## Extremal inequalities in Sobolev spaces and quasiconformal mappings

In dieser Arbeit wird die  $L_p$ -Norm eines zweidimensionalen Hilbert-Operators abgeschätzt. Diese Abschätzungen führen zu einer Reihe von Ungleichungen aus der Theorie der Sobolevschen Räume. In einigen Fällen werden mit Hilfe quasikonformer Abbildungen bestmögliche Abschätzungen erzielt.

В работе оценивается  $L_p$ норма двумерного оператора Гильберта. Эта задача связана с некоторыми неравенствами из теории пространств Соболева. В некоторых случаях, используя методы теории квазиконформных отображений, получаются найлучшие оценки.

We estimate the norm of a two-dimensional Hilbert operator in  $L_p$ -spaces. This problem leads to inequalities of the theory of Sobolev spaces. In certain cases, by using methods of the theory of quasiconformal mappings, we get best possible estimates.

The  $L_p$  estimations for functions of Sobolev spaces are central both to the theory of partial differential equations with discontinuous coefficients and to the theory of nonlinear differential equations. While the singular integral operators play a fundamental part in these, in practice we very often need to know the best estimations. Quasiconformal maps, especially in the two dimensional cases, may be used as a tool for attacking the problem of extremal inequalities and they suggest a way of formulating these inequalities properly. However, some problems of quasiconformal mapping theory lead to difficult questions in the theory of Sobolev spaces and partial differential equations. In this paper we discuss a few special cases of this.

Let us illustrate the general idea on an example of a non-linear system of partial differential equations in two variables which are strongly elliptic in the sense of Lavrent'ev. For future use it is convenient to introduce the complex variable  $z = x + iy$  and the complex differential operators

$$
D_z = \frac{1}{2} (D_x - iD_y), \qquad D_{\bar{z}} = \frac{1}{2} (D_x + iD_y).
$$

Then the system of Lavrent'ev reduces to one complex non-linear Beltrami equation

$$
w_{\overline{z}}=q(z,w,w_{\overline{z}})\,w_z,
$$

where the ellipticity conditions reads

$$
|q(z, w, \xi_1) \xi_1 - q(z, w, \xi_2) \xi_2| \leq \beta |\xi_1 - \xi_2|, \quad \beta < 1
$$

for  $\xi_1, \xi_2 \in \mathbb{C}$ ,  $w \in \mathbb{C}$ ,  $z \in \Omega \subset \mathbb{C}$ , (see [2]). We call  $\beta$  the *ellipticity constant* of (1). By a solution of (1) we mean a function w from the Sobolev space  $W_{2,loc}^1(\Omega)$  for which equation (1) holds almost everywhere in  $\Omega$ .

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T. Iwaniec -

 $(1)$ 

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It has been proved by B0JARSKI [1] that every solution of (1) actually belongs to  $W_{p,loc}^1(\Omega)$  for some  $p>2$ . The exponent p depends on  $\beta$  only.

Generalizing, we may say that when the coefficients of differential equations are discontinuous, it is correct to ask about the integrability exponent of derivatives of. solutions. Bojarski's result is one of the first in this direction. For further results 2 T. IWANIEC<br>
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Definition 1: A m<br>  $w \in W_{2,loc}^1(\Omega)$  and It has been proved by BOJARSKI [1] that every solution of (1) actually belongs to  $\frac{1}{p}$ ,  $loc(\Omega)$  for some  $p > 2$ . The exponent  $p$  depends on  $\beta$  only.<br>Generalizing, we may say that when the coefficients of differenti <sup>2</sup> **T. IWANIEC**<br>
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Definition 1: A mapping<br>  $w \in W$ 1 by BOJARSKI [1] that every solution of (1) actually belongs to<br>  $> 2$ . The exponent  $p$  depends on  $\beta$  only.<br>
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Generalizing, we may say that when the coefficients of differential

$$
|w_{\tilde{z}}(z)| \leq \beta |w_{z}(z)| \quad \text{almost everywhere in } \Omega.
$$

When  $w$  is homeomorphic on  $\Omega$ , then we call it  $\beta$ -quasiconformal. The number  $|w \in W_{2,loc}(2^2)$  and<br>  $|w_2(z)| \leq \beta$ <br>
When *w* is homeo<br>  $K = (1 + \beta) (1 - \text{conformal}, \text{when no})$  $K = (1 + \beta) (1 - \beta)^{-1}$  is called the *maximal dilatation* of *w*. We also call *w K*-quasi-conformal, when no confusion occurs.

