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Nonlinear noncoercive equations and applications

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Es werden periodische Lösungen der nichtlinearen Balkengleichung

$$\beta u_t + u_{tt} + u_{xxxx} - \lambda u + \varphi(u) = f$$

in Abhängigkeit von einer nichtlinearen Funktion $\varphi: \mathbf{R} \to \mathbf{R}$ betrachtet. Die Untersuchungen schließen sich an eine Arbeit von Fučík [6] an und enthalten gegenüber dieser einige neue Ergebnisse.

Рассматриваются периодические решения нелинейного уравнения балки

$$\beta u_t + u_{tt} + u_{xxxx} - \lambda u + \varphi(u) = f$$

в зависимости от нелинейной функции $\varphi: \mathbf{R} \to \mathbf{R}$. Исследования примыкают к работе S. Fučík [6] и включают в себя некоторые новые результаты, не содержащиеся в этой работе.

This paper deals with the periodic solvability of the nonlinear beam equation

$$\beta u_t + u_{tt} + u_{xxxx} - \lambda u + \varphi(u) = f,$$

which depends on non-linear $\varphi: \mathbf{R} \to \mathbf{R}$. This paper continues the subject of the paper by S. Fučík [6]. We present some new methods and results which are not included in [6].

1. Introduction

This paper continues the subject of the paper by S. Fučík [6]. We shall study, as in [6], problems which have their abstract formulation as an equation:

$$Tu = f, \tag{1.1}$$

where T is operator acting from a Banach space X into a Banach space Z, T being of the form

$$Tu = Lu + Su, \tag{1.2}$$

where L is linear and S is nonlinear. We are interested in the case when T does not satisfy the coercivity condition

$$\lim_{\|u\|_X \to +\infty} \|Tu\|_Z = +\infty. \tag{1.3}$$

Typical examples of the operator equation (1.1) with condition (1.2) are the following problems. Let λ be a real number, $\omega > 0$, $\beta > 0$, and let φ be a real valued continuous function.

Boundary value problems for ordinary differential equations:

$$\left.\begin{array}{ll} -u''(x) - \lambda u(x) + \varphi(u(x)) = f(x), & x \in (0, \pi) \\ u(0) = u(\pi) = 0. & \end{array}\right\} \tag{1.4}$$

Periodic problems for ordinary differential equations:

$$\left.\begin{array}{ll} -u''(x) - \lambda u(x) + \varphi(u(x)) = f(x), & x \in (0, \pi) \\ u(0) = u(\pi), u'(0) = u'(\pi). & \end{array}\right\} \tag{1.5}$$

Boundary value problems for partial differential equations of elliptic type:

$$\left.\begin{array}{ll} -\Delta u(x) - \lambda u(x) + \varphi(u(x)) = f(x), & x \in \Omega \\ u(x) = 0, \quad x \in \partial\Omega, & \end{array}\right\} \tag{1.6}$$

where Ω is a sufficiently smooth bounded domain in N-dimensional space.

One can consider higher order equations of the type (1.4)—(1.6) and also another type of boundary conditions than Dirichlet ones in (1.6).

Periodic solutions of the boundary value problem for the nonlinear heat equation:

$$\left.\begin{array}{l} u_t(t, x) - u_{xx}(t, x) - \lambda u(t, x) + \varphi(u(t, x)) \\ = f(t, x), \ (t, x) \in Q := (-\infty, +\infty) \times (0, \pi) \\ u(t, 0) = u(t, \pi) = 0, \quad t \in (-\infty, +\infty), \\ u(t + \omega, x) = u(t, x), \quad (t, x) \in Q. \end{array}\right\} \tag{1.7}$$

Periodic solutions of the nonlinear telegraph equation:

$$\left.\begin{array}{l} \beta u_t(t, x) + u_{tt}(t, x) - u_{xx}(t, x) - \lambda u(t, x) + \varphi(u(t, x)) \\ = f(t, x), \ t, x \in (-\infty, +\infty), \\ u(t, x) = u(t + 2\pi, x) = u(t, x + 2\pi), \quad t, x \in (-\infty, +\infty). \end{array}\right\} \tag{1.8}$$

Periodic solutions of the nonlinear beam equation:

$$\left.\begin{array}{l} \beta u_t(t, x) + u_{tt}(t, x) + u_{xxxx}(t, x) - \lambda u(t, x) + \varphi(u(t, x)) = f(t, x), \\ u(t + 2\pi, x + 2\pi) = u(t + 2\pi, x) = u(t, x + 2\pi) \\ = u(t, x) \ (t, x \in (-\infty, +\infty)). \end{array}\right\} \tag{1.9}$$

In the previous examples, the nonlinear operator S is given by the nonlinear part $\varphi(u)$ of the problem considered and the operator L is defined by the linear part, i.e. it is given by

$$u \mapsto -u'' - \lambda u \text{ in (1.4) and (1.5)},$$

$$u \mapsto -\Delta u - \lambda u \text{ in (1.6)},$$

$$u \mapsto u_t - u_{xx} - \lambda u \text{ in (1.7)},$$

$$u \mapsto \beta u_t + u_{tt} - u_{xx} - \lambda u \text{ in (1.8)},$$

$$u \mapsto \beta u_t + u_{tt} + u_{xxxx} - \lambda u \text{ in (1.9)}.$$

We present some methods and results (which are not included in [6]) about the solvability of the previous types of nonlinear equations. As in [6] we choose (1.9) as the model for the explanation of these methods. The methods used here for solving (1.9) can be applied also for (1.4)—(1.8). The reason for choosing (1.9) is the same as in [6].

2. Preliminaries

In the sequel we shall denote by $\mathbf{I}$ the open interval $(0, 2\pi)$. Further, $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ will denote the set of positive integers, integers and real numbers, respectively. Put

$$\mathbf{I}^2 = \mathbf{I} \times \mathbf{I}$$

and analogously for other sets.

