Zeitschrift für Analysis
und ihre Anwendungen
Bd. 1 (4) (1982), S. 37–45

Factorization of operators mapping (F)-spaces into (DF)-spaces

H. JUNEK

Die Arbeit behandelt die Faktorisierung linearer Operatoren von (F)-Räumen in (DF)-Räume durch Banachräume und durch Operatoren gegebener Operatorenideale. Grob gesprochen wird die Frage untersucht, in welchem Ausmaß globale Eigenschaften solcher Operatoren durch ihr Verhalten auf den beschränkten Teilmengen bestimmt werden. Die Resultate werden zur Charakterisierung der geometrischen Struktur der Nullumgebungen in (F)-Räumen durch die Geometrie ihrer beschränkten Mengen verwendet. Ferner gestatten sie eine neue Einsicht in die Theorie nuklearer (F)-Räume.

Исследована факторизация линейных операторов, отображающих пространства типа (F) в пространства типа (DF), через Банаховые пространства и через операторы данных идеалов операторов. Это равносильно вопросу, в какой мере глобальные свойства таких операторов определены их поведением на ограниченных подмножествах. Результаты использованы для характеризации геометрической структуры окрестностей нуля в пространствах типа (F) геометрией их ограниченных множеств и позволяют новый взгляд на теорию ядерных пространств типа (F).

This paper deals with the factorization of linear operators mapping (F)-spaces into (DF)-spaces through Banach spaces and through operators of given operator ideals. Roughly speaking, we answer the question of, to what extend global properties of such operators are determined by their behaviour on the bounded subsets. The results are used to characterize the geometric structure of the neighbourhoods of zero in (F)-spaces by the geometry of their bounded subsets. Moreover, they allow further insights into the theory of nuclear (F)-spaces.

Introduction

A specific situation appearing in the study of locally convex spaces in opposite to Banach spaces is given by the interplay between the neighbourhoods and the bounded subsets of these spaces. It turns out that a lot of different hard problems in the theory of locally convex spaces can be solved by answering the crucial question of, roughly speaking, to what extend global properties of linear operators acting in these spaces are determined by the behaviour of these operators on the bounded subsets of the spaces. This question was treated in the papers [3, 7] for linear operators mapping (DF)-spaces into (F)-spaces. Here we are dealing with the case of operators mapping (F)- into (DF)-spaces. Although in both cases the main tool is the well developed theory of operator ideals in Banach spaces, the results and the methods which are used in obtaining them are very different.

In Section 1, after some basic notations, we will give the strong formulation of our basic problem. In Section 2 we present the main results (Theorem 2.5). Section 3 is devoted to some applications of the factorization theorems.

1. The weak and the strong $\mathscr{A}$-property

Let us recall some definitions concerning the theory of operator ideals. For more details we refer to [6]. Only for the sake of simplicity, we assume in all the following the completeness of the locally convex spaces considered. If E, F are locally convex spaces, then by $\mathscr{L}(E, F)$ we denote the set of all linear continuous operators from E into F, and by $\mathscr{F}(E, F)$ the set of all finite dimensional operators.

1.1. Definition ([6]): An *operator ideal* $\mathscr{A}$ (BAN-*ideal*) is a class of linear continuous operators such that the following holds:

(*i*) The components $\mathscr{A}(X, Y) = \mathscr{A} \cap \mathscr{L}(X, Y)$ are linear subspaces of $\mathscr{L}(X, Y)$ containing $\mathscr{F}(X, Y)$ for all Banach spaces X, Y.

(*ii*) If $R \in \mathscr{L}(X, X_0)$, $T \in \mathscr{A}(X_0, Y_0)$, $Q \in \mathscr{L}(Y_0, Y)$ then $QTR \in \mathscr{A}(X, Y)$ for all Banach spaces X, X_0, Y, Y_0.

