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The Galerkin approximation for quasilinear elliptic equations with rapidly (or slowly) increasing coefficients

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In der vorliegenden Arbeit wird die Konvergenz des Galerkinverfahrens für quasilineare elliptische Differentialgleichungen mit stark (oder schwach) wachsenden Koeffizienten gezeigt. Die Untersuchungen erfolgen auf der Grundlage der von Gossez entwickelten Theorie pseudomonotoner Operatoren in komplementären Systemen von Sobolev-Orliczräumen.

В работе доказывается сходимость метода Галеркина для квазилинейных эллиптических дифференциальных уравнений с быстро (или медленно) растущими коэффициентами. Бадис исследований-теория Госсец псевдомонотонных операторов в дополнительных системах пространств Соболева-Орлича.

In this paper the convergence of Galerkin's method is proved for quasilinear elliptic partial differential equations with rapidly (or slowly) increasing coefficients. The investigations are based on Gossez's theory of pseudomonotone operators in complementary systems of Sobolev-Orlicz spaces.

1. Introduction

We are going to consider the following boundary value problem (BVP):

$$\begin{aligned} G: &\quad \sum_{|\alpha|\leqq m} (-1)^{|\alpha|} D^\alpha A_\alpha\big(x, Du(x)\big) = f(x), \\ \partial G: &\quad D^\beta u(x) = 0 \quad \text{for} \quad \beta: 0 \leqq |\beta| \leqq m-1, \end{aligned} \tag{1}$$

where the coefficients have *rapid* (or *slow*) *growth* ($G \subset \mathbf{R}^N$, G open and bounded, $Du = (D^\beta u)_{|\beta|\leqq m}$, $m \geqq 1$).

Example 1 ("rapid growth"):

$$\begin{aligned} G: &\quad \sum_{i=1}^{N} - D_i[D_i u(x) \exp |D_i u(x)|] = f(x), \\ \partial G: &\quad u(x) = 0. \end{aligned}$$

Example 2 ("slow growth"):

$$\begin{aligned} G: &\quad \sum_{i=1}^{N} - D_i\Big[\big(\operatorname{sign} u(x)\big) \ln \big(1 + |D_i u(x)|\big)\Big] = f(x), \\ \partial G: &\quad u(x) = 0. \end{aligned}$$

In the last few years the existence theory of such equations and similiar ones was investigated by a number of authors, e.g. Ball [4, Ch. 6, 7: sequential weak continuity of mappings on Orlicz-Sobolev spaces, existence theorems], Browder [6: quasilinear elliptic differential equations with strongly nonlinear lower order terms, existence theory for generalized pseudomonotone operators with domains dense in the underlying Banach space, weak coerciveness hypotheses], DONALDSON [8: use of complementary systems of Sobolev-Orlicz spaces], DONALDSON [9: inhomogeneous Sobolev-Orlicz spaces, complementary systems, investigation of parabolic differential equations], GOSSEZ [11: monotone operators of dense type, connections with convex analysis, applications to semilinear elliptic differential equations with strong nonlinearities in Orlicz spaces], GOSSEZ [12, 14: general existence theory for quasilinear elliptic equations in Sobolev-Orlicz spaces, very weak coerciveness hypotheses], GOSSEZ [13: proofs for the existence of solutions of quasilinear elliptic differential equations using the principle of topological degree], HESS [15, 16: elliptic differential equations with strongly nonlinear lower order terms, use of two Sobolev spaces], LACROIX [20, 21: traces of functions from Sobolev-Orlicz spaces, inhomogeneous boundary value problems], LANDES [22: investigation of quasilinear elliptic differential equations involving strongly nonlinear terms in Sobolev spaces, Euler equations of convex variational problems], LANGENBACH [23, Ch. 3, § 4: variational problems in Sobolev-Orlicz spaces], SIMADER [30: strongly nonlinear elliptic differential equations in Sobolev spaces, also for unbounded domains], VIŠIK [32, 33: a-priori estimates for the derivatives of order $(m + 1)$, compactness arguments].

It seems that there is only one paper concerning the approximate solution of problems in Orlicz- and Sobolev-Orlicz spaces: ROBERT [26]. In that paper approximation schemes in the sense of AUBIN [2, 3] are established and applied to the approximate solution of Hammerstein integral equations.

It is our aim to prove the convergence of the Galerkin approximation to equation (1). In abstract form our result reads as follows:

(i) the operator equation $Au = b$ is uniquely solvable,
(ii) the approximate equations $A_n u_n = b_n$ are uniquely solvable,
(iii) we have convergence $u_n \to u$ in an appropriate sense.

The investigations in this paper use some considerations of SCHUMANN and ZEIDLER [29] where the convergence of the finite difference method for equation (1) was proved under the hypothesis that the coefficients A_α have polynomial growth. Allowing rapidly (or slowly) increasing coefficients, however, we have to replace the Sobolev spaces used in [29] by Sololev-Orlicz spaces. This gives rise to some serious complications since, in general, Sobolev-Orlicz spaces are neither separable nor reflexive. In general, both the C_0^∞-functions and the bounded functions fail to form a dense subset in Sobolev-Orlicz spaces. In this case one cannot expect the approximate solutions to converge to the exact solution of the boundary value problem in the norm of the underlying Sobolev-Orlicz space, yet we succeed in proving the convergence in a "weaker" norm. For the simpler case where the Young function H^* characterizing the target space satisfies the additional growth restriction $H^* \in \Delta_2$ (cf. Sect. 2) an analogous result was presented by the author at a summer school held in Berlin 1979 (cf. SCHUMANN [28]) using an approximation result of ROCKAFELLAR [27].

2. Orlicz spaces and Sobolev-Orlicz spaces

In this section we collect the definitions and the main properties of Orlicz- and Sobolev-Orlicz spaces which are needed to restate the BVP (1) as an operator equation and to prove the convergence result. As for the proofs we refer to the literature, e.g. ADAMS [1], GOSSEZ [12, 14], KUFNER, JOHN and FUČIK [19], KRASNOSELSKIJ and RUTICKIJ [18].

2.1. Young functions and Orlicz spaces

Let G be an open and bounded subset of $\mathbf{R}^N$ ($N \geqq 1$), $G \neq \emptyset$. A *Young function* H is a function of the form

$$H(t) = \int_0^{|t|} \varphi(s)\, ds \quad \text{for all} \quad t \in \mathbf{R}, \tag{2}$$

where φ: $[0, +\infty[\to [0, +\infty[$ is a continuous and strictly increasing function with $\varphi(0) = 0$, $\lim\limits_{s \to +\infty} \varphi(s) = +\infty$.

