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A regularization method in the Cauchy problem for holomorphic functions

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In der Arbeit wird ein Regularisierungsverfahren beim Cauchy-Problem für holomorphe Funktionen im Einheitskreis vorgeschlagen, welches auf der Einbettung des Problems in eine Schar von Carlemanschen Kopplungsproblemen beruht.

В работе предложен метод регуляризации у задачи Коши для голоморфных функций в единичном круге, который основывается на вложении задачи в семейство краевых задач Карлемана.

In the paper a regularization method in the Cauchy problem for holomorphic functions in the unit disk is proposed, which relies on the embedding of this problem into a family of Carleman conjugacy problems.

Introduction

The Cauchy problem for analytic functions or equivalently for harmonic functions in the plane is one of the most important incorrectly posed problems of mathematical physics. It possesses applications in geophysics, hydrodynamics, plasma physics, electron optics, and medicine. Furthermore, its solution is basic for the solution of the Cauchy problem for more general elliptic equations and systems (cf. [4a, 4b, 6, 7, 27, 28]).

Fundamental results on the continuous dependence of the solution to the Cauchy problem for the Laplace equation upon the data, the estimation of the solution and the approximative solution of the problem are given in a classical paper of T. Carleman [2] (cf. also [14]), in the pioneering papers of F. John [10, 11], M. M. Lavrentiev [13, 14] (cf. also the monographs [15, 15a, 16]), and C. Pucci [21, 22], by L. E. Payne [20], V. K. Ivanov [8, 9], and R. Lattès and J.-L. Lions [12]. Also we refer to the recent review paper by G. Talenti [23]. New constructive solution methods are developed by J. R. Cannon and P. Du Chateau [1] (cf. also H. D. Mittelmann [18a, b]), P. Colli Franzone, L. Guerri, B. Taccardi, and C. Viganotti [3], P. N. Vabiščevič [25], P. N. Vabiščevič, V. B. Glasko, and J. A. Kriksin [26], and M. V. Urev [24].

In this note a further approximation method in the Cauchy problem for analytic functions will be proposed. The Cauchy problem is dealt with in the normal form, in which a holomorphic function in the unit disk is to be determined from its boundary values on the upper semi-circle. Cauchy problems for general simply connected domains may be reduced to this normal form by conformal mapping. The Cauchy problem is embedded into a family of simple Carleman conjugacy problems for the unit disk (cf. [17]), which can be solved in explicit manner by transforming them to well-known Hilbert conjugacy problems for a slit on the real axis. Assuming the existence of a solution to the Cauchy problem being Hölder continuous on the closed unit disk, we obtain the uniform convergence of the approximate solutions to the exact solution in the interior of the unit disk and on the circle, respectively. No numerical results are given in this paper.

1. Statement of problem and regularization method

Let G be the unit disk $G: |z| < 1$ with boundary $\Gamma: |z| = 1$ and $\Gamma_1: |z| = 1$, $0 \leqq \arg z \leqq \pi$, and $\Gamma_2: |z| = 1$, $-\pi \leqq \arg z \leqq 0$, the upper and lower semi-circle, respectively. We deal with the following *Cauchy problem*:

To find a holomorphic function $F(z)$ in G being Hölder continuous in $\bar{G}$, which takes on prescribed Hölder continuous boundary values $f(t)$ on $\Gamma_1: t = e^{is}$, $0 \leqq s \leqq \pi$:

$$F^+(t) = f(t), \qquad t \in \Gamma_1. \tag{1}$$

We make the *basic assumption* that there exists a (uniquely determined) solution $F(z)$ of the problem, i.e., the given Hölder continuous function $f(t)$ on Γ_1 is boundary function of a holomorphic function $F(z)$ in G with Hölder continuous boundary values $F^+(t)$ on Γ.