Before starting a precise definition of a periodic solution of the nonlinear beam equation

$$\beta u_t + u_{tt} + u_{xxxx} + \varphi(u) = h(t, x), \tag{2.1}$$

we introduce, in the same way as in [6], the suitable function spaces.

Denote by $\mathbf{H}$ the space of all measurable real valued functions $u(t, x)$ defined almost everywhere on $\mathbf{R}^2$ which are 2π-periodic in the variables t and x, i.e.

$$u(t + 2\pi, x + 2\pi) = u(t + 2\pi, x) = u(t, x + 2\pi) = u(t, x)$$

for almost all $(t, x) \in \mathbf{R}^2$, and which are square integrable over $\mathbf{I}^2$. Introducing the inner product

$$\langle h, k\rangle = \int_{\mathbf{I}^2} h(t, x)\, k(t, x)\, dt\, dx \qquad (h, k \in \mathbf{H}),$$

$\mathbf{H}$ becomes a Hilbert space. Its norm we denote by $\|\cdot\|_{\mathbf{H}}$, i.e.

$$\|h\|_{\mathbf{H}} = \langle h, h\rangle^{1/2}, \qquad h \in \mathbf{H}.$$

Let $\tilde{\mathbf{H}} = \mathbf{H} + i\mathbf{H}$ be the complexification of the space $\mathbf{H}$. It is easy to see that

$$\{e^{i(mx+nt)}: (m, n) \in \mathbf{Z}^2\}$$

forms a complete orthogonal system in $\tilde{\mathbf{H}}$. Thus arbitrary $h \in \mathbf{H}$ can be expressed by

$$h(t, x) = \sum_{(m,n)\in\mathbf{Z}^2} h_{m,n} e^{i(mx+nt)}$$

(the convergence is in the space $\tilde{\mathbf{H}}$), where

$$\sum_{(m,n)\in\mathbf{Z}^2} |h_{m,n}|^2 < \infty, \qquad h_{m,n} = \overline{h_{-m,-n}}.$$

Let p, r be nonnegative integers. By $\mathbf{C}_{2\pi}^{p,r}$ we mean the set of all continuous functions $u(t, x)$ defined on $\mathbf{R}^2$ which are 2π-periodic in both variables, and such that the partial derivatives by t up to the order p and the partial derivatives by x up to the order r are continuous on $\mathbf{R}^2$. With the norm

$$\|u\|_{\mathbf{C}_{2\pi}^{0,0}} = \max_{(t,x)\in\mathbf{R}^2} |u(t, x)|$$

$\mathbf{C}_{2\pi}^{0,0}$ becomes a Banach space.

Definition 2.1: Let $p, r \in \mathbf{N} \cup \{0\}$. Define

$$\mathbf{H}^{p,r} = \left\{h \in \mathbf{H}: \sum_{(m,n)\in\mathbf{Z}^2} (m^{2r} + n^{2p})\, |h_{m,n}|^2 < \infty\right\}.$$

$\mathbf{H}^{p,r}$ with the norm

$$\|h\|_{\mathbf{H}^{p,r}} = \left(\sum_{(m,n)\in\mathbf{Z}^2} (m^{2r} + n^{2p})\, |h_{m,n}|^2\right)^{1/2}$$

is a Banach space which is nothing other than a *Sobolev space of periodic* functions (see [10]).

Definition 2.2 (*Generalized periodic solutions*): Let φ be a continuous function defined on $\mathbf{R}$. Suppose that there exist $\alpha_1, \alpha_2 \geqq 0$ such that

$$|\varphi(z)| \leqq \alpha_1 + \alpha_2 |z|, \qquad z \in \mathbf{R}. \tag{2.2}$$

Let $\beta > 0$ and $h \in \mathbf{H}$. A *generalized periodic solution* (GPS) of (2.1) is a real function $u \in \mathbf{H}$ such that for all $v \in \mathbf{C}_{2\pi}^{2,4}$ one has

$$\langle u, -\beta v_t + v_{tt} + v_{xxxx} \rangle = \langle h - \varphi(u), v \rangle. \tag{2.3}$$

Remark 2.1: (i) The growth condition (2.2) is necessary for a Němytskij's operator

$$u \mapsto \varphi(u)$$

acting from $\mathbf{H}$ into $\mathbf{H}$ (see e.g. [6, 7]).

(ii) Using integration by parts in (2.3) we can prove that if $u \in \mathbf{C}_{2\pi}^{2,4}$ is a GPS of (2.1), then the equation (2.1) is fulfilled on $\mathbf{R}^2$ (i.e. u is "a classical solution" of (2.1)). On the other hand, an arbitrary GPS $u \in \mathbf{H}$ has better properties (see [6: Th. 2.4, 2.5]).

Let us denote

$$\sigma = \left\{q \in \mathbf{N} \cup \{0\} : q^{1/4} \in \mathbf{N}\right\}.$$

The following two theorems are proved in [6]:

Theorem 2.1: *Let $\lambda \in \mathbf{R}$. Then the equation*

$$\beta u_t + u_{tt} + u_{xxxx} - \lambda u = h \tag{2.4}$$

has for arbitrary $h \in \mathbf{H}$ a unique GPS *$u \in \mathbf{H}$ if and only if $\lambda \notin \sigma$.*

Theorem 2.2: *Let $\lambda = q \in \sigma$. Denote by $\mathscr{H}_q$ and $\mathscr{H}_q{}^\perp$ two closed orthogonal subspaces of $\mathbf{H}$ with the following properties:*

$$\mathscr{H}_q = \{h \in \mathbf{H} : h_{0,q} = h_{0,-q} = 0\};$$

$$\mathscr{H}_q{}^\perp = \textit{linear hull of } \{\sin q^{1/4}x, \cos q^{1/4}x\} \textit{ provided } q \neq 0;$$

$$\mathscr{H}_0{}^\perp = \textit{linear hull of constant functions.}$$

Then for an arbitrary $h \in \mathscr{H}_q$ there exists a unique GPS *$u \in \mathscr{H}_q$ of* (2.4).

