Zeitschrift für Analysis und ihre Anwendungen
Bd. 1 (6) (1982), S. 85-44

A regularization method in the Cauchy problem for holomorphic functions

E. WAGNER and L. v. WOLFERSDORF

In der Arbeit wird ein Regularisierungsverfahren beim Cauchy-Problem für holomorphe Funktionen im Einheitskreis vorgeschlagen, welches auf der Einbettung des Problems in eine Schar von Carlemanschen Kopplungsproblemen beruht.

В работе предложен метод регуляризации у задачи Коши для голоморфных функций в единичном круге, который основывается на вложении задачи в семейство красвых эадач Карлемана.

In the paper a regularization method in the Cauchy problem for holomorphic functions in the unit disk is proposed, which relies on the embedding of this problem into a family of Carleman conjugacy problems.

Introduction

34

The Cauchy problem for analytic functions or equivalently for harmonic functions in the plane is one of the most important incorrectly posed problems of mathematical physics. It possesses applications in geophysics, hydrodynamics, plasma physics, electron optics, and medicine. Furthermore, its solution is basic for the solution of the Cauchy problem for more general elliptic equations and systems (cf. $[4a, 4b, 6, 7, 27, 28]$.

Fundamental results on the continuous dependence of the solution to the Cauchy problem for the Laplace equation upon the data, the estimation of the solution and the approximative solution of the problem are given in a classical paper of T. CARLE-MAN $\left[2\right]$ (cf. also $\left[14\right]$), in the pioneering papers of F: JOHN $\left[10, 11\right]$, M. M. LAVREN-TIEV $(13, 14)$ (cf. also the monographs [15, 15a, 16]), and C. Pucci [21, 22], by L. E. PAYNE [20], V. K. IVANOV [8, 9], and R. LATTES and J.-L. LIONS [12]. Also we refer to the recent review paper by G. TALENTI [23]. New constructive solution methods are developed by J. R. CANNON and P. DU CHATEAU [1] (cf. also H. D. MITTELMANN [18a, b]), P. COLLI FRANZONE, L. GUERRI, B. TACCARDI, and C. VIGA-NOTTI [3], P. N. VABIŠČEVIČ [25], P. N. VABIŠČEVIČ, V. B. GLASKO, and J. A. KRIKSIN [26], and M. V. UREV [24].

In this note a further approximation method in the Cauchy problem for analytic functions will be proposed. The Cauchy problem is dealt with in the normal form, in which a holomorphic function in the unit disk is to be determined from its boundary values on the upper semi-circle. Cauchy problems for general simply connected domains may be reduced to this normal form by conformal mapping. The Cauchy problem is embedded into a family of simple Carleman conjugacy problems for the unit disk (cf. [17]), which can be solved in explicit manner by transforming them to well-known Hilbert conjugacy problems for a slit on the real axis. Assuming the existence of a solution to the Cauchy problem being Hölder continuous on the closed unit disk, we obtain the uniform convergence of the approximate solutions to the exact solution in the interior of the unit disk and on the circle, respectively. No numerical results are given in this paper. **1. Statement of problem and regularization method**
 1. State

 $\mathbf r$

 (1)

Let G be the unit, disk $G: |z| < 1$ with boundary $\Gamma: |z| = 1$ and $\Gamma_1: |z| = 1$, $0 \le \arg z \le \pi$, and Γ_2 : $|z| = 1$, $-\pi \le \arg z \le 0$, the upper and lower semi-circle, respectively. We deal with the following *Cauchy problem*:

To find a holomorphic function $F(z)$ in G being Hölder continuous in G, which takes on prescribed Hölder continuous boundary values $f(t)$ on Γ_1 : *t* Ey. We deal with the formulation of a holomorphic function prescribed Hölder contracts π :

F⁺(t) = f(t), $t \in \Gamma_1$.

$$
F^+(t) = f(t), \qquad t \in \Gamma_1.
$$

We make the *basic assumption* that there exists a (uniquely determined) solution $F(z)$ of the problem, i.e., the given Hölder continuous function $f(t)$ on Γ_1 is boundary function of a holomorphic function $F(z)$ in G with Hölder continuous boundary values $F^+(t)$ on Γ .

> Remark: There are simple sufficient conditions for this basic assumption, eg. the development of the real part of $f(t)$ in a sufficiently rapidly convergent cosinus (resp., sinus) series in $[0, \pi]$ and of the imaginary part of $f(t)$ in the sinus (resp., eosin'us) serie's with the same (resp., the same with opposite sign) coefficients. But this basic assumption may be generally regarded as fulfilled from the physical viewpoint; without such an existence assumption the problem of regularization $F^+(t) = f(t)$, $t \in \Gamma_1$.
We make the basic assumption th
 $F(z)$ of the problem, i.e., the given
function of a holomorphic functi
values $F^+(t)$ on Γ .
Remark: There are simple suf
the development of the real part (resp., $\begin{array}{l} \text{ficient} \ \text{of} \ \text{f}(t) \ \text{p., the} \ \text{general} \ \text{tence} \ \text{y } \ \text{proj} \end{array}$ function of a holomorphic function $F(z)$ in G with Hölder continue
values $F^+(t)$ on Γ .
Remark: There are simple sufficient conditions for this basic asset
the development of the real part of $f(t)$ in a sufficiently **Figure 1.1 FCC FCC**

has obviously no sense.
As it is well known, the Cauchy problem is incorrectly posed. To construct an approximate solution of it for only approximately given data */* we embed it into the following family of *Carleman conjgacy problems* (cf. [17:. *§* 13]): by sublik and sense.

So well known, the Cauchy problem is incorrectly posed. To

the solution of it for only approximately given data f we em

family of *Carleman conjugacy problems* (cf. [17: § 13]):

d a holomorphic fu ed it into the

s in \overline{G} , which

s in \overline{G} , which

(2)

e limit $\varepsilon \to 0$.

. 159 – 161]).

(3)
 $L = [-2, 2]$

. to the lower

To find a holomorphic function $F_c(z)$ in G being Hölder continuous in \overline{G} , which satisfies the conjugacy condition

$$
F_{\epsilon}^+(t) + \epsilon F_{\epsilon}^+(1/t) = f(t), \qquad t \in \Gamma_1.
$$

Here ε is a positive parameter, the Cauchy problem corresponds to the limit $\varepsilon \to 0$.

The *conjugacy problem* (2) can be solved in explicit manner (cf. [17: pp. 159—161]).
The Joucovsky mapping

$$
\omega(z) = z + \frac{1}{z} \tag{3}
$$

maps the unit disk G in the *z* plane onto the ω plane cut along the slit $L = [-2, 2]$ on the real axis, where the upper₁(lower) semi-circle $\Gamma_1(\Gamma_2)$ corresponds to the lower (upper) boundary $L^{-}(L^{+})$ of the slit, respectively. The inverse mapping to (3) is $F_t^*(t) + \varepsilon F_t^*(1|t) = f(t),$ $t \in \Gamma_1$. (2)

a positive parameter, the Cauchy problem corresponds to the limit $\varepsilon \to 0$.

injugacy problem (2) can be solved in explicit manner (cf. [17: pp. 159-161]).

ovsky mapping
 $\omega(z) =$

$$
z(\omega)=\frac{1}{2}\left(\omega-\sqrt{\omega^2-4}\right),\,
$$

A regularization method in the Cauchy problem
\nwhere arg
$$
\sqrt{\omega^2 - 4} = 0
$$
 for real $\omega > 2$, with the boundary values
\n
$$
z^+(t) = \frac{1}{2} \left(t - i \sqrt{4 - \tau^2} \right) = \frac{1}{t} \in \Gamma_2 \text{ for } \tau \in L^+,
$$
\n(5a)
\n
$$
z^-(\tau) = \frac{1}{2} \left(\tau + i \sqrt{4 - \tau^2} \right) = t \in \Gamma_1 \text{ for } \tau \in L^-,
$$
\n(5b)
\nwhere $\sqrt{4 - \tau^2} > 0$ for $|\tau| < 2$. Therefore, putting
\n
$$
\Phi_t(\omega) = F_t(z) = F_t \left(\frac{1}{2} \left[\omega - \sqrt{\omega^2 - 4} \right] \right)
$$
\n(6)

$$
\tau^-(\tau) = \frac{1}{2} \left(\tau + i \sqrt{4 - \tau^2} \right) = t \in \Gamma_1 \quad \text{for} \quad \tau \in L^-, \tag{5b}
$$