To formulate more exactly what properties of operators we want to consider in this paper and what factorizations of operators we have in mind, we give the following definition:

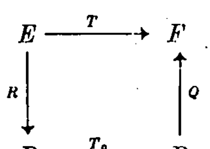
Let $\mathscr{A}$ be a BAN-ideal and let E and F be any two locally convex spaces. An operator $T \in \mathscr{L}(E, F)$ is said to have

(*i*) the *weak $\mathscr{A}$-property,* if for all Banach spaces B_1, B_2 and all operators $R \in \mathscr{L}(B_1, E)$, $Q \in \mathscr{L}(F, B_2)$ the product

$$QTR : B_1 \to E \to F \to B_2$$

belongs to $\mathscr{A}(B_1, B_2)$.

(*ii*) the *strong $\mathscr{A}$-property,* if there are Banach spaces B_1, B_2 and a factorization



of T with operators $R \in \mathscr{L}(E, B_1)$, $Q \in \mathscr{L}(B_2, F)$ and $T_0 \in \mathscr{A}(B_1, B_2)$.

By $\mathscr{A}^w$ and $\mathscr{A}^s$ we denote the classes of all operators having the weak or the strong $\mathscr{A}$-property, respectively. Of course, the strong $\mathscr{A}$-property implies the weak $\mathscr{A}$-property, i.e. $\mathscr{A}^s \subseteqq \mathscr{A}^w$. However, it was shown in [3] that there is a space E such that for any ideal $\mathscr{A}$ we have the inequality $\mathscr{A}^s(E, E) \neq \mathscr{A}^w(E, E)$. Therefore, the main problem has to be stated as follows:

Problem: *Find sufficiently large classes $\mathscr{K}_1$ and $\mathscr{K}_2$ of locally convex spaces and general assumptions on the ideal $\mathscr{A}$, such that $\mathscr{A}^w(F, E) = \mathscr{A}^s(F, E)$ holds for all $F \in \mathscr{K}_1$ and $E \in \mathscr{K}_2$.*

Theorems of this type are called factorization theorems. We will prove such theorems in Section 2. Here we give a simple but more interior criterion for the weak and the strong $\mathscr{A}$-property. Simultaneously, this criterion shows explicitly, that the weak $\mathscr{A}$-property of an operator $T : E \to F$ only depends on its behaviour on the bounded subsets of E, while the strong $\mathscr{A}$-property depends on the global structure of T. Let E be quasicomplete locally convex space. By $\mathscr{B}(E)$ we denote the directed from below system of all bounded, closed and absolutely convex subsets of E. By $\mathscr{U}(E)$ we denote the directed from above system of all absolutely convex neighbourhoods of zero of E. Let $A \in \mathscr{B}(E)$ und $U \in \mathscr{U}(E)$ be given. By p_A and p_U we denote their gauge functionals. Let us define the linear spaces

$$E(A) = \bigcup_{n=1}^{\infty} nA \quad \text{and} \quad E/U = E/p_U^{-1}(0),$$

which can be normed by $\|x\|_A = p_A(x)$ and $\|\hat{x}\|_U = p_U(x)$, respectively. Their completions we denote by E_A and E_U. There are canonical linear continuous mappings

$$C_A : E_A \to E \quad \text{and} \quad C_U : E \to E_U,$$

which are defined by $C_A x = x$ and $C_U x = \hat{x}$, respectively.

The product of these operators we denote by $C_{AU} = C_U C_A : E_A \to E_U$. Furthermore, if $A, B \in \mathscr{B}(E)$ with $A \subseteq \varrho B$ and $U, V \in \mathscr{U}(E)$ with $V \subseteq \varrho U$ for some $\varrho > 0$, then there exist canonical mappings $C_{AB} : E_A \to E_B$ and $C_{VU} : E_V \to E_U$. These mappings are uniquely defined by the equations $C_B C_{AB} = C_A$ and $C_{VU} C_V = C_U$.

Proposition: *Let $\mathscr{A}$ be a* BAN-*ideal. An operator $T \in \mathscr{L}(E, F)$ belongs to $\mathscr{A}^w(E, F)$ iff for all $A \in \mathscr{B}(E)$ and all $U \in \mathscr{U}(F)$ the product $C_U T C_A$ belongs to $\mathscr{A}(E_A, F_U)$. The operator T belongs to $\mathscr{A}^s(E, F)$ iff there are $U \in \mathscr{U}(E)$, $A \in \mathscr{L}(F)$ and $T_0 \in \mathscr{A}(E_U, F_A)$ such that $T = C_A T_0 C_U$.*

We omit the simple proof.