Put $\mathbf{C}_q = \mathbf{C}_{2\pi}^{0,0} \cap \mathscr{H}_q$ and define the mapping

$$\tilde{T}_q : \mathscr{H}_q \to \mathscr{H}_q, \ \tilde{T}_q : h \mapsto u,$$

where u is the unique GPS of (2.4). Then

(i) $\tilde{T}_q$ is linear, $\operatorname{Im} \tilde{T}_q \subset \mathbf{C}_q$;

(ii) The mappings $\tilde{T}_q : \mathscr{H}_q \to \mathscr{H}_q$, $\tilde{T}_q : \mathscr{H}_q \to \mathbf{C}_q$, $\tilde{T}_q|_{\mathbf{C}_q} : \mathbf{C}_q \to \mathbf{C}_q$, are completely continuous (where the norm $\|\cdot\|_{\mathbf{C}_{2\pi}^{0,0}}$ is introduced in the space $\mathbf{C}_q$ and the norm $\|\cdot\|_{\mathbf{H}}$ in $\mathscr{H}_q$).

(iii) If $p, r \in \mathbf{N} \cup \{0\}$ then

$$\tilde{T}_q(\mathbf{H}^{p,r} \cap \mathscr{H}_q) \subset (\mathbf{H}^{p+1,r+1} \cap \mathscr{H}_q).$$

3. Bounded nonlinearities

Assumptions: We prove the existence and multiplicity of GPSs of

$$\beta u_t + u_{tt} + u_{xxxx} + \varphi(u) = h, \tag{3.1}$$

where $\beta > 0$, $h \in \mathbf{H}$ and $\varphi: \mathbf{R} \to \mathbf{R}$ is a continuous function with finite limits

$$\varphi(\pm\infty) = \lim_{z \to \pm\infty} \varphi(z).$$

Moreover let us suppose

$$\varphi(z)\, z \geqq 0, \qquad \varphi(-\infty) \leqq 0 \leqq \varphi(+\infty)$$

(the case $\varphi(z)\, z \leqq 0$, $\varphi(+\infty) \leqq 0 \leqq \varphi(-\infty)$ can be treated similarly). Suppose there exists $\delta > 0$ such that

$$\varphi(z) \geqq \varphi(+\infty), \qquad z \geqq \delta,$$
$$\varphi(z) \leqq \varphi(-\infty), \qquad z \leqq -\delta.$$

A typical example of such a function φ is

$$\varphi(z) = z e^{-z^2}, \qquad z \in \mathbf{R}.$$

In contrast to [6], we make no assumptions about the limits

$$\lim_{z \to \pm\infty} \big(\varphi(z) - \varphi(\pm\infty)\big)\, z.$$

Remark 3.1: Denote by $\mathbf{P}_0$ the orthogonal projection from $\mathbf{H}$ onto $\mathscr{H}_0{}^{\perp}$. Put

$$P_0{}^c: u \mapsto u - \mathbf{P}_0 u, \qquad u \in \mathbf{H}.$$

The mapping $P_0{}^c$ is the orthogonal projection from $\mathbf{H}$ onto $\mathscr{H}_0$. Then for each $h \in \mathbf{H}$ there exists an $s \in \mathbf{R}$ and an $h_1 \in \mathscr{H}_0$ such that

$$h = s + h_1,$$

$s = P_0 h$, $h_1 = P_0{}^c h$.

Theorem 3.1: *For each $h_1 \in \mathscr{H}_0$ there exist real numbers $T_1 \leqq 0 \leqq T_2$ such that*

(i) *the equation* (3.1) *has at least one* GPS *for $h = s + h_1$ with $s \in (T_1, T_2)$ in the case $T_1 < T_2$, moreover* (3.1) *has at least one* GPS *for $h = h_1$;*

(ii) *the equation* (3.1) *has at least two distinct* GPSs *for $h = s + h_1$ with $s \in \big(T_1, \varphi(-\infty)\big) \cup \big(\varphi(+\infty), T_2\big)$ in the case $T_1 < T_2$.*

Proof: Put

$$G: u \mapsto \varphi(u), \qquad u \in \mathbf{H}.$$

Then it is easy to see that the equation (3.1) is solvable (in the sense of Definition 2.2) if and only if the following bifurcation system is solvable:

$$\left.\begin{aligned} v + \tilde{T}_0 P_0{}^c G(w + v) &= \tilde{T}_0 P_0{}^c h, \\ P_0 G(w + v) &= P_0 h, \end{aligned}\right\} \tag{3.2}$$

where $w = P_0 u$, $v = P_0{}^c u$, $u \in \mathbf{H}$ (see e.g. [6]).

Let us denote $w_1(t, x) = \frac{1}{4\pi^2}$, for $(t, x) \in \mathbf{I}^2$. Then

$$\int_{\mathbf{I}^2} w_1(t, x)\, dt\, dx = 1$$

and for each $w \in \mathscr{H}_0{}^\perp$ there exists a real number τ such that

$$w = \tau w_1 .$$

Let $h_1 \in \mathscr{H}_0$ be arbitrary but fixed. Because the function φ is continuous and bounded on $\mathbf{R}$ then for a possible solution v of the first equation in (3.2),

$$\|v\|_{\mathbf{C}^{0,0}_{2\pi}} \leqq c \tag{3.3}$$

(see Theorem 2.2 (ii)), where the constant $c > 0$ is independent of w. Let us consider a ball $\mathbf{B}_R(0)$ with its centre at the origin and with sufficiently large radius $R > 0$. Then for each $w \in \mathscr{H}_0{}^\perp$ and $v \in \partial\mathbf{B}_R(0)$

$$v + \tilde{T}_0 P_0{}^c G(w + v) - \tilde{T}_0 P_0{}^c h \neq 0 .$$

By a standard application of the Leray-Schauder degree theory we can prove (see [5]) that for each $w \in \mathscr{H}_0{}^c$ there is at least one $v \in \mathscr{H}_0$ satisfying the first equation in (3.2). Let us define

$$\mathbf{S} = \{(\tau, v) \in \mathbf{R} \times \mathscr{H}_0 : w = \tau w_1 \text{ and } v \text{ satisfies the first equation in (3.2)}\} .$$

