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Extremal inequalities in Sobolev spaces and quasiconformal mappings

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In dieser Arbeit wird die L_p-Norm eines zweidimensionalen Hilbert-Operators abgeschätzt. Diese Abschätzungen führen zu einer Reihe von Ungleichungen aus der Theorie der Sobolevschen Räume. In einigen Fällen werden mit Hilfe quasikonformer Abbildungen bestmögliche Abschätzungen erzielt.

В работе оценивается L_p-норма двумерного оператора Гильберта. Эта задача связана с некоторыми неравенствами из теории пространств Соболева. В некоторых случаях, используя методы теории квазиконформных отображений, получаются наилучшие оценки.

We estimate the norm of a two-dimensional Hilbert operator in L_p-spaces. This problem leads to inequalities of the theory of Sobolev spaces. In certain cases, by using methods of the theory of quasiconformal mappings, we get best possible estimates.

The L_p estimations for functions of Sobolev spaces are central both to the theory of partial differential equations with discontinuous coefficients and to the theory of nonlinear differential equations. While the singular integral operators play a fundamental part in these, in practice we very often need to know the best estimations. Quasiconformal maps, especially in the two dimensional cases, may be used as a tool for attacking the problem of extremal inequalities and they suggest a way of formulating these inequalities properly. However, some problems of quasiconformal mapping theory lead to difficult questions in the theory of Sobolev spaces and partial differential equations. In this paper we discuss a few special cases of this.

Let us illustrate the general idea on an example of a non-linear system of partial differential equations in two variables which are strongly elliptic in the sense of Lavrent'ev. For future use it is convenient to introduce the complex variable $z = x + iy$ and the complex differential operators

$$D_z = \frac{1}{2}(D_x - iD_y), \qquad D_{\bar{z}} = \frac{1}{2}(D_x + iD_y).$$

Then the system of Lavrent'ev reduces to one complex non-linear Beltrami equation

$$w_{\bar{z}} = q(z, w, w_{\bar{z}})\, w_z, \tag{1}$$

where the ellipticity conditions reads

$$|q(z, w, \xi_1)\, \xi_1 - q(z, w, \xi_2)\, \xi_2| \leqq \beta\, |\xi_1 - \xi_2|, \qquad \beta < 1$$

for $\xi_1, \xi_2 \in \mathbf{C}$, $w \in \mathbf{C}$, $z \in \Omega \subset \mathbf{C}$, (see [2]). We call β the *ellipticity constant* of (1). By a solution of (1) we mean a function w from the Sobolev space $W^1_{2,loc}(\Omega)$ for which equation (1) holds almost everywhere in Ω.

It has been proved by BOJARSKI [1] that every solution of (1) actually belongs to $W^1_{p,loc}(\Omega)$ for some $p > 2$. The exponent p depends on β only.

Generalizing, we may say that when the coefficients of differential equations are discontinuous, it is correct to ask about the integrability exponent of derivatives of solutions. Bojarski's result is one of the first in this direction. For further results see [3].

Definition 1: A mapping $w: \Omega \to \mathbf{C}$ is said to be *β-quasiregular*, $0 \leqq \beta < 1$ iff $w \in W^1_{2,loc}(\Omega)$ and

$$|w_{\bar{z}}(z)| \leqq \beta\, |w_z(z)| \quad \text{almost everywhere in } \Omega. \tag{2}$$

When w is homeomorphic on Ω, then we call it *β-quasiconformal*. The number $K = (1+\beta)(1-\beta)^{-1}$ is called the *maximal dilatation* of w. We also call w K-quasiconformal, when no confusion occurs.

Any solution of (1) is β-quasiregular with β being the ellipticity constant of (1). We shall consider the function

$$p(\beta) = \sup\{p\colon \text{any } \beta\text{-quasiconformal mapping belongs to } W^1_{p,loc}(\Omega)\}. \tag{3}$$

On taking the mapping $w(z) = z\,|z|^{-2\beta/(1+\beta)}$ we immediately deduce that $p(\beta)$ does not exceed $1 + 1/\beta$. In other words

$$p(\beta) - 1 \leqq \frac{1}{\beta}. \tag{4}$$

In the recent work of W. GOLDSTEIN [6], he announced a proof of equality $p(\beta) = 1 + 1/\beta$, which was conjectured by GEHRING in [4].

The problem mentioned above is one of many which are related to the two dimensional Hilbert operator S. This operator is defined by a singular integral of the Calderon and Zygmund type

$$(Sf)(z) = -\frac{1}{\pi}\iint \frac{f(t)\, d\sigma_t}{(z-t)^2}, \quad \text{for } f \in L_p(\mathbf{C}).$$

Our main interest is in its L_p norm

$$\|S\|_p = \sup_{f \in L_p(\Omega)} \frac{\|Sf\|_p}{\|f\|_p}, \qquad 1 < p < \infty$$

and its relations to the special kinds of boundary value problems for elliptic systems of partial differential equations in 2-dimensional domains. Let us remark that one can formulate an opposite inequality for $p(\beta)$ in terms of the norm $\|S\|_p$ (see [1]):

$$p(\beta) \geqq \sup\left\{p\colon \|S\|_p < \frac{1}{\beta}\right\}. \tag{5}$$

1. Two dimensional Hilbert transform

Let us recall that S changes $D_{\bar{z}}$ into D_z, i.e. $S(D_{\bar{z}}w) = D_z w$, for $w \in W_p^1(\mathbf{C})$. We then have

$$\|S\|_p = \sup_{w \in \mathring{W}_p^1(\Omega)} \frac{\|w_z\|_p}{\|w_{\bar{z}}\|_p}. \tag{6}$$

For

$$w(z) = \begin{cases} z\,|z|^{-2u/p} & \text{if } |z| < 1 \\ 1/\bar{z} & \text{if } |z| \geqq 1, \end{cases} \qquad 0 < u < 1$$

one obtains

$$\|S\|_p \geqq \left[\frac{(p-1)(p-u)^p}{(p-1)\,u^p + (1-u)\,p^p}\right]^{1/p}.$$

The right hand side tends to $p - 1$ as u approaches 1. Therefore $\|S\|_p \geqq p - 1$. Similarly, by considering the function $\bar{w}$ we deduce $\|S\|_p \geqq 1/(p-1)$. This proves the following:

$$\|S\|_p \geqq \begin{cases} p - 1 & \text{if } p \geqq 2 \\ \dfrac{1}{p-1} & \text{if } 1 < p \leqq 2. \end{cases} \tag{7}$$

Both (4) and (5) suggest the following statement.