We shall consider the function  $\cdot$ Any solution of (1) is  $\beta$ -quasiregular with  $\beta$  being the ellipticity constant of (1).<br>We shall consider the function<br> $p(\beta) = \sup \{p : \text{any } \beta\text{-quasiconformal mapping belongs to } W_{p,loc}(\Omega)\}.$  (3)<br>On taking the mapping  $w(z) = z |z|^{-2\beta/(1+\beta)}$  we immediat

p() = sup {p: any -quasiconformal mapping-belong to *W, <sup>1</sup> ( Q)} . ,.:(3)* 

(4)

 $(6)$ 

not exceed  $1 + 1/\beta$ . In other words

$$
p(\beta)-1\leqq \frac{1}{\beta}.
$$

In the recent work of W. GOLDSTEIN [6], he announced a proof of equality  $p(\beta)$  $t = 1 + 1/\beta$ , which was conjectured by GEHRING in [4].

The problem mentioned above is one of many which are related to the two dimen. The problem mentioned above is one of many which are related to the two dimensional Hilbert operator S. This operator is defined by a singular integral of the Calderon and Zygmund type<br>  $(Sf)(z) = -\frac{1}{\pi} \int \int \frac{f(t) d\sigma_t}{(z-t)^$ Any solution of (1) is  $\beta$ -quasiregular with  $\beta$  being the<br>
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$$
(Sf)(z) = -\frac{1}{\pi} \int \int \frac{f(t) d\sigma_t}{(z-t)^2}, \text{ for } t \in L_p(0).
$$

$$
\|S\|_p = \sup_{f \in L_p(Q)} \frac{\|Sf\|_p}{\|f\|_p}, \qquad 1 < p < \infty
$$

and its relations to the special kinds of boundary value problems for elliptic systems of partial differential equationsin 2-dimensional domains. Let us remark that one can formulate an opposite inequality for  $p(\beta)$  in terms of the norm  $||S||_p$  (see [1]): Our main inte<br>  $||S||_p$ <br>
and its relation<br>
of partial different properties<br>  $p(\beta)$ (SI)  $(z) = -\frac{1}{\pi} \int \int \frac{1}{(z - t)^2}$ , for  $t \in L_p(v)$ .<br>
Our main interest is in its  $L_p$  norm<br>  $||S||_p = \sup_{f \in L_p(Q)} \frac{||Sf||_p}{||f||_p}$ ,  $1 < p < \infty$ <br>
and its relations to the special kinds of boundary value problems for elliptic sys  $||S||_p = \sup_{f \in L_p(Q)} \frac{||Sf||_p}{||f||_p},$   $1 < p < \infty$ <br>
and its relations to the special kinds of boundary value proficial differential equations in 2-dimensional domains<br>
can formulate an opposite inequality for  $p(\beta)$  in terms of

$$
p(\beta) \ge \sup \left\{ p : ||S||_p < \frac{1}{\beta} \right\}.
$$
 (5)

Let us recall that S changes  $D_i$  into  $D_i$ , i.e.  $S(D_iw) = D_iw$ , for  $w \in W_p^1(\mathbb{C})$ . We then

$$
\|\mathcal{S}\|_p = \sup_{w \in \mathring{W}_p^{-1}(\varOmega)} \frac{\|w_z\|_p}{\|w_{\bar{z}}\|_p}.
$$

 

 $0 \lt u \lt 1$ 

For

$$
w(z) = \begin{cases} z |z|^{-2u/p} & \text{if} \quad |z| < 1 \\ 1/\overline{z} & \text{if} \quad |z| \geq 1, \end{cases}
$$

one obtains

$$
|S||_p \geqq \left[\frac{(p-1) (p-u)^p}{(p-1) u^p + (1-u) p^p}\right]^{1/p}.
$$