where $\sqrt{4 - \tau^2} > 0$ for $|\tau| < 2$. Therefore, putting

A regularization method in the Cauchy problem
\nwhere arg
$$
\sqrt{\omega^2 - 4} = 0
$$
 for real $\omega > 2$, with the boundary values
\n $z^+(t) = \frac{1}{2} (t - i\sqrt{4 - t^2}) = \frac{1}{t} \in \Gamma_2$ for $\tau \in L^+$, (5a)
\n $z^-(\tau) = \frac{1}{2} (t + i\sqrt{4 - t^2}) = t \in \Gamma_1$ for $\tau \in L^-$, (5b)
\nwhere $\sqrt{4 - \tau^2} > 0$ for $|\tau| < 2$. Therefore, putting
\n $\Phi_{\epsilon}(\omega) = F_{\epsilon}(z) = F_{\epsilon} \left(\frac{1}{2} [\omega - \sqrt{\omega^2 - 4}] \right)$ (6)
\nand
\n $\varphi(\tau) = f(t) = f \left(\frac{1}{2} [\tau + i\sqrt{4 - \tau^2}] \right)$, (7)
\nthe Carleman conjugacy condition (2) goes over into the Hilbert conjugacy condition
\n $\Phi_{\epsilon}^+(\tau) + \frac{1}{\epsilon} \Phi_{\epsilon}^-(\tau) = \frac{1}{\epsilon} \varphi(\tau)$, $\tau \in L$, (8)
\nfor the sectionally holomorphic function $\Phi_{\epsilon}(\omega)$ in the ω plane regular at infinity and
\n-bounded at the endpoints of the cut L .
\nThe Hilbert problem (8) has the uniquely determined solution (cf. [19: Chap. 4,

$$
\varphi(\tau) = f(t) = f\left(\frac{1}{2}\left[\tau + i\sqrt{4-\tau^2}\right]\right),\tag{7}
$$

the Carleman conjugacy condition (2) goes over into the *Hilbert conjugacy condition*

$$
\Phi_{\epsilon}{}^{+}(\tau) + \frac{1}{\epsilon} \, \dot{\Phi}_{\epsilon}{}^{-}(\tau) = \frac{1}{\epsilon} \, \varphi(\tau), \qquad \tau \in L,
$$

for the sectionally holoniorphic function 0,(w) in the *co* plane regular at infinity and ____ The *Hubert problem* (8) has the uniquely determined solution (cf. [19: Chap. 4, the Carleman conjugacy condition (2) goes over into the Hilbert con
 $\Phi_t^+(\tau) + \frac{1}{\epsilon} \Phi_t^-(\tau) = \frac{1}{\epsilon} \varphi(\tau), \qquad \tau \in L,$

for the sectionally holomorphic function $\Phi_t(\omega)$ in the ω plane regula

bounded at the endpoints

A regularization method in the Cauchy problem
\nwhere arg
$$
\sqrt{\omega^2 - 4} = 0
$$
 for real $\omega > 2$, with the boundary values
\n
$$
z^+(r) = \frac{1}{2} (r - i\sqrt{4 - r^2}) = \frac{1}{t} \in \Gamma_2
$$
 for $\tau \in L^+$, (5a)
\n
$$
z^-(r) = \frac{1}{2} (r + i\sqrt{4 - r^2}) = t \in \Gamma_1
$$
 for $\tau \in L^-$, (5b)
\nwhere $\sqrt{4 - r^2} > 0$ for $|r| < 2$. Therefore, putting
\n
$$
\Phi_t(\omega) = F_t(z) = F_t \left(\frac{1}{2} [\omega - \sqrt{\omega^2 - 4}] \right)
$$
 (6)
\nand
\n
$$
\varphi(r) = f(t) = f \left(\frac{1}{2} [r + i\sqrt{4 - r^2}] \right),
$$
 (7)
\nthe Carleman conjugacy condition (2) goes over into the *Hilbert conjugacy* condition
\n
$$
\Phi_t^+(r) + \frac{1}{e} \Phi_t^-(r) = \frac{1}{e} \varphi(r), \qquad r \in L,
$$
 (8)
\nfor the sectionally holomorphic function $\Phi_t(\omega)$ in the ω plane regular at infinity and
\n–bounded at the endpoints of the cut L .
\nThe *Hilbert problem* (8) has the uniquely determined solution (cf. [19: Chap. 4,
\n§ 80])
\n
$$
\Phi_t(\omega) = -\frac{\sqrt{\omega^2 - 4}}{2\pi} \left(\frac{\omega - 2}{\omega + 2}\right)^{-\delta_t} e^{\delta\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\varphi(r)}{\sqrt{4 - r^2}(r - \omega)} \left(\frac{2 - \tau}{2 + r}\right)^{\delta_t} dr
$$
 (9)
\nwith arg $(\omega - 2) = \arg (\omega + 2) = 0$ for real $\omega > 2$ and arg $(2 - \tau) = \arg (2 + \tau)$
\n= 0 on *L*. The boundary values of $\Phi_t(\omega)$ on *L* are

for the sectionally holomorphic function
$$
\Phi_{\epsilon}(\omega)
$$
 in the ω plane regular at infinity and bounded at the endpoints of the cut L .
\nThe *Hilbert problem* (8) has the uniquely determined solution (cf. [19: Chap. 4,
\n§ 80])
\n
$$
\Phi_{\epsilon}(\omega) = -\frac{\sqrt{\omega^2 - 4}}{2\pi} \left(\frac{\omega - 2}{\omega + 2}\right)^{-\delta_i} e^{\delta \pi} \int_{-2}^{2} \frac{\varphi(\tau)}{\sqrt{4 - \tau^2} (\tau - \omega)} \left(\frac{2 - \tau}{2 + \tau}\right)^{\delta_i} d\tau
$$
(9)
\nwith arg $(\omega - 2) = \arg (\omega + 2) = 0$ for real $\omega > 2$ and arg $(2 - \tau) = \arg (2 + \tau)$
\n $= 0$ on *L*. The boundary values of $\Phi_{\epsilon}(\omega)$ on *L* are
\n
$$
\Phi_{\epsilon}(\tau) = \frac{\varphi(\tau)}{2} - \frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{\varphi(\zeta) d\zeta}{X(\zeta) (\zeta - \tau)},
$$
(10a)

$$
\oint_{\epsilon} S(0) \quad \Phi_{\epsilon}(\omega) = -\frac{\sqrt{\omega^{2}-4}}{2\pi} \left(\frac{\omega-2}{\omega+2} \right)^{-\delta_{i}} e^{\delta_{\pi}} \int_{-2}^{2} \frac{\varphi(\tau)}{\sqrt{4-\tau^{2}}(\tau-\omega)} \left(\frac{2-\tau}{2+\tau} \right)^{\delta_{i}} d\tau \quad (9)
$$
\nwith arg $(\omega-2) = \arg(\omega+2) = 0$ for real $\omega > 2$ and arg $(2-\tau) = \arg(2+\tau)$
\n $= 0$ on *L*. The boundary values of $\Phi_{\epsilon}(\omega)$ on *L* are\n
$$
\Phi_{\epsilon}(\tau) = \frac{\varphi(\tau)}{2} - \frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{\varphi(\zeta) d\zeta}{X(\zeta) (\zeta-\tau)}, \qquad (10a)
$$
\n
$$
\Phi_{\epsilon}^{+}(\tau) = \frac{1}{\epsilon} \left[\frac{\varphi(\tau)}{2} + \frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{\varphi(\zeta) d\zeta}{X(\zeta) (\zeta-\tau)} \right], \qquad (10b)
$$
\nwhere $\delta = (1/2\pi) \ln[1/\epsilon] > 0$ and\n
$$
X(\tau) = \sqrt{4-\tau^{2}} \left(\frac{2-\tau}{2+\tau} \right)^{-\delta_{i}}.
$$
\nHence the solution to the *Carleman problem* (2) is given by

$$
X(\tau) = \sqrt{4 - \tau^2} \left(\frac{2 - \tau}{2 + \tau}\right)^{-\delta t}.
$$
\n(11)