Conjecture 1: *For $p > 1$ it holds that* $\|S\|_p = \max\left(p - 1, \dfrac{1}{p-1}\right)$.

Although we cannot evaluate $\|S\|_p$ exactly, we have succeded in proving some inequalities related to the problem.

Observe that Conjecture 1 implies

$$\left\|w_z - \frac{\bar{z}}{z}\,w_{\bar{z}}\right\|_p \leqq \begin{cases} p\,\|w_{\bar{z}}\|_p & \text{for } p \geqq 2 \\ \dfrac{p}{p-1}\,\|w_z\|_p & \text{for } 1 < p \leqq 2 \end{cases} \tag{8}$$

for $w \in C_0^\infty(\mathbf{C})$.

If (8) were true then, in particular, we would get the following statement.

Conjecture 2: $\lim\limits_{p\to1}(p-1)\,\|S\|_p = \lim\limits_{p\to\infty}\dfrac{\|S\|_p}{p} = 1$.

Theorem 1: *The inequality (8) holds for every function of the form $w(z) = f\big(zu(|z|)\big)$, where $u \in C_0^1(\mathbf{R})$ and $f(\xi)$ is an analytic function of C^1 class on $\{\xi\colon |\xi| \leqq \sup |zu(|z|)|\}$ and $(f')^{p/2}$ is single-valued*[1]). *Moreover, the constants p and $p/(p-1)$ are the best possible. There is no function w such that equality occurs in (8).*

Proof: Given the assumption about f, we can write the following Taylor expansion

$$\big(f'(\xi)\big)^{p/2} = \sum_{n=0}^{\infty} a_n \xi^n.$$

After a simple calculation, (8) becomes equivalent to

$$\big\|u(|z|)\,f'\big(zu(|z|)\big)\big\|_p \leqq \frac{p}{2}\,\big\|zu'(|z|)\,f'\big(zu(|z|)\big)\big\|_p \quad \text{if } p \geqq 2$$

and

$$\big\|z^{-2}v(|z|)\,f'\big(\bar{z}^{-1}v(|z|)\big)\big\|_p \leqq \frac{p}{2p-2}\,\big\|z^{-1}v'(|z|)\,f'\big(\bar{z}^{-1}v(|z|)\big)\big\|_p \quad \text{if } 1 < p \leqq 2$$

[1]) This assumption always holds if $p = 2, 4, \ldots$ or $f'(\xi) \neq 0$.

where $v(r)$ denotes $r^2u(r)$. We notice next that the above are equivalent to

$$\iint |u(|z|)|^p \left|\sum a_n z^n u^n(|z|)\right|^2 d\sigma_z \leqq (p/2)^p \iint |zu'(|z|)|^p \left|\sum a_n z^n u^n(|z|)\right|^2 d\sigma_z \quad \text{if} \quad p \geqq 2,$$

and

$$\iint |z^{-2}v(|z|)|^p \left|\sum a_n \bar{z}^{-n} v^n(|z|)\right|^2 d\sigma_z \leqq \left(\frac{p}{2p-2}\right)^p \iint |z^{-1}v'(|z|)|^p \left|\sum a_n \bar{z}^{-n} v^n(|z|)\right|^2 d\sigma_z \quad \text{if} \quad 1 < p \leqq 2.$$

In polar coordinates these inequalities take the form

$$\sum |a_n|^2 \int_0^\infty r^{2n+1} |u(r)|^{2n+p}\, dr \leqq \left(\frac{p}{2}\right)^p \sum |a_n|^2 \int_0^\infty r^{2n+p+1} |u(r)|^{2n} |u'(r)|^p\, dr,$$

and

$$\sum |a_n|^2 \int_0^\infty r^{1-2n-2p} |v(r)|^{2n+p}\, dr \leqq \left(\frac{p}{2p-2}\right)^p \sum |a_n|^2 \int_0^\infty r^{1-2n-p} |v(r)|^{2n} |v'(r)|^p\, dr.$$

To show that we appeal to two complex versions of the Hardy inequality, namely

$$\int_0^\infty r^{2n+1} |u(r)|^{2n+p}\, dr \leqq \left(\frac{2n+p}{2n+2}\right)^p \int_0^\infty r^{2n+p+1} |u(r)|^{2n} |u'(r)|^p\, dr \tag{9}$$

and

$$\int_0^\infty r^{1-2n-2p} |v(r)|^{2n+p}\, dr \leqq \left(\frac{2n+p}{2n+2p-2}\right)^p \int_0^\infty r^{1-2n-p} |v(r)|^{2n} |v'(r)|^p\, dr.$$

Notice that

$$\sup_{n\geqq 0} \frac{2n+p}{2n+2} = \left.\frac{2n+p}{2n+2}\right|_{n=0} = \frac{p}{2}, \quad \text{for} \quad p \geqq 2$$

and

$$\sup_{n\geqq 0} \frac{2n+p}{2n+2p-2} = \left.\frac{2n+p}{2n+2p-2}\right|_{n=0} = \frac{p}{2p-2}, \quad \text{for} \quad 1 < p \leqq 2.$$

