_{n=1}^{\infty} \int_a^{b-2\varepsilon} \int_{\partial\Omega} \exp\left(-\lambda_n(t - s)\right) v_n(\xi)\, u(s, \xi)\, dS_\xi\, ds \right\rangle dy(t).$$

Since $t - s \geqq \varepsilon$ the series in the brackets is uniformly convergent, thus after changing the order of integration and summation there is obtained

$$\langle y, Au \rangle (0, T) = \beta \int_a^{b-2\varepsilon} \int_{\partial\Omega} \sum_{n=1}^{\infty} \langle \varphi, v_n \rangle v_n(\xi) \int_t^T \exp\left(-\lambda_n(s - t)\right) dy(s)\, u(t, \xi)\, dS_\xi\, dt$$

$$= [z, u](\Gamma),$$

since $u \in H$. This, however, implies $(A'y)(t, x) = z(t, x)$ on $[a, b - 2\varepsilon)$, and $\varepsilon \to 0$ proves the statement ∎

4. The boundary control problem

4.1. Existence of optimal controls and optimality conditions

Let there be given $z(\cdot) \in C(\bar{\Omega})$, $c_i^1(\cdot)$ and $c_i^2(\cdot) \in C[0, T]$ $(i = 1, 2, \ldots, k)$ with $c_i^2(t) - c_i^1(t) > 0$ on $[0, T]$, and linear functionals $\varphi_i \in C'(\bar{\Omega})$. Then the control problem is

$$\|Su - z\|_\infty (\bar{\Omega}) = \min! \qquad \text{subject to } u \in L_\infty(\Gamma), \tag{4.1}$$

$$\|u_\infty\| (\Gamma) \leqq 1, \tag{4.2}$$

$$c_i^1(t) \leqq \langle \varphi_i, w(u; t, \cdot) \rangle \leqq c_i^2(t) \qquad (t \in [0, T];\ i = 1, 2, \ldots, k). \tag{4.3}$$

The presence of the state-constraints (4.3) is the main point of this paper and makes the problem different to that investigated in [4]. The reason to impose this general type of constraints is the fact that several kinds of state constraints for heating problems (especially certain stress-constraints) can be expressed in the form (4.3).

Remark: All the theory is written down for the L_∞-norm in (4.1). The same results hold for the L_2-norm, which is even easier to handle. Then only $\langle \cdot, \cdot \rangle$ is to be replaced by $[\cdot, \cdot]\,(\Omega)$ and $C'(\bar{\Omega})$ by $L_2(\Omega)$.

Theorem 4.1: *If there is at least one $u \in L_p(\Gamma)$ such that (4.2) and (4.3) are fulfilled, then there exists an optimal control.*

Proof: The set $\{u \mid \|u\|_\infty\,(\Gamma) \leqq 1\}$ is weakly compact in $L_p(\Gamma)$. The set of all $u \in L_p(\Gamma)$ which fulfil (4.3) is weakly closed in $L_p(\Gamma)$ and the functional (4.1) convex and continuous, hence weakly l.s.c. Thus existence is assured by the theorem of Weierstrass ∎

Define now linear operators A_i by

$$(A_i u)(t) := \langle \varphi_i, w(u; t, \cdot) \rangle \quad (i = 1, 2, \ldots, k).$$

All A_i are linear and continuous mappings from $L_p(\Gamma)$ into $C[0, T]$. An optimal control of problem (4.1—3) is now supposed to exist and denoted with u_0. Then the following optimality conditions are obtained:

Theorem 4.2: *Suppose that there exists $\bar{u}$ with $\|\bar{u}\|_\infty\,(\Gamma) \leqq 1$ and*

$$c_i^1(t) < (A_i \bar{u})\,(t) < c_i^2(t) \qquad (t \in [0, T];\ i = 1, 2, \ldots, k) \tag{4.4}$$

("Slater-condition"). *If u_0 is an optimal control and $\|Su_0 - z\|_\infty\,(\bar{\Omega}) > 0$, then there exist $2k$ non-decreasing Lagrange multipliers y_i^1, y_i^2 from* NBV$[0, T]$ *and a non-zero $\alpha \in C'(\bar{\Omega})$ such that*

$$\min_{|u| \leqq 1} \left\{ (S'\alpha)\,(t, x) + \sum_{i=1}^{k} \left(A_i'(y_i^2 - y_i^1) \right) (t, x) \right\} u \tag{4.5}$$

is achieved by $u_0(t, x)$ a.e. on Γ and

$$\int_0^T (A_i u_0 - c_i^2)\,(t)\, dy_i^2(t) = \int_0^T (A_i u_0 - c_i^1)\,(t)\, dy_i^1(t) = 0 \qquad (i = 1, 2, \ldots, k) \tag{4.6}$$

("complementary slackness conditions").