Then the solutions of (3.1) are such $u = \tau w_1 + v$ that $(\tau, v) \in \mathbf{S}$ and

$$\psi(\tau, v) = \langle h, w_1 \rangle ,$$

where

$$\psi(\tau, v) = \int_{\mathbf{I}^2} \varphi(\tau w_1 + v)\, w_1$$

is a real continuous function defined on S. For fixed $\tau \in \mathbf{R}$ put

$$\begin{aligned} &\underline{\tau}(\tau) = \inf_{(\tau, v) \in \mathbf{S}} \psi(\tau, v) \quad \text{and} \quad \overline{\tau}(\tau) = \sup_{(\tau, v) \in \mathbf{S}} \psi(\tau, v) ; \\ &T_1 = \inf_{\tau \in \mathbf{R}} \overline{\tau}(\tau) \quad \text{and} \quad T_2 = \sup_{\tau \in \mathbf{R}} \underline{\tau}(\tau) . \end{aligned} \tag{3.4}$$

Let us remark that if for some $v \in \mathscr{H}_0$ there exists $\tau \in \mathbf{R}$ such that $(\tau, v) \in \mathbf{S}$ then

$$\|v\|_{\mathbf{C}^{0,0}_{2\pi}} \leqq c$$

(see (3.3)). So the assumptions at the beginning of this section guarantee the inequality $T_1 \leqq 0 \leqq T_2$.

Suppose, now, that $T_1 < T_2$ and $s \in (T_1, T_2)$. Then according to (3.4) there exist $\tau_1, \tau_2 \in \mathbf{R}$ such that

$$\psi(\tau_1, v) > s \quad \text{and} \quad \psi(\tau_2, v) < s$$

for all $(\tau_1, v) \in \mathbf{S}$, $(\tau_2, v) \in \mathbf{S}$. To prove that the equation (3.1) has at least one GPS for $h = s + h_1$, we need the following lemma.

Lemma 3.1: *For each real number $\overline{\beta} > 0$ there exists a connected subset $\mathbf{S}_{\overline{\beta}} \subset \mathbf{S}$ such that $proj_{\mathbf{R}} \mathbf{S}_{\overline{\beta}} \supset [-\overline{\beta}, \overline{\beta}]$ (where $proj_{\mathbf{R}} \mathbf{S}_{\overline{\beta}}$ denotes the projection of the set $\mathbf{S}_{\overline{\beta}}$ onto $\mathbf{R}$).*

For the proof of Lemma 3.1 see [1, 4].

Having the assertion of Lemma 3.1, we can choose $\bar{\beta} > 0$ such that

$$\bar{\beta} > \max \{|\tau_1|, |\tau_2|\}.$$

The fact that ψ is continuous on the connected subset $S_{\bar{\beta}}$ implies the existence of at least one pair $(\tau, v) \in S_{\bar{\beta}}$ such that

$$\psi(\tau, v) = s.$$

Then $u = \tau w_1 + v$ is the desired solution of (3.1). According to the assumptions on φ there exists $\tau_0 \in \mathbf{R}$ such that

$$\psi(\tau_0, v) \leqq 0 \qquad \text{and} \qquad \psi(-\tau_0, v) \geqq 0,$$

for all $(\tau_0, v) \in S$, $(-\tau_0, v) \in S$. Using Lemma 3.1 we prove the existence of at least one GPS of (3.1) with $s = 0$ and the assertion (i) is proved.

We shall prove, now, the assertion (ii). Let $T_1 < T_2$ and $s \in (T_1, \varphi(-\infty))$. Then according to (3.4) there exists $\tau_3 \in \mathbf{R}$ such that $\varphi(\tau_3, v) < s$ for all $(\tau_3, v) \in S$. It is sufficient to prove the existence of $\tau_4, \tau_5 \in \mathbf{R}$ such that $\psi(\tau_i, v) > s$ for all $(\tau_i, v) \in S$, $i = 4, 5$. Then using Lemma 3.1 we obtain at least two distinct solutions of (3.1). Put

$$\mathbf{I}^2_{n,\tau} = \{(t, x) \in \mathbf{I}^2 \colon \tau w_1(t, x) + v(t, x) \leqq n \quad \text{for all} \quad (\tau, v) \in S\},$$

$$\mathbf{I}^2_{-n,\tau} = \{(t, x) \in \mathbf{I}^2 \colon \tau w_1(t, x) + v(t, x) \geqq -n \quad \text{for all} \quad (\tau, v) \in S\}.$$

It is easy to see that

$$\lim_{t \to \pm\infty} \operatorname{meas} \mathbf{I}^2{}_{\pm n, t} = 0 \quad \text{for each} \quad n \in \mathbf{N}$$