Since any Young function is convex *Jensen's inequality* holds:

$$H\left(\sum_{i=1}^{n} \alpha_i u_i\right) \leqq \sum_{i=1}^{n} \alpha_i H(u_i) \tag{3}$$

for all $u_1, \ldots, u_n \in \mathbf{R}$; $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ with $\alpha_i \geqq 0$, $\sum_{i=1}^{n} \alpha_i = 1$ (cf. Kufner, John and Fučík [19, p. 133]).

A Young function H is said to satisfy *condition* Δ_2 if there exist numbers $t_0, c > 0$ such that $H(2t) \leqq cH(t)$ for all $t \geqq t_0$. If H is a Young function we define the *conjugate Young function* H^* of H by

$$H^*(t) = \int_0^{|t|} \varphi^{-1}(s)\, ds \text{ for all } t \in \mathbf{R}.$$

This definition implies *Young's inequality*

$$tt^* \leqq H(t) + H^*(t^*) \qquad \text{for all } t, t^* \in \mathbf{R}. \tag{4}$$

The *Orlicz class* $\tilde{L}_H(G)$ is defined to be the set of all measurable functions u: $G \to \mathbf{R}$ with $\varrho_H(u) = \int_G H(u(x))\, dx$.

The *Orlicz space* is defined as the set of all u: $G \to \mathbf{R}$ such that $\alpha(u)\, u \in \tilde{L}_H(G)$ for some real number $\alpha(u)$ depending on u. Lastly, $E_H(G)$ denotes the set of all $u \in L_H(G)$ with $\alpha u \in \tilde{L}_H(G)$ for any real number $\alpha > 0$. As in the definition of the Lebesgue spaces $L_p(G)$ functions having equal values almost everywhere on G are not distinguished. The Orlicz space $L_H(G)$ is a Banach space with respect to the *Orlicznorm*

$$\|u\|_H = \sup_v \int_G |u(x)\, v(x)|\, dx \tag{5}$$

where the supremum is taken over all $v \in \tilde{L}_{H^*}(G)$ such that $\varrho_{H^*}(v) \leqq 1$. An equivalent norm on $L_H(G)$ is the *Luxemburg norm*

$$\|u\|_{(H)} = \inf \left\{k > 0\colon \int_G H(k^{-1}u(x))\, dx \leqq 1\right\}. \tag{6}$$

The relation between both norms is given by

$$\|u\|_{(H)} \leqq \|u\|_H \leqq 2\, \|u\|_{(H)} \qquad \text{for all } u \in L_H(G). \tag{7}$$

The set $E_H(G)$ is a closed and separable subspace of $L_H(G)$; furthermore $E_H(G) = \operatorname{cl} L_\infty(G) = \operatorname{cl} C_0^\infty(G)$ (closure in $\|\cdot\|_H$). Note that $E_H(G) = L_H(G)$ if and only if $H \in \Delta_2$.

The following *generalized Hölder inequality* plays an important role in our estimates:

$$\int_G |u(x)\, v(x)|\, dx \leqq \|u\|_H\, \|v\|_{H^*} \tag{8}$$

for all $u \in L_H(G)$, $v \in L_{H^*}(G)$.

We will also need some comparison results for Orlicz spaces. Let H_1, H_2 be Young functions. Then we write

(i) $H_1 < H_2$ if there are numbers $t_0, c > 0$ such that $H_1(t) \leqq H_2(ct)$ for all $t \geqq t_0$;
(ii) $H_1 \ll H_2$ if $\lim_{t\to\infty} H_1(t)/H_2(\lambda t) = 0$ for any $\lambda > 0$.

With these preliminaries we may formulate a lemma.

Lemma 1: (i) $L_{H_1}(G) \subset L_{H_2}(G)$ *if* $H_2 < H_1$. ($\subset$ *denotes continuous imbedding; thus in our case* $L_{H_1}(G) \subset L_{H_2}(G)$ *and* $\|u\|_{H_2} \leqq c\,\|u\|_{H_1}$ *for all* $u \in L_{H_1}(G)$, $c > 0$ *constant.*)

(ii) $L_{H_1}(G) \subset E_{H_2}(G)$ *if* $H_2 \ll H_1$.

(iii) *Let* $H_2 \ll H_1$. *Suppose a sequence* $(u_n)_{n\geqq 1} \subset L_{H_1}(G)$ *satisfies* $\lim_{n\to\infty} \varrho_{H_1}(u_n) = 0$. *Then* $\lim_{n\to\infty} \|u_n\|_{H_2} = 0$.

Proof: Cf. ADAMS [1, p. 234–237], KRASNOSELSKIJ and RUTICKIJ [18, p. 130], KUFNER, JOHN and FUČIK [19, p. 185–192] ∎

2.2. Sobolev-Orlicz spaces

Let us now turn to *Sobolev-Orlicz spaces*. For a Young function H and an integer $m \geqq 1$ we denote by $W^mL_H(G)$ the set of those functions $u \in L_H(G)$ whose generalized derivatives $D^\alpha u$ belong to $L_H(G)$ up to order m. Analogously, $W^mE_H(G)$ is the set of all functions $u \in E_H(G)$ with $D^\alpha u \in E_H(G)$ if $|\alpha| \leqq m$.

The spaces $W^mL_H(G)$ and $W^mE_H(G)$ are Banach spaces with respect to the norm

$$|u|_{m,H} = \Big(\sum_{|\alpha|\leqq m} \|D^\alpha u\|_H{}^2\Big)^{1/2}. \tag{9}$$

We identify $W^mL_H(G)$ with a subspace of the product space $L = \prod_{|\alpha|\leqq m} L_H(G)$ via $u \in W^mL_H(G) \mapsto (D^\alpha u)_{|\alpha|\leqq m} \in L$. Furthermore we define Sobolev-Orlicz spaces with "zero boundary values" by

$$\mathring{W}^mL_H(G) = \sigma\big(\Pi L_H(G), \Pi E_{H^*}(G)\big) - \mathrm{cl}\; C_0^\infty(G) \qquad \text{in } W^mL_H(G)$$