The right hand side tends to  $p-1$  as u approaches 1. Therefore  $||S||_p \geq p-1$ . Similarly, by considering the function  $\overline{w}$  we deduce  $||S||_p \geq 1/(p-1)$ . This proves the following:

$$
\|\mathcal{S}\|_{p} \geq \begin{cases} p-1 & \text{if } p \geq 2 \\ \frac{1}{p-1} & \text{if } 1 < p \leq 2. \end{cases}
$$
 (7)

Both (4) and (5) suggest the following statement.

Conjecture 1: For 
$$
p > 1
$$
 it holds that  $||S||_p = \max\left(p-1, \frac{1}{p-1}\right)$ .

Although we cannot evaluate  $||S||_p$  exactly, we have succeded in proving some inequalities related to the problem.

Observe that Conjecture 1 implies

$$
\left\|w_{z}-\frac{\overline{z}}{z}w_{\overline{z}}\right\|_{p}\leq\begin{cases}p\left\|w_{\overline{z}}\right\|_{p} & \text{for } p\geq 2\\ \frac{p}{p-1}\left\|w_{z}\right\|_{p} & \text{for } 1\n(8)
$$

for  $w \in C_0^{\infty}(\mathbb{C})$ .

If (8) were true then, in particular, we would get the following statement.

Conjecture 2:  $\lim_{p \to 1} (p-1) ||S||_p = \lim_{p \to \infty} \frac{||S||_p}{p} = 1.$ 

Theorem 1: The inequality (8) holds for every function of the form  $w(z) = f(zu(|z|)),$ where  $u \in C_0^1(\mathbf{R})$  and  $f(\xi)$  is an analytic function of  $C^1$  class on  $\{\xi : |\xi| \leq \sup |zu(|z|)\}$ . and  $(f')^{p/2}$  is single-valued<sup>1</sup>). Moreover, the constants p and  $p/(p-1)$  are the best possible. There is no function w such that equality occurs in (8).

Proof: Given the assumption about  $f$ , we can write the following Taylor expansion

$$
(f'(\xi))^{p/2}=\sum_{n=0}^{\infty}a_n\xi^n.
$$

After a simple calculation, (8) becomes equivalent to

$$
||u(|z|) f'(zu(|z|))||_p \leq \frac{p}{2} ||zu'(|z|) f'(zu(|z|)||_p \quad \text{if} \quad p \geq 2
$$

 $4$  and

$$
\|z^{-2}v(|z|)\,f'(\bar{z}^{-1}v(|z|))\|_p\leqq\frac{p}{2p-2}\,\|z^{-1}v'(|z|)\,f'(\bar{z}^{-1}v(|z|))\|_p\quad\text{if}\quad 1
$$

<sup>1</sup>) This assumption always holds if  $p = 2, 4, ...$  or  $f'(\xi) \neq 0$ .

## T. **TWANIEC**

 $\frac{1}{4}$ 

where  $v(r)$  denotes  $r^2u(r)$ . We notice next that the above are equivalent to

$$
\iint |u(|z|)|^p |\sum a_n z^n u^n(|z|)|^2 d\sigma_z
$$
  
\n
$$
\leq (p/2)^p \iint |z u'(|z|)|^p |\sum a_n z^n u^n(|z|)|^2 d\sigma_z \quad \text{if} \quad p \geq 2,
$$
  
\n
$$
\iint |z^{-2}v(|z|)|^p |\sum a_n \overline{z}^{-n}v^n(|z|)|^2 d\sigma_z
$$
  
\n
$$
\leq \left(\frac{p}{2p-2}\right)^p \iint |z^{-1}v'(|z|)|^p |\sum a_n \overline{z}^{-n}v^n(|z|)|^2 d\sigma_z \quad \text{if} \quad 1 < p \leq 2.
$$
  