Proof: Define $U = \{u \in L_p(\Gamma) \mid u$ fulfils (4.2—3)$\}$. Thus $\|Su_0 - z\|_\infty\,(\bar{\Omega}) = \inf_{u \in U} \|Su - z\|_\infty\,(\bar{\Omega})$. Since $\|Su_0 - z\|_\infty\,(\bar{\Omega}) > 0$ there is a non-zero $\alpha \in C'(\bar{\Omega})$ with $\langle \alpha, Su_0 - z \rangle = \inf_{u \in U} \langle \alpha, Su - z \rangle$. (See [4], (4.3)). Thus u_0 solves the linear continuous programming problem

$$\begin{aligned} &[S'\,\alpha, u]\,(\Gamma) = \min, \qquad \|u\|_\infty\,(\Gamma) \leqq 1, \\ &c_i^1(t) \leqq (A_i u)\,(t) \leqq c_i^2(t) \qquad (i = 1, 2, \ldots, k). \end{aligned} \tag{4.7}$$

The Slater-condition (4.4) assures that u_0 solves this program if and only if there exist Lagrange multipliers mentioned in the theorem such that (u_0, y_0) with $y_0 = (y_1^1, \ldots, y_k^1, y_1^2, \ldots, y_k^2)$ is a saddle point of $\mathscr{L}(u, y)$ defined in (2.1), i.e.

$$\mathscr{L}(u_0, y_0) = \min_{\|u\|_\infty(\Gamma) \leqq 1} \mathscr{L}(u, y_0) = \max_{y \geqq 0} \mathscr{L}(u_0, y) \tag{4.8}$$

($y \geqq 0$ means that all y_i^j are non-decreasing), see Luenberger [7], Corollary 1, p. 219, and Theorem 2, p. 221. Writing down the expression for $\mathscr{L}(u, y)$ one finds that the first equation of (4.8) implies (4.5) and the second one (4.6) ∎

4.2. Generalized Bang-Bang-Principle

For problems without the state-constraint (4.3) the so called bang-bang-principle can be shown to hold for the optimal control. The presence of state-constraints changes this behaviour. However, a natural generalization of the bang-bang-principle can be obtained. To prove this generalization the following *growth-condition* is essential:

If $\tau > 0$ and $\{b^n\} \subseteq R^k$ is a sequence of unit vectors, whose components do not change their sign for all $n \geqq n_0$, then

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{k} b_i^n \langle \varphi_i, v_n \rangle \right)^2 \exp(\lambda_n \tau) = +\infty. \tag{4.9}$$

If only one functional constraint is given ($k = 1$), then (4.9) is equivalent to the simpler condition

$$\sum_{n=1}^{\infty} \langle \varphi_1, v_n \rangle^2 \exp(\lambda_n \tau) = +\infty, \qquad \tau > 0.$$

Theorem 4.3: *Suppose for an optimal control u_0 that*

(i) $\|Su_0 - z\|_\infty (\bar{\Omega}) > 0$,

(ii) *the Slater-condition* (4.4) *is fulfilled,*

(iii) *the growth-condition* (4.9) *is met, and*

(iv) $\alpha \notin \operatorname{span} \{\varphi_1, \ldots, \varphi_k\}$, *where $\alpha \in C'(\bar{\Omega})$ is defined in Theorem* 4.2.