(i.e. closure of C_0^∞ in $W^mL_H(G) \subset L$ with respect to the topology on $W^mL_H(G)$ induced by the weak *-topology on L),

$$\mathring{W}_mE_H(G) = \mathrm{cl}\; C_0^\infty(G) \quad \text{in} \quad W^mE_H(G) \qquad \text{(norm closure)}.$$

The spaces $\mathring{W}^mL_H(G)$ and $\mathring{W}^mE_H(G)$ are Banach spaces, too. On these spaces the norm $|u|_{m,H}$ and $\|u\|_{m,H} = \Big(\sum_{|\alpha|=m} \|D^\alpha u\|_H{}^2\Big)^{1/2}$ are equivalent. (Cf. ADAMS [1, p. 246], GOSSEZ [12, 14], KUFNER, JOHN and FUČIK [19, Ch. 7].) Let us now give the definitions of two distribution spaces:

$$W^{-m}L_{H^*}(G) = \Big\{f \in \mathscr{D}'(G)\colon f = \sum_{|\alpha|\leqq m} D^\alpha f_\alpha \quad \text{with} \quad f_\alpha \in L_{H^*}(G)\Big\},$$

$$W^{-m}E_{H^*}(G) = \Big\{f \in \mathscr{D}'(G)\colon f = \sum_{|\alpha|\leqq m} D^\alpha f_\alpha \quad \text{with} \quad f_\alpha \in E_{H^*}(G)\Big\}.$$

We define a pairing between $u \in \mathring{W}^mL_H(G)$ and $f \in W^{-m}L_{H^*}(G)$ by

$$\langle u, f\rangle = \int_G \sum_{|\alpha|\leqq m} f_\alpha(x)\, D^\alpha u(x)\, dx. \tag{10}$$

In the next section we shall need an imbedding theorem. We write $X \subset\subset Y$ if $X \subset Y$ and the imbedding operator $X \to Y$ is compact.

Lemma 2 (*Sobolev-Orlicz-imbedding theorem*):
(i) *Suppose G has Lipshitz boundary*: $G \in \mathscr{C}^{0,1}$ (cf. ADAMS [1, p. 66], KUFNER, JOHN and FUČIK [19, p. 204]). *Then* $W^mL_H(G) \subset\subset W^{m-1}E_H(G)$.
(ii) *If* $G \subset \mathbf{R}^N$ *is an arbitrary open and bounded set then* $\mathring{W}^mL_H(G) \subset\subset \mathring{W}^{m-1}E_H(G)$.

Proof: Cf. ADAMS [1, p. 247–258], DONALDSON and TRUDINGER [10], GOSSEZ [12, Prop. 4.13, Lemma 4.14], KUFNER, JOHN and FUČIK [19, p. 352–369].

2.3. Complementary systems

The BVP (1) will be formulated as an operator equation in a complementary system of Sobolev-Orlicz spaces. Therefore we give the following definition (cf. GOSSEZ [12, 14]):

Let Y and Z be (real) Banach spaces; $\langle \cdot, \cdot \rangle$: $Y \times Z \to \mathbf{R}$ denotes a continuous bilinear form such that the following conditions are satisfied:
(i) $\langle y, z \rangle = 0$ for all $z \in Z$ implies $y = 0$,
(ii) $\langle y, z \rangle = 0$ for all $y \in Y$ implies $z = 0$.
Suppose $Y_0 \subset Y$ and $Z_0 \subset Z$ are linear subspaces of Y and Z, respectively. Then the quadruple $(Y, Y_0; Z, Z_0)$ is said to be a *complementary system* if, by means of $\langle \cdot, \cdot \rangle$, Y_0^* can be identified with Z and Z_0^* with Y; i.e. there exist linear homeomorphisms γ_1: $Y_0^* \xrightarrow{\text{onto}} Z$, γ_2: $Z_0^* \xrightarrow{\text{onto}} Y$ such that

$$f(y) = \langle y, \gamma_1 f \rangle \qquad \text{for all } y \in Y_0, \quad f \in Y_0^*$$

and

$$g(z) = \langle \gamma_2 g, z \rangle \qquad \text{for all } z \in Z_0, \quad g \in Z_0^*.$$

For this situation we write shortly: $Y_0^* \cong Z$, $Z_0^* \cong Y$.

Examples: (i) $(L_H(G), E_H(G); L_{H^*}(G), E_{H^*}(G))$ is a complementary system with respect to the pairing: $u \in L_H(G)$, $v \in L_{H^*}(G) \mapsto \int_G uv\, dx$.
(ii) $(\mathring{W}^mL_H(G), \mathring{W}^mE_H(G); W^{-m}L_{H^*}(G), W^{-m}E_{H^*}(G))$ is a complementary system with respect to the pairing (10) (cf. GOSSEZ [12, 14]) provided that G has the segment property (cf. ADAMS [1, p. 66]).

3. Generalized solutions of the BVP and convergence theorem

Now we are looking for generalized solutions of our BVP (1).

Problem $(\mathscr{P})$: Let H be a Young function; suppose $f \in E_{H^*}(G)$ is a given function. A function $u \in D(A) \subset Y = \mathring{W}^mL_H(G)$ is said to be a *generalized solution* of (1) if

$$a(u, v) = b(v) \quad \text{for any} \quad v \in Y_0 = \mathring{W}^mE_H(G) \tag{11}$$

where
$$a(u, v) = \int_G \sum_{|\alpha| \leq m} A_\alpha(x, Du(x))\, D^\alpha v(x)\, dx,$$
$$D(A) = \{u \in Y: A_\alpha(\cdot, Du) \in L_{H^*}(G) \quad \text{for all} \quad |\alpha| \leq m\},$$
$$b(v) = \int_G f(x)\, v(x)\, dx.$$

To solve Problem $(\mathscr{P})$ approximatively by Galerkin's method we replace the space Y_0 in equation (11) by spaces Y_n $(n = 1, 2, \ldots)$ from an increasing sequence $Y_1 \subset Y_2 \subset \cdots$ of finite-dimensional subspaces of $Y_0 = \mathring{W}_mE_H(G)$ whose union

$V = \bigcup_{n=1}^{\infty} Y_n$ is dense in Y_0. Under the norm $\|\cdot\|_{m,H}$ the spaces Y_n are (finite dimensional) Banach spaces. Thus we are led to consider the following sequence of problems:

Problem $(\mathscr{P}_n)$: Find a function $u_n \in Y_n$ such that

$$a(u_n, v_n) = b(v_n) \quad \text{for any} \quad v_n \in Y_n. \tag{11'}$$

Now we can state the convergence theorem.