Of the two inequalities stated in (9), we verify only the first one, the second can be shown in a similar manner. Integrating by parts and applying the Hölder inequality we obtain

$$\int_0^\infty r^{2n+1} |u(r)|^{2n+p}\, dr = \frac{-1}{2n+2} \int_0^\infty r^{2n+2} \left(|u(r)|^{2n+p}\right)' dr$$

$$= \frac{-2n-p}{2n+2} \int_0^\infty r^{2n+2} |u(r)|^{2n+p-2} \operatorname{Re} \overline{u(r)}\, u'(r)\, dr$$

$$\leqq \frac{2n+p}{2n+2} \int_0^\infty (r^{1+(2n+1)/p} |u|^{2n/p} |u'|) (r^{2n+1-(2n+1)/p} |u|^{2n+p-(2n+p)/p}) \, dr$$

$$\leqq \frac{2n+p}{2n+2} \left(\int_0^\infty r^{2n+p+1} |u|^{2n} |u'|^p \, dr \right)^{1/p} \left(\int_0^\infty r^{2n+1} |u|^{2n+p} \, dr \right)^{(p-1)/p}.$$

Hence

$$\left(\int_0^\infty r^{2n+1} |u(r)|^{2n+p} \, dr \right)^{1/p} \leqq \frac{2n+p}{2n+2} \left(\int_0^\infty r^{2n+p+1} |u|^{2n} |u'|^p \, dr \right)^{1/p}$$

as claimed.

From the proof presented it follows that if the extremal function which equalizes (8) exists, then it must be of the form $w(z) = zu(|z|)$, because it should be such that $(f'(\xi))^{p/2} = a_0 = \text{const}$. Thus, we are led to the following equalities

$$\left(\int_0^\infty |u(r)|^p \, r \, dr \right)^{1/p} = \frac{p}{2} \left(\int_0^\infty |ru'(r)|^p \, r \, dr \right)^{1/p}, \quad \text{if} \quad p \geqq 2, \tag{10}$$

and

$$\left(\int_0^\infty |r^{-1}v(r)|^p \, r \, dr \right)^{1/p} = \frac{p}{2p-2} \left(\int_0^\infty |r^{-1}v'(r)|^p \, r \, dr \right)^{1/p}, \quad \text{if} \quad 1 < p \leqq 2.$$

For simplicity we only explore the first one. Investigating again the proof of inequality (9), in the case $n = 0$, we easily find some necessary conditions for a function u which would satisfy (10). Such a function must equalize all inequalities which appeared in the proof of (9). So, the first condition in question is

$$\operatorname{Re} u'(r) \, \overline{u(r)} = -2 \, |u(r)| \, |u'(r)| .$$

The second one is a result of an application of the Hölder inequality, giving

$$|u'(r)|^p \, r^{p+1} = \text{const.} \; r \, |u(r)|^p .$$

Both of these conditions imply a differential equation

$$u'(r) = -c \, \frac{u(r)}{r}, \qquad c > 0$$

which is neither solvable in the space $C_0^1(\mathbf{R})$, nor in the space $W_p^1(\mathbf{R})$. All the solutions of this equation have the form

$$u(r) = \text{const.} \; r^{-c}. \tag{11}$$

Thus the equality in (9) does not hold for any function u, which implies the same is true for (8).

The case $c = 2/p$ is of an extremal character. Namely, for $u(r) = r^{-2/p}$, both integrals in (10) are not convergent either at 0 or at ∞. By refining the function $u(r) = r^{-2/p}$ near the points 0 and ∞, one can construct examples which show that the inequality (8) cannot be improved. This completes the proof of the theorem ∎

Proposition 1: *Let $R > 1$. Then we have*

$$\sup_{u \in \mathring{W}_p{}^1([1,R])} \frac{\left(\int_1^R |u(r)|^p \, r \, dr\right)^{1/p}}{\left(\int_1^R |ru'(r)|^p \, r \, dr\right)^{1/p}} = A_p < \frac{p}{2}, \quad p \geq 2.$$

Proof: The inequality $A_p \leq p/2$ follows immediately from (10). On the other hand there exists an extremal function, say $u_0 \in \mathring{W}_p{}^1([1, R])$, which realizes the supremum. This statement is a consequence of the compactness of the mapping

$$r^{1+1/p}u'(r) \to r^{1/p}u(r)$$

acting from $\mathring{W}_p{}^1([1, R])$ into $L_p([1, R])$. So, if the equality $A_p = p/2$ were true, then the function u_0 would be extremal for (10). This contradicts the original assumption ∎

For our further investigations it is important to extend the class $\mathring{W}_p{}^1([1, R])$ in order to get the equality $A_p = p/2$ for the extended class. For this purpose let us observe that inequality (9), with $n = 0$, and its proof remain true for functions $u \in W_p{}^1([1, R])$ which satisfy the boundary condition $|u(R)| = R^{-2/p} |u(1)|$.

Proposition 2: *Let $u \in W_p{}^1([1, R])$, $|u(R)| = R^{-2/p} |u(1)|$. Then it follows that*

$$\left(\int_1^R |u(r)|^p \, r \, dr\right)^{1/p} \leq \frac{p}{2} \left(\int_1^R |ru'(r)|^p \, r \, dr\right)^{1/p}.$$

The functions which give equality have the form $u(r) =$ const. $r^{-2/p}$.

2. Modified Hilbert transform

In this section we define a boundary value problem for elliptic systems which arose in the study of the estimation of the operator S in L_p spaces. We begin with the simplest case.

Let $D_R = \{z: 1 < |z| < R\}$ and $f \in L_p(D_R)$. Consider the problem

$$\begin{cases} w_{\bar{z}} = f \\ w(Re^{i\theta}) = R^{1-2/p} w(e^{\theta i}), \quad \theta \in [0, 2\pi] \end{cases} \tag{11}$$

where $p \in (1, \infty)$, and w is an unknown function from $W_p{}^1(D_R)$.