2. Factorization Theorems

In this section we solve the main problem formulated in Section 1. As was mentioned in the introduction, in [3] the case was investigated where $\mathscr{K}_1$ is the class of (DF)-spaces and $\mathscr{K}_2$ is the class of (F)-spaces. In this paper we are dealing with the converse situation. Let us recall the definition of these spaces.

2.1. Definition: A locally convex space is called an (F)-space if it is complete and metrizable. A locally convex space E is called a (DF)-*space*, if it has a countable increasing fundamental system of bounded convex sets $\{B_n\}$ and if the intersection $V = \bigcap_{n=1}^{\infty} U_n$ of any countable system of neighbourhoods $U_n \in \mathscr{U}(E)$ is a neighbourhood of zero, assuming that V absorbs each bounded subset of E.

We remember that the strong dual space of an (F)-space is a (DF)-space and that the strong dual space of a (DF)-space is an (F)-space. Before we present the results let us show by an example that a positive answer to our problem is far from being trivial.

2.2. Let F be Köthe's example of a Frechet-Montel-space having the Banach space l^1 as a quotient space ([1, II.3], [4, 35.5]). Let $T : F \to l^1$ be the quotient map. The Montel property of F implies $T \in \mathscr{K}^w(F, l^1)$, where $\mathscr{K}$ denotes the ideal of the compact operators. But T cannot be compact because T is an open mapping. Therefore, it does not admit any factorization through a compact operator, thus it fails to have the strong $\mathscr{K}$-property. A similar situation holds if one replaces $\mathscr{K}$ by the ideal $\mathscr{W}$ of weakly compact operators. Let us mention a further counterexample, without going too much into the details. Let F be the nuclear space s of all rapidly decreasing sequences. We consider the ideal $\mathscr{N}_0$ of all strongly nuclear operators.

These are those operators T having a representation

$$T = \sum_{i=1}^{\infty} \lambda_i a_i \otimes y_i \quad \text{with} \quad \|a_i\|, \|y_i\| \leqq 1 \quad \text{and} \quad (\lambda_i) \in s.$$

Because the dual space s' of s is strongly nuclear, all operators $F_A \to F \to F_U$ belong to $\mathcal{N}_0$, i.e. $C_U \in \mathcal{N}_0{}^w(F, F_U)$ for all neighbourhoods $U \in \mathcal{U}(F)$. On the other hand, not all mappings C_U are factorizable through an operator belonging to $\mathcal{N}_0$, because then $F = s$ would be a strongly nuclear space, which is not true. Therefore, there is a neighbourhood U such that $C_U \notin \mathcal{N}_0{}^w(F, F_U)$.

2.3. In contrast to these counterexamples, there is the following first positive solution to our problem. It was mentioned in [1] without a complete proof.

Proposition: *Let F be an* (F)*-space and E a complete* (DF)*-space. If T is a linear operator from F into E which maps bounded sets into bounded sets then there is a neighbourhood V in F which is mapped by T into a bounded set. In other words, T is continuous factorizable through a Banach space.*

Proof: Let $\{U_n\} \subseteqq \mathcal{U}(F)$ and $\{A_n\} \subseteq \mathcal{B}(E)$ be a decreasing and an increasing countable fundamental system, respectively. Suppose that there is no natural number n such that $P_{A_n}(TU_n) \leqq n$. Then there are sequences (x_n) and (a_n') such that $x_n \in U_n$, $a_n' \in A_n{}^0$ and $|\langle Tx_n, a_n'\rangle| > n$ for all $n \in \mathbf{N}$. We define linear continuous functionals $g_n(a') = \langle Tx_n, a'\rangle$ on E'. For each fixed $a' \in E'$ there is a neighbourhood $V \in \mathcal{U}(E)$ such that $a' \in V^0$. Because T maps the bounded sequence (x_n) into a bounded set, we obtain $|g_n(a')| \leqq p_V(Tx_n)\, p_{V^0}(a') \leqq C$ for all $n \in \mathbf{N}$. Therefore, the family (g_n) is pointwise bounded on the (F)-space E_b'. By the Uniform Boundedness Principle it then follows that (g_n) is uniformly continuous on E_b'. This means that there is a bounded set A in E such that $|g_n(A^0)| \leqq C$ for all $n \in \mathbf{N}$. From $A_n{}^0 \subseteqq A^0$ for large n, this contradicts $|g_n(a_n')| = |\langle Tx_n, a_n'\rangle| > n$. Therefore, there is an $n \in \mathbf{N}$ such that $T(U_n) \subseteqq (n+1) A_n$, thus $T(U_n)$ is bounded. The continuous factorization of T is given by $T : F \to F_{U_n} \to E$. ∎