Remark: There are simple sufficient conditions for this basic assumption, e.g. the development of the real part of $f(t)$ in a sufficiently rapidly convergent cosinus (resp., sinus) series in $[0, \pi]$ and of the imaginary part of $f(t)$ in the sinus (resp., cosinus) series with the same (resp., the same with opposite sign) coefficients. But this basic assumption may be generally regarded as fulfilled from the physical viewpoint; without such an existence assumption the problem of regularization has obviously no sense.

As it is well known, the Cauchy problem is incorrectly posed. To construct an approximate solution of it for only approximately given data f we embed it into the following family of *Carleman conjugacy problems* (cf. [17: § 13]):

To find a holomorphic function $F_\varepsilon(z)$ in G being Hölder continuous in $\bar{G}$, which satisfies the conjugacy condition

$$F_\varepsilon^+(t) + \varepsilon F_\varepsilon^+(1/t) = f(t), \qquad t \in \Gamma_1. \tag{2}$$

Here ε is a positive parameter, the Cauchy problem corresponds to the limit $\varepsilon \to 0$.

The *conjugacy problem* (2) can be solved in explicit manner (cf. [17: pp. 159–161]). The Joucovsky mapping

$$\omega(z) = z + \frac{1}{z} \tag{3}$$

maps the unit disk G in the z plane onto the ω plane cut along the slit $L = [-2, 2]$ on the real axis, where the upper (lower) semi-circle $\Gamma_1(\Gamma_2)$ corresponds to the lower (upper) boundary $L^-(L^+)$ of the slit, respectively. The inverse mapping to (3) is

$$z(\omega) = \frac{1}{2}\left(\omega - \sqrt{\omega^2 - 4}\right), \tag{4}$$

where $\arg\sqrt{\omega^2-4}=0$ for real $\omega>2$, with the boundary values

$$z^+(\tau)=\frac{1}{2}\left(\tau-i\sqrt{4-\tau^2}\right)=\frac{1}{t}\in\Gamma_2\quad\text{for}\quad\tau\in L^+,\tag{5a}$$

$$z^-(\tau)=\frac{1}{2}\left(\tau+i\sqrt{4-\tau^2}\right)=t\in\Gamma_1\quad\text{for}\quad\tau\in L^-,\tag{5b}$$

where $\sqrt{4-\tau^2}>0$ for $|\tau|<2$. Therefore, putting

$$\Phi_\varepsilon(\omega)=F_\varepsilon(z)=F_\varepsilon\left(\frac{1}{2}\left[\omega-\sqrt{\omega^2-4}\right]\right)\tag{6}$$

and

$$\varphi(\tau)=f(t)=f\left(\frac{1}{2}\left[\tau+i\sqrt{4-\tau^2}\right]\right),\tag{7}$$

the Carleman conjugacy condition (2) goes over into the *Hilbert conjugacy condition*

$$\Phi_\varepsilon^+(\tau)+\frac{1}{\varepsilon}\Phi_\varepsilon^-(\tau)=\frac{1}{\varepsilon}\varphi(\tau),\qquad\tau\in L,\tag{8}$$

for the sectionally holomorphic function $\Phi_\varepsilon(\omega)$ in the ω plane regular at infinity and bounded at the endpoints of the cut L.

The *Hilbert problem* (8) has the uniquely determined solution (cf. [19: Chap. 4, § 80])

$$\Phi_\varepsilon(\omega)=-\frac{\sqrt{\omega^2-4}}{2\pi}\left(\frac{\omega-2}{\omega+2}\right)^{-\delta i}e^{\delta\pi}\int_{-2}^{2}\frac{\varphi(\tau)}{\sqrt{4-\tau^2}\,(\tau-\omega)}\left(\frac{2-\tau}{2+\tau}\right)^{\delta i}d\tau\tag{9}$$

with $\arg(\omega-2)=\arg(\omega+2)=0$ for real $\omega>2$ and $\arg(2-\tau)=\arg(2+\tau)=0$ on L. The boundary values of $\Phi_\varepsilon(\omega)$ on L are