Then

$$\operatorname{mes} \{(t, x) \in \Gamma \mid |u_0(t, x)| < 1 \quad \textit{and} \quad c_i^1(t) < (A_i u_0)(t) < c_i^2(t)$$
$$(i = 1, 2, \ldots, k)\} = 0.$$

The assumptions of this theorem will be discussed by means of examples in Section 6. For the proof of Theorem 4.3. the following statement is used:

Lemma 4.1: *If $f(t)$ is continuous non-increasing and $y \in \mathrm{NBV}[a, b]$ is non-decreasing, then*

$$\int_a^b f(t)\, dy(t) \geqq f(b) \big(y(b) - y(a)\big).$$

Proof: Take any partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$. Then

$$\sum_{i=0}^{n} f(t_i) \big(y(t_i) - y(t_{i-1})\big) \geqq f(b) \sum_{i=0}^{n} \big(y(t_i) - y(t_{i-1})\big) = f(b) \left(y(b) - y(a)\right).$$

The proof is finished letting $n \to \infty$ ∎

Proof of Theorem 4.3: Define

$$M := \{t \in (0, T) \mid c_i^1(t) < (A_i u_0)(t) < c_i^2(t)\}.$$

M is union of countably many open intervals (components), since all $(A_i u_0)(t)$ are continuous in t. Now suppose that the theorem is not true. Thus there is a component $(a, b) \subseteq M$ and $\varepsilon > 0$ sufficiently small such that $\sigma = \{(t, x) \in (a, b - 2\varepsilon] \times \partial\Omega \mid |u_0(t, x)| < 1\}$ has positive measure. Hence (4.5) gives

$$\left(S'\alpha + \sum_{i=1}^{k} A_i'(y_i^2 - y_i^1)\right)(t, x) = 0 \tag{4.10}$$

almost everywhere on σ. Moreover (4.6) implies for $t \in (a, b)$

$$y_i^j(t) = y_i^j(a) \qquad (i = 1, 2, \ldots, k;\ j = 1, 2). \tag{4.11}$$

Now put $y_i := y_i^2 - y_i^1$, $\tau := b - 3\varepsilon/2$ and define $\alpha_1, \alpha_2 \in C'(\bar{\Omega})$ by

$$\langle \alpha_1, v_n \rangle := \langle \alpha, v_n \rangle \exp\big(-\lambda_n(T - \tau)\big),$$

$$\langle \alpha_2, v_n \rangle := \sum_{i=1}^{k} \langle \varphi_i, v_n \rangle \int_{b-\varepsilon}^{T} \exp\big(-\lambda_n(s - \tau)\big)\, dy_i(s).$$

α_1 and α_2 are uniquely determined linear continuous functionals on $L_2(\Omega) \supseteq C(\bar{\Omega})$, since the system $\{v_n\}$ is a base for $L_2(\Omega)$ and $\|v_k\|_\infty (\bar{\Omega}) = \mathcal{O}(k^m)$. Furthermore on $[0, \tau) \times \bar{\Omega}$ there is introduced

$$p_\tau(t, x) := \sum_{n=1}^{\infty} \langle \alpha_1 + \alpha_2, v_n \rangle \exp\big(-\lambda_n(\tau - t)\big)\, v_n(x). \tag{4.12}$$

Because of (4.11) the representation of the A_i' is given on $(a, \tau]$ by Lemma 3.2., (3.12), and after extracting the terms $\exp\big(-\lambda_n(\tau - t)\big)$ in the series for S' and A_i' it is found out that (4.10) implies

$$p_\tau(t, x) = 0 \tag{4.13}$$

a.e. on σ (remark that according to (4.11) the integration in A_i' is only to stretch over $[b - \varepsilon, T]$). Applying Lemma 3.1. with $T := \tau$ shows that $p_\tau(t, x)$ solves the equation $-\dfrac{\partial}{\partial t} p = Lp$ on $[0, \tau) \times \Omega$ together with $B_\xi p_\tau(t, \xi)|_\sigma = 0$. Thus (4.13) gives $\partial p_\tau(t, \xi)/\partial n(\xi) = 0$ on σ, and hence $p_\tau(t, x) \equiv 0$ on $[0, \tau) \times \bar{\Omega}$ is obtained using the result of Schmidt and Weck [9], Corollary 2.3., which has already been mentioned in Section 3.3. After integration of $v_n(x)\, p_\tau(t, x)$ one finds $\langle \alpha_1 + \alpha_2, v_n \rangle = 0$ $(n = 1, 2, \ldots)$ and therefore

$$\langle \alpha, v_n \rangle = -\sum_{i=1}^{k} \langle \varphi_i, v_n \rangle \int_{b-\varepsilon}^{T} \exp\big(\lambda_n(T - s)\big)\, dy_i(s) \tag{4.14}$$

is obtained for $n = 1, 2, \ldots$ Define $t_0 \geqq b$ to be the first point, where at least one $y_i^2(y_i^1)$ is increasing, i.e. $y_i^2(t) > y_i^2(b - \varepsilon)$ $\big(y_i^1(t) > y_i^1(b - \varepsilon)\big)$ for $t > t_0$. Assume at first $t_0 < T$:

Because of (4.6) there must hold $(A_i u_0)(t_0) = c_i^2(t_0)$ $\big(= c_i^1(t_0)\big)$, and the continuity of $(A_i u_0)(t)$ assures $(A_i u_0)(t) > c_i^1(t)$ $\big(< c_i^2(t)\big)$ on $[t_0, t_0 + 2\delta]$ with $\delta > 0$, $t_0 + 2\delta < T$. Applying oncemore (4.6) it is found that $y_i^1(t)$ $\big(y_i^2(t)\big)$ is still identically constant on $[t_0, t_0 + 2\delta]$. Define $h_i := y_i(t_0 + \delta) - y_i(b - \varepsilon)$ $(i = 1, 2, \ldots, k)$. Now, suppose that y_i^2 is increasing in t_0. Applying Lemma 4.1.

$$\begin{aligned} d_i^n &:= \int_{b-\varepsilon}^{T} \exp\big(\lambda_n(T - s)\big)\, dy_i(s) \\ &\geqq \int_{b-\varepsilon}^{t_0+\delta} \exp\big(\lambda_n(T - s)\big)\, dy_i^2(s) + 0 - \left| \int_{t_0+2\delta}^{T} \exp\big(\lambda_n(T - s)\big)\, dy_i(s) \right| \\ &\geqq \exp\big(\lambda_n(T - t_0 - \delta)\big)\, h_i - \exp\big(\lambda_n(T - t_0 - 2\delta)\big) \operatorname*{var}_{t_0+2\delta}^{T} y_i(\cdot) \\ &\geqq \exp\big(\lambda_n(T - t_0 - \delta)\big)\, h_i/2 \end{aligned} \tag{4.15}$$

is obtained for $n \geqq n_0$ sufficiently large. If y_i^1 increases the same holds for $-d_i^n$ and $-h_i$. Thus for $n \geqq n_0$ the d_i^n do not change their sign. Applying the same method to the remaining Lagrange multipliers, which increase at first in $t_i > t_0$, it is found that all the d_i^n do not change their sign from a certain n_0 on. Define $b^n := d^n/|d^n|_k$. Then by (4.14), (4.15), and assumption (iii) for $t > 2(t_0 + \delta) - T$, $\tau := T - 2(t_0 + \delta) + t > 0$, and $p(t, x)$ defined by (3.10)

$$\beta^{-1}\langle \alpha, p(t, \cdot)\rangle = \sum_{n=1}^{\infty} \langle \alpha, v_n\rangle^2 \exp\left(-\lambda_n(T-t)\right)$$
$$\geqq \sum_{n=n_0}^{\infty} \left(\sum_{i=1}^{k} \langle \varphi_i, v_n\rangle\, b_i^n\right)^2 |d^n|_k^2 \exp\left(-\lambda_n(T-t)\right)$$
$$\geqq h \sum_{n=n_0}^{\infty} \left(\sum_{i=1}^{n} \langle \varphi_i, v_n\rangle\, b_i^n\right)^2 \exp(\lambda_n \tau) = +\infty$$

is obtained with certain $h > 0$, contradicting $p(\cdot, \cdot) \in C([0, T) \times \bar{\Omega})$. Hence the assumption $t_0 < T$ was false. It remains to discuss $t_0 = T$. Now only jumps are possible in t_0. Put $d_i := y_i(T) - y_i(b - \varepsilon)$. Then (4.14) gives $\left\langle \alpha + \sum_{i=1}^{k} d_i \varphi_i, v_n\right\rangle = 0$, $n \in \mathrm{N}$, and $\operatorname{cl}\operatorname{span}\{v_n\} = C(\bar{\Omega})$ implies $\alpha = -\sum_{i=1}^{k} d_i \varphi_i$ in contrary to assumption (iv). Thus the theorem must be true ∎

After this paper was accepted for publication in this journal some new results on generalized bang-bang-principles were found by MACKENROTH [15]. He extended the author's method to time-optimal control of parabolic equations in Sobolev spaces as well as to convex constraints.