Proposition 3: *For every $f \in L_p(D_R)$, $p \neq 2$, there exists exactly one solution of (11). If $p = 2$, the solution is unique up to a constant.*

Proof: For the proof we shall show that an analytic function with boundary condition (11) must be a constant, zero when $p \neq 2$. Let $\sum a_n z^n$ be its Laurent's expansion. The boundary condition becomes

$$\sum a_n R^n e^{in\theta} = \sum a_n R^{1-2/p} e^{in\theta}.$$

Hence, by comparing the coefficients we get

$$a_n = 0 \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots \quad \text{if} \quad p \neq 2,$$

$$a_n = 0 \quad \text{for} \quad n = \pm 1, \pm 2, \pm 3, \ldots \quad \text{if} \quad p = 2.$$

The existence follows from the explicit formula

$$w(z) = (T_R f)(z) = \frac{-1}{\pi} \iint\limits_{D_R} \left[\frac{1}{t - z} + K(z, t)\right] f(t)\, d\sigma_t, \tag{12}$$

where the kernel $K(z, t)$ is given by

$$K(z, t) = t^{-1} \sum_{n=0}^{\infty} \frac{z^n t^{-n}}{R^{n+2/p-1} - 1} + t^{-1} \sum_{-\infty}^{-1} \frac{z^n t^{-n}}{1 - R^{1-n-2/p}}.$$

Both series converge uniformly on compact subsets of $D_R \times D_R$, therefore $K(z, t)$ is analytic in $D_R \times D_R$. We shall show that $w(z)$, as defined by (12), is in $W_p^1(D_R)$ and that it satisfies the boundary condition (11). First we assume that $\operatorname{supp}(f) \subset D_R$. Hence we obviously have $w \in W_p^1(D_R)$, and $w_{\bar z} = f$, because the kernel $K(z, t)$ is analytic and $-1/\pi(t - z)$ is a fundamental solution of the Cauchy-Riemann equation. Moreover

$$\begin{aligned} w(Re^{i\theta}) &= \frac{-1}{\pi} \iint\limits_{D_R} [(t - Re^{i\theta})^{-1} + K(Re^{i\theta}, t)]\, f(t)\, d\sigma_t \\ &= \frac{-R^{1-2/p}}{\pi} \iint\limits_{D_R} [(t - e^{i\theta})^{-1} + K(e^{i\theta}, t)]\, f(t)\, d\sigma_t = R^{1-2/p} w(e^{i\theta}). \end{aligned}$$

The last equality is a consequence of the following property of the kernel $K(z, t)$:

$$\begin{aligned} &R^{1-2/p} K(e^{i\theta}, t) - K(Re^{i\theta}, t) \\ &= \frac{R^{1-2/p}}{t} \left[\sum_{n=0}^{\infty} \frac{t^{-n} e^{in\theta}}{R^{n+2/p-1} - 1} + \sum_{-\infty}^{-1} \frac{t^{-n} e^{in\theta}}{1 - R^{1-n-2/p}}\right] \\ &\quad - \frac{1}{t} \left[\sum_{n=0}^{\infty} \frac{R^n t^{-n} e^{in\theta}}{R^{n+2/p-1} - 1} + \sum_{-\infty}^{-1} \frac{R^n t^{-n} e^{in\theta}}{1 - R^{1-n-2/p}}\right] \\ &= \frac{-1}{t} \sum_{n=0}^{\infty} \frac{(R^n - R^{1-2/p})\, t^{-n} e^{in\theta}}{R^{n+2/p-1} - 1} - \frac{1}{t} \sum_{-\infty}^{-1} \frac{(R^n - R^{1-2/p})\, t^{-n} e^{in\theta}}{1 - R^{1-n-2/p}} \\ &= \frac{-R^{1-2/p}}{t - e^{i\theta}} - \frac{1}{t - Re^{i\theta}}. \end{aligned}$$

Therefore $w(z)$ is the solution of the problem if $\operatorname{supp}(f) \subset D_R$. Now we define the following modified Hilbert transform

$$(S_R f)(z) = (T_R f)_z(z) = \frac{-1}{\pi} \iint\limits_{D_R} [(t - z)^{-2} + K_z(z, t)]\, f(t)\, d\sigma_t.$$

It remains to prove that S_R is a bounded operator in $L_p(D_R)$.

Theorem 2: *We have*

$$\sup_{f \in L_p(D_R)} \frac{\|S_R f\|_p}{\|f\|_p} = \|S\|_p \tag{13}$$

and

$$\inf_{f\in L_p(D_R)} \frac{\|S_R f\|_p}{\|f\|_p} = \frac{1}{\|S\|_p}. \tag{14}$$

Proof: Letting $M_p{}^1(D_R) = \{w;\, w \in W_p{}^1(D_R),\, w(Re^{i\theta}) = R^{1-2/p}w(e^{i\theta})\}$ gives

$$\|S\|_p = \sup_{w\in W_p{}^1(\mathbf{C})} \frac{\|w_z\|_p}{\|w_{\bar z}\|_p} = \sup_{w\in \hat W_p{}^1(D_R)} \frac{\|w_z\|_p}{\|w_{\bar z}\|_p} \leqq \sup_{w\in M_p{}^1(D_R)} \frac{\|w_z\|_p}{\|w_{\bar z}\|_p} = \sup_{f\in L_p(D_R)} \frac{\|S_R f\|_p}{\|f\|_p}$$

and

$$\frac{1}{\|S\|_p} = \inf_{w\in W_p{}^1(\mathbf{C})} \frac{\|w_z\|_p}{\|w_{\bar z}\|_p} = \inf_{w\in \hat W_p{}^1(D_R)} \frac{\|w_z\|_p}{\|w_{\bar z}\|_p} \geqq \inf_{M_p{}^1(D_R)} \frac{\|w_z\|_p}{\|w_{\bar z}\|_p} = \inf_{f\in L_p(D_R)} \frac{\|S_R f\|_p}{\|f\|_p}.$$

Now we shall prove the opposite inequalities. For every positive integer n we construct an extension of any $w \in M_p{}^1(D_R)$ as follows:

$$\tilde w(z) = \begin{cases} 0 & \text{for } |z| \leqq 1, \\ \dfrac{|z|-1}{R-1}\, w(z) & \text{for } 1 \leqq |z| \leqq R, \\ R^{j(1-2/p)} w(zR^{-j}) & \text{for } R^j \leqq |z| \leqq R^{j+1}, \quad j = 1, 2, \ldots, n, \\ \dfrac{R^{n+2}-|z|}{R^{n+2}-R^{n+1}}\, w(zR^{-n-1}) & \text{for } R^{n+1} \leqq |z| \leqq R^{n+2}, \\ 0 & \text{for } R^{n+2} \leqq |z|. \end{cases}$$