Hence the solution to the *Carleman problem* (2) is given by

$$
X(\tau) = \sqrt{4 - \tau^2} \left(\frac{2 - \tau}{2 + \tau} \right) \tag{11}
$$
\n
$$
\text{the solution to the } \text{Carleman problem (2) is given by}
$$
\n
$$
F_c(z) = \frac{1 - z^2}{2\pi} \left(\frac{1 + z}{1 - z} \right)^{2\delta i} e^{\delta \pi} \int_{-2}^{2} \frac{\varphi(\tau)}{\sqrt{4 - \tau^2} (1 + z^2 - \tau z)} \left(\frac{2 - \tau}{2 + \tau} \right)^{\delta i} d\tau, \tag{12}
$$

where arg $(1 + z) = \arg (1 - z) = 0$ for real $z \in G$. Analogous expressions hold for the boundary values $\widetilde{F}_t^+(t) = \Phi_t^-(\tau)$, $t \in \Gamma_1$, and $F_t^+(1/t) = \Phi_t^+(\tau)$, $(1/t) \in \Gamma_2$. In particular, one has

$$
F_{\epsilon}^{+}(\pm 1) = f(\pm 1)/(1 + \epsilon)
$$
\n(13)

and

$$
F_c(0) = \frac{e^{\delta \pi}}{2\pi} \int_{-2}^{2} \frac{\varphi(\tau)}{\sqrt{4 - \tau^2}} \left(\frac{2 - \tau}{2 + \tau}\right)^{\delta i} d\tau.
$$
 (14)

In terms of the given function $f = f(s)$ on Γ_1 the formula (12) for $F_e(z)$ writes

$$
F_{\epsilon}(z) = \frac{1-z^2}{2\pi} \left(\frac{1+z}{1-z}\right)^{2\delta i} e^{\delta \pi} \int_{0}^{z} \frac{f(s)}{2\cos s \cdot z - 1 - z^2} \left(\frac{1-\cos s}{1+\cos s}\right)^{\delta i} ds. \tag{12'}
$$

Remark 1: Taking $\psi(t) = F^+(t)$, $t \in \Gamma_2$, as unknown function, from the Cauchy formula

$$
F(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(t) \, dt}{t - z} + \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\psi(t) \, dt}{t - z}, \qquad z \in G,
$$
\n⁽¹⁵⁾

and the Plemelj-Sochozki formulae (cf. [19: Chap. 1, § 16]) for the boundary values of $F(z)$ on Γ_1 and Γ_2 we obtain the following *integral equations* each of them equivalent to the Cauchy problem (1) :

$$
\frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\psi(\tau) d\tau}{\tau - t} = \frac{f(t)}{2} - \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\tau) d\tau}{\tau - t}, \qquad t \in \Gamma_1,\tag{16}
$$

$$
\frac{\psi(t)}{2} - \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{\psi(\tau) d\tau}{\tau - t} = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\tau) d\tau}{\tau - t}, \qquad t \in \Gamma_2.
$$
 (17)

(Cf. [15: Chap. II, § 1], where the equation (16) is used in suitably modified form.) Introducing the Carleman problem (2), the equations (16), (17) are embedded in the family of integral equations for $\psi_{\epsilon}(t) = F_{\epsilon}^{\dagger}(t), t \in \Gamma_2$:

$$
\frac{1}{2\pi i} \int_{\Gamma_{\mathbf{t}}} \frac{\psi_{\epsilon}(\tau) d\tau}{\tau - t} + \varepsilon \left[\frac{1}{2} \psi_{\epsilon}(1/t) \right] - \frac{1}{2\pi i} \int_{\Gamma_{1}} \frac{\psi_{\epsilon}(1/\tau)}{\tau - t} d\tau \right]
$$
\n
$$
= \frac{f(t)}{2} - \frac{1}{2\pi i} \int_{\Gamma_{1}} \frac{f(\tau) d\tau}{\tau - t}, \qquad t \in \Gamma_{1}, \qquad (16_{0})
$$

$$
\frac{\psi_{\epsilon}(t)}{2} - \frac{1}{2\pi i} \int\limits_{\Gamma_{\mathbf{a}}} \frac{\psi_{\epsilon}(\tau) d\tau}{\tau - t} + \frac{\varepsilon}{2\pi i} \int\limits_{\Gamma_{\mathbf{a}}} \frac{\psi_{\epsilon}(1/\tau) d\tau}{\tau - t}
$$
\n
$$
= \frac{1}{2\pi i} \int\limits_{\Gamma_{\mathbf{a}}} \frac{f(\tau) d\tau}{\tau - t}, \qquad t \in \Gamma_{\mathbf{a}}; \tag{17}_{0}
$$

respectively, where in the Cauchy formula for $F_{\epsilon}(z)$ the conjugacy condition (2) has been used in the form

$$
F_{\epsilon}^+(t) = f(t) - \varepsilon \psi_{\epsilon}(1/t), \qquad t \in \Gamma_1. \tag{2'}
$$

As is easily seen by means of the identity theorem for analytic functions and the uniqueness property of the Cauchy problem, the solution to each of the equations (16) – $(17₀)$ is uniquely determined.

38

. Remark 2: Obviously, the function $\Phi_{\epsilon}(\omega)$ in (9) may be regarded as an approximation to the solution $\Phi(\omega)$ of the corresponding Cauchy problem for a sectionally holomorphic function in the ω plane with the slit $L = [-2, 2]$ and given boundary values $\Phi^{-}(\tau)$ on one side of it. In the same way this Cauchy problem for an arbitrary simple smooth arc L can be regularized (cf. again $[19: Chap. 4]$).

2. Asymptotic behaviour of the solution to the Carleman. problem

Under the above basic assumption of the existence of the solution $F(z)$ to the Cauchy problem (1) the solution $F_{\ell}(z)$ and its boundary values $F_{\ell}(t)$ on Γ_1 and Γ_2 converge problem (1) the solution $F_i(z)$ and its boundary values $F_i^*(t)$ on F_1 and F_2 converges to $F(z)$ in G and $f(t)$ on Γ_1 , $F^+(t)$ on Γ_2 , respectively, as ε goes to zero. More precisely, the following theorem holds. Momorphic function in the ω plane with the slit $L = [-\text{hom}(F)]$ becomen in the ω plane with the slit $L = [-\text{Hom}(F)]$ and $\Phi^{-1}(\tau)$ on one side of it. In the same way this Cauchy apple smooth arc L can be regularized (c *Find the complement* ϵ *F*(*z*) on one side of it. In the ω plan ϵ *F*(*z*) on one side of it. In the involting to the solution $F_c(z)$ and if G and $f(t)$ on Γ_1 , $F^+(t)$ on F is G and $f(t)$ on Γ_1 , *z* with the slit $L = |\$
same way this Caud
zed (cf. again [19: C
zed (cf. again [19: C
ion to the Carleman
f the existence of the
s boundary values i
2, respectively, as ε
 $z \in G$, basic assumption of the existence of the solution $F(z)$ to the cattery

colution $F_t(z)$ and its boundary values $F_t^+(t)$ on T_1 and T_2 converge

(*t*) on T_1 , $F^+(t)$ on T_2 , respectively, as ε goes to zero. *F*_{*a*} *f*_{*k*} *f*_{*a*} *f*_{*a*} *f*_{*f*} *f*_**

$$
F_{\epsilon}(z) = F(z) + \epsilon(\epsilon^{\lambda(z)}), \qquad z \in G, \qquad (18)
$$

where

$$
\lambda(z) = \frac{1}{2} + \frac{1}{\pi} \arg \frac{1+z}{1-z} > 0 \tag{19}
$$

with. (1/2) $\leq \lambda(z) < 1$ for $\text{Im } z \geq 0$, $0 < \lambda(z) \leq (1/2)$ for $\text{Im } z \leq 0$;
 $F^+(t) = f(t) - F^+(1/t) \epsilon + g(\epsilon) = f(t) + O(\epsilon)$. $t \in \Gamma$.

$$
\dot{F}_t^{\ t}(t) = F^{\ t}(t) + c(1), \qquad t \in \Gamma_2.
$$
\n(20b)

*F*_{*t*}(*z*) = *F*(*z*) + *c*(*z*¹(*z*), $z \in G$, (18)
 *F*_{*t*}(*z*) = *F*(*z*) + *c*(*z*^{1(*z*}), $z \in G$, (18)
 A(*z*) = $\frac{1}{2} + \frac{1}{\pi} \arg \frac{1+z}{1-z} > 0$ (19)
 $\leq \lambda(z) < 1$ for Im $z \geq 0$, $0 < \lambda(z) \leq (1/2)$ for Im Proof: We work with the function $\Phi_{\epsilon}(\omega)$ in (9). Applying the residue theorem to the integral in (9) for a domain bounded by the slit L and a circle with sufficiently large radius *R* going to infinity, we first obtain the relation