Assumption (iv) is in general difficult to verify, since α is only known, if u_0 is determined. However, in the case of the L_2-norm as performance functional condition (iv) is automatically fulfilled, if (i) holds and the φ_i cannot be extended from $C(\bar{\Omega})$ to the space $L_2(\Omega)$. This follows from $\alpha = \text{const}\,(Su_0 - z) \in L_2(\Omega)$ after identifying $L_2'(\Omega)$ with $L_2(\Omega)$.

There can be constructed counter-examples, which already for $k = 1$ show that an assumption of the type (4.9) on the growth of the φ_i cannot be omitted.

5. Control only time-dependent

An important class of parabolic boundary control problems is obtained, if the control $u(t, x)$ has the form $g(x)\,\tilde{u}(t)$ with fixed function g. Here the theory will be extended to this class along the lines of [4]. The following problem is regarded: Define now $w(u; t, x)$ to be the generalized solution of

$$\frac{\partial}{\partial t} w(t, x) = L_x w(t, x), \qquad (t, x) \in (0, T] \times \Omega, \tag{5.1}$$

$$w(0, x) = 0, \qquad x \in \bar{\Omega}, \tag{5.2}$$

$$B_\xi w(t, \xi) = g(\xi)\, u(t), \qquad (t, \xi) \in (0, T] \times \partial\Omega, \tag{5.3}$$

where $g(\cdot) \in L_\infty(\partial\Omega)$ is given fixed. According to (3.4) this means

$$w(u; t, x) = \beta \int_0^t \sum_{n=1}^{\infty} v_n(x)\, g_n \exp\left(-\lambda_n(t-s)\right) u(s)\, ds \tag{5.4}$$

with $g_n := \int_{\partial\Omega} v_n(\xi)\, g(\xi)\, dS_\xi$. Analogously to Section 3 the operator S is defined by $(Su)(x) := w(u; T, x)$, which is now a continuous mapping from $L_p[0, T]$ into $C(\bar{\Omega})$, if $p > N + 1$. The control problem is

$$\|Su - z\|_\infty(\bar{\Omega}) = \min! \tag{5.5}$$

subject to $\|u\|_\infty([0, T]) \leqq 1$ and (4.3).

The existence of an optimal control is proved analogously to Theorem 4.1. Define now the set $K(g) := \{j \in \mathbf{N} \mid g_j \neq 0\}$ and the subspace $L(g) := \operatorname{span}\{v_j \mid j \in K(g)\}$. For $\varphi \in C'(\bar{\Omega})$ define φ_L to be the restriction of φ to $\operatorname{cl} L(g)$.

There holds

Theorem 5.1: *Let u_0 be an optimal control for problem* (5.5). *Suppose that*

(i) $\|Su_0 - z\|_\infty(\bar{\Omega}) > 0$,

(ii) *the Slater-condition* (4.4) *is fulfilled,*

(iii) *condition* (4.9) *is met with* $\sum_{n \in K(g)}$ *instead of* $\sum_{n=1}^{\infty}$,

(iv) $\alpha_L \notin \operatorname{span}\{(\varphi_1)_L, \ldots, (\varphi_k)_L\}$,

(v) $\lambda_i \neq \lambda_j \ (i, j \in K(g))$.

Then the set

$$\{t \in [0, T] \mid |u_0(t)| < 1, \quad c_i^1(t) < (A_i u_0)(t) < c_i^2(t) \qquad (i = 1, 2, \ldots, k)\}$$

has measure zero.

Proof: The proof is only briefly scetched, as its main ideas are the same as those for validating Theorem 4.3. If Theorem 5.1. would not hold, then instantly

$$\langle \alpha, v_n \rangle = -\sum_{i=1}^{k} \langle \varphi_i, v_n \rangle \int_{b-\varepsilon}^{T} \exp\left(\lambda_n (T - s)\right) d\left(y_i^2(s) - y_i^1(s)\right)$$

is obtained for $n \in K(g)$, which corresponds to (4.14). Using the expression $(S'\alpha)(t) = \sum_{n \in K(g)} \langle \alpha, v_n \rangle g_n \exp\left(-\lambda_n (T - t)\right)$ there is obtained analogously to the further proof of Theorem 4.3

$$\left\langle \alpha + \sum_{i=1}^{k} d_i \varphi_i, v_n \right\rangle = 0, \qquad n \in K(g).$$

Since $\operatorname{cl} L(g) = \{v \in C(\bar{\Omega}) \mid \langle v, v_n \rangle = 0, n \in K(g)\}$ (cf. [4], (3.2)-Lemma 7) this implies $\alpha_L \in \operatorname{span}\{(\varphi_1)_L, \ldots, (\varphi_k)_L\}$, contradicting (iv) ∎

Using analyticity arguments as in Tröltzsch [13] the stronger result can be proved that $u_0(t)$ has at most countably many switching points on

$$M = \{t \in [0, T] \mid c_i^1(t) < (A_i u_0)(t) < c_i^2(t) \qquad (i = 1, 2, \ldots, k)\}$$

and accumulation points of switches can be only located at the right ends of components of M.