From the definition of $M_p{}^1(D_R)$, it follows that $w \in W_p{}^1(\mathbf{C})$. Moreover

$$\tilde w_t(z) = \begin{cases} 0 & \text{for } |z| \leqq 1, \\ \dfrac{|z|-1}{R-1}\, w_t(z) + \dfrac{\bar z w(z)}{z\,|z|\,(R-1)} & \text{for } 1 < |z| \leqq R, \\ R^{-2j/p} w_t(R^{-j}z) & \text{for } R^j < |z| \leqq R^{j+1}, \quad j = 1, 2, \ldots, n \\ \dfrac{(R^{n+2}-|z|)\, w_t(R^{-n-1}z) - |z|\, z^{-2} R^{n+1} w(R^{-n-1}z)}{(R-1)\, R^{(1+2/p)(n+1)}} & \text{for } R^{n+1} < |z| \leqq R^{n+2}, \\ 0 & \text{for } R^{n+2} < |z|. \end{cases}$$

We have to estimate the integral $\iint |\tilde w_t(z)|^p\, d\sigma_z$. Therefore we first transform it into an integral over the ring D_R, and then we decompose this integral into a finite sum of integrals over the rings in which $\tilde w(z)$ is defined. By using a natural substitution we reduce each of these integrals to an integral over D_R. Accordingly we write

$$\begin{aligned} &\iint |\tilde w_z(z)|^p\, d\sigma_z \\ &= n \iint_{D_R} |w_z(z)|^p\, d\sigma_z + \iint_{D_R} \left| \frac{|z|-1}{R-1}\, w_z + \frac{\bar z w}{z\,|z|\,(R-1)} \right|^p d\sigma_z \\ &\quad + \iint_{D_R} \left| \frac{R-|z|}{R-1}\, w_z - \frac{\bar z w}{z\,|z|\,(R-1)} \right|^p d\sigma_z \\ &= n \iint_{D_R} |w_z(z)|^p\, d\sigma_z + \mathcal{O}(1). \end{aligned}$$

Similarly we can prove that

$$\iint |\tilde{w}_{\bar z}(z)|^p \, d\sigma_z = n \iint_{D_R} |w_{\bar z}(z)|^p \, d\sigma_z + \mathcal{O}(1).$$

Letting n approach infinity we get

$$\frac{\displaystyle\iint_{D_R} |w_z(z)|^p \, d\sigma_z}{\displaystyle\iint_{D_R} |w_{\bar z}(z)|^p \, d\sigma_z} = \lim_{n\to\infty} \frac{\displaystyle\iint |\tilde{w}_z(z)|^p \, d\sigma_z + \mathcal{O}\left(\frac{1}{n}\right)}{\displaystyle\iint |\tilde{w}_{\bar z}(z)|^p \, d\sigma_z + \mathcal{O}\left(\frac{1}{n}\right)}$$

$$= \begin{cases} \leqq \sup\limits_{w\in W_p^1(\mathbf{C})} \dfrac{\|w_z\|_p{}^p}{\|w_{\bar z}\|_p{}^p} = \|S\|_p{}^p, \\ \geqq \inf\limits_{w\in W_p^1(\mathbf{C})} \dfrac{\|w_z\|_p{}^p}{\|w_{\bar z}\|_p{}^p} = \dfrac{1}{\|S\|_p{}^p}. \end{cases}$$

Since w was chosen arbitrarily, we can write

$$\sup_{f\in L_p(D_R)} \frac{\|S_R f\|_p}{\|f\|_p} \leqq \|S\|_p \quad \text{and} \quad \inf_{f\in L_p(D_R)} \frac{\|S_R f\|_p}{\|f\|_p} \geqq \frac{1}{\|S\|_p}.$$

The proof of the theorem is complete ∎

Our theorem implies that $\|S_R\|_p = \|S\|_p$. There are many reasons why it has a practical value. It is much easier to study the modified Hilbert transform than the original one. For example, $\frac{z}{\bar z} S_R$ has a very simple spectral decomposition.

Theorem 3: *The operator* $\frac{z}{\bar z} S_R$ *has a point spectrum only. The numbers*

$$\beta_{m,n} = \frac{1 - \dfrac{2}{p} + m + \dfrac{2\pi i n}{\ln R}}{1 - \dfrac{2}{p} - m + \dfrac{2\pi i n}{\ln R}} \qquad (m, n = 0, \pm 1, \pm 2, \ldots) \tag{15}$$

are its eigenvalues and the corresponding eigenfunctions

$$f_{m,n} = z^{m+1}(|z|)^{\frac{-2}{p} - m + \frac{2\pi i n}{\ln R} - 1} \qquad (m, n = 0, \pm 1, \pm 2, \ldots) \tag{16}$$

form a complete system in $L_p(D_R)$, $p \geqq 2$.