\nare coordinates these inequalities take the form  
\n
$$
\sum |a_n|^2 \int r^{2n+1} |u(r)|^{2n+p} dr \leq \left(\frac{p}{2}\right)^p \sum |a_n|^2 \int r^{2n+p+1} |u(r)|^{2n} |u'(r)| d\sigma_z
$$

In polar coordinates these inequalities take the form

$$
\begin{aligned}\n&= (2p-2)^{r} \int_{0}^{\infty} e^{-\lambda |z|} \left( \frac{|z|}{2} \right)^{r} \left( \frac{|z|}{2} \right)^{r} \left( \frac{1}{2} \right)^{r} \\
&= 1, \\
&= 1, \\
&= 1, \\
&= 1, \\
&= 1, \\
&= 1.\n\end{aligned}
$$
\ncoordinates these inequalities take the form

\n
$$
\sum |a_n|^2 \int_{0}^{\infty} r^{2n+1} |u(r)|^{2n+p} dr \leq \left( \frac{p}{2} \right)^p \sum |a_n|^2 \int_{0}^{\infty} r^{2n+p+1} |u(r)|^{2n} |u'(r)|^p dr,
$$

and

and

• 

$$
\sum |a_n|^2 \int_{0}^{\infty} r^{2n+1} |u(r)|^{2n+p} dr \leq \left(\frac{p}{2}\right)^p \sum |a_n|^2 \int_{0}^{\infty} r^2
$$
  
and  

$$
\sum |a_n|^2 \int_{0}^{\infty} r^{1-2n-2p} |v(r)|^{2n+p} dr
$$

$$
\leq \left(\frac{p}{2p-2}\right)^p \sum |a_n|^2 \int_{0}^{\infty} r^{1-2n-p} |v(r)|^{2n} |v'(r)|^p dr.
$$
To show that we appeal to two complex versions of the  

$$
\int_{0}^{\infty} r^{2n+1} |u(r)|^{2n+p} dr \leq \left(\frac{2n+p}{2n+2}\right)^p \int_{0}^{\infty} r^{2n+p+1} |u(r)|^{2n+p} dr
$$
  
and  

$$
\int_{0}^{\infty} r^{1-2n-2p} |v(r)|^{2n+p} dr \leq \left(\frac{2n+p}{2n+2p-2}\right)^p \int_{0}^{\infty} r^{1-2n+p} dr
$$
  
Notice that

To show that we appeal to **two** complex versions of the Hardy inequality, namely

$$
\sum |a_n|^2 \int r^{2n+1} |u(r)|^{2n+p} dr \leq \left(\frac{1}{2}\right) \sum |a_n|^2 \int r^{2n+p+1} |u(r)|^{2n} |u'(r)|^p dr,
$$
\nand\n
$$
\sum |a_n|^2 \int r^{1-2n-2p} |v(r)|^{2n+p} dr
$$
\n
$$
\leq \left(\frac{p}{2p-2}\right)^p \sum |a_n|^2 \int r^{1-2n-p} |v(r)|^{2n} |v'(r)|^p dr.
$$
\nTo show that we appeal to two complex versions of the Hardy inequality, namely\n
$$
\int_0^\infty r^{2n+1} |u(r)|^{2n+p} dr \leq \left(\frac{2n+p}{2n+2}\right)^p \int_0^\infty r^{2n+p+1} |u(r)|^{2n} |u'(r)|^p dr \qquad (9)
$$
\nand\n
$$
\int_0^\infty r^{1-2n-2p} |v(r)|^{2n+p} dr \leq \left(\frac{2n+p}{2n+2p-2}\right)^p \int_0^\infty r^{1-2n-p} |v(r)|^{2n} |v'(r)|^p dr.
$$
\nNotice that\n
$$
\sup_{n\geq 0} \frac{2n+p}{2n+2} = \frac{2n+p}{2n+2} \bigg|_{n=0} = \frac{p}{2}, \text{ for } p \geq 2
$$
\nand\n
$$
\sup_{n\geq 0} \frac{2n+p}{2n+2p-2} = \frac{2n+p}{2n+2p-2} \bigg|_{n=0} = \frac{p}{2p-2}, \text{ for } 1, < p \leq 2.
$$
\nOf the two inequalities stated in (9), we verify only the first one, the second can be