Theorem 1: *Let G be an open bounded subset of the Euclidean space $\mathbf{R}^N$ $(N \geqq 1)$, $G \neq \emptyset$, with Lipshitz boundary: $G \in \mathscr{C}^{0,1}$. Let H and Ψ be Young functions such that $\Psi \ll H$. Furthermore, assume that the following conditions are satisfied:*

a) *Carathèodory condition: For all α: $|\alpha| \leqq m$ let*

$$\begin{aligned} &A_\alpha\colon G \times \mathbf{R}^\mu \to \mathbf{R} \text{ be a function such that} \\ &x \mapsto A_\alpha(x, D) \text{ is measurable on } G \text{ for all } D = (D^\beta) \in \mathbf{R}^\mu \text{ and} \\ &D \mapsto A_\alpha(x, D) \text{ is continuous on } \mathbf{R}^\mu \text{ for almost all } x \in G. \end{aligned} \tag{12}$$

(μ is the cardinal number of the set $\{\alpha: |\alpha| \leqq m\}$.)

b) *Growth condition:*

$$|A_\alpha(x, D)| \leqq g(x) + c_1 \sum_{|\beta| \leqq m} (H^*)^{-1} H(\bar{c}_1 D^\beta) \tag{13}$$

for all $X \in G$, $D = (D^\beta)_{|\beta| \leqq m} \in \mathbf{R}^\mu$ where $g \in E_{H^*}(G)$, $c_1, \bar{c}_1 > 0$ constant, $|\alpha| \leqq m$.

c) *Monotonicity:*

$$\sum_{|\alpha| \leqq m} \big(A_\alpha(x, D) - A_\alpha(x, \mathbf{D}')\big)\,(D^\alpha - D'^\alpha) > 0 \tag{14a}$$

for all $x \in G$, $D = (D^\beta)_{|\beta| \leqq m}$, $D' = (D'^\beta)_{|\beta| \leqq m} \in \mathbf{R}^\mu$ with $D \neq D'$.

$$\sum_{|\alpha| \leqq m} A_\alpha(x, D)\, D^\alpha \geqq c_0 \sum_{|\beta| = m} H(aD^\beta) - K(x) \tag{14b}$$

for all $x \in G$, $D \in \mathbf{R}^\mu$ where $K \in L_1(G)$, $c_0, a > 0$ constant.
Then:

(i) *Problem $(\mathscr{P})$ has exactly one solution $u \in D(A)$.*
(ii) *Problem $(\mathscr{P}_n)$ has exactly one solution $u_n \in Y_n$ for all $n = 1, 2, \ldots$*
(iii) *$D^\alpha u_n \to D^\alpha u$ in $E_H(G)$ as $n \to \infty$ for all α: $|\alpha| \leqq m - 1$.*
(iv) *$D^\alpha u_n \to D^\alpha u$ in $E_\Psi(G)$ as $n \to \infty$ for all α: $|\alpha| \leqq m$.*
(v) *There exists a real number $\gamma > 0$ such that*

$$\varrho_H\big(\gamma(D^\alpha u_n - D^\alpha u)\big) \to 0 \quad \text{for all} \quad \alpha: |\alpha| = m, \quad \text{as} \quad n \to \infty.$$

Corollary: *If $H \in \Delta_2$ then $\|u_n - u\|_{m,H} \to 0$ as $n \to \infty$.*

Proof: If $H \in \Delta_2$ then $L_H = E_H$ and $\varrho_H(v_n - v) \to 0$ if and only if $\|v_n - v\|_H \to 0$ (cf. KUFNER, JOHN and FUČIK [19, p. 159]). Therefore the corollary follows from Theorem 1 (iii), (v).

Examples: Let us consider a simple but typical application. Suppose φ: $\mathbf{R} \to \mathbf{R}$ is a continuous, odd and strictly increasing function such that $\lim_{t\to\infty} \varphi(t) = \infty$. We assume that G satisfies the hypotheses of Theorem 1. Consider the BVP

$$-\sum_{i=1}^{N} D_i\big(\varphi(D_i u(x))\big) = f(x) \quad \text{in } G$$

$$u(x) = 0 \quad \text{on } \partial G.$$

Then all assertions of Theorem 1 hold.

Now we return to the examples mentioned in the introduction.

For Example 1 we use $\varphi(t) = t \exp |t|$; the associated Young function $H(t) = (|t| - 1) \exp |t| + 1$ has exponential growth and does not satisfy condition Δ_2 (cf. KRASNOSELSKIJ and RUTICKIJ [18, p. 38]; KUFNER, JOHN and FUČIK [19, p. 138]). If we set $\Psi(t) = |t|^p$ then $\Psi \ll H$ for all $p > 1$.

As for Example 2 we choose $\varphi(t) = \operatorname{sign} t \ln(1 + |t|)$; the associated Young function $H(t) = (1 + |t|)(\ln(1 + |t|) - |t|$ satisfies $H \in \Delta_2$ (cf. KRASNOSELSKIJ and RUTICKIJ [18, p. 41]). Therefore the corollary is applicable, too.

Proof of Theorem 1: The proof is based on Gossez's theory of pseudomonotone operators in complementary systems of Sobolev-Orlicz spaces (cf. GOSSEZ [12, 14]). We shall work in the complementary system $(Y, Y_0; Z, Z_0)$ where $Y = \mathring{W}^m L_H(G)$, $Y_0 = \mathring{W}^m E_H(G)$, $Z = W^{-m} L_{H^*}(G)$, $Z_0 = W^{-m} E_{H^*}(G)$.

(I) *Operator A:* We define an operator $A : D(A) \subset Y \to Z$ assigning to any $u \in D(A)$ the element $Au \in Y_0^* \cong Z$ with

$$\langle v, Au \rangle = a(u, v) \qquad \text{for all} \qquad v \in Y_0. \tag{15}$$

Thus (11) is equivalent to the operator equation

$$Au = b. \tag{16}$$

Note that $Y_0 \subset D(A)$ by virtue of (13). We intend to show that (16) has a solution $u \in D(A)$ for any $b \in Z_0$. The method is to prove the existence and uniform boundedness of the solutions of Galerkin's equation (11') and then to go to the limit using pseudomonotonicity of A.