Proof: We have to find the solutions of the boundary value problem

$$\left.\begin{aligned} z w_z(z) &= \beta \bar z w_{\bar z}(z), & 1 < |z| < R, \\ w(Re^{i\theta}) &= R^{1-2/p} w(e^{i\theta}), & 0 \leqq \theta \leqq 2\pi. \end{aligned}\right\} \tag{17}$$

It is convenient to work with polar coordinates (r, θ):

$$\left.\begin{aligned} &\left(w_r - \frac{i}{r} w_\theta\right) = \beta \left(w_r + \frac{i}{r} w_\theta\right), \\ &w(R, \theta) = R^{1-2/p} w(1, \theta). \end{aligned}\right\} \tag{18}$$

The boundary condition implies that the function $r^{-1+2/p}w(r, \theta)$ is 2π-periodic in θ and $(R - 1)$-periodic in r. Therefore we can expand $w(r, \theta)$ into the Fourier series

$$w(r, \theta) = \sum \sum a_{mn} w_{mn}(r, \theta),$$

where

$$w_{mn}(r,\theta) = r^{-\frac{2}{p}+1+\frac{2\pi in}{\ln R}} e^{im\theta} = z^m |z|^{1-\frac{2}{p}-m+\frac{2\pi in}{\ln R}}.$$

Inserting this into (18) we get

$$\sum_{m,n} a_{mn}\left(1-\frac{2}{p}+m+\frac{2\pi in}{\ln R}\right) r^{\frac{2\pi in}{\ln R}-\frac{2}{p}} e^{im\theta}$$

$$= \beta \sum_{m,n} a_{mn}\left(1-\frac{2}{p}-m+\frac{2\pi in}{\ln R}\right) r^{\frac{2\pi in}{\ln R}-\frac{2}{p}} e^{im\theta}.$$

By comparing the coefficients of these series we immediately obtain (15) and (16) with $f_{m,n} = (w_{mn})_{\bar z} \cdot \mathrm{const.}$ ∎

The eigenvalue of $\frac{z}{\bar z} S_R$ with the greatest modul is $\beta_{1,0} = 1-p$, which corresponds to $w_{1,0}(z) = z\,|z|^{-2/p}$, a quasiformal solution of (17). That with the smallest modul is $\beta_{-1,0} = (1-p)^{-1}$ which corresponds to $w_{-1,0}(z) = \overline{w_{1,0}(z)} = \bar z\,|z|^{-2/p}$. We do not know whether $|\beta_{1,0}| = p-1$ is the norm of S_R in $L_p(D_R)$, but there is strong evidence for expecting it. Namely, if one writes the Euler equations

$$\begin{cases} (|w_z|^{p-2}\,\overline{w_z})_z = \beta^p(|w_{\bar z}|^{p-2}\,\overline{w_{\bar z}})_{\bar z}, \\ w(Re^{i\theta}) = R^{1-2/p} w(e^{i\theta}), \end{cases}$$

for the variational problem

$$\beta = \sup_{w\in M_p^1(D_R)} \frac{\|w_z\|_p}{\|w_{\bar z}\|_p},$$

then it turns out that the $w_{m,n}$ are the solutions corresponding to $\beta = |\beta_{m,n}|$. This again suggests Conjecture 1 and provides another reason to study the following problem.

Generalized eigenvalues problem: Let $p > 2$. Find $w \in W_p^1(D_R)$ such that

$$\begin{cases} w_z(z) = \beta(z)\, w_{\bar z}(z), & 1 < |z| < R, \\ w(Re^{i\theta}) = R^{1-2/p} w(e^{i\theta}) \end{cases} \tag{19}$$

where $\beta(z)$ is a measurable function which satisfies the ellipticity condition

$$|\beta(z)| \geqq \frac{1}{\beta} > 1 \quad \text{or} \quad |\beta(z)| \leqq \beta < 1.$$

It is clear that this problem admits only the trivial solution if $\beta\,\|S\|_p = \beta\,\|S_R\|_p < 1$. On the other hand we shall prove the following statement.

Theorem 4: *A necessary condition for the existence of a non-trivial solution of* (19) *is* $\beta \geqq (p-1)^{-1}$.

In the proof we use the following:

Definition 2: By a *ring* D we mean a domain the boundary of which consists of two Jordan curves $\mathscr{C}_1$, $\mathscr{C}_2$. The *modulus* $M(D)$ of the ring D is

$$M(D) = \inf_q \iint_{\mathbf{C}} q^2(z)\, d\sigma_z$$

where the infimum is taken over all Borel measurable functions $q(z) \geqq 0$ such that

$$\int_l q(z)\,|dz| \geqq 1$$

for every locally rectifiable curve $l \subset D$ which joins the boundary components $\mathscr{C}_1$ and $\mathscr{C}_2$.

For the spherical ring $D_R = \{z: 1 < |z| < R\}$ we have $M(D_R) = \dfrac{2\pi}{\ln R}$.

Lemma 1: *Let G be a ring whose boundary components $\mathscr{C}_1$ and $\mathscr{C}_2$ satisfy*

$$\mathscr{C}_2 = \{az: z \in \mathscr{C}_1\}, \qquad a > 1.$$

Then $M(G) \geqq \dfrac{2\pi}{\ln a}$.

Proof: Let q be an arbitrary admissible function. For every angle $\theta \in [0, 2\pi]$ we denote by $l_\theta = \{z: z = re^{i\theta}, r_1 \leqq r \leqq r_2\}$ the radial segment contained in G which joins the boundary components $\mathscr{C}_1$ and $\mathscr{C}_2$.

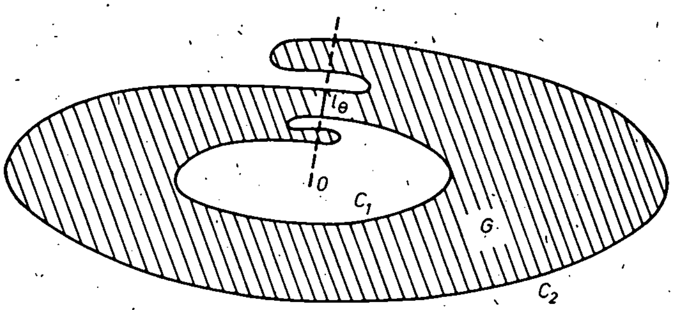

Abb. 1

So, for every $\theta \in [0, 2\pi]$ we have

$$1 \leqq \int_{l_\theta} q(z)\,|dz| \leqq \int_0^\infty q(re^{i\theta})\,dr.$$

Integrating over all θ and applying the Hölder inequality one can obtain

$$2\pi \leqq \int_0^{2\pi}\int_0^\infty q(re^{i\theta})\,dr\,d\theta \leqq \left(\int_0^{2\pi}\int_0^\infty q^2(re^{i\theta})\,r\,dr\,d\theta\right)^{1/2}\left(\int_0^{2\pi}\int_0^\infty \frac{\chi_G(re^{i\theta})}{r}\,dr\,d\theta\right)^{1/2}$$