Theorem 1:
$$
As \varepsilon \to 0
$$

\n $F_{\epsilon}(z) = F(z) + e(\varepsilon^{1/2})$, $-z \in G$, (18)
\nwhere
\n $\lambda(z) = \frac{1}{2} + \frac{1}{\pi} \arg \frac{1+z}{1-z} > 0$ (19)
\nwith $(1/2) \le \lambda(z) < 1$ for $\text{Im } z \ge 0$, $0 < \lambda(z) \le (1/2)$ for $\text{Im } z \le 0$, $0 - \lambda(z)$
\n $F_{\epsilon}^+(t) = f(t) - F^+(1/t) \varepsilon + e(\varepsilon) = f(t) + O(\varepsilon)$, $t \in \Gamma_1$, (20a)
\nProof: We work with the function $\Phi_{\epsilon}(\omega)$ in (9). Applying the residue theorem to
\nthe integral in (9) for a domain bounded by the slit *L* and a circle with sufficiently
\nlarge radius *R* going to infinity, we first obtain the relation
\n $\Phi_{\epsilon}(\omega) = \Phi(\omega) + \frac{\sqrt{\omega^2 - 4}}{2\pi} \left(\frac{\omega - 2}{\omega + 2} \right)^{-\delta i} e^{-\pi \delta} \int_{-2}^{2} \frac{\Phi^+(z)}{\sqrt{4 - z^2}(\tau - \omega)} \left(\frac{2 - z}{2 + \tau} \right)^{\delta i} dt$,
\nwhere in accordance with (6)
\n $\Phi(\omega) = F(z) = F \left(\frac{1}{2} [\omega - \sqrt{\omega^2 - 4}] \right)$ (6')
\nand $\Phi^+(z) = F^+(1/t)$, $(1/t) \in \Gamma_2$. Performing the substitution $e^z = (2 - \tau)/(2 + \tau)$
\nin the integral term in (21) and applying the Riemann-Leesgue lemma, one sees
\nthat this term tends to zero as $\delta \to \infty$. Besides
\n
$$
\left| \left(\frac{\omega - 2}{\omega + 2} \right)^{\delta i} \right| = \exp \left(\delta \arg \frac{\omega - 2}{\omega + 2} \right) = \exp \left(-2\delta \arg \frac{1 + z}{1 - z} \right)
$$

$$
\varPhi(\omega) = F(z) = F\left(\frac{1}{2}\left[\omega - \sqrt{\omega^2 - 4}\right]\right) \tag{6'}
$$

and $\Phi^+(r) = F^+(1/t)$, $(1/t) \in \Gamma_2$. Performing the substitution $e^t = (2 - \tau)/(2 + \tau)$ in the integral term in (21) and applying the Riemann-Lebesgue lemma, one sees that this term tends to zero as $\delta \to \infty$. Besides $\sqrt{(\omega+2)}$, $e^{i\omega}$, $\sqrt{4-2}$
 $-\sqrt{2^2-4}$
 $\sqrt{(\omega^2-4)}$
 $\sqrt{(\omega^2-4)}$
 $\sqrt{(\omega^2-4)}$
 $\sqrt{(\omega^2-4)}$
 $\sqrt{(\omega+2)}$
 \approx $\sqrt{(\omega+2)^2}$
 \approx \approx $\sqrt{(\omega+2)^2}$
 \approx \approx $\sqrt{(\omega+2)^2}$
 \approx \approx \approx $\sqrt{(\omega+2)^2}$
 \approx \approx $\Phi(\omega) = F(z) = F\left(\frac{1}{2}\right)$
 $\Phi(\tau) = F^+(1|t), (1|t) \in \Gamma$

integral term in (21) are

is term tends to zero as
 $\left|\left(\frac{\omega - 2}{\omega + 2}\right)^{i}\right| = \exp\left(\delta\right)$

er with the relation $e^{-2\pi i}$ $[\omega - \sqrt{\omega^2 - 4}]$ (6')

2. Performing the substitution $e^{\epsilon} = (2 - \tau)/(2 + \tau)$

2. Performing the Riemann-Lebesgue lemma, one sees
 $\delta \to \infty$. Besides
 $\arg \frac{\omega - 2}{\omega + 2} = \exp \left(-2\delta \arg \frac{1 + z}{1 - z}\right)$

3. $\delta = \epsilon$ this vields the asse Let α be the integral correlation $\Phi(\omega) = F(z) = F\left(\frac{1}{2}\left[\omega - \sqrt{\omega^2 - 4}\right]\right)$

d $\Phi^+(\tau) = F^+(1/t)$, $(1/t) \in \Gamma_2$. Performing the the integral term in (21) and applying the Riat this term tends to zero as $\delta \to \infty$. Besides
 p(*w*) = *F*(*z*) = *F*($\frac{1}{2}$ [*w*² - *V*_{*w*² - 4])
 2, - F⁺(1*lt*), (1*lt*) \in *F*₂. Performing the substitution $e^z = (2 - \tau)/\ell$ egral term in (21) and applying the Riemann-Lebesgue lemma, of therm tends}

$$
\left|\left(\frac{\omega-2}{\omega+2}\right)^{\delta i}\right|=\exp\left(\delta \arg \frac{\omega-2}{\omega+2}\right)=\exp\left(-2\delta \arg \frac{1+z}{1-z}\right).
$$

Together with the relation
$$
e^{-2\pi\delta} = \epsilon
$$
 this yields the assertion (18).
To proof (20a, b) we first show that

$$
\frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{\Phi^+(\zeta) d\zeta}{X(\zeta) (\zeta - \tau)} \to -\frac{1}{2} \Phi^+(\tau)
$$
(22)

40 E. WAGNER und L. v. WOLFERSDORF

for $\tau \in (-2,2)$ as $\delta \to \infty$, where $X(\tau)$ is given by (11). The Hilbert problem (8) $\epsilon = 1/(1 + \epsilon)$. Therefore, L. WAGNER und L. v. WOLFERSDORF

-2, 2) as $\delta \to \infty$, where $X(\tau)$ is given by (11). The Hil

right-hand side $\varphi = 1$ has the (uniquely determined
 ϵ). Therefore,
 $\frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{d\zeta}{X(\zeta)(\zeta - \tau)} = \frac{1}{2} - \$ for τ
with $\tau = 1/2$
 \ldots
and t

with the right-hand side
$$
\varphi = 1
$$
 has the (uniquely determined) solution $\Psi_{\epsilon}(\omega)$
\n
$$
= 1/(1 + \epsilon). \text{ Therefore,}
$$
\n
$$
\frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{d\zeta}{X(\zeta) (\zeta - \tau)} = \frac{1}{2} - \Psi_{\epsilon}(\tau) = -\frac{1}{2} + \frac{\epsilon}{1 + \epsilon}
$$
\n(23)
\nand the expression on the left-hand side of (22) is equal to
\n
$$
\left[-\frac{1}{2} + \frac{\epsilon}{1 + \epsilon} \right] \Phi^{+}(\tau) + \frac{X(\tau)}{2\pi i} \int_{-\infty}^{2} \frac{\Phi^{+}(\zeta) - \Phi^{+}(\tau)}{X(\zeta) (\zeta - \tau)} d\zeta.
$$

and the expression on the left-hand side of (22) is equal to

the right-hand side
$$
\varphi = 1
$$
 has the (uniquely deté)
\n
$$
\frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{d\zeta}{X(\zeta) (\zeta - \tau)} = \frac{1}{2} - \Psi_{\epsilon}(\tau) = -\frac{1}{2} + \frac{\epsilon}{1 + \tau}
$$
\nthe expression on the left-hand side of (22) is equal to\n
$$
\left[-\frac{1}{2} + \frac{\epsilon}{1 + \epsilon} \right] \Phi^{+}(\tau) + \frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{\Phi^{+}(\zeta) - \Phi^{+}(\tau)}{X(\zeta) (\zeta - \tau)} d\zeta
$$