$$\Phi_\varepsilon^-(\tau)=\frac{\varphi(\tau)}{2}-\frac{X(\tau)}{2\pi i}\int_{-2}^{2}\frac{\varphi(\zeta)\,d\zeta}{X(\zeta)\,(\zeta-\tau)},\tag{10a}$$

$$\Phi_\varepsilon^+(\tau)=\frac{1}{\varepsilon}\left[\frac{\varphi(\tau)}{2}+\frac{X(\tau)}{2\pi i}\int_{-2}^{2}\frac{\varphi(\zeta)\,d\zeta}{X(\zeta)\,(\zeta-\tau)}\right],\tag{10b}$$

where $\delta=(1/2\pi)\ln[1/\varepsilon]>0$ and

$$X(\tau)=\sqrt{4-\tau^2}\left(\frac{2-\tau}{2+\tau}\right)^{-\delta i}.\tag{11}$$

Hence the solution to the *Carleman problem* (2) is given by

$$F_\varepsilon(z)=\frac{1-z^2}{2\pi}\left(\frac{1+z}{1-z}\right)^{2\delta i}e^{\delta\pi}\int_{-2}^{2}\frac{\varphi(\tau)}{\sqrt{4-\tau^2}\,(1+z^2-\tau z)}\left(\frac{2-\tau}{2+\tau}\right)^{\delta i}d\tau,\tag{12}$$

where $\arg(1+z)=\arg(1-z)=0$ for real $z\in G$. Analogous expressions hold for the boundary values $F_\varepsilon^+(t)=\Phi_\varepsilon^-(\tau)$, $t\in\Gamma_1$, and $F_\varepsilon^+(1/t)=\Phi_\varepsilon^+(\tau)$, $(1/t)\in\Gamma_2$. In particular, one has

$$F_\varepsilon^+(\pm 1) = f(\pm 1)/(1+\varepsilon) \tag{13}$$

and

$$F_\varepsilon(0) = \frac{e^{\delta\pi}}{2\pi} \int\limits_{-2}^{2} \frac{\varphi(\tau)}{\sqrt{4-\tau^2}} \left(\frac{2-\tau}{2+\tau}\right)^{\delta i} d\tau. \tag{14}$$

In terms of the given function $f = f(s)$ on Γ_1 the formula (12) for $F_\varepsilon(z)$ writes

$$F_\varepsilon(z) = \frac{1-z^2}{2\pi} \left(\frac{1+z}{1-z}\right)^{2\delta i} e^{\delta\pi} \int\limits_0^\pi \frac{f(s)}{2\cos s \cdot z - 1 - z^2} \left(\frac{1-\cos s}{1+\cos s}\right)^{\delta i} ds. \tag{12'}$$

Remark 1: Taking $\psi(t) = F^+(t)$, $t \in \Gamma_2$, as unknown function, from the Cauchy formula

$$F(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(t)\,dt}{t-z} + \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\psi(t)\,dt}{t-z}, \qquad z \in G, \tag{15}$$

and the Plemelj-Sochozki formulae (cf. [19: Chap. 1, § 16]) for the boundary values of $F(z)$ on Γ_1 and Γ_2 we obtain the following *integral equations* each of them equivalent to the Cauchy problem (1):

$$\frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\psi(\tau)\,d\tau}{\tau - t} = \frac{f(t)}{2} - \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\tau)\,d\tau}{\tau - t}, \qquad t \in \Gamma_1, \tag{16}$$

$$\frac{\psi(t)}{2} - \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\psi(\tau)\,d\tau}{\tau - t} = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\tau)\,d\tau}{\tau - t}, \qquad t \in \Gamma_2. \tag{17}$$