where χ_G is the characteristic function of G. The integral $\displaystyle\int_0^\infty \frac{\chi_G(re^{i\theta})}{r}\,dr$ does not depend on the angle θ and is equal to $\ln a$ (we omit the proof of this). Hence we get

$$2\pi \leqq \left(\iint q^2(z)\,d\sigma_z\right)^{1/2}(2\pi \ln a)^{1/2},$$

and finally

$$M(G) = \inf_q \iint q^2(z)\,d\sigma_z \geqq \frac{2\pi}{\ln a} \blacksquare$$

Proof of Theorem 4: For simplicity we examine the case $|\beta(z)| \geqq \frac{1}{\beta} > 1$ only. Every non-trivial solution of (19) is a β-quasiregular map. Suppose that w is β-quasiconformal. It transforms the spherical ring D_R onto a domain G which satisfies the assumptions of Lemma 1 with $a = R^{1-2/p}$. It is a fundamental result of the theory of quasiconformal mappings that

$$M(G) \leqq KM(D_R) = \frac{1+\beta}{1-\beta} M(D_R). \tag{20}$$

Hence, we have

$$\frac{2\pi}{\ln R^{1-2/p}} \leqq \frac{1+\beta}{1-\beta} \frac{2\pi}{\ln R},$$

so that

$$\beta \geqq \frac{1}{p-1}.$$

The inequality (20) remains true, in its proper formulation, for any quasiregular mapping (see [7]). This permits us to prove the theorem completely ∎

As a result of this theorem we obtain, in particular, a solution of the following extremal problem:

Problem: In the class of functions $w \in M_p^1(D_R)$ find a quasiconformal mapping with minimal dilatation.

Actually, the minimal dilatation β is at least $\frac{1}{p-1}$ (because of Theorem 4). On the other hand, the map $w(z) = z(|z|)^{-2/p}$ has the minimal dilatation $\beta = 1/(p-1)$,

$$\beta(z) = \frac{w_{\bar{z}}}{w_z} = \frac{-1}{p-1} \frac{z^2}{|z|^2}.$$

Thus $w(z)$ is extremal; it is Teichmüller's quasiconformal map.

3. Asymptotic behaviour of $\|S\|_p$

The operator S is of the type (1,1), i.e.

$$\text{meas}\,\{z: |(Sf)(z)| > \alpha\} \leqq \frac{A}{\alpha} \|f\|_1, \quad \text{for every } \alpha > 0, \tag{21}$$

where A is a constant which does not depend on $f \in L_1(\mathbf{C})$. The smallest of such constants defines a norm of the operator S. We will denote this also by A. Using the method presented in the book of E. Stein [8], it can be proved that

$$(\ln 2)^{-1} < A < 30.$$

We shall use the following general lemma.

Lemma 2: *Assume that S is an arbitrary linear operator which satisfies* (21) *and is bounded in some L_{p_0}, $p_0 > 1$. Then for every $p \in (1, p_0)$ the operator S is bounded in L_p and*

$$\limsup_{p\to 1} (p-1) \|S\|_p \leqq A.$$

This supplements the well known lemma of interpolation theory for operators in L_p spaces. The proof only requires a slight modification. A careful examination of it yields that

$$\|S\|_p{}^p \leqq \frac{pA}{(p-1)\,t} + \frac{p\,\|S\|_{p_0}^{p_0}}{(p_0 - p)\,(1-t)^{p_0}}, \quad \text{for every } t \in (0, 1),$$

whence the lemma. The adjoint Hilbert operator S^* has the conjugate kernel

$$(S^*f)(z) = \overline{(S\bar{f})(z)} = \frac{-1}{\pi} \iint \frac{f(t)\,d\sigma_t}{(\bar{z} - \bar{t})^2},$$

from which it follows that

$$\|S\|_p = \|S^*\|_p = \|S\|_{\frac{p}{p-1}}.$$

Therefore,

$$\limsup_{p\to 1}\,(p-1)\,\|S\|_p = \limsup_{p\to 1} \frac{p-1}{p}\,\|S\|_{\frac{p}{p-1}} = \limsup_{q\to\infty} \frac{\|S\|_q}{q}.$$

Let the

$$\liminf_{p\to\infty} \frac{\|S\|_p}{p}$$

be denoted by a. We have proved that $1 \leqq a \leqq A < 30$. There has been conjectured that $a = 1$.

Theorem 5: *The extremal exponent function* $p = p(\beta)$ *satisfies*[2])

$$\frac{1+\beta}{\beta} \geqq p(\beta) \geqq \frac{2(1+\beta)^a}{(1+\beta)^a - (1-\beta)^a} \geqq \frac{1+\beta}{\beta} \cdot 2^{1-a}. \tag{22}$$

Proof: The proof is based on the following inequality

$$p\left(\frac{\beta' + \beta''}{1 + \beta'\beta''}\right) \geqq \frac{p(\beta')\,p(\beta'')}{p(\beta') + p(\beta'') - 2} \quad \text{for} \quad 0 \leqq \beta', \quad \beta'' < 1. \tag{23}$$

Therefore we first prove this inequality.

Let $w = w(z)$ be a $\dfrac{\beta' + \beta''}{1 + \beta'\beta''}$-quasiconformal map in the domain $\Omega \subset \mathbf{C}$ and p be an arbitrary exponent strictly less than $\dfrac{p(\beta')\,p(\beta'')}{p(\beta') + p(\beta'') - 2}$. Our aim is to prove that $w \in W^1_{p,loc}(\Omega)$. By applying the existence theorems for the Beltrami system, the following decomposition property can be proved:

$$w = g \circ h$$

where h is a β''-quasiconformal map in Ω and g is a β'-quasiconformal map in the domain $h(\Omega)$. According to the definition of the extremal exponents, we have $h \in W^1_{p'',loc}(\Omega)$ and $g \in W^1_{p',loc}\big(h(\Omega)\big)$ for every $p' < p(\beta')$ and $p'' < p(\beta'')$, respectively. Without loss of generality we may omit the symbol "loc" by eventually considering a compact subset of Ω instead of Ω itself.