2.4. Definition ([6, 6.1.1]): A BAN-ideal $\mathcal{A}$ is called *p-normed* $(0 < p \leqq 1)$, if there is a non negative functional α defined on $\mathcal{A}$ such that the following holds:

(*i*) If $1_{\mathbf{R}}$ denotes the identity of the space of real numbers, then $\alpha(1_{\mathbf{R}}) = 1$.

(*ii*) If (T_n) is a sequence of operators $T_n \in \mathcal{A}(X, Y)$ such that $\sum_{n=1}^{\infty} \alpha(T_n)^p < \infty$, then $T = \sum_{n=1}^{\infty} T_n \in \mathcal{A}(X, Y)$ and $\alpha(T)^p \leqq \sum_{n=1}^{\infty} \alpha(T_n)^p$.

(*iii*) For $T \in \mathcal{A}(X, Y)$, $R \in \mathcal{L}(X_0, X)$, $Q \in \mathcal{L}(Y, Y_0)$ it holds $\alpha(QRT) \leqq \|Q\|\,\|R\|\,\alpha(T)$.

As an easy consequence of the axioms of the ideal p-norm α and Auerbach's lemma [6, B4.8], we get the estimation

$$\alpha(T) \leqq \|T\| (\dim T)^{1/p} \quad \text{for all } T \in \mathcal{F}(X, Y).$$

2.5. The following property is shown to be essential to obtain an affirmative answer to our main problem.

Definition ([6, 8.7.4]): Let $(\mathcal{A}, \alpha)$ be any p-normed BAN-ideal. The *maximal hull* $(\mathcal{A}^{\max}, \alpha^{\max})$ *of* $(\mathcal{A}, \alpha)$ is the BAN-ideal of all operators $T \in \mathcal{L}(X, Y)$ such that for all finite dimensional Banach spaces X_0, Y_0 and all operators $R \in \mathcal{L}(X_0, X)$, $S \in \mathcal{L}(Y, Y_0)$ the estimation $\alpha(STR) \leqq \varrho\, \|S\|\, \|R\|$ holds, for some constant ϱ only depending on T. Furthermore, $\alpha^{\max}(T)$ is defined by the infimum of all such constants ϱ. The ideal $(\mathscr{A}, \alpha)$ is said to be *maximal*, if $\mathscr{A} = \mathscr{A}^{\max}$. The p-norms α and $\alpha^{\max}$ are then automatically equivalent.

The ideals $\mathscr{K}$ and $\mathscr{W}$ used in our counterexample are not maximal, the ideal $\mathscr{N}_0$ is not p-normed. The largest maximal normed ideal is $\mathscr{L}$, of course. Further examples are given in section 3. Now we state the main results of this section.

Theorem: *Let F be an* (F)*-space and let E be a Banach space or a (complete) semi-reflexive* (DF)*-space. If $(\mathscr{A}, \alpha)$ is a maximal p-normed ideal, then $\mathscr{A}^s(F, E) = \mathscr{A}^w(F, E)$.*

The proof of this theorem is based on some lemmata.