(Cf. [15: Chap. II, § 1], where the equation (16) is used in suitably modified form.) Introducing the Carleman problem (2), the equations (16), (17) are embedded in the *family of integral equations* for $\psi_\varepsilon(t) = F_\varepsilon^+(t)$, $t \in \Gamma_2$:

$$\frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\psi_\varepsilon(\tau)\,d\tau}{\tau - t} + \varepsilon \left[\frac{1}{2}\,\psi_\varepsilon(1/t) - \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\psi_\varepsilon(1/\tau)}{\tau - t}\,d\tau\right]$$
$$= \frac{f(t)}{2} - \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\tau)\,d\tau}{\tau - t}, \qquad t \in \Gamma_1, \tag{16_0}$$

$$\frac{\psi_\varepsilon(t)}{2} - \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{\psi_\varepsilon(\tau)\,d\tau}{\tau - t} + \frac{\varepsilon}{2\pi i} \int\limits_{\Gamma_1} \frac{\psi_\varepsilon(1/\tau)\,d\tau}{\tau - t}$$
$$= \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\tau)\,d\tau}{\tau - t}, \qquad t \in \Gamma_2; \tag{17_0}$$

respectively, where in the Cauchy formula for $F_\varepsilon(z)$ the conjugacy condition (2) has been used in the form

$$F_\varepsilon^+(t) = f(t) - \varepsilon\psi_\varepsilon(1/t), \qquad t \in \Gamma_1. \tag{2'}$$

As is easily seen by means of the identity theorem for analytic functions and the uniqueness property of the Cauchy problem, the solution to each of the equations (16)—(17_0) is uniquely determined.

Remark 2: Obviously, the function $\Phi_\varepsilon(\omega)$ in (9) may be regarded as an approximation to the solution $\Phi(\omega)$ of the corresponding Cauchy problem for a sectionally holomorphic function in the ω plane with the slit $L = [-2, 2]$ and given boundary values $\Phi^-(\tau)$ on one side of it. In the same way this Cauchy problem for an arbitrary simple smooth arc L can be regularized (cf. again [19: Chap. 4]).

2. Asymptotic behaviour of the solution to the Carleman problem

Under the above basic assumption of the existence of the solution $F(z)$ to the Cauchy problem (1) the solution $F_\varepsilon(z)$ and its boundary values $F_\varepsilon^+(t)$ on Γ_1 and Γ_2 converge to $F(z)$ in G and $f(t)$ on Γ_1, $F^+(t)$ on Γ_2, respectively, as ε goes to zero. More precisely, the following theorem holds.

Theorem 1: *As* $\varepsilon \to 0$

$$F_\varepsilon(z) = F(z) + o(\varepsilon^{\lambda(z)}), \qquad z \in G, \tag{18}$$

where

$$\lambda(z) = \frac{1}{2} + \frac{1}{\pi} \arg \frac{1+z}{1-z} > 0 \tag{19}$$

with $(1/2) \leqq \lambda(z) < 1$ *for* $\operatorname{Im} z \geqq 0$, $0 < \lambda(z) \leqq (1/2)$ *for* $\operatorname{Im} z \leqq 0$;

$$F_\varepsilon^+(t) = f(t) - F^+(1/t)\,\varepsilon + o(\varepsilon) = f(t) + O(\varepsilon), \qquad t \in \Gamma_1, \tag{20a}$$

$$F_\varepsilon^+(t) = F^+(t) + o(1), \qquad t \in \Gamma_2. \tag{20b}$$

Proof: We work with the function $\Phi_\varepsilon(\omega)$ in (9). Applying the residue theorem to the integral in (9) for a domain bounded by the slit L and a circle with sufficiently large radius R going to infinity, we first obtain the relation

$$\Phi_\varepsilon(\omega) = \Phi(\omega) + \frac{\sqrt{\omega^2-4}}{2\pi} \left(\frac{\omega-2}{\omega+2}\right)^{-\delta i} e^{-\pi\delta} \int_{-2}^{2} \frac{\Phi^+(\tau)}{\sqrt{4-\tau^2}\,(\tau-\omega)} \left(\frac{2-\tau}{2+\tau}\right)^{\delta i} d\tau, \tag{21}$$