In the right-hand integral we again perform the substitutions $e^{\eta} = (2 - \zeta)/(2 + \zeta)$ and $e^t = (2 - \tau)/(2 + \tau)$ and apply the Riemann-Lebesgue lemma taking into account the assumed Hölder continuity of $\Phi^{\dagger}(\tau)$. This proves (22). Now we apply the residue theorem to the integral in $(10a, b)$ in a domain bounded by the slit *L* with sufficiently small semicircles encircling the point $\tau \in (-2, 2)$ and a circle with sufficiently large radius. After taking the limits, we obtain' d side of (22) is equal to
 $\frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{\Phi^+(\zeta) - \Phi^+(\tau)}{X(\zeta) (\zeta - \tau)} d\zeta$.

in perform the substitutions $e^{\eta} = (2 - \zeta)/(2 + \zeta)$

ply the Riemann-Lebesgue lemma taking, into

nuity of $\Phi^+(\tau)$. This proves (22). *znt* $\frac{1}{2}$ $\frac{2\pi i}{\lambda(\zeta)(\zeta - \tau)}$
 A(ζ) ($\zeta - \tau$)
 A(ζ) ($\zeta + \zeta$)
 A(ζ) + c) and apply the Riemann-Lebesgue lemma taking into
 med Hölder continuity of $\Phi^+(\tau)$. This proves (22). Now we apply

$$
\varPhi_{\epsilon}(\tau) = \varphi(\tau) + \varepsilon \left[-\frac{1}{2} \varPhi^+(\tau) + \frac{X(\tau)}{2\pi i} \int_{-2}^2 \frac{\varPhi^+(\zeta) d\zeta}{X(\zeta) (\zeta - \tau)} \right]. \tag{24}
$$

On account of (22) this implies

$$
\Phi_{\epsilon}^-(\tau) = \varphi(\tau) - \epsilon[\Phi^+(\tau) + \mathfrak{c}(1)],\tag{25}
$$

which is equivalent to the asymptotic relation (20a) for interior points of Γ_{1} . Moreover, according to the conjugacy condition (8)

$$
\Phi_{\epsilon}^{-}(\tau) = \varphi(\tau) - \epsilon[\Phi^{+}(\tau) + c(1)], \qquad (25)
$$
\nequivalent to the asymptotic relation (20a) for interior points of Γ_{1} . More-
\nording, to the conjugacy condition (8)\n
$$
\Phi_{\epsilon}^{-}(\tau) = \frac{1}{\epsilon} [\varphi(\tau) - \Phi_{\epsilon}^{-}(\tau)] = \Phi^{+}(\tau) + o(1), \qquad (26)
$$

i.e., also the relation (20b) holds for interior points of Γ_2 . Finally, because of (13) the validity of (20a, b) in the endpoints of Γ_1 and Γ_2 is trivial \blacksquare

3. Estimation of the approximate solution **to the** Cauchy problem

Now let $f_{\gamma}(t)$, $t \in \Gamma_1$, be a known approximation of the unknown exact Cauchy data 3. Estimation
Now let $f_y(t)$, if
 $f(t)$, where • ation of the approximate solution to the Cauchy problem
 $f_{\gamma}(t)$, $t \in \Gamma_1$, be a known approximation of the unknown exact Cauchy data
 $f_{\gamma}(t)$, $t \in \Gamma_1$, be a known approximation of the unknown exact Cauchy data
 $\$ tion of the state $f(t)$, $t \in I$
 $\max_{t \in \Gamma_1} |f(t)|$

the given

$$
\max_{t \in \Gamma_1} |f(t) - f_{\gamma}(t)| < \gamma. \tag{27}
$$

Like $f(t)$ the given function $f_\gamma(t)$ shall satisfy a Hölder condition on \varGamma_1 . As *approximate* solution $\tilde{F}(z)$ of the Cauchy problem (1) the solution of the Carleman problem⁻(2) with right-hand side $f_{\nu}(t)$ will be taken:

the relation (20b) holds for interior points of
$$
\Gamma_2
$$
. Finally, because of (13)
\nility of (20a, b) in the endpoints of Γ_1 and Γ_2 is trivial \blacksquare \n\n $f_{\gamma}(t)$, $t \in \Gamma_1$, be a known approximation of the unknown exact Cauchy data\n\nre\n $\max_{t \in \Gamma_1} |f(t) - f_{\gamma}(t)| < \gamma.$ \n\nthe given function $f_{\gamma}(t)$ shall satisfy a Hölder condition on Γ_1 . As approximate $\tilde{F}(z)$ of the Cauchy problem (1) the solution of the Carleman problem (2)
\nht-hand side $f_{\gamma}(t)$ will be taken:\n
$$
\tilde{F}(z) = F_{\epsilon,\gamma}(z) = \frac{1 - z^2}{2\pi} \left(\frac{1 + z}{1 - z} \right)^{2\delta i} e^{\delta \pi} \int_{-2}^{2} \frac{\varphi_{\gamma}(t)}{\sqrt{4 - \tau^2} (1 + z^2 - \tau z)} \left(\frac{2 - \tau}{2 + \tau} \right)^{\delta i} d\tau,
$$
\n(28)

A regularization method in the Cauchy problem

where in accordance with (7)

$$
\varphi_r(\tau) = f_r(t) = f_r\left(\frac{1}{2}\left[\tau + i\sqrt{4-\tau^2}\right]\right). \tag{7'}
$$

and the sufficiently small positive parameter ε has to be chosen in dependence on γ . We estimate the difference between $\tilde{F}(z)$ and $F_{\epsilon}(z)$. Because of (27) one has

$$
|\tilde{F}(z) - F_{\epsilon}(z)| < \gamma \frac{|1 - z^2|}{2\pi} e^{\delta \pi} \exp\left(-2\delta \arg \frac{1 + z}{1 - z}\right)
$$

$$
\times \int_{-2}^{2} \frac{d\tau}{\sqrt{4 - \tau^2} |1 + z^2 - \tau z|} \leq \frac{\gamma}{\epsilon} e^{\lambda(z)} \beta(z) \frac{1}{2\pi} \int_{-2}^{2} \frac{d\tau}{\sqrt{4 - \tau^2}},
$$

i.e.,

 $|\tilde{F}(z) - F_{\epsilon}(z)| < \frac{1}{2} \beta(z) \frac{\gamma}{\varepsilon} \varepsilon^{\lambda(z)},$

where $\lambda(z)$ is defined in (19) and

$$
\beta(z) = \frac{|1 - z^2|}{\alpha(z)}, \qquad \alpha(z) = \min_{-2 \leq z \leq 2} |1 + z^2 - rz|.
$$
\n(30)

Lemma: The function $\alpha(z)$ fulfils the inequality

$$
\alpha(z) \geq (1-|z|)^2, \qquad z \in G;
$$
\n⁽³¹⁾

more precisely,

$$
\alpha(z) = \begin{cases}\n|1 - z|^2 & \text{if } \operatorname{Re}\left[z + (1/z)\right] \ge 2 \\
|1 + z|^2 & \text{if } \operatorname{Re}\left[z + (1/z)\right] \le -2 \\
|z| \operatorname{Im}\left[z + (1/z)\right] & \text{if } -2 \le \operatorname{Re}\left[z + (1/z)\right] \le 2.\n\end{cases} \tag{32}
$$

Proof: In the first two cases of (32) the minimum of $|1 + z^2 - zz|$ is attained for $\tau = +2$ and -2, respectively. In the third case this minimum is attained for $\tau = \text{Re} [z + (1/z)]$ and

$$
\alpha^2(z) \ge r^2 \left(r - \frac{1}{r}\right)^2 \left[1 - 4\left(r + \frac{1}{r}\right)^{-2}\right] = (1 - r)^2 \left[(1 + r)^2 - 4\frac{(1 + r)^2}{\left(r + \frac{1}{r}\right)^2}\right]
$$

\n
$$
\ge (1 - r)^2 \left[(1 + r)^2 - 4r\right] = (1 - r)^4 = (1 - |z|)^4,
$$

\nbecause of $(1 + r)^2 \le r \left(r + \frac{1}{r}\right)^2$, where $r = |z| \blacksquare$

From (29) and the lemma we obtain the desired estimation

$$
|\tilde{F}(z) - F_{\epsilon}(z)| < \frac{1}{2} \frac{|1 - z^2|}{(1 - |z|)^2} \frac{\gamma}{\varepsilon} \varepsilon^{\lambda(z)}, \qquad z \in G. \tag{33}
$$

In particular, the first part of Theorem 1 and the estimation (33) imply the following result.