2.6. Lemma: *If E is a semireflexive locally convex space then for every closed, bounded set $B \in \mathscr{B}(E)$ there is a natural isometry $E_B \cong (E'_{B^\circ})'$.*

Proof: Let $B \in \mathscr{B}(E)$ be given. Then an isometric embedding $E_B \to (E'_{B^\circ})'$ is given by $x \mapsto \varphi_x(C_{B^\circ}x') = \langle x, x' \rangle$ (see [5, 0.11.4]). We will show that this map is onto. Each functional $\varphi \in (E'_{B^\circ})'$ defines a functional $\Phi = \varphi \cdot C_{B^\circ} \in E'' = E$ and we have $\Phi(B^0) = \varphi(C_{B^\circ}B^0) \leqq \|\varphi\|$. Therefore, $\Phi \in \|\varphi\| B^{00}$, and by the bipolar theorem it follows that $\Phi \in E(B)$ ∎

The next lemma is crucial for the proof of the theorem, because it states a connection between the finite dimensional structure of the bounded sets and the finite dimensional structure of the neighbourhoods for semireflexive spaces. The proof of this lemma is based on the following equivalent version of the principle of local reflexivity [6, E 3.1].

2.7. Version of the principle of local reflexivity: Given any linear bounded operator S from a dual Banach space X' into a finite dimensional space Y, for any finite dimensional subspace $H \subseteqq X'$ and any $\varepsilon > 0$ there is a w^*-continuous linear bounded operator R from X' into Y coinciding on H with S such that $\|R\| \leqq (1 + \varepsilon) \|S\|$.

2.8. Lemma: *Let E be a semireflexive locally convex space and let $B \in \mathscr{B}(E)$. Then given any linear bounded operator S from E_B into a finite dimensional space Y, for every finite dimensional subspace $H \subseteqq E(B)$ and any $\varepsilon > 0$ there is linear continuous operator $Q \in \mathscr{L}(E, Y)$ such that $\|QC_B\| \leqq \|S\| (1 + \varepsilon)$ and* $\sup \{\|Sx - Qx\| : x \in H \cap B\} \leqq \varepsilon$.

Proof: We put $X = E'_{B^\circ}$. By Lemma 2.6 we have $X' = E_B$. Now, we apply the principle of local reflexivity to $S : X' \to Y$. The w^*-continuous operator $R : X' \to Y$ must be of the form $R = \sum_{i=1}^{n} a_i \otimes y_i$ for some elements $a_i \in E'_{B^\circ}$ and $y_i \in Y$. This operator can be approximated uniformly on B^0 by an operator $R_0 = \sum_{i=1}^{n} C_{B^\circ}x_i' \otimes y_i$ for some $x_i' \in E'$. Now it is obvious that the operator $Q = \sum_{i=1}^{n} x_i' \otimes y_i$ has the desired properties ∎

2.9. Lemma: *Let F be any locally convex space, let $U \in \mathscr{U}(F)$. Then for any linear bounded operator R from a finite dimensional space X into F_U and any $\varepsilon > 0$ there is a linear continuous operator $P : X \to F$ such that*

$$\|C_U P\| \leqq \|R\| (1 + \varepsilon) \quad \text{and} \quad \|R - C_U P\| < \varepsilon.$$

Proof: Let $R = \sum_{i=1}^{n} x_i' \otimes \tilde{x}_i$ for some $x_i' \in X'$ and some $\tilde{x}_i \in F_U$. We approximate the elements $\tilde{x}_i$ by elements $C_U x_i$ with $x_i \in F$ in the space F_U. It is simple to check that the operator $P = \sum_{i=1}^{n} x_i' \otimes x_i$ then has the desired properties ∎

2.10. Proof of the theorem in 2.5: Let F be an (F)-space and let E be a (complete) semireflexive (DF)-space or a Banach space. Let $T \in \mathscr{A}^w(F, E)$ be given. We choose countable fundamental systems $\{U_n\} \subseteqq \mathscr{U}(F)$ and $\{B_n\} \subseteqq \mathscr{B}(E)$ such that $2U_{n+1} \subseteqq U_n$ and $2B_n \subseteqq B_{n+1}$ for all $n \in \mathbf{N}$. In the case where E is a Banach space, we choose the B_n to be the multiples $2^n B_0$ of the unit ball B_0 of E. From Proposition 2.3 we conclude the existence of a factorization

$$\begin{array}{ccc} F & \xrightarrow{\ T\ } & E \\ {\scriptstyle c_{U_n}}\downarrow & & \uparrow{\scriptstyle c_{B_n}} \\ F_{U_n} & \xrightarrow{\ T_n\ } & E_{B_n} \end{array}, \tag{1}$$