where in accordance with (6)

$$\Phi(\omega) = F(z) = F\left(\frac{1}{2}\left[\omega - \sqrt{\omega^2-4}\right]\right) \tag{6'}$$

and $\Phi^+(\tau) = F^+(1/t)$, $(1/t) \in \Gamma_2$. Performing the substitution $e^\xi = (2-\tau)/(2+\tau)$ in the integral term in (21) and applying the Riemann-Lebesgue lemma, one sees that this term tends to zero as $\delta \to \infty$. Besides

$$\left|\left(\frac{\omega-2}{\omega+2}\right)^{\delta i}\right| = \exp\left(\delta \arg \frac{\omega-2}{\omega+2}\right) = \exp\left(-2\delta \arg \frac{1+z}{1-z}\right).$$

Together with the relation $e^{-2\pi\delta} = \varepsilon$ this yields the assertion (18).

To proof (20a, b) we first show that

$$\frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{\Phi^+(\zeta)\, d\zeta}{X(\zeta)\,(\zeta-\tau)} \to -\frac{1}{2}\,\Phi^+(\tau) \tag{22}$$

for $\tau \in (-2, 2)$ as $\delta \to \infty$, where $X(\tau)$ is given by (11). The Hilbert problem (8) with the right-hand side $\varphi \equiv 1$ has the (uniquely determined) solution $\Psi_\varepsilon(\omega) = 1/(1+\varepsilon)$. Therefore,

$$\frac{X(\tau)}{2\pi i}\int_{-2}^{2}\frac{d\zeta}{X(\zeta)(\zeta-\tau)} = \frac{1}{2} - \Psi_\varepsilon^-(\tau) = -\frac{1}{2} + \frac{\varepsilon}{1+\varepsilon} \tag{23}$$

and the expression on the left-hand side of (22) is equal to

$$\left[-\frac{1}{2} + \frac{\varepsilon}{1+\varepsilon}\right]\Phi^+(\tau) + \frac{X(\tau)}{2\pi i}\int_{-2}^{2}\frac{\Phi^+(\zeta)-\Phi^+(\tau)}{X(\zeta)(\zeta-\tau)}\,d\zeta.$$

In the right-hand integral we again perform the substitutions $e^\eta = (2-\zeta)/(2+\zeta)$ and $e^\xi = (2-\tau)/(2+\tau)$ and apply the Riemann-Lebesgue lemma taking into account the assumed Hölder continuity of $\Phi^+(\tau)$. This proves (22). Now we apply the residue theorem to the integral in (10a, b) in a domain bounded by the slit L with sufficiently small semicircles encircling the point $\tau \in (-2, 2)$ and a circle with sufficiently large radius. After taking the limits, we obtain

$$\Phi_\varepsilon^-(\tau) = \varphi(\tau) + \varepsilon\left[-\frac{1}{2}\Phi^+(\tau) + \frac{X(\tau)}{2\pi i}\int_{-2}^{2}\frac{\Phi^+(\zeta)\,d\zeta}{X(\zeta)(\zeta-\tau)}\right]. \tag{24}$$

On account of (22) this implies

$$\Phi_\varepsilon^-(\tau) = \varphi(\tau) - \varepsilon[\Phi^+(\tau) + o(1)], \tag{25}$$

which is equivalent to the asymptotic relation (20a) for interior points of Γ_1. Moreover, according to the conjugacy condition (8)

$$\Phi_\varepsilon^+(\tau) = \frac{1}{\varepsilon}[\varphi(\tau) - \Phi_\varepsilon^-(\tau)] = \Phi^+(\tau) + o(1), \tag{26}$$

i.e., also the relation (20b) holds for interior points of Γ_2. Finally, because of (13) the validity of (20a, b) in the endpoints of Γ_1 and Γ_2 is trivial ∎