Theorem 2: If the function $f_{\nu}(t)$ is Hölder continuous on Γ_1 and satisfies the inequality (27) with $\gamma = c(\varepsilon)$, the approximate solution $\tilde{F}(z)$ fulfils the asymptotic relation.

$$
\widetilde{F}(z) = F(z) + o(\varepsilon^{\lambda(z)}), \qquad z \in G,
$$

 (34)

as
$$
\varepsilon \to 0
$$
. Whereas for the choice $\varepsilon \sim \gamma$ in (2)

42 E. Waoser und L. v. WolfersDOBF
\nas
$$
\varepsilon \to 0
$$
. Whereas for the choice $\varepsilon \sim \gamma$ in (2)
\n $\tilde{F}(z) = F(z) + \mathcal{O}(\gamma^{\lambda(z)})$, $z \in G$, (35)
\nas $\gamma \to 0$.

sdorf

y in (2)

z
e G,

imate solution \tilde{F}

= $\lambda(z)$] Moreover 42 E. WAGNER und L. v

as $\varepsilon \to 0$. Whereas for the ch
 $\tilde{F}(z) = F(z) + \mathcal{O}(\gamma)$

as $\gamma \to 0$.

Remark: Generally, the

if $\varepsilon \sim \gamma^{\mu(z)}$ with $0 < \mu(z)$

term in (21) in analogous Remark: Generally, the approximate solution $\bar{F}(z)$ converges to $F(z)$ as $\gamma \to 0$ if $\varepsilon \sim \gamma^{\mu(z)}$ with $0 < \mu(z) < 1/[1 - \lambda(z)]$. Moreover, if we estimate the integral term in (21) in analogous way as the difference between $\tilde{F}(z)$ and $F_{\epsilon}(z)$ (cp. (21) 42 E. WAGNER und L. v. WOLFERSDORF

as $\varepsilon \to 0$. Whereas for the choice $\varepsilon \sim \gamma$ in (2)
 $\tilde{F}(z) = F(z) + \mathcal{O}(\gamma^{l(z)}), \quad z \in G$,

as $\gamma \to 0$.

Remark: Generally, the approximate solution $\tilde{F}(z)$ conver

if $\varepsilon \sim \gamma^{\mu(z)}$ $\epsilon \to 0$. Whereas for the $\tilde{F}(z) = F(z) + \gamma \to 0$.

Remark: Generally
 $\epsilon \sim \gamma^{\mu(z)}$ with $0 <$

rm in (21) in analog

th (9)), we obtain
 $|F_{\epsilon}(z) - F(z)|$

d together with (33) e, choice $\varepsilon \sim \gamma$ in
 $\mathcal{O}(\gamma^{\lambda(z)})$, $z \in$

the approxima
 $\iota(z) < 1/[1 - \lambda(\zeta)$

us way as the
 $\langle \frac{1}{2} \frac{|1 - z^2|}{(1 - |z|)^2} \rangle$

this implies the *z*)]. Moreover, if we esting
difference between $\tilde{F}(z)$ and
 $M \varepsilon^{\lambda(z)}$, $M = \max_{t \in \Gamma_z} |F^+(t)|$, $\begin{aligned} \text{Remark: Generally, the approximate solution } \tilde{F}(z) \text{ converges} \\ \epsilon &\sim \gamma^{\mu(z)} \text{ with } 0 < \mu(z) < 1/(1-\lambda(z)). \text{ Moreover, if we can}\\ \text{erm in (21) in analogous way as the difference between } \tilde{F}(z) \\ \text{with (9)), we obtain} \\ |\tilde{F}_\epsilon(z) - F(z)| &< \frac{1}{2} \frac{|1-z^2|}{(1-|z|)^2} M \epsilon^{\lambda(z)}, \quad M = \max_{t \in \Gamma_t} |F^+| \\ \text{and together with (33) this implies the estimation} \end{aligned}$ as $\varepsilon \to 0$. Whereas for the choice $\varepsilon \sim \gamma$ in (2)
 $\tilde{F}(z) = F(z) + \mathcal{O}(\gamma^{\lambda(z)})$, $z \in G$,

as $\gamma \to 0$.

Remark: Generally, the approximate solution $\tilde{F}(z)$ con

if $\varepsilon \sim \gamma^{\mu(z)}$ with $0 < \mu(z) < 1/[1 - \lambda(z)]$. Moreover, (35)
 $\tilde{F}(z)$ converges to $F(z)$ as $\gamma \to 0$

ver, if we estimate the integral

between $\tilde{F}(z)$ and $F_{\epsilon}(z)$ (cp. (21)
 $I = \max_{t \in \Gamma_{\epsilon}} |F^+(t)|$, (18')
 $\iota(\epsilon)$, $z \in G$. (36)

oon the variable $z \in G$ may be seen as $\gamma \to 0$.

Remark: Generally, the approximate solution $\tilde{F}(z)$ convert

if $\varepsilon \sim \gamma^{\mu(z)}$ with $0 < \mu(z) < 1/[1 - \lambda(z)]$. Moreover, if we esterm in (21) in analogous way as the difference between $\tilde{F}(z)$

with (9)), we

$$
|F_{\epsilon}(z) - F(z)| < \frac{1}{2} \frac{|1 - z^2|}{(1 - |z|)^2} M \epsilon^{\lambda(z)}, \quad M = \max_{t \in \Gamma_{\epsilon}} |F^+(t)|,
$$
\n(18')

\nthen with (33) this implies the *estimation*

$$
|F_{\epsilon}(z) - F(z)| < \frac{1}{2} \frac{1}{(1 - |z|)^2} M \epsilon^{\lambda(z)}, \quad M = \max_{t \in \Gamma_{\epsilon}} |F^+(t)|, \tag{18'}
$$
\nthere with (33) this implies the *estimation*

\n
$$
|\tilde{F}(z) - F(z)| < \frac{1}{2} \frac{|1 - z^2|}{(1 - |z|)^2} \left[M + \frac{\gamma}{\epsilon} \right] \epsilon^{\lambda(z)}, \quad z \in G. \tag{36}
$$
\nwhere $|F(z) - F(z)| < \frac{1}{2} \frac{|1 - z^2|}{(1 - |z|)^2} \left[M + \frac{\gamma}{\epsilon} \right] \epsilon^{\lambda(z)}, \quad z \in G.$

\nwhere $|F(z) - F(z)| < \frac{1}{2} \frac{|1 - z^2|}{(1 - |z|)^2} \left[M + \frac{\gamma}{\epsilon} \right] \epsilon^{\lambda(z)}, \quad z \in G$.

\nwhere $|F(z)|$ and $|F(z)|$ is the same.

\nwhere $|F(z)|$ and $|F$

From (36) the dependence of the $\mathcal O$ term in (35) upon the variable $z \in G$ may be seen

Further, we will estimate the difference of the *boundary values* of $\tilde{F}(z)$ and $F(z)$ on Γ_1 and Γ_2 , respectively. For this end we assume additionally the inequality

$$
| [f(t_1) - f_{\gamma}(t_1)] - [f(t_2) - f_{\gamma}(t_2)] | \leq A_{\gamma} |t_1 - t_2|^{\sigma} \text{ on } \Gamma_1
$$
 (37)

Further, we will estimate the difference of the *boundary values* of $\tilde{F}(z)$ and $F_z(z)$ on Γ_1 and Γ_2 , respectively. For this end we assume additionally the inequality $|[f(t_1) - f_y(t_1)] - [f(t_2) - f_y(t_2)]| \leq A\gamma |t_1 - t_$ $\Phi_i(\omega)$ on the slit $L = [-2, 2]$ of the ω plane, the assumption (37) will be used in the for the Hölder continuous functions $f(t)$, $f_{\gamma}(t)$ with Hölder exponent σ , $0 < \sigma \le 1$, say. Working with the boundary values of the corresponding functions $\tilde{\Phi}(\omega)$ and $\Phi_{\epsilon}(\omega)$ on the slit $L = [-2, 2]$ of the $\$ ^{[*x*}(*x*) - *x*(*x*) | \leq $\frac{1}{2}$ $\frac{1}{(1 - |z|)^2} \left[\frac{2a}{(1 - |z|)^2} + \frac{1}{z} \right]$ \leq \cdots $\$ for the Hölder c
say. Working w
 $\Phi_{\epsilon}(\omega)$ on the slit
form
 $|g(\tau_1) -$
for the function
follows from (3'
 $(a, b \ge 0)$. Now
 $|\tilde{\Phi}^{-}(\tau)|$
 $|\tilde{\Phi}^{+}(\tau)|$
with
 $K(\tau) =$

$$
|g(\tau_1)-g(\tau_2)| \leq A\gamma |\tau_1-\tau_2|^{\sigma/2} \quad \text{on } L \tag{38}
$$