(see (3.3)). According to the assumptions on φ we can choose for arbitrary $\varepsilon > 0$ such $\tau_0 \in \mathbf{R}$ and $n \in \mathbf{N}$ that for $\tau_4 = -\tau_0$

$$\left| \int_{\mathbf{I}^2 \setminus \mathbf{I}^2_{-n,\tau_4}} \varphi(\tau_4 w_1 + v)\, w_1 - \varphi(-\infty) \right| < \frac{\varepsilon}{2}, \qquad \left| \int_{\mathbf{I}^2_{-n,\tau_4}} \varphi(\tau_4 w_1 + v)\, w_1 \right| < \frac{\varepsilon}{2}, \tag{3.5}$$

and for $\tau_5 = \tau_0$

$$\left| \int_{\mathbf{I}^2 \setminus \mathbf{I}^2_{n,\tau_5}} \varphi(\tau_5 w_1 + v)\, w_1 - \varphi(+\infty) \right| < \frac{\varepsilon}{2}, \qquad \left| \int_{\mathbf{I}^2_{n,\tau_5}} \varphi(\tau_5 w_1 + v)\, w_1 \right| < \frac{\varepsilon}{2}, \tag{3.6}$$

for all $(\tau_i, v) \in S$, $i = 4, 5$. From (3.5), (3.6) we obtain

$$\left| \int_{\mathbf{I}^2} \varphi(\tau_4 w_1 + v)\, w_1 - \varphi(-\infty) \right| < \varepsilon \quad \text{and} \quad \left| \int_{\mathbf{I}^2} \varphi(\tau_5 w_1 + v)\, w_1 - \varphi(+\infty) \right| < \varepsilon$$

for all $(\tau_i, v) \in S$, $i = 4, 5$. Put $\varepsilon = \dfrac{\varphi(-\infty) - s}{2}$. Then the last two inequalities imply that the function ψ has desired property, i.e. $\psi(\tau_i, v) > s$ for all $(\tau_i, v) \in S$, $i = 4, 5$. If $s \in (\varphi(+\infty), T_2)$, the proof of the assertion (ii) is analogous. This completes the proof of Theorem 3.1 ∎

Remark 3.2: Let us consider the equation

$$\beta u_t + u_{tt} + u_{xxxx} + ue^{-u^2} = h \qquad (h = s + h_1). \tag{3.1'}$$

Then from Theorem 3.1' it follows (by an easy calculation) that for each $h_1 \in \mathscr{H}_0$ there exist $T_1(h_1) < 0 < T_2(h_1)$ such that (3.1') has at least one GPS if $s \in (T_1, T_2)$ and (3.1') has at least two distinct GPSs if $s \in (T_1, 0) \cup (0, T_2)$.

Remark 3.3: The existence of the solution of the boundary value problem for second order elliptic partial differential equations with analogous nonlinearity $\varphi(u)$ is proved in [5]. Existence and multiplicity results of such problems are proved in [2].

4. Superlinear nonlinearities

In this section we shall consider the generalized periodic solvability of the equation

$$\beta u_t + u_{tt} + u_{xxxx} - \lambda u + \varphi(u) = h, \tag{4.1}$$

where $\lambda > 0$, φ is a continuous real valued function which is bounded on the interval $(-\infty, 0]$ and

$$\lim_{z \to +\infty} \frac{\varphi(z)}{z} = +\infty. \tag{4.2}$$

Adding suitable constants on both sides of the equation (4.1), we may assume without loss of generality that

$$\varphi(z) \geqq 0 \quad \text{for all} \quad z \in \mathbf{R}.$$

As an example we can present the function $\varphi(z) = e^z$, $z \in \mathbf{R}$. Since there are no restrictions on the growth of φ in $+\infty$, we must slightly modify the definition of a GPS (see Definition 2.2).

Definition 4.1: Let $\beta > 0$ and $h \in \mathbf{H}$. A *generalized periodic solution* (GPS) of (4.1) is a real function $u \in \mathbf{H}_1$ such that for all $v \in \mathbf{C}_{2\pi}^{2,4}$ we have

$$\langle u, -\beta v_t + v_{tt} + v_{xxxx} \rangle = \int_{\mathbf{I}^2} hv - \int_{\mathbf{I}^2} \varphi(u)\, v + \int_{\mathbf{I}^2} \lambda uv \tag{4.3}$$

where

$$\mathbf{H} \supset \mathbf{H}_1 := \left\{ u \in \mathbf{H} : \int_{\mathbf{I}^2} \varphi(u) < +\infty \right\}.$$

The main assertion of this section is the following theorem.

Theorem 4.1: *Let $h \in \mathbf{H}$, $h = s + h_1$. Then there exist real numbers $T_1(h_1) \leqq T_2(h_1)$ and a bounded set $\mathbf{M}(h_1) \subset [T_1(h_1), T_2(h_1)]$, $T_2(h_1) \in \mathbf{M}(h_1)$ such that*

(i) *the equation* (4.1) *has at least two distinct* GPSs *if* $s > T_2(h_1)$;

(ii) *the equation* (4.1) *has at least one* GPS *if* $s \in \mathbf{M}(h_1)$;

(iii) *the equation* (4.1) *has no* GPS *if* $s < T_1(h_1)$.

Proof: Let $h_1 \in \mathscr{H}_0$ be arbitrary but fixed. We shall prove, first, that for fixed $\tau \in \mathbf{R}$ there exists at least one $v(\tau) \in \mathscr{H}_0$ such that

$$\langle v(\tau), -\beta v_t + v_{tt} + v_{xxxx} \rangle = \int_{\mathbf{I}^2} h_1 v - \int_{\mathbf{I}^2} \varphi(\tau w_1 + v(\tau))\, v + \lambda \int_{\mathbf{I}^2} v(\tau)\, v \tag{4.4}$$

holds for each $v \in \mathbf{C}_{2\pi}^{2,4} \cap \mathscr{H}_0$.

We insert now two lemmata.

Lemma 4.1: *Let $\mathscr{I} \subset \mathbf{R}$ be a bounded interval. Then there exists a constant $r > 0$ such that for $u \in \mathscr{H}_0 \cap \mathbf{C}^{2,4}_{2\pi}$, $\|u\|_{\mathbf{H}} > r$ and $\tau \in \mathscr{I}$*

$$u \neq \sigma \tilde{T}_0(h_1 - P_0^c G(\tau w_1 + u) + \lambda u),$$

for all $\sigma \in [0, 1]$.