6. Examples

In this section some types of state-constraints are discussed, which may be expressed in terms of a linear functional. It will be shown that in these cases the growth-condition (4.9) is met.

Throughout this section take $N \geqq 2$, $\Omega :=$ unit ball of R^N, and $L := \Delta_N$, the Laplace operator. It is convenient to introduce spherical coordinates $(\vartheta, r) = (\vartheta_1, \ldots, \vartheta_{N-1}, r)$ and to put $w(u; t, x) = y(u; t, \vartheta, r)$, where y is solution of the corresponding initial-boundary value problem in spherical coordinates. The normalized eigenfunctions of problem (3.6—7) are given by

$$v_{i,l,m}(x) = S_m^{(l)}(\vartheta)\, c_i^{(l+N/2-1)} r^{-N/2+1} J_{l+N/2-1}(x_i^{(l+N/2-1)} r),$$

$i = 1, 2, \ldots$; $l = 0, 1, \ldots$; $m = 1, 2, \ldots, V_N(l)$; with the number $V_N(l)$ of linearly independent spherical harmonics $S_m^{(l)}$ of order l, the Bessel-function J_ν of order ν and normalizing constants

$$c_i^{(\nu)} := \left(\int_0^1 r J_\nu^2(x_i^{(\nu)} r)\, dr \right)^{-1}.$$

The eigenvalues $\lambda_{l,i}$ are given by $\lambda_{l,i} = (x_i^{(l+N/2-1)})^2$, where $x_i^{(\nu)}$ is the k-th solution of

$$x J_\nu'(x) + (\beta + N/2 - 1)\, J_\nu(x) = 0,$$

and have multiplicity $V_N(l) = \binom{l+N-1}{N-1} - \binom{l+N-3}{N-1}$. According to [1], 7.10.4, (49), and to the equation for the $x_i^{(\nu)}$ one finds

$$c_i^{(\nu)} = 2(x_i^{(\nu)})^2 / \big((x_i^{(\nu)})^2 + \beta(\beta + 2\nu)\big)\, J_\nu^2(x_i^{(\nu)}).$$

Example 1: Regard the state-constraint $\left| \int_\Omega w(u; t, x)\, dx \right| \leqq c$, $c > 0$, where w is defined by (3.4). This constraint fits in the scheme of problem (4.1—3) introducing $k = 1$, $c_1^1 = -c$, $c_1^2 = c$, and defining φ_1 by $\langle \varphi_1, v \rangle := \int_\Omega v(x)\, dx$. Now (4.9) will be shown to hold. With $\nu := N/2 - 1$, $x_i := x_i^{(\nu)}$, $c_i := c_i^{(\nu)}$ one obtains for $i = 1, 2, \ldots$

$$|\langle \varphi_1, v_{i,0,1} \rangle| = \left| \int_0^1 r^{-\nu} J_\nu(x_i r)\, r^{N-1}\, dr c_i \right| = \left| \int_0^1 r^{\nu+1} J_\nu(x_i r)\, dr c_i \right|$$
$$= \frac{2(\beta + 2\nu)}{\big(\beta(\beta + 2\nu) + x_i^2\big)\, |J_\nu(x_i)|} \geqq \frac{2(\beta + 2\nu)}{\beta(\beta + 2\nu) + x_i^2}$$

using [1], 7.2.8., (55), 7.7.1., (1), and the equation for x_i. The remaining scalar products are vanishing, as $S_1^{(0)}(\vartheta) \equiv 1$ is orthogonal to all other spherical harmonics. Taking into account the asymptotic behaviour of x_i one sees that

$$\sum_{n=1}^\infty \langle \varphi_1, v_n \rangle^2 \exp(\lambda_n \tau) = \sum_{i=1}^\infty \langle \varphi_1, v_{i,0,1} \rangle^2 \exp(x_i^2 \tau) = +\infty,$$

if $\tau > 0$. Thus (4.9) holds.