3. Estimation of the approximate solution to the Cauchy problem

Now let $f_\gamma(t)$, $t \in \Gamma_1$, be a known approximation of the unknown exact Cauchy data $f(t)$, where

$$\max_{t\in\Gamma_1}|f(t) - f_\gamma(t)| < \gamma. \tag{27}$$

Like $f(t)$ the given function $f_\gamma(t)$ shall satisfy a Hölder condition on Γ_1. As *approximate solution* $\tilde{F}(z)$ of the Cauchy problem (1) the solution of the Carleman problem (2) with right-hand side $f_\gamma(t)$ will be taken:

$$\tilde{F}(z) \equiv F_{\varepsilon,\gamma}(z) = \frac{1-z^2}{2\pi}\left(\frac{1+z}{1-z}\right)^{2\delta i} e^{\delta\pi}\int_{-2}^{2}\frac{\varphi_\gamma(\tau)}{\sqrt{4-\tau^2}\,(1+z^2-\tau z)}\left(\frac{2-\tau}{2+\tau}\right)^{\delta i} d\tau, \tag{28}$$

where in accordance with (7)

$$\varphi_\gamma(\tau) = f_\gamma(t) = f_\gamma\left(\frac{1}{2}\left[\tau + i\sqrt{4-\tau^2}\right]\right), \tag{7'}$$

and the sufficiently small positive parameter ε has to be chosen in dependence on γ.

We estimate the difference between $\tilde{F}(z)$ and $F_\varepsilon(z)$. Because of (27) one has

$$|\tilde{F}(z) - F_\varepsilon(z)| < \gamma \frac{|1-z^2|}{2\pi} e^{\delta\pi} \exp\left(-2\delta \arg \frac{1+z}{1-z}\right)$$

$$\times \int_{-2}^{2} \frac{d\tau}{\sqrt{4-\tau^2}\,|1+z^2-\tau z|} \leqq \frac{\gamma}{\varepsilon} \varepsilon^{\lambda(z)} \beta(z) \frac{1}{2\pi} \int_{-2}^{2} \frac{d\tau}{\sqrt{4-\tau^2}},$$

i.e.,

$$|\tilde{F}(z) - F_\varepsilon(z)| < \frac{1}{2} \beta(z) \frac{\gamma}{\varepsilon} \varepsilon^{\lambda(z)}, \tag{29}$$

where $\lambda(z)$ is defined in (19) and

$$\beta(z) = \frac{|1-z^2|}{\alpha(z)}, \qquad \alpha(z) = \min_{-2\leqq\tau\leqq 2} |1+z^2-\tau z|. \tag{30}$$

Lemma: *The function $\alpha(z)$ fulfils the inequality*

$$\alpha(z) \geqq (1-|z|)^2, \qquad z \in G; \tag{31}$$

more precisely,

$$\alpha(z) = \begin{cases} |1-z|^2 & \text{if} \quad \operatorname{Re}[z + (1/z)] \geqq 2 \\ |1+z|^2 & \text{if} \quad \operatorname{Re}[z + (1/z)] \leqq -2 \\ |z|\,|\operatorname{Im}[z + (1/z)]| & \text{if} \quad -2 \leqq \operatorname{Re}[z + (1/z)] \leqq 2. \end{cases} \tag{32}$$