Proof: We argue by contradiction. Suppose for all $n \in \mathbf{N}$ there exists $\tilde{u}_n \in \mathscr{H}_0 \cap \mathbf{C}^{2,4}_{2\pi}$, $\|\tilde{u}_n\|_{\mathbf{H}} \geqq n$, $\tau_n \in \mathscr{I}$ and $\sigma_n \in [0, 1]$ such that

$$\tilde{u}_n = \sigma_n \tilde{T}_0(h_1 - P_0^c G(\tau_n w_1 + \tilde{u}_n) + \lambda \tilde{u}_n). \tag{4.5}$$

Applying $\tilde{T}_0^{-1}$ to both sides of (4.5) and taking the inner product with $\tilde{u}_n$ we obtain

$$\langle \tilde{T}_0^{-1}\tilde{u}_n, \tilde{u}_n\rangle = \sigma_n[\langle h_1, \tilde{u}_n\rangle - \langle G(\tau_n w_1 + \tilde{u}_n), \tilde{u}_n\rangle + \lambda\langle \tilde{u}_n, \tilde{u}_n\rangle].$$

Letting $u_n = \dfrac{\tilde{u}_n}{\|\tilde{u}_n\|_{\mathbf{H}}}$ and dividing by $\|\tilde{u}_n\|_{\mathbf{H}}^2$ we get

$$\begin{aligned}&\langle u_n, -\beta(u_n)_t + (u_n)_{tt} + (u_n)_{xxxx}\rangle + \sigma_n \|\tilde{u}_n\|_{\mathbf{H}}^{-1} \langle G(\tau_n w_1 + \tilde{u}_n), u_n\rangle - \sigma_n\lambda \\ &= \sigma_n \|\tilde{u}_n\|_{\mathbf{H}}^{-1} \langle h_1, u_n\rangle.\end{aligned} \tag{4.6}$$

This means that $\langle u_n, -\beta(u_n)_t\, (u_n)_{tt} + (u_n)_{xxxx}\rangle$ is a real number and so $\langle u_n, -\beta(u_n)_t + (u_n)_{tt} + (u_n)_{xxxx}\rangle \geqq$ const. > 0 for each $n \in \mathbf{N}$. From (4.6) we obtain

$$\begin{aligned}&\langle u_n, -\beta(u_n)_t + (u_n)_{tt} + (u_n)_{xxxx}\rangle \\ &+ \sigma_n \|\tilde{u}_n\|_{\mathbf{H}}^{-1} \langle G(\tau_n w_1 + \tilde{u}_n), u_n\rangle - \sigma_n\lambda \leqq \|h_1\|_{\mathbf{H}} \|\tilde{u}_n\|_{\mathbf{H}}^{-1}.\end{aligned}$$

Since $\langle G(\tau_n w_1 + \tilde{u}_n), u_n\rangle$ is bounded below (we assume $\varphi \geqq 0$ and φ is bounded on $(-\infty, 0]$),

$$\langle u_n, -\beta(u_n)_t + (u_n)_{tt} + (u_n)_{xxxx}\rangle \leqq (\lambda + 1) \tag{4.7}$$

for sufficiently large $n \in \mathbf{N}$. We prove that there exists an $\alpha \in (0, 1)$ such that $\|u_n^+\|_{\mathbf{H}} \geqq \alpha$ for sufficiently large n. Suppose on the contrary that there is a subsequence of $\{u_n\}_{n=1}^\infty$, which we shall also denote by $\{u_n\}_{n=1}^\infty$, such that $\|u_n^+\|_{\mathbf{H}} \to 0$. From (4.7) we obtain that $\{u_n\}_{n=1}^\infty$ is bounded in the space $\mathbf{H}^{2,1}$. Since $\mathbf{H}^{2,1} \bigcirc\!\bigcirc\, \mathbf{H}$ (compact imbedding) we can suppose, after possibly passing to a suitable subsequence, that $u_n \xrightarrow{H} u_0$, $\|u_0\|_{\mathbf{H}} = 1$ and $u_0 \leqq 0$ a.e. in $\mathbf{I}^2$. But this is a contradiction to the fact $u_0 \in \mathscr{H}_0$.

Note that there exists such a constant $\gamma > 0$ that

$$\varphi(z) \geqq \frac{\lambda}{\alpha^2} z - \gamma, \quad \text{for all } z \in \mathbf{R}. \tag{4.8}$$

Further note that

$$\langle G(\tau_n w_1 + \tilde{u}_n), u_n\rangle \geqq \langle G(\tau_n w_1 + \tilde{u}_n), u_n^+\rangle - c_1,$$

because φ is bounded on $(-\infty, 0]$, $w_1 \in L^\infty(\mathbf{I}^2)$ and $\{\tau_n\}$ is bounded. From (4.6) and (4.8) we obtain

$$\begin{aligned}\|h_1\|_{\mathbf{H}} &\geqq \text{const. } \|\tilde{u}_n\|_{\mathbf{H}} + \sigma_n\langle G(\tau_n w_1 + \tilde{u}_n), u_n^+\rangle - \sigma_n c_1 - \sigma_n\lambda \|\tilde{u}_n\|_{\mathbf{H}} \\ &\geqq \text{const. } \|\tilde{u}_n\|_{\mathbf{H}} + \sigma_n \left[\frac{\lambda}{\alpha^2} \int_{\mathbf{I}^2} (\tau_n w_1 + \|\tilde{u}_n\|_{\mathbf{H}}\, u_n)\, u_n^+ - \gamma \int_{\mathbf{I}^2} u_n^+\right] \\ &\quad - \sigma_n c_1 - \sigma_n\lambda \|\tilde{u}_n\|_{\mathbf{H}}\end{aligned}$$

$$\geqq \text{const.}\ \|\tilde{u}_n\|_{\mathbf{H}} + \sigma_n \frac{\lambda}{\alpha^2} \alpha^2 \|\tilde{u}\|_{\mathbf{H}} - c_2 - \sigma_n \lambda \|\tilde{u}_n\|_{\mathbf{H}}$$

$$\geqq \text{const.}\ \|\tilde{u}_n\|_{\mathbf{H}} - c_2 .$$

The final inequality implies the boundedness of $\|\tilde{u}_n\|_{\mathbf{H}}$, which is a contradiction. This proves the Lemma ∎

Lemma 4.2: Let $\mathscr{I} \subset \mathbf{R}$ *be a bounded interval. Then there exists a constant* $r > 0$ *that for each* $\tau \in \mathscr{I}$ *there exists* $v(\tau) \in \mathscr{H}_0$ *such that* (4.4) *holds for each* $v \in \mathbf{C}_{2\pi}^{2,4} \cap \mathscr{H}_0$ *and moreover* $\|v(\tau)\|_{\mathbf{H}} \leqq r$.