(II) *Operator A_n:* For any $u_n \in Y_n$ we define $A_n u_n \in Y_n^*$ by

$$\langle v_n, A_n u_n \rangle = a(u_n, v_n) \qquad \text{for all} \qquad v_n \in Y_n$$

(here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between Y_n and Y_n^*). The operators A_n are continuous by virtue of Gossez [12, Lemma 4.3]. Thus by condition (14a) $A_n : Y_n \to Y_n^*$ is a one-to-one continuous mapping from Y_n onto the range $R(A_n)$. From the Brouwer theorem on invariance of domain we conclude that $R(A_n)$ is an open set of Y_n^*. We prove that $R(A_n)$ is closed in Y_n^*. Then $R(A_n) = Y_n^*$ and assertion (ii) is verified (cf. PETRYSHYN [25]).

In fact, let (y_j^*) be a sequence from $R(A_n)$, i.e. $y_j^* = A_n u_j (u_j \in Y_n)$ such that $y_j^* \to y^*$ in Y_n^*. We intend to show that

$$\sup_j \|u_j\|_{Y_n} < \infty. \tag{17}$$

In view of dim $Y_n < \infty$ (17) implies the existence of a subsequence $(u_{j'}) \subset (u_j)$ such that $u_{j'} \to u$ in Y_n. Since A_n is continuous we get $y_{j'}^* = A_n u_{j'} \to A_n u = y^*$, i.e. $y^* \in R(A_n)$. Now we turn to the proof of (17). We have

$$\langle u_j, A_n u_j \rangle = a(u_j, u_j) = y_j^*(u_j). \tag{18}$$

Let us investigate the linear functionals $y_j^* \in Y_n^*$. Remember that $(Y, Y_0; Z, Z_0)$ is a complementary system; $Y_n \subset Y$ is a finite dimensional (and therefore closed) subspace of Y_0. We use Chapter 1 of GOSSEZ [12, p. 166] to generate a new complementary system $(Y_n, \cdot; \cdot, \cdot)$. In the terminology of Gossez we have $E = Y_n, E_0 = Y_n$, $F = Z/Y_n{}^\perp, F_0 = \{z + Y_n{}^\perp : z \in Z_0\}$ where $Y_n{}^\perp = \{z \in Z: \langle y, z\rangle = 0 \text{ for all } y \in Y_n\}$. Lemma 1.2 of GOSSEZ [12] proves that $(Y_n, Y_n; F, F_0)$ is a complementary system. Furthermore $F = F_0$. The norm on F_0 is the quotient norm:

$$\|[z]\|_{F_0} = \inf \{\|z + y_n{}^\perp\|_Z : y_n{}^\perp \in Y_n{}^\perp\}.$$

There exists a linear homeomorphism $\gamma\colon Y_n^* \to F_0$ (cf. Sect. 2.3) and we may assume that the norm on Y_n^* is given by $y_n^* \in Y_n^* \mapsto \|\gamma y_n^*\|_{F_0}$. Let us write $y_j^*(u_j) = (y_j^* - y^*)(u_j) + y^*(u_j)$. Choose $z_j \in Z_0$ such that $[z_j] = \gamma(y_j^* - y^*)$ and $w_j \in [z_j]$, $w_j \in Z_0$ such that

$$\|w_j\|_Z \leqq \|y_j^* - y^*\|_{Y_n} + 2^{-j}. \tag{19}$$

Furthermore we suppose that $\gamma y^* = [z]$ $(z \in Z_0)$. In view of (14b) and (18) we have

$$\begin{aligned} c_0 \int_G \sum_{|\beta|=m} H(aD^\beta u_j)\,dx - \int_G K(x)\,dx &\leqq \int_G \sum_{|\alpha|\leqq m} A_\alpha(x, Du_j)\,D^\alpha u_j\,dx \\ &= y_j^*(u_j) = \langle u_j, w_j\rangle + \langle u_j, z\rangle \\ &= \int_G \sum_{|\alpha|\leqq m} w_j{}^\alpha D^\alpha u_j\,dx + \int_G \sum_{|\alpha|\leqq m} z^\alpha D^\alpha u_j\,dx \end{aligned}$$

where $(w_j{}^\alpha), (z^\alpha) \in \prod_{|\alpha|\leqq m} E_{H^*}(G)$ represent the elements $w_j \in Z_0$ and $z \in Z_0$, respectively and the representations of the elements w_j are choosen such that $\|w_j{}^\alpha\|_{H^*} \leqq \|w_j\|_Z + 2^{-j}$ for all $|\alpha| \leqq m$, $j = 1, 2, \ldots$ Since $K \in L_1(G)$ we may use Young's inequality (4) to conclude

$$\begin{aligned} &\int_G \sum_{|\beta|=m} H(aD^\beta u_j)\,dx - \text{const.} \\ &\leqq c_0{}^{-1} \int_G \sum_{|\alpha|\leqq m} \big(H^*(\gamma^{-1}w_j{}^\alpha) + H(\gamma^{-1}z^\alpha)\big)\,dx + 2c_0{}^{-1} \int_G \sum_{|\alpha|\leqq m} H(\gamma D^\alpha u_j)\,dx \end{aligned} \tag{20}$$

for any $\gamma > 0$.
The generalization of Friedrich's inequality to Sobolev-Orlicz spaces (cf. GOSSEZ [12, Lemma 5.7]) gives

$$\int_G \sum_{|\alpha|\leqq m} H(\gamma D^\alpha u_j)\,dx \leqq c_2 \int_G \sum_{|\beta|\leqq m} H(c_3\gamma D^\beta u_j)\,dx$$

where $c_2, c_3 > 0$ are constants.