$\|T_n\| \leqq 1$, for some n. By shortening the system, if necessary, we may assume that (1) holds for all $n \geqq 1$. Now, we assume $T \notin \mathscr{A}^s(F, E)$. Obviously, this implies $T_n \notin \mathscr{A}(F_{U_n}, E_{B_n})$ for all $n \in \mathbf{N}$. Taking in account the maximality of $\mathscr{A}$, there are for each $n \in \mathbf{N}$ finite dimensional spaces X_n and Y_n and operators $R_n \in \mathscr{L}(X_n, F_{U_n})$, $S_n \in \mathscr{L}(E_{B_n}, Y_n)$ such that $\|R_n\| = \|S_n\| = 1$ and $\alpha(S_n T_n R_n) > 2n$. Now, for given $\varepsilon > 0$ we will construct linear continuous operators P_n and Q_n in the non-commutative diagramm

$$\begin{array}{ccccccc} & \swarrow^{P_n} & F & \xrightarrow{\ T\ } & E & \searrow^{Q_n} & \\ & & {\scriptstyle c_{U_n}}\downarrow & & \downarrow{\scriptstyle c_{B_n}} & & \\ X_n & \xrightarrow{\ R_n\ } & F_{U_n} & \xrightarrow{\ T_n\ } & E_{B_n} & \xrightarrow{\ S_n\ } & Y_n \end{array}$$

such that

$$\|C_{U_n} P_n\| \leqq 1 + \varepsilon, \qquad \|Q_n C_{B_n}\| \leqq 1 + \varepsilon \quad \text{and} \quad \alpha(Q_n T P_n) > n \tag{2}$$

holds for all $n \in \mathbf{N}$. To this end we fix n, put $d_n = 2 \dim X_n + 2 \dim Y_n$ and choose $0 < \delta < \varepsilon$ such that $2d_n \delta^p < (2^p - 1) n^p$. Now, we choose the operators P_n and Q_n according to the Lemmas 2.8 and 2.9 with respect to δ and $H = TP_n(X_n) = T_n C_{U_n} P_n X_n \subseteqq E(B_n)$. If E is a Banach space, we simple put $Q_n = S_n$. From

$$S_n T_n R_n - Q_n T P_n = S_n T_n (R_n - C_{U_n} P_n) + (S_n - Q_n C_{B_n}) T_n C_{U_n} P_n$$

and 2.4 it follows

$$\begin{aligned} \alpha(S_n T_n R_n - Q_n T P_n)^p &\leqq \alpha\big(S_n T_n (R_n - C_{U_n} P_n)\big)^p + \alpha\big((S_n - Q_n C_{B_n}) T_n C_{U_n} P_n\big)^p \\ &\leqq \|S_n\|^p \|T_n\|^p \alpha(R_n - C_{U_n} P_n)^p + \alpha\big((S_n - Q_n C_{B_n}) T_n C_{U_n} P_n\big)^p \\ &\leqq d_n \|R_n - C_{U_n} P_n\|^p + d_n \|(S_n - Q_n C_{B_n}) T_n C_{U_n} P_n\|^p = 2d_n \delta^p. \end{aligned}$$

Therefore, we have

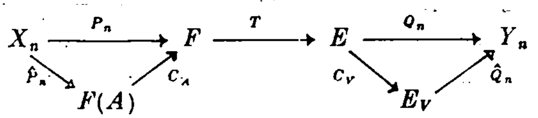
$$\alpha(Q_n T P_n)^p \geqq \alpha(S_n T_n R_n)^p - \alpha(S_n T_n R_n - Q_n T P_n)^p \geqq (2n)^p - 2d_n \delta^p > n^p.$$

This proves the result (2).