Proof: In the first two cases of (32) the minimum of $|1+z^2-\tau z|$ is attained for $\tau = +2$ and -2, respectively. In the third case this minimum is attained for $\tau = \operatorname{Re}[z + (1/z)]$ and

$$\alpha^2(z) \geqq r^2\left(r - \frac{1}{r}\right)^2 \left[1 - 4\left(r + \frac{1}{r}\right)^{-2}\right] = (1-r)^2 \left[(1+r)^2 - 4\frac{(1+r)^2}{\left(r+\frac{1}{r}\right)^2}\right]$$

$$\geqq (1-r)^2 [(1+r)^2 - 4r] = (1-r)^4 = (1-|z|)^4,$$

because of $(1+r)^2 \leqq r\left(r + \frac{1}{r}\right)^2$, where $r = |z|$ ∎

From (29) and the lemma we obtain the desired *estimation*

$$|\tilde{F}(z) - F_\varepsilon(z)| < \frac{1}{2} \frac{|1-z^2|}{(1-|z|)^2} \frac{\gamma}{\varepsilon} \varepsilon^{\lambda(z)}, \qquad z \in G. \tag{33}$$

In particular, the first part of Theorem 1 and the estimation (33) imply the following result.

Theorem 2: *If the function $f_\gamma(t)$ is Hölder continuous on Γ_1 and satisfies the inequality (27) with $\gamma = o(\varepsilon)$, the approximate solution $\tilde{F}(z)$ fulfils the asymptotic relation*

$$\tilde{F}(z) = F(z) + o(\varepsilon^{\lambda(z)}), \qquad z \in G, \tag{34}$$

as $\varepsilon \to 0$. *Whereas for the choice* $\varepsilon \sim \gamma$ *in* (2)

$$\tilde{F}(z) = F(z) + \mathcal{O}(\gamma^{\lambda(z)}), \quad z \in G, \tag{35}$$

as $\gamma \to 0$.

Remark: Generally, the approximate solution $\tilde{F}(z)$ converges to $F(z)$ as $\gamma \to 0$ if $\varepsilon \sim \gamma^{\mu(z)}$ with $0 < \mu(z) < 1/[1 - \lambda(z)]$. Moreover, if we estimate the integral term in (21) in analogous way as the difference between $\tilde{F}(z)$ and $F_\varepsilon(z)$ (cp. (21) with (9)), we obtain

$$|F_\varepsilon(z) - F(z)| < \frac{1}{2} \frac{|1 - z^2|}{(1 - |z|)^2} M \varepsilon^{\lambda(z)}, \quad M = \max_{t \in \Gamma_1} |F^+(t)|, \tag{18'}$$

and together with (33) this implies the *estimation*

$$|\tilde{F}(z) - F(z)| < \frac{1}{2} \frac{|1 - z^2|}{(1 - |z|)^2} \left[M + \frac{\gamma}{\varepsilon}\right] \varepsilon^{\lambda(z)}, \quad z \in G. \tag{36}$$

From (36) the dependence of the $\mathcal{O}$ term in (35) upon the variable $z \in G$ may be seen in more concrete manner.

Further, we will estimate the difference of the *boundary values* of $\tilde{F}(z)$ and $F_\varepsilon(z)$ on Γ_1 and Γ_2, respectively. For this end we assume additionally the inequality

$$|[f(t_1) - f_\gamma(t_1)] - [f(t_2) - f_\gamma(t_2)]| \leqq A\gamma |t_1 - t_2|^\sigma \quad \text{on } \Gamma_1 \tag{37}$$

for the Hölder continuous functions $f(t)$, $f_\gamma(t)$ with Hölder exponent σ, $0 < \sigma \leqq 1$, say. Working with the boundary values of the corresponding functions $\tilde{\Phi}(\omega)$ and $\Phi_\varepsilon(\omega)$ on the slit $L = [-2, 2]$ of the ω plane, the assumption (37) will be used in the form

$$|g(\tau_1) - g(\tau_2)| \leqq A\gamma |\tau_1 - \tau_2|^{\sigma/2} \quad \text{on } L \tag{38}$$

for the function $g(\tau) = \varphi(\tau) - \varphi_\gamma(\tau)$ with φ, φ_γ given by (7), (7'). The relation (38) follows from (37) by means of the elementary inequality $|a^{1/2} - b^{1/2}| \leqq |a - b|^{1/2}$ $(a, b \geqq 0)$. Now by (10a) and (10b) one has