0 *•*

0 ' ' ⁰

• -•

5
5
5

- **0**

.'0

0 **0**

0

0'

for the function $g(\tau) = \varphi(\tau) - \varphi_r(\tau)$ with φ , φ_r given by (7), (7'). The relation (38) form
 $|g(\tau_1) - g(\tau_2)| \leq A\gamma |\tau_1 - \tau_2|^{\sigma/2}$ on L (38)

for the function $g(\tau) = \varphi(\tau) - \varphi_r(\tau)$ with φ , φ_r given by (7), (7'). The relation (38)

follows from (37) by means of the elementary inequality $|a^{1/2} - b^{1/$ $(a, b \ge 0)$. Now by (10a) and (10b) one has follows from (37) by means of the elementary inequality $|a^{1/2} - b^{1/2}| \leq |a - b|^{1/2}$.

form
\n
$$
|g(\tau_1) - g(\tau_2)| \leq A\gamma |\tau_1 - \tau_2|^{\sigma/2} \quad \text{on } L
$$
\nfor the function $g(\tau) = \varphi(\tau) - \varphi_r(\tau)$ with φ , φ , given by (7), (7'). The relation (38
\nfollows from (37) by means of the elementary inequality $|a^{1/2} - b^{1/2}| \leq |a - b|^1$
\n $|a, b \geq 0$). Now by (10a) and (10b) one has
\n
$$
|\tilde{\Phi}^-(\tau) - \Phi_{\epsilon}^-(\tau)| = \left| \frac{1}{2} g(\tau) - g(\tau) K(\tau) - L(\tau) \right|,
$$
\n
$$
|\tilde{\Phi}^+(\tau) - \Phi_{\epsilon}^+(\tau)| = \frac{1}{\epsilon} \left| \frac{1}{2} g(\tau) + g(\tau) K(\tau) + L(\tau) \right|,
$$
\nwith
\n
$$
K(\tau) = \frac{X(\tau)}{2\pi i} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{d\zeta}{X(\zeta) (\zeta - \tau)} = -\frac{1}{2} + \frac{\epsilon}{1 + \epsilon}
$$
\nby (23) and
\n
$$
L(\tau) = \frac{X(\tau)}{2\pi i} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{g(\zeta) - g(\tau) d\zeta}{X(\zeta) (\zeta - \tau)}.
$$
\nOn account of (27) and (38) this yields the inequalities

• -

with
\n
$$
K(\tau) = \frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{d\zeta}{X(\zeta) \cdot (\zeta - \tau)} = -\frac{1}{2} + \frac{\varepsilon}{1 + \varepsilon}
$$
\nby (23) and
\n
$$
L(\tau) = \frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{[g(\zeta) - g(\tau)] d\zeta}{X(\zeta) \cdot (\zeta - \tau)}.
$$
\nOn account of (27) and (38) this yields the inequalities
\n
$$
|\tilde{\Phi}^{-}(\tau) - \Phi_{\epsilon}^{-}(\tau)| \leq \frac{\gamma}{1 + \varepsilon} + A\gamma \sqrt{4 - \tau^{2}} I(\tau),
$$

•

$$
K(\tau) = \frac{X(\tau)}{2\pi i} \int_{-2}^{\infty} \frac{d\zeta}{X(\zeta) \cdot (\zeta - \tau)} = -\n \text{by (23) and}
$$
\n
$$
L(\tau) = \frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{[g(\zeta) - g(\tau)] d\zeta}{X(\zeta) (\zeta - \tau)}.
$$
\nOn second of (27) and (28) this yields the

$$
K(\tau) = \frac{X(\tau)}{2\pi i} \int_{-2}^{\tau} \frac{d\zeta}{X(\zeta) \cdot (\zeta - \tau)} = -\frac{1}{2} + \frac{\varepsilon}{1 + \varepsilon}
$$

by (23) and

$$
L(\tau) = \frac{X(\tau)}{2\pi i} \int_{-2}^{2} \frac{[g(\zeta) - g(\tau)] d\zeta}{X(\zeta) (\zeta - \tau)}.
$$

On account of (27) and (38) this yields the inequalities

$$
|\tilde{\Phi}^-(\tau) - \Phi_{\epsilon}^-(\tau)| \le \frac{\gamma}{1 + \varepsilon} + A\gamma \sqrt{4 - \tau^2} I(\tau),
$$

$$
|\tilde{\Phi}^+(\tau) - \Phi_{\epsilon}^+(\tau)| \le \frac{\gamma}{1 + \varepsilon} + A\frac{\gamma}{\varepsilon} \sqrt{4 - \tau^2} I(\tau),
$$

with the integral

A regular
integral

$$
I(\tau) = \frac{1}{2\pi} \int_{2}^{2} \frac{|\zeta - \tau|^{o/2 - 1}}{\sqrt{4 - \zeta^2}} d\zeta.
$$

A regularization method in the Cauch.
 $\frac{1}{2\pi} \int_{-2}^{2} \frac{|\zeta - \tau|^{q/2-1}}{\sqrt{4-\zeta^2}} d\zeta.$

a be expressed as a linear combination of two Gause.

Chap. II) and estimated by an expression of the This integral can be expressed as a linear combination of two Gauss hypergeometric functions (cf. $[5: Chap. II]$) and estimated by an expression of the form

$$
B'_{1}(4-\tau^{2})^{\sigma/2-1/2}+B_{2}(2-\tau)^{\sigma/2-1/2}+B_{3}(2+\tau)^{\sigma/2-1/2}\leq B_{0}(4-\tau^{2})^{-1/2}
$$

with constants $B_k = B_k(\sigma)$, $k = 0, ..., 3$. Hence there follow the *estimations*

A regularization method in the Cauchy problem 43
\nwith the integral
\n
$$
I(\tau) = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{|\zeta - \tau|^{g/2-1}}{\sqrt{4-\zeta^2}} d\zeta.
$$
\nThis integral can be expressed as a linear combination of two Gauss hypergeometric functions (cf. [5: Chap. II]) and estimated by an expression of the form
\n
$$
B'_1(4 - \tau^2)^{g/2-1/2} + B_2(2 - \tau)^{g/2-1/2} + B_3(2 + \tau)^{g/2-1/2} \leq B_0(4 - \tau^2)^{-1/2}
$$
\nwith constants $B_k = B_k(\sigma)$, $k = 0, ..., 3$. Hence there follow the *estimations*
\n
$$
|\tilde{F}^+(\t) - F_\epsilon^+(\t)| \leq \frac{\gamma}{1+\epsilon} + AB_0\gamma, \quad t \in \Gamma_1,
$$
\n(39a)
\n
$$
|\tilde{F}^+(\t) - F_\epsilon^+(\t)| \leq \frac{\gamma}{1+\epsilon} + AB_0\frac{\gamma}{\epsilon}, \quad t \in \Gamma_2.
$$
\n(39b)
\nTogether with the second part of Theorem 1 this implies the following theorem.
\nTheorem 3: If the function $f_\gamma(t)$ is Hölder continuous on Γ_1 and satisfies the in-
\n*equilities* (27) and (37) with $\gamma = c(\epsilon)$, the boundary values of the approximate solution
\n
$$
\tilde{F}^+(\t) = f(t) - F^+(\t) + c(1), \quad t \in \Gamma_2,
$$
\n(40b)
\nas $\epsilon \to 0$. For $\gamma = Q(\epsilon)$ in (27) and (37)
\n
$$
\tilde{F}^+(\t) = f(t) + O(\epsilon), \quad t \in \Gamma_1.
$$
\nFurthermore, (40b) holds also for the choice $\epsilon \sim \gamma$ in (2), if $A = A(\gamma) = c(1)$ in (37)
\nas $\gamma \to 0$.

Together with the second part of Theorem 1 this implies the following theorem.