Now, we put $\varepsilon = 1$ and define the sets

$$A = \operatorname{cl\,conv} \bigcup_{n=1}^{\infty} P_n S_{X_n} \subseteqq F \quad \text{and} \quad V = \bigcap_{n=1}^{\infty} Q_n^{-1}(S_{Y_n}) \subseteqq E,$$

where S_{X_n} and S_{Y_n} denote the unit balls of X_n and Y_n, respectively. To show the boundedness of A in F, let U_k be any element of our neighbourhood basis $\{U_n\}$. Because the $P_n(S_{X_n})$ are bounded subsets of F, the set $\bigcup_{n=1}^{k} P_n(S_{X_n})$ is absorbed by U_k. For all $n > k$ we have $P_n(S_{X_n}) \subseteqq 2U_n \subseteqq U_k$. This shows that the whole set A is absorbed by U_k. Therefore, A is bounded. Now, let us prove that V is a neighbourhood of zero. By the definition of a (DF)-space it is sufficient to show that V absorbs each bounded subset of E. Let $B_k \in \{B_n\}$ be given. Then it is absorbed by the finite intersection $\bigcap_{n=1}^{k} Q_n^{-1}(S_{Y_n})$. For $n > k$ we have $B_k \subseteqq B_n$ and $Q_n(B_k) \subseteqq Q_n(B_n) \subseteqq 2S_{Y_n}$. This means $B_k \subseteqq 2Q_n^{-1}(S_{Y_n})$ for all $n > k$. Therefore, V absorbs B_k. Now we finish the proof of the theorem by defining mappings $\hat{P}_n$ and $\hat{Q}_n$ in the commutative diagram



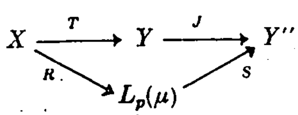
by $C_A\hat{P}_n = P_n$ and $\hat{Q}_nC_V = Q_n$. From $\hat{P}_n(S_{X_n}) \subseteqq A$ and $\hat{Q}_n(V) \subseteqq S_{Y_n}$ it follows that the mappings P_n and Q_n are correctly defined and that $\|\hat{P}_n\| \leqq 1$ and $\|\hat{Q}_n\| \leqq 1$. Finally, we have $\alpha(\hat{Q}_n \cdot C_VTC_A \cdot \hat{P}_n) = \alpha(Q_nTP_n) > n$ for all $n \in \mathbf{N}$ by (2). But this contradicts $T \in \mathscr{A}^{u}(F, E)$ ∎

2.11. Remark: a) As was shown by our counterexamples, the maximality of the ideal $\mathscr{A}$ seems to be necessary to prove the conclusion of the theorem. There is also another argument which confirms that the maximality, or a similar property, is needed. In fact, it is somewhat surprising, that no representation of the theory of nuclear spaces is known in the literature, which does not need the integral of the absolutely summing operators behind the nuclear ones to develop central parts of this theory. Actually, these operators are used to prove, among other things, that the strong dual of a nuclear (DF)-space is also nuclear (cf. Section 3). Taking in account the fact that the ideal of the integral operators is the maximal hull of the ideal of the nuclear operators ([6, 8.7.6]), this curiosity is explained in some sense by our theorem.

b) In contrast to the maximality of the ideal, it is conjectured that the assumption of the semireflexivity is superfluous. Without proof we mention that the statement of the theorem is also true if E is an arbitrary (complete) (DF)-space and $\mathscr{A}$ is an injective maximal ideal (concerning the injectivity see [6, 8.4]).

3. Applications

3.1. We apply the theorem to the maximal normed ideal $(\mathscr{L}_p, \lambda_p)$, $(1 \leqq p \leqq \infty)$. Recall that a Banach space operator $T \in \mathscr{L}(X, Y)$ belongs to $\mathscr{L}_p(X, Y)$ if there is a factorization



through some $L_p(\mu)$-space (cf. [6, 19.3.1]). The norm $\lambda_p(T)$ is given by $\lambda_p(T) = \inf \|R\|\,\|S\|$, where the infimum is taken over all possible factorizations.