Proof: Let $\tau \in \mathscr{I}$ be fixed. We use the Galerkin method to prove the existence of $v(\tau)$. Choose $\mathbf{V}_n \subset \mathscr{H}_0 \cap \mathbf{C}_{2\pi}^{\infty,\infty}$ such that $\dim \mathbf{V}_n = n$, $\mathbf{V}_n \subset \mathbf{V}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathbf{V}_n$ is dense in $\mathscr{H}_0$. A function $u_n \in \mathbf{V}_n$ is called the Galerkin solution of (4.4) if

$$u_n = \tilde{T}_0(h_1 - P_0{}^c G(\tau w_1 + u_n) + \lambda u_n). \tag{4.9}$$

From Lemma 4.1 and from the homotopy invariance property of the Brouwer degree, we obtain for each $n \in \mathbf{N}$ the existence of the Galerkin solution u_n of (4.4) such that $\|u_n\|_{\mathbf{H}} \leqq r$. Then $\|h_1 - P_0{}^c G(\tau w_1 + u_n) + \lambda u_n\|_{\mathbf{H}} \leqq \text{const.}$ (we use the continuity of $\tilde{T}_0{}^{-1}$). After possibly passing to subsequences, we can suppose that

$$h_1 - P_0{}^c G(\tau w_1 + u_n) + \lambda u_n \rightharpoonup u_0 \in \mathscr{H}_0 .$$

According to the complete continuity of $\tilde{T}_0$. (Th. 2.2(ii)) and (4.9) we can suppose that $\{u_n\}_{n=1}^{\infty}$ is convergent in the norm $\|\cdot\|_{\mathbf{H}}$. So there exists $v(\tau) \in \mathscr{H}_0$ such that $\lim_{n \to +\infty} \|u_n - v(\tau)\|_{\mathbf{H}} = 0$ and

$$\|v(\tau)\|_{\mathbf{H}} \leqq r .$$

It remains to prove that $\varphi(\tau w_1 + v(\tau)) \in L^1(\mathbf{I}^2)$ and

$$\varphi(\tau w_1 + u_n) \xrightarrow{L^1(\mathbf{I}^2)} \varphi(\tau w_1 + v(\tau)).$$

Since $u_n \in \mathbf{V}_n$ is the Galerkin solution of (4.4) for each $n \in \mathbf{N}$, we obtain from (4.9)

$$\int_{\mathbf{I}^2} |u_n \varphi(\tau w_1 + u_n)| \leqq K \|u_n\|_{\mathbf{H}}^2 + \lambda \|u_n\|_{\mathbf{H}} + \|h_1\|_{\mathbf{H}} \|u_n\|_{\mathbf{H}} \leqq \text{const.} \tag{4.10}$$

(where K is a constant independent of n).

We have proved that $u_n \xrightarrow{H} v(\tau)$ and so

$$u_n \to v(\tau) \quad \text{and} \quad u_n \varphi(\tau w_1 + u_n) \to v(\tau)\, \varphi(\tau w_1 + v(\tau)) \quad \text{a.e. in } \mathbf{I}^2.$$

Having (4.10) Fatou's lemma implies that

$$v(\tau)\, \varphi(\tau w_1 + v(\tau)) \in L^1(\mathbf{I}^2).$$

Then for each $k \in \mathbf{N}$ and $\varepsilon > 0$ there exists such a $\delta > 0$ that for each $\Omega \subset \mathbf{I}^2$, meas $\Omega < \delta$ we obtain

$$\int_{\Omega \cap \{|u_n| \leqq k\}} |\varphi(\tau w_1 + u_n)| < \frac{\varepsilon}{2}, \quad \frac{1}{k} \int_{\Omega \cap \{|u_n| > k\}} |u_n \varphi(\tau w_1 + u_n)| < \frac{\varepsilon}{2}.$$

The final two inequalities imply

$$\int_{\Omega} |\varphi(\tau w_1 + u_n)| \leqq \int_{\Omega \cap \{|u_n| < k\}} |\varphi(\tau w_1 + u_n)| + \frac{1}{k} \int_{\Omega \cap \{|u_n| > k\}} |u_n \varphi(\tau w_1 + u_n)| < \varepsilon .$$

Using Vitali's theorem we have

$$\varphi(\tau w_1 + v(\tau)) \in L^1(\mathbf{I}^2) \quad \text{and} \quad \varphi(\tau w_1 + u_n) \xrightarrow{L^1(\mathbf{I}^2)} \varphi(\tau w_1 + v(\tau)).$$

This means that $(\tau, v(\tau))$ fulfill (4.4) and the proof of Lemma 4.2 is completed (essentially the same procedure can be found in STRAUSS [9]) ∎

We go on proving Theorem 4.1. Put

$$\mathrm{S} = \{(\tau, v(\tau)) \in \mathbf{R} \times \mathscr{H}_0 : (\tau, v(\tau)) \text{ fulfill } (4.4)\},$$

$$\mathrm{S}_n = \{(\tau, u_n) \in \mathbf{R} \times \mathbf{V}_n : u_n \text{ is the Galerkin solution of } (4.4)\}.$$