$$|\tilde{\Phi}^-(\tau) - \Phi_\varepsilon^-(\tau)| = \left|\frac{1}{2} g(\tau) - g(\tau) K(\tau) - L(\tau)\right|,$$

$$|\tilde{\Phi}^+(\tau) - \Phi_\varepsilon^+(\tau)| = \frac{1}{\varepsilon} \left|\frac{1}{2} g(\tau) + g(\tau) K(\tau) + L(\tau)\right|,$$

with

$$K(\tau) = \frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{d\zeta}{X(\zeta)(\zeta - \tau)} = -\frac{1}{2} + \frac{\varepsilon}{1 + \varepsilon}$$

by (23) and

$$L(\tau) = \frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{[g(\zeta) - g(\tau)]\, d\zeta}{X(\zeta)(\zeta - \tau)}.$$

On account of (27) and (38) this yields the inequalities

$$|\tilde{\Phi}^-(\tau) - \Phi_\varepsilon^-(\tau)| \leqq \frac{\gamma}{1 + \varepsilon} + A\gamma \sqrt{4 - \tau^2}\, I(\tau),$$

$$|\tilde{\Phi}^+(\tau) - \Phi_\varepsilon^+(\tau)| \leqq \frac{\gamma}{1 + \varepsilon} + A \frac{\gamma}{\varepsilon} \sqrt{4 - \tau^2}\, I(\tau),$$

with the integral

$$I(\tau) = \frac{1}{2\pi} \int_{-2}^{2} \frac{|\zeta - \tau|^{\sigma/2-1}}{\sqrt{4 - \zeta^2}} \, d\zeta.$$

This integral can be expressed as a linear combination of two Gaúss hypergeometric functions (cf. [5: Chap. II]) and estimated by an expression of the form

$$B_1(4 - \tau^2)^{\sigma/2-1/2} + B_2(2 - \tau)^{\sigma/2-1/2} + B_3(2 + \tau)^{\sigma/2-1/2} \leqq B_0(4 - \tau^2)^{-1/2}$$

with constants $B_k = B_k(\sigma)$, $k = 0, \ldots, 3$. Hence there follow the *estimations*

$$|\tilde{F}^+(t) - F_\varepsilon^+(t)| \leqq \frac{\gamma}{1 + \varepsilon} + AB_0\gamma, \qquad t \in \Gamma_1, \tag{39a}$$

$$|\tilde{F}^+(t) - F_\varepsilon^+(t)| \leqq \frac{\gamma}{1 + \varepsilon} + AB_0 \frac{\gamma}{\varepsilon}, \qquad t \in \Gamma_2. \tag{39b}$$

Together with the second part of Theorem 1 this implies the following theorem.

Theorem 3: *If the function $f_\gamma(t)$ is Hölder continuous on Γ_1 and satisfies the inequalities* (27) *and* (37) *with $\gamma = o(\varepsilon)$, the boundary values of the approximate solution $\tilde{F}(z)$ fulfil the asymptotic relations*

$$\tilde{F}^+(t) = f(t) - F^+(1/t)\,\varepsilon + o(\varepsilon), \qquad t \in \Gamma_1, \tag{40a}$$

$$\tilde{F}^+(t) = F^+(t) + o(1), \qquad t \in \Gamma_2, \tag{40b}$$

as $\varepsilon \to 0$. For $\gamma = O(\varepsilon)$ in (27) *and* (37)

$$\tilde{F}^+(t) = f(t) + O(\varepsilon), \qquad t \in \Gamma_1. \tag{41}$$

Furthermore, (40b) *holds also for the choice $\varepsilon \sim \gamma$ in* (2), *if $A = A(\gamma) = o(1)$ in* (37) *as $\gamma \to 0$.*

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