Without loss of generality we may assume that $4c_0{}^{-1}c_2 > 1$. Thus the last term on the right-hand side of (20) is not greater than $1/2 \int_G \sum_{|\beta|=m} H(4c_0{}^{-1}c_2c_3\gamma D^\beta u_j)\,dx$ (cf. KUFNER and JOHN, FUČIK [19, p. 128]). Now we choose $\gamma = \gamma_0$ with $\gamma_0 = 1/4c_0c_2{}^{-1}c_3{}^{-1}\,a$ to get

$$\begin{aligned} &\int_G \sum_{|\beta|=m} H(aD^\beta u_j) \\ &\leqq \text{const.} + 2c_0{}^{-1} \int_G \sum_{|\alpha|\leqq m} H^*(\gamma_0{}^{-1}w_j{}^\alpha)\,dx + 2c_0{}^{-1} \int_G \sum_{|\alpha|\leqq m} H^*(\gamma_0{}^{-1}z^\alpha)\,dx. \end{aligned} \tag{21}$$

The last term on the right-hand side of (21) does not depend on j. In view of (19) we have

$$\|w_j{}^\alpha\|_{H^*} \leqq \|w_j\|_Z + 2^{-j} \leqq \|y_j^* - y^*\|_{Y_n} + 2^{1-j} \to 0 \quad \text{as} \quad j \to \infty.$$

Therefore there exists an integer $j_0 \geqq 1$ such that $\|\gamma_0^{-1} w_j^\alpha\|_{H^*} < 1$ for all $j \geqq j_0$, $|\alpha| \leqq m$. From a well known inequality (cf. Kufner, John and Fučík [19, p. 154]) we conclude that $\varrho_{H^*}(\gamma_0^{-1} w_j^\alpha) \leqq \|\gamma_0^{-1} w_j^\alpha\|_{H^*} \to 0$ as $j \to \infty$ ($|\alpha| \leqq m$). Thus $\sup_j \int_G \sum_{|\beta|=m} H(aD^\beta u_j) < \infty$. This implies $\sup_j \|u_j\|_{m,H} < \infty$ and (17) is proved.

(III) *Uniform boundedness:* We intend to show that the solutions u_n of the Galerkin equations (11′) are uniformly bounded: $\sup_n \|u_n\|_{m,H} < \infty$. Since

$$\int_G \sum_{|\alpha| \leqq m} A_\alpha(x, Du_n)\, D^\alpha u_n\, dx = \int_G f u_n\, dx \qquad \text{for all } n = 1, 2, \ldots$$

hypothesis (14b) implies

$$c_0 \int_G \sum_{|\beta| \leqq m} H(aD^\beta u_n)\, dx \leqq \int_G f u_n\, dx + \int_G K\, dx.$$

By virtue of Young's inequality we get

$$c_0 \int_G \sum_{|\beta|=m} H(aD^\beta u_n) \leqq \text{const.} + \int_G H(\gamma^{-1} f)\, dx + \int_G H(\gamma u_n)\, dx$$

for any $\gamma > 0$. Because of $f \in E_{H^*}(G)$ we may proceed as in (II) to conclude $\sup_n \int_G \sum_{|\beta|=m} H(aD^\beta u_n)\, dx < \infty$. This, of course, implies $\sup_n \|u_n\|_{m,H} < \infty$.

(IV) *We prove assertion* (i): Problem ($\mathscr{P}$) has exactly one solution. By condition (14a) the solution is unique if it exists. Since $\sup \|u_n\|_{m,H} < \infty$ where the elements u_n are the solutions of the Galerkin equation (11′) and Z_0 is separable we may select a subsequence from (u_n) denoted by (u_n) again such that

$$u_n \to u \in Y \quad \text{in} \quad \sigma(Y, Z_0) \qquad \text{as } n \to \infty \tag{22}$$

(cf. Dieudonné [7, Theorem 12.15.9]). Since $V = \bigcup_n Y_n$ it follows from

$$\langle v, Au_n \rangle = a(u_n, v) \to b(v) \quad \text{as} \quad n \to \infty \qquad \text{for any } v \in V$$

that

$$Au_n \to b \quad \text{in} \quad \sigma(Z, V) \qquad \text{as } n \to \infty. \tag{23}$$

Moreover

$$\langle u_n, Au_n \rangle = a(u_n, u_n) = b(u_n) \to \langle u, b \rangle \qquad \text{as } n \to \infty \tag{24}$$

because of (22) and $b \in Z_0$.

The reasoning of Gossez [12, proof of Theorem 4.1, p. 188–189] shows that we can assume again passing to a subsequence that

$$A_\alpha(\cdot, Du_n) \to A_\alpha(\cdot, Du) \quad \text{in} \quad \sigma(L_{H^*}, E_H) \qquad \text{as } n \to \infty \quad (|\alpha| \leqq m) \tag{25}$$

and

$$u \in D(A),\ Au = b. \tag{26}$$

Thus assertion (i) is proved. Since the solution of $Au = b$ is unique an argument concerning subsequences (cf. Zeidler [34, Band I, p. 117] shows that (22), (23), (24), (25) hold for the entire sequence (u_n).

From (22) and the Sobolev-Orlicz imbedding theorem (cf. Sect. 2.2, Lemma 2) one obtains assertion (iii). Assertion (iv) for $|\alpha| \leqq m - 1$ immediately follows from assertion (iii) and Lemma 1.

(V) *We prove assertion* (V): Suppose $u \in D(A)$ is the solution of (16). We introduce the sets

$$G_k = \{x \in G\colon |D^\alpha u(x)| \leqq k \quad \text{for all } |\alpha| \leqq m\}, \qquad k = 1, 2, \ldots$$

Let χ_k denote the characteristic function of G_k. Thus the truncated functions $\chi_k D^\alpha u$ belong to $E_H(G)$ for all $|\alpha| \leqq m$, $k = 1, 2, \ldots$ Moreover $\chi_k D^\alpha u \to D^\alpha u$ in $\sigma(L_H, L_{H^*})$ as $k \to \infty$ (cf. KUFNER, JOHN and FUČIK [19, p. 181]).

(V_1): Consider

$$\Delta_{n,k} = \int_G \sum_{|\alpha| \leqq m} \big(A_\alpha(x, \chi_k Du) - A_\alpha(x, Du_n)\big)\,(\chi_k D^\alpha u - D^\alpha u_n)\,dx. \tag{27}$$

Now by virtue of (22) to (26) we obtain

$$\Delta_{n,k} \to \int_G \sum_{|\alpha| \leqq m} \big(A_\alpha(x, \chi_k Du) - A_\alpha(x, Du)\big)\,(\chi_k D^\alpha u - D^\alpha u)\,dx \equiv d_k \tag{28}$$

as $n \to \infty$ for k fixed. But

$$d_k = \int_{G - G_k} \sum_{|\alpha| \leqq m} \big(A_\alpha(x, 0) - A_\alpha(x, Du)\big)\,D^\alpha u\,dx.$$