It is easy to see that the GPSs of (4.1) are such $u = \tau w_1 + v(\tau)$ that $(\tau, v(\tau)) \in \mathrm{S}$ and

$$-\lambda\tau + \int_{\mathbf{I}^2} \varphi(\tau w_1 + v(\tau))\, w_1 = s.$$

Let us define a continuous function (see [3, 4, 8])

$$F: \mathscr{S} = \mathrm{S} \cup \left(\bigcup_{n=1}^{\infty} S_n\right) \to \mathbf{R}$$

by the relation

$$F(\tau, v) = -\lambda\tau + \int_{\mathbf{I}^2} \varphi(\tau w_1 + v)\, w_1 .$$

Since $\varphi(z) \geqq 0$ ($z \in \mathbf{R}$) and $w_1 > 0$ on $\mathbf{I}^2$, we obtain

$$F(\tau, v) \geqq -\lambda\tau, \tag{4.11}$$

for all $(\tau, v) \in \mathscr{S}$. Using (4.8) we obtain

$$F(\tau, v) \geqq -\lambda\tau + \int_{\mathbf{I}^2} \frac{\lambda}{\alpha^2} (\tau w_1 + v)\, w_1 - \gamma \int_{\mathbf{I}^2} w_1 = \left(\frac{\lambda}{\alpha^2} - \lambda\right)\tau - \gamma \int_{\mathbf{I}^2} w_1, \tag{4.12}$$

for all $(\tau, v) \in \mathscr{S}$. Since $0 < \alpha < 1$, (4.11) and (4.12) imply

$$\lim_{\tau \to \pm\infty} F(\tau, v) = +\infty, \tag{4.13}$$

uniformly with respect to such $v \in \mathscr{H}_0$ that $(\tau, v) \in \mathscr{S}$.

Put

$$T_2(h_1) = \sup_{\substack{(\tau,v) \in \cup \mathrm{S}_n \\ \tau \in [-1,1]}} F(\tau, v). \tag{4.14}$$

From Lemma 4.1 we obtain $T_2(h_1) < +\infty$. Suppose $s > T_2(h_1)$. By (4.13) there exists $\tau_0 \in \mathbf{R}$ such that

$$\inf_{\substack{(\tau,v) \in \cup \mathrm{S}_n \\ \tau \in (-\infty, -\tau_0] \cup [\tau_0, +\infty)}} F(\tau, v) > s.$$

Using the assertion of Lemma 3.1 (see also [3, 4]), we obtain for each $n \in \mathbf{N}$ the existence of a connected subset $\tilde{\mathrm{S}}_n \subset \mathrm{S}_n$ such that $[-\tau_0, \tau_0] \subset proj_{\mathbf{R}}\, \tilde{\mathrm{S}}_n$. Then according to the definition of $T_2(h_1)$ by (4.14), for each $n \in \mathbf{N}$ there exists $\tau_n^1 \in (-\infty, -1)$, $\tau_n^2 \in (1, +\infty)$ and $v_n^1, v_n^2 \in \mathscr{H}_0$ such that

$$(\tau_n^i, v_n^i) \in S_n \ (i = 1, 2) \quad \text{and} \quad F(\tau_n^i, v_n^i) = s.$$

By (4.13), $\{\tau_n^1\}_{n=1}^{\infty} \subset (-\infty, -1)$, $\{\tau_n^2\}_{n=1}^{\infty} \subset (1, +\infty)$ are bounded sequences and Lemma 4.1 implies that $\{v_n^i\}_{n=1}^{\infty} \subset \mathscr{H}_0$ ($i = 1, 2$) are also bounded. After possibly passing to a suitable subsequence, we may suppose (by the same argument as in the proof of Lemma 4.2) that

$$\tau_n^i \to \tau^i, \qquad v_n^i \to v^i$$

and by the same procedure as in the proof of Lemma 4.2 we prove that $(\tau^i, v^i) \in \mathrm{S}$ ($i = 1, 2$). Since $\tau^1 \neq \tau^2$ and $F(\tau^i, v^i) = s$ ($i = 1, 2$), the functions $u_i = \tau^i w_1 + v^i$ are two distinct GPSs of (4.1). This fact proves the assertion (i) of Theorem 4.1.

Put

$$T_1(h_1) = \inf_{(\tau,v)\in \mathrm{S}} F(\tau, v).$$

From (4.11), (4.12) we obtain $T_1(h_1) > -\infty$. If $s < T_1(h_1)$ then (4.1) has no GPS, which proves the assertion (iii).

By the assertion (i) there is a sequence $\{s_m\}_{m=1}^{\infty} \subset (T_2(h_1), +\infty)$, $s_m \to T_2(h_1)$ such that there exist bounded sequences $\{\tau_m\}_{m=1}^{\infty} \subset \mathbf{R}$, $\{v_m\}_{m=1}^{\infty} \subset \mathscr{H}_0$ such that $(\tau_m, v_m) \in \mathbf{V}_m$, $F(\tau_m, v_m) = s_m$. After possibly passing to a subsequence, we can suppose that

$$\tau_m \to \tau_0, \qquad v_m \xrightarrow{H} v_0 \in \mathscr{H}_0.$$

By the same procedure as in the proof of Lemma 4.2 we prove that $(\tau_0, v_0) \in \mathrm{S}, F(\tau_0, v_0) = T_2(h_1)$. Then $T_2(h_1) \in \mathbf{M}(h_1)$, which proves the assertion (ii). The proof of Theorem 4.1 is completed ∎

Remark 4.1: There are no restrictions to the growth of φ in $+\infty$ and so the Němytskij's operator

$$u \mapsto \varphi(u)$$

is not always acting from **H** into **H**. This is the reason why we use the Galerkin method and the properties of Brouwer degree (instead of the Leray-Schauder degree) to prove the existence and multiplicity of solutions.

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