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[2]) A similar result has also been proved in [5].

Let E be an arbitrary measurable subset of Ω. We estimate the measure of the set $h(E)$ as follows

$$\operatorname{meas} h(E) = \iint_E \left(|h_z(z)|^2 - |h_{\bar z}(z)|^2\right) d\sigma_z$$
$$\leq (\operatorname{meas} E)^{1-2/p''} \left(\iint_E |h_z(z)|^{p''} d\sigma_z\right)^{2/p''} \leq \text{const.} (\operatorname{meas} E)^{1-2/p''}.$$

As above we also deduce that

$$\operatorname{meas} w(E) = \operatorname{meas} g\big(h(E)\big) \leq \text{const.} \big(\operatorname{meas} h(E)\big)^{1-2/p'}.$$

Takes together these yield the inequality

$$\operatorname{meas} w(E) \leq \text{Const.} (\operatorname{meas} E)^{(1-2/p')(1-2/p'')}.$$

The left-hand side of this inequality is nothing other than the integral over E of the Jacobian $J(z) = |w_z(z)|^2 - |w_{\bar z}(z)|^2$ of the map w. Thus it reads

$$\iint_E J(z)\, d\sigma_z \leq \text{const.} (\operatorname{meas} E)^{(1-2/p')(1-2/p'')}.$$

We utilize this inequality for estimating the measure of the set

$$E_T = \{z \in \Omega : J(z) > T\},$$

where T is an arbitrary positive parameter.

$$\operatorname{meas}(E_T) = \iint_{E_T} d\sigma_z \leq \frac{1}{T} \iint_{E_T} J(z)\, d\sigma_z \leq \frac{C}{T} (\operatorname{meas} E_T)^{(1-2/p')(1-2/p'')}.$$

Hence

$$\operatorname{meas} E_T \leq \text{const.}\ T^{\frac{-p'p''}{2(p'+p''-2)}}.$$

On the other hand it is well known that

$$\iint_\Omega J(z)^{p/2} d\sigma_z = \frac{p}{2} \int_0^\infty T^{p/2-1} \operatorname{meas} E_T\, dT$$
$$\leq \frac{p \operatorname{meas}(\Omega)}{2} \int_0^1 T^{(p-2)/2}\, dT$$
$$+ \text{const.} \int_1^\infty T^{(p-2)/2}\, T^{\frac{-p'p''}{2(p'+p''-2)}}\, dT < \infty.$$

This last follows from the assumption $p < \dfrac{p'p''}{p'+p''-2}$. So, we have that $J(z) \in L_{p/2}(\Omega)$. By the quasiconformality of w we finally get $w \in W_p^1(\Omega)$. This completes the proof of (23).

Remark: We have shown that it always holds that $p(\beta) \leq \dfrac{1+\beta}{\beta}$. Notice that equality occurs in (23) when $p(\beta) = \dfrac{1+\beta}{\beta}$.

Now we shall prove (22). We begin by defining the preparatory function

$$F(s) = -\ln\left(1 - \frac{2}{p(\beta)}\right), \quad \text{for} \quad s = \ln\frac{1+\beta}{1-\beta} \geqq 0.$$

From $p(0) = \infty$ it follows that $F(0) = 0$. Now, inequality (23) becomes

$$F(s_1 + s_2) \leqq F(s_1) + F(s_2). \tag{24}$$

Indeed,

$$F\big(s(\beta') + s(\beta'')\big) = F\left(\ln\frac{1+\beta'}{1-\beta'} + \ln\frac{1+\beta''}{1-\beta''}\right) = F\left(\ln\frac{1 + \dfrac{\beta'+\beta''}{1+\beta'\beta''}}{1 - \dfrac{\beta'+\beta''}{1+\beta'\beta''}}\right)$$

$$= F\left(s\left(\frac{\beta'+\beta''}{1+\beta'\beta''}\right)\right) = -\ln\left(1 - \frac{2}{p\left(\dfrac{\beta'+\beta''}{1+\beta'\beta''}\right)}\right)$$

$$\leqq -\ln\left(1 - \frac{2p(\beta') + 2p(\beta'') - 4}{p(\beta')\,p(\beta'')}\right)$$

$$= -\ln\left(1 - \frac{2}{p(\beta')}\right)\left(1 - \frac{2}{p(\beta'')}\right) = F\big(s(\beta')\big) + F\big(s(\beta'')\big).$$

Inequality (24) implies that the function $\dfrac{F(s)}{s}$ is decreasing in $s \in (0, \infty)$. In particular we get

$$\frac{F(r)}{r} \leqq \liminf_{s\to 0} \frac{F(s)}{s} = \liminf_{s\to 0} \frac{e^s+1}{e^s-1}\,\frac{e^{F(s)}-1}{2e^{F(s)}} = \liminf_{\beta\to 0} \frac{1}{\beta p(\beta)}.$$

But from (5) it follows that $\dfrac{1}{\beta p(\beta)} \leqq \dfrac{S_{p(\beta)}}{p(\beta)}$. Hence,

$$\frac{F(r)}{r} \leqq \liminf_{\beta\to 0} \frac{S_{p(\beta)}}{p(\beta)} = \liminf_{p\to\infty} \frac{\|S\|_p}{p} = a.$$

By the definition of F we immediately get

$$1 - \frac{2}{p(\beta)} = e^{-F\left(\ln\frac{1+\beta}{1-\beta}\right)} \geqq e^{-a\ln\frac{1+\beta}{1-\beta}} = \left(\frac{1-\beta}{1+\beta}\right)^a.$$

From this we obtain (22). We remark that the equality $\liminf\limits_{p\to\infty} \dfrac{\|S\|_p}{p} = 1$ conjectured previously, taken together with (22), implies $p(\beta) = 1 + 1/\beta$ ∎

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