Example 2: Choose a fixed $x_0 \in \partial\Omega$ and $g(\xi) = 1$ on $\partial\Omega$. Regard the constraint $|w(u; t, 0) - w(u; t, x_0)| \leqq c$ for the control problem with u only time-dependent, i.e. w is now defined by (5.4). Now the functional φ_1, $\langle \varphi_1, v \rangle := v(0) - v(x_0)$ is to be used. There holds $K(g) = \{(i, l, m) \mid l = 0, m = 1, i \in \mathrm{N}\}$. Using the same notation as in Example 1 one has

$$v_{i,0,1}(0) - v_{i,0,1}(x_0) = c_i\big(x_i^\nu / (2^\nu \Gamma(\nu + 1)) - J_\nu(x_i)\big).$$

Since $|c_i| \geqq 1$ for sufficiently large i and $J_\nu(x_i) \to 0$ for $i \to \infty$, there holds $|v_{i,0,1}(0) - v_{i,0,1}(x_0)| \geqq 1$ for $i \geqq i_0$. Therefore with $\tau > 0$ obviously

$$\sum_{n \in K(g)} \langle \varphi_1, v_n \rangle^2 \exp(\lambda_n \tau) \geqq \sum_{i=i_0}^{\infty} 1 \cdot \exp(x_i^2 \tau) = +\infty$$

is fulfilled.

Example 3: Choose the notation as in Example 2, take $N := 3$ and impose the constraints $|w(u; t, 0)| \leqq c$, and $|w(u; t, x_0)| \leqq c$ (control u only time-dependent). Now there is to define $k := 2$, $\langle \varphi_1, v \rangle := v(0)$, $\langle \varphi_2, v \rangle := v(x_0)$. Since now $\nu = 1/2$ one has $J_\nu(x) = \big(2/(\pi x)\big)^{1/2} \sin x$ and $\beta^{-1} x_i + \tan x_i = 0$ $(i = 1, 2, \ldots)$. Thus $x_i \in \big((i-1)\pi, i\pi\big)$, $x_i \sim \pi/2 + \pi(i-1)$, and $\operatorname{sgn} J_\nu(x_i) = (-1)^{i-1}$.

Take now a sequence $\{b^{(n)}\} = \{(b_1^{(n)}, b_2^{(n)})\}$ with $(b_1^{(n)})^2 + (b_2^{(n)})^2 = 1$ and assume without limitation of generality $b_1^{(n)} b_2^{(n)} \geqq 0$ for all n. Then

$$K := \sum_{n \in K(g)} (b_1^{(n)} \langle \varphi_1, v_n \rangle + b_2^{(n)} \langle \varphi_2, v_n \rangle)^2 \exp(\lambda_n \tau)$$

$$= \sum_{i=1}^{\infty} \big(b_1^{(i)} x_i^\nu / (2^\nu \Gamma(\nu + 1)) + b_2^{(i)} J^\nu(x_i)\big)^2 c_i^2 \exp(x_i^2 \tau)$$

$$\geqq \sum_{j=j_0}^{\infty} \big((b_1^{(2j-1)})^2 (x_{2j-1}^\nu / (2^\nu \Gamma(\nu + 1)))^2 + \big(1 - (b_1^{(2j-1)})^2\big) (2\pi x_{2j-1})^{-1}$$
$$\times \exp(x_{2j-1}^2 \tau)$$

for sufficiently large j_0 (using $\operatorname{sgn} J_\nu(x_{2j-1}) = 1$ and the estimations $|J_\nu(x_i)| \geqq (2\pi x_i)^{-1/2}$ and $|c_i| \geqq 1$, which hold for sufficiently large i), and hence

$$K = \text{const} + \sum_{j=j_1}^{\infty} (2 x_{2j-1} \pi)^{-1} \exp(x_{2j-1}^2 \tau) = +\infty$$

is obtained for sufficiently large j_1 and $\tau > 0$. Thus (4.9) holds, if $b_1^{(n)} b_2^{(n)} \geqq 0$. If $b_1^{(n)} b_2^{(n)} \leqq 0$, then the proof is carried out setting $i := 2j$, $j = 1, 2, \ldots$ ∎

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