Since meas $(G - G_k) \to 0$ as $k \to \infty$ and $A_\alpha(\cdot, 0), A_\alpha(\cdot, Du) \in L_{H^*}$ we get

$$d_k \to 0 \qquad \text{as } k \to \infty. \tag{29}$$

(V_2): From (28) and (29) we derive the existence of a sequence (n_k), $n_k \to \infty$ as $k \to \infty$ such that

$$\Delta_{n_k,k} \to 0 \qquad \text{as } k \to \infty. \tag{30}$$

Now we use an argument that is often employed in the theory of monotone operators (cf. BROWDER [5, p. 29], LIONS [24, p. 184]). Define

$$F_k(x) = \sum_{|\alpha| \leqq m} \big(A_\alpha(x, \chi_k Du(x)) - A_\alpha(x, Du_{n_k}(x))\,(\chi_k D^\alpha u(x)) - D^\alpha u_{n_k}(x)\big) \qquad (x \in G).$$

By monotonicity (14a) it follows that $F_k(x) \geqq 0$ $(x \in G)$ and (30) implies $F_k \to 0$ in $L_1(G)$. Therefore we may assume passing to a subsequence that

$$F_k(x) \to 0 \qquad \text{a.e. on } G \text{ as } k \to \infty. \tag{31}$$

From assertion (iii) we know that

$$D^\alpha u_{n_k}(x) \to D^\alpha u(x) \qquad \text{a.e. on } G \text{ for } |\alpha| \leqq m - 1. \tag{32}$$

(again after passing to a subsequence).

Let $M \subset G$ be a set of measure zero such that (31), (32) hold for all $x \in G - M$. From (13) and (14b) we derive

$$\begin{aligned} F_k(x) &\geqq c_0 \sum_{|\beta| = m} H\big(aD^\beta u_{n_k}(x)\big) - K(x) \\ &\quad + \sum_{|\alpha| \leqq m} A_\alpha\big(x, \chi_k Du(x)\big)\big(\varkappa_k D^\alpha u(x) - D^\alpha u_{n_k}(x)\big) \\ &\quad - \sum_{|\alpha| \leqq m} A_\alpha\big(x, Du_{n_k}(x)\big)\,\chi_k D^\alpha u(x) \\ &\geqq c_0 \sum_{|\beta| = m} H\big(aD^\beta u_{n_k}(x)\big) - K(x) \\ &\quad - c(x)\Big\{1 + \sum_{|\alpha| \leqq m} |D^\alpha u_{n_k}(x)| + \sum_{|\alpha| \leqq m} (H^*)^{-1} H\big(\bar{c}_1 D^\alpha u_{n_k}(x)\big)\Big\} \equiv R_k(x) \end{aligned}$$

where $c(x) > 0$ is a number depending only on x. Fix $x \in G - M$ and suppose that $\eta^\alpha(x)$ is any limit of the sequence $D^\alpha u_{n_k}(x)$ ($|\alpha| = m$). It is easy to see that $|\eta^\alpha(x)| < \infty$. Indeed, if we had $|\tilde{u}_{n_k}(x)| \to \infty$ for some subsequence $(\tilde{u}_{n_k}) \subset (u_{n_k})$ then a short consideration of the growth behavior of the functions H and $(H^*)^{-1}$ yields $R_k(x) \to \infty$, i.e. $F_k(x) \to \infty$, too, contrary to (31). Therefore $|\eta^\alpha(x)| < \infty$ for all $x \in G - M$. Combining this and (31) we see that

$$\sum_{|\alpha| \leq m} \big(A_\alpha(x, Du(x)) - A_\alpha(x, \eta(x))\big)\big(D^\alpha u(x) - \eta^\alpha(x)\big) = 0 \qquad (x \in G - M).$$

In view of (14a) we get $D^\alpha u(x) = \eta^\alpha(x)$ for all $x \in G - M$, i.e.

$$D^\alpha u_{n_k}(x) \to D^\alpha u(x) \qquad \text{a.e. on } G \text{ for } |\alpha| \leq m. \tag{33}$$

(V_3): Define

$$w_k(x) = K(x) + \sum_{|\alpha| \leq m} A_\alpha\big(x, Du_{n_k}(x)\big)\, D^\alpha u_{n_k}(x),$$

$$w(x) = K(x) + \sum_{|\alpha| \leq m} A_\alpha\big(x, Du(x)\big)\, D^\alpha u(x) \qquad (k = 1, 2, \ldots;\ x \in G).$$

By (14b) $w_k(x) \geqq 0$, $w(x) \geqq 0$ for all $x \in G$. From (33) it follows that

$$w_k(x) \to w(x) \quad \text{a.e. on } G \text{ as } k \to \infty. \tag{34}$$

Observe that

$$\int_G w_k\, dx = \int_G K\, dx + \int_G \sum_{|\alpha| \leq m} A_\alpha(x, Du_{n_k})\, D^\alpha u_{n_k}\, dx$$
$$\to \int_G K\, dx + \int_G \sum_{|\alpha| \leq m} A_\alpha(x, Du)\, D^\alpha u\, dx = \int_G w\, dx,$$

i.e. $\|w_k\|_{L_1(G)} \to \|w\|_{L_1(G)}$. This fact and (34) imply $w_k \to w$ in $L_1(G)$ (cf. Dieudonné [7, Ch. 13.11], Hewitt and Stromberg [17, p. 208]). Passing to a subsequence we may assume that $w_k(x) \leqq h(x)$ a.e. on G ($k = 1, 2, \ldots$) where $h \in L_1(G)$ (cf. Kufner, John and Fučík [19, p. 74]). Then by (14b)

$$c_0 \sum_{|\beta| = m} H\big(aD^\beta u_{n_k}(x)\big) \leqq w_k(x) \leqq h(x) \quad \text{a.e. on } G \qquad (k = 1, 2, \ldots).$$

Therefore Jensen's inequality (3) gives for $|\beta| = m$

$$H\big(2^{-1}a(D^\beta u(x) - D^\beta u_{n_k}(x))\big) \leqq 2^{-1}H\big(aD^\beta u(x)\big) + 2^{-1}H\big(aD^\beta u_{n_k}(x)\big)$$
$$\leqq 2^{-1}H\big(aD^\beta u(x)\big) + 2^{-1}c_0^{-1}h(x) \quad \text{a.e. on } G \qquad (k = 1, 2, \ldots). \tag{35}$$

Furthermore it follows from (14b) that $H(aD^\beta u) \in L_1(G)$. Thus the right-hand side of (35) belongs to $L_1(G)$. Therefore by Lebesgue's theorem on dominated convergence

$$\int_G H\big(2^{-1}a(D^\beta u(x) - D^\beta u_{n_k}(x))\big)\, dx \to 0 \quad \text{for all } |\beta| = m. \tag{36}$$

Applying the already mentioned argument concerning subsequences (36) immediately proves assertion (v).