Proposition: *If F is an (F)-space having a fundamental system $\{A_\alpha\} \subseteqq \mathscr{B}(F)$ such that the spaces F_{A_α} are $L_p(\mu)$-spaces for some measures μ, than F is a subspace of a countable projective limit of $L_p(\mu)$-spaces.*

Proof: Let $U \in \mathscr{U}(F)$ be given. The assumption implies $C_U \in \mathscr{L}_p{}^w(F, F_U)$. Applying the theorem, there is a neighbourhood $V \in \mathscr{U}(F)$ such that $C_{VU} \in \mathscr{L}_p(F_U, F_V)$. Then we get a factorization

$$\begin{array}{ccc} F_U{}'' & \longrightarrow & F_V{}'' \\ & \searrow \quad \nearrow & \\ & L_p(\mu) & \end{array}$$

Now, F is a subspace of proj lim $\{F_U{}'' : U \in \mathscr{U}(F)\}$. Therefore, it can also be considered as a subspace of the projective limit of the $L_p(\mu)$-spaces ∎

Corollary: *An* (F)*-space F has a fundamental system $\{A_\alpha\} \subseteqq \mathscr{B}(F)$ of the bounded subsets such that the spaces F_{A_α} are Hilbert spaces if and only if the topology of F can be generated by semiscalar products.*

Proof: The sufficiency of the given condition follows from the proposition. The necessity could be derived from results of [2], but here we give another direct proof. Let $\{p_n\}$ be a countable system of seminorms, defining the topology of F. Assuming, that the p_n are generated by semiscalar products. Let A be any bounded subset of F. We put $c_n = \sup_{x \in A} p_n(x)$ and choose a sequence (α_n) of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n{}^2 c_n{}^2 \leqq 1$. We consider the functional

$$q(x) = \left(\sum_{n=1}^{\infty} \alpha_n{}^2 p_n(x)^2 \right)^{1/2}, \qquad x \in F,$$

which can take the value ∞. Then $B = \{x \in F : q(x) \leqq 1\}$ is a bounded set and its gauge functional q performs the parallelogram equality, because the p_n's do it. Therefore, q is generated by a scalarproduct defined on $F(B)$. Finally, for $x \in A$ it follows $q(x) \leqq 1$. This shows $A \subseteqq B$ ∎

3.2. Recall, that a locally convex space F is called *conuclear* if for any $A \in \mathscr{B}(F)$ there is a $B \in \mathscr{B}(F)$ such that the mapping C_{AB} is absolutely summing, and it is called *nuclear* if for any $U \in \mathscr{U}(F)$ there is a $V \in \mathscr{U}(F)$ such that C_{VU} is absolutely summing (cf. [5, 4.1]). It was proved in [6, 8.7.8], that the ideal $\mathscr{P}$ of the absolutely summing operators is a normed maximal operators is a normed maximal operator ideal. Applying Theorem 2.5, we get a new proof of the fact that every conuclear (F)-space is nuclear. This shows that the whole theory of the nuclear (F)- and (DF)-spaces can be founded on the theory of operator ideals.

REFERENCES

[1] GROTHENDIECK, A.: Sur les espaces (F) et (DF). Summa Brasil Math. vol. 3 fasc. 6 (1954), 57–121.

[2] JUNEK, H.: On dual spaces of locally convex spaces defined by operator ideals. Serd. bulg. math. publ. 3 (1977), 227–235.

[3] JUNEK, H.: Factorization of linear operators, mapping (DF)-spaces into (F)-spaces. (To appear.)

[4] KÖTHE, G.: Topologische lineare Räume I. Berlin–Göttingen–Heidelberg 1960.

[5] PIETSCH, A.: Nuclear Locally Convex Spaces. Berlin 1972.

[6] PIETSCH, A.: Operator Ideals. Berlin 1978.
[7] NELIMARKKA, E.: On operator ideals and locally convex $\mathscr{A}$-spaces with applications to λ-nuclearity. Ann. Acad. Sci. Fenn., Math. Diss. **13**. Helsinki 1977.

Manuskripteingang: 21. 07. 1981

VERFASSER:

Dr. Heinz Junek
Sektion Mathematik/Physik der Pädagogischen Hochschule „Karl Liebknecht"
DDR-1500 Potsdam, Am Neuen Palais