Theorem 3: If the function $f_r(t)$ is Holder continuous; on Γ_1 and satisfies the in-*'equalities* (27) *and* (37) *with* $\gamma = o(\varepsilon)$, *the boundary values of the approximate solution* $\tilde{F}(z)$ *fulfil the asymptotic relations.* s the following theorer
by $\sum_{i=1}^{n}$ on $\prod_{i=1}^{n}$ and satisfies the solution of the approximate solution with the second part of Theorem 1 this implies the following theorem.

em 3: If the function $f_{\gamma}(t)$ is Hölder continuous, on Γ_1 and satisfies the in-

(27) and (37) with $\gamma = o(\varepsilon)$, the boundary values of the appr Together with the second part of Theorem 1 this implies the following

Theorem 3: If the function $f_y(t)$ is Hölder continuous, on Γ_1 and sequalities (27) and (37) with $\gamma = (e, e)$, the boundary values of the approx
 $\$

$$
\tilde{F}^{+}(t) = f(t) - F^{+}(1/t) \epsilon + o(\epsilon), \qquad t \in \Gamma_{1}, \qquad t = 0.40a
$$
\n
$$
\tilde{F}^{+}(t) = F^{+}(t) + o(1), \qquad t \in \Gamma_{2}, \qquad (40b)
$$

 $as \epsilon \rightarrow 0$. For $\gamma = O(\epsilon)$ in (27) and (37) ϵ

$$
\tilde{F}^+(t) = f(t) + O(\varepsilon), \qquad t \in \Gamma_1.
$$
\n(41)

Furthermore, (40b) *holds also for the choice* $\varepsilon \sim \gamma$ *in (2), if* $A = A(\gamma) = o(1)$ *in (37)* . .

REFERENCES

- **[1] CANN0N, ^S J.** R., and P. Du CHATEAU: Approximating the solution to the Cauchy probhm for Laplace's equation. SIAM J. Num. **Anal.** 14 (1977), 473-483.
-
- $\tilde{F}^+(t) = F^+(t) + o(1)$, $t \in \Gamma_2$,
 $s \epsilon \to 0$. For $\gamma = O(\epsilon)$ in (27) and (37)
 $\tilde{F}^+(t) = f(t) + O(\epsilon)$, $t \in \Gamma_1$.
 $\tilde{F}^+(t) = f(t) + O(\epsilon)$, $t \in \Gamma_1$.
 $\tilde{F}^+(t) = f(t) + O(\epsilon)$, $t \in \Gamma_1$.
 $\tilde{F}^+(t) = f(t) + O(\epsilon)$, $t \in \Gamma_1$.
 [3] COLLI FRANZONE, P., GUERRI, L., TACCARDI, B., and C. VIGANOTTI: The direct and inverse potential problems in electrocardiology. Numerical aspects of some regularization methods and 'application to data collected. in isolated dog heart experiments. Labor. Anal. Num. Consiglio Naz. Richerche, Pubbl. N 222, Pavia 1979. Usines. 3. K., and Y. D. University: Approximating the solution of Maplace's equation. SIAM J. Num. Anal. 14 (1977), 473-483.
CARLEMAN, T.: Les fonctions quasi analytiques. Paris 1926.
COLLI FRANZONE, P., GUEREI, L., TACCA
- {4a) **COLTON,** D,: Cauchy's problem for almost linear elliptic equations in two independent variables. J. Approx. Theory 3 (1970), $66-71$.
- [4 b] **COLTON,** D.: Cauchy's problem for almost linear elliptic equations 'in two independent. variables II. J. Approx. Theory 4 (1971), 288-294.
- [5] ERDELYI, A., MAONUS, W., OBERHETTINGER, F., and F. G. TRICOMI: Higher trans. cendental functions, Vol. I. New York 1953. Anal. Num. Consiglio Naz. Rich

Colorov, D.: Cauchy's problem

variables. J. Approx. Theory 3

| Colorov, D.: Cauchy's problem

variables II. J. Approx. Theory

EFDELYI, A., MAONUS, W., Olorodental functions, Vol. I. New

- **[6] GARABEDIAN, P.** R.: Partial differential equations. New York 1964.
- **[7] HENRICI, P.: A survey of I. N. Vekua's theory of elliptic partial differential equations with analytic coefficients. ZAMP 8 (1957), 169—203.

[8] Иванов, В. К.: О некорректно поставленных задачах. Матем. сборник 61**
- [8] Иванов, В. К.: О некорректно поставленных задачах. Матем. сборник 61 (103), vekua s theory of
P 8 (1957), 169—203
но поставленных з
для vравнения Лаг ren
0pi
* . . **1963**), 211-223.
[9] Ивлнов, В. К.: Задача Коши для уравнения Лапласа в бесконечной полосе. Дифф.
- уравнения **1** (1965), 131-136.

E. WAGNER und L. v. WOLFERSDORF

- [10] JOHN, F.: A note on "improper" problems in partial differential equations. Comm. Pure Appl. Math. 8 (1955), 591-594.
- [11] JOHN, F.: Continuous dependence on data for solutions of partial differential equations with a prescribed bound. Comm. Pure Appl. Math. 13 (1960), $551-585$.

[12] LATTES, R., et J.-L. LIONS: Methode de quasi-réversibilité et applications. Paris 1967.

- [13] ЛАВРЕНТЬЕВ, М. М.: О задаче Коши для уравнения Лапласа. Докл. Акад. Наук CCCP 102 (1955), $205-206$.
- [14] ЛАВРЕНТЬЕВ, М. М.: О задаче Коши для уравнения Лапласа. Изв. Акад. Наук СССР, сер. матем. 20 (1956), 819-842.
- [15] ЛАВРЕНТЬЕВ, М. М.: О некоторых некорректных задачах математической физики. Новосибирск 1962.
- [15a] LAVRENTIEV, M. M.: Some improperly posed problems of mathematical physics. Berlin-Heidelberg-New York 1967.
- [16] ЛАВРЕНТЬЕВ, М. М., Романов, В. Г., и С. П. Шишатский: Некорректные задачи математической физики и анализа. Москва 1980.
- [17] Литвинчук, Г. С.: Краевые задачи и сингулярные интегральные уравнения со сдвигом. Москва 1977.
- [18a] MITTELMANN, H. D.: Einige Bemerkungen zum Cauchy-Problem für die Laplace-Gleichung. In: Inkorrekt gestellte Probleme I (Ed.: R. GORENFLO). FU Berlin 1977, \sqrt{S} . 187 – 196.
- [18b] MITTELMANN, H. D.: Zur numerischen Behandlung des Cauchy-Problems der Laplace-Gleichung (Numerische Beispiele). In: Inkorrekt gestellte Probleme II (Ed.: R. GOREN-FLO). FU Berlin 1978, S. 75-91.
- [19] MUSCHELISCHWILI, N. I.: Singuläre Integralgleichungen. Berlin 1965.
- [20] PAYNE, L. E.: Bounds in the Cauchy problem for the Laplace equation. Arch. Rational Mech. Anal. 5 (1960), $35-45$.
- [21] Pucci, C.: Sui problemi di Cauchy non "ben posti". Rend. Acc. Naz. Lincei (8) 18 (1955), $473 - 477.$
- [22] Pucci, C.: Discussione del problema di Cauchy per le equazioni di tipo ellittico. Ann. mat pura appl. 46 (1958), 131-153.
- [23] TALENTI, G.: Sui problemi mal posti. Boll. Un. Mat. Ital., ser. 5, 15 A (1978), 1-29.
- [24] Урев, М. В.: Об осесимметричной задаче Коши для уравнения Лапласа. Ж. вычисл. матем. и матем. физ. 20 (1980), 939-947.
- [25] Вльищевич, П. Н.: О решении задачи Коши для уравнения Лапласа в двухсвязной области. Докл. Акад. Наук СССР 241 (1978), 1257-1260.
- [26] ВАБИЩЕВИЧ, П. Н., Гласко, В. Б., и Ю. А. Криксин: О решении одной задачи-Адамара с помощью регуляризующего по А. Н. Тихонову алгоритма. Ж. вычисл. матем. и матем. физ. 19 (1979), 1462-1470.
- [27] Yu, C.-L.: Cauchy problem and analytic continuation for systems of first order elliptic equations with analytic coefficients. Trans. Amer. Math. Soc. 185 (1973), 429-443.
- [28] Yu, C.-L.: Cauchy problem for systems of first order analytic equations in the plane. SIAM J. Math. Anal. 8 (1977), 719-740.

Manuskripteingang: 4.08.1981

VERFASSER:

DOZ. Dr. EBERHARD WAGNER Sektion Mathematik der Martin-Luther-Universität Halle-Wittenberg. DDR-4010 Halle, Universitätsplatz 6

Prof. Dr. LOTHAR V. WOLFERSDORF Sektion Mathematik der Bergakademie Freiberg DDR-9200 Freiberg, Bernhard-v.-Cotta-Str. 2