(VI) *The end of the proof:* Lemma 1, (iii) and our hypothesis $\Psi \ll H$ prove assertion (iv) for $|\alpha| = m$ ∎

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REFERENCES

[1] Adams, R. A.: Sobolev spaces. New York—San Francisco—London 1975.

[2] Aubin, J. P.: Approximation des espaces de distribution et des opérateurs differentials Bull. Soc. Math. France Mém. 12 (1967), 1—138.

[3] Aubin, J. P.: Approximation of elliptic boundary value problems. Wiley Interscience: New York 1972.

[4] Ball, J. M.: Convexity Conditions and Existence Theorems in Nonlinear Elasticity. Arch. Rat. Mech. Anal. 63 (1976/77), 337—403.

[5] Browder, F. E.: Existence theorems for nonlinear partial differential equations. In: Global analysis. Proc. Symp. Pure Math. 16 (1970), 1—60. Amer. Math. Soc.: Providence, R. I.

[6] Browder, F. E.: Existence theory for boundary value problems for quasilinear elliptic systems with strongly nonlinear lower order terms. In: Partial differential equations. Proc. Symp. Pure Math. 23 (1973), 269—286. Amer. Math. Soc.: Providence, R. I.

[7] Dieudonné, J.: Grundzüge der modernen Analysis. Band 2. VEB Deutscher Verlag der Wiss.: Berlin 1975.

[8] Donaldson, T.: Nonlinear elliptic boundary value problems in Orlicz-Sobolev spaces. J. Diff. Equations 10 (1971), 507—528.

[9] Donaldson, T.: Inhomogeneous Orlicz-Sobolev spaces and nonlinear parabolic initial value problems. J. Diff. Equations 16 (1974), 201—256.

[10] Donaldson, T., and N. S. Trudinger: Orlicz-Sobolev spaces and imbedding theorems. J. Funct. Anal. 8 (1971), 52—75.

[11] Gossez, J. P.: Opérateurs monotones nonlinéaires dans les espaces de Banach non réflexifs. J. Math. Anal. Appl. 34 (1971), 371—395.

[12] Gossez, J. P.: Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. Transact. Amer. Math. Soc. 190 (1974), 163—205.

[13] Gossez, J. P.: Surjectivity results for pseudomonotone mappings in complementary systems. J. Math. Anal. Appl. 53 (1976), 484—494.

[14] Gossez, J. P.: Orlicz-Sobolev spaces and nonlinear elliptic boundary value problems. In: Fučik, S., and A. Kufner (eds.): Nonlinear analysis, function spaces and applications. Proc. of a Spring School held in Horni Bradlo, 1978. Teubner-Texte zur Mathematik. Teubner Verlag: Leipzig 1979.

[15] Hess, P.: On nonlinear mappings of monotone type with respect to two Banach spaces. J. Math. Pures Appl. 52 (1973), 13—26.

[16] Hess, P.: A strongly nonlinear elliptic boundary value problem. J. Math. Anal. Appl. 43 (1973), 241—249.

[17] Hewitt, E., and K. Stromberg: Real and abstract analysis. Springer Verlag: Berlin—Heidelberg—New York 1965.

[18] Красносельский, М. А., и Я. Б. Рутицкий: Выпуклые функции и пространства Орлича. Москва 1958.

[19] Kufner, A., John, A., and S. Fučik: Function spaces. Academia: Prague 1977.

[20] Lacroix, M. T.: Etude des conditions au bord pour des problèmes elliptiques fortement non lineaires. Annali Mat. Pura ed Appl. 109 (1976), 203—220.

[21] Lacroix, M. T.: Espaces de trace des espaces de Sobolev-Orlicz. J. Math. Pures Appl. 53 (1974), 439—458.

[22] Landes, R.: Quasilineare elliptische Differentialoperatoren mit starkem Wachstum in den Termen höchster Ordnung. Math. Z. 157 (1977), 23—36.

[23] Langenbach, A.: Monotone Potentialoperatoren in Theorie und Anwendungen. VEB Deutscher Verlag der Wissenschaften: Berlin 1976.

[24] Lions, J. L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod-Gauthier-Villars: Paris 1969.

[25] Petryshyn, W. V.: Nonlinear equations involving nocompact operators. In: Nonlinear Functional Analysis. Proc. Symp. Pure Math. 18 (1970), 206—233. Amer. Math. Soc.: Providence, R. I.

[26] Robert, J.: Approximations des espaces d'Orlicz et applications. Numer. Math. 17 (1971), 338—356.

[27] Rockafellar, R. T.: On the maximal monotonicity of subdifferential mappings. Pacific J. Math. 33 (1970), 209—216.

[28] Schumann, R.: The finite difference method for quasilinear elliptic equations with rapidly increasing coefficients. In: R. Kluge (ed.): Nonlinear Analysis. Theory and Applications. Proc. of a Summer School held in Berlin 1979. Akademie-Verlag, Berlin 1981, 399—402.

[29] Schumann, R., and E. Zeidler: The finite difference method for quasilinear elliptic equations of order $2m$. J. Numer. Funct. Anal. Optim. 1 (1979), 161—194.

[30] Simader, C. G.: Über schwache Lösungen des Dirichletproblems für streng nichtlineare elliptische Differentialgleichungen. Math. Z. 150 (1976), 1—26.

[31] Trudinger, N. S.: On imbeddings into Orlicz spaces and some applications. J. Math. Mech. 17 (1967), 473—484.

[32] Вишик, М. И.: Квазилинейные сильно эллиптические системы дифференциальных уравнений. Труды Моск. матем. об-ва 12 (1963), 125—184.

[33] Вишик, М. И.: О разрешимости первой краевой задачи для квазилинейных уравнений с быстро растущими коэффициентами в классах Орлича. Докл. Акад. Наук СССР 151 (1963), 758—761.

[34] Zeidler, E.: Vorlesungen über nichtlineare Funktionalanalysis. Teil I: Fixpunktsätze; Teil II: Monotone Operatoren; Teil III: Variationsmethoden und Optimierung. Teubner-Texte zur Mathematik. Teubner-Verlag: Leipzig 1976, 1977, 1978. (English translation in preparation.)

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