$$
\int_{0}^{\infty} r^{2n+1} |u(r)|^{2n+p} dr \leq \left(\frac{2n+p}{2n+2}\right)^{p} \int_{0}^{\infty} r^{2n+p+1} |u(r)|^{2n} |u'(r)|^{p} dr
$$
\nand\n
$$
\int_{0}^{\infty} r^{1-2n-2p} |v(r)|^{2n+p} dr \leq \left(\frac{2n+p}{2n+2p-2}\right)^{p} \int_{0}^{\infty} r^{1-2n-p} |v(r)|^{2n} |v'(r)|^{p} dr.
$$
\nNotice that\n
$$
\sup_{n\geq 0} \frac{2n+p}{2n+2} = \frac{2n+p}{2n+2} \bigg|_{n=0} = \frac{p}{2}, \text{ for } p \geq 2
$$
\nand\n
$$
\sup_{n\geq 0} \frac{2n+p}{2n+2p-2} = \frac{2n+p}{2n+2p-2} \bigg|_{n=0} = \frac{p}{2p-2}, \text{ for } 1, < p \leq 2.
$$
\nOf the two inequalities stated in (9), we verify only the first one, the second c shown in a similar manner. Integrating by parts and applying the Hölder inequality we obtain\n
$$
\int_{0}^{\infty} r^{2n+1} |u(r)|^{2n+p} dr = \frac{-1}{2n+2} \int_{0}^{\infty} r^{2n+2} (|u(r)|^{2n+p})' dr'
$$

Notice that

$$
\sup_{n\geq 0}\frac{2n+p}{2n+2}=\frac{2n+p}{2n+2}\bigg|_{n=0}=\frac{p}{2},\quad \text{for}\quad p\geq 2.
$$

$$
\int_{\substack{n\geq 0}}^{\infty} \frac{2n+p}{2n+2} = \frac{2n+p}{2n+2}\bigg|_{n=0} = \frac{p}{2}, \text{ for } p \geq 2
$$
\n
$$
\sup_{n\geq 0} \frac{2n+p}{2n+2p-2} = \frac{2n+p}{2n+2p-2}\bigg|_{n=0} = \frac{p}{2p-2}, \text{ for } 1, p \leq 2.
$$

shown in a similar manner. Integrating by parts and applying the Holder inequality

$$
\sup_{n\geq 0} \frac{2n+p}{2n+2} = \frac{2n+p}{2n+2}\Big|_{n=0} = \frac{p}{2}, \text{ for } p \geq 2
$$
  
and  

$$
\sup_{n\geq 0} \frac{2n+p}{2n+2p-2} = \frac{2n+p}{2n+2p-2}\Big|_{n=0} = \frac{p}{2p-2}, \text{ for } 1, < p \leq 2.
$$
  
Of the two inequalities stated in (9), we verify only the first one, the second can be  
shown in a similar manner. Integrating by parts and applying the Hölder inequality  
we obtain  

$$
\int_{0}^{\infty} r^{2n+1} |u(r)|^{2n+p} dr = \frac{-1}{2n+2} \int_{0}^{\infty} r^{2n+2} |u(r)|^{2n+p} dr
$$

$$
= \frac{-2n-p}{2n+2} \int_{0}^{\infty} r^{2n+2} |u(r)|^{2n+p-2} \text{Re } \overline{u(r)} u'(r) dr
$$

Extremal inequalities and quasiconformal mappings 

Extremal inequalities and quasiconformal mappings  
\n
$$
\leq \frac{2n+p}{2n+2} \int_{0}^{\infty} (r^{1+(2n+1)/p} |u|^{2n/p} |u'|) (r^{2n+1-(2n+1)/p} |u|^{2n+p-(2n+p)/p}) dr
$$
\n
$$
\leq \frac{2n+p}{2n+2} \left( \int_{0}^{\infty} r^{2n+p+1} |u|^{2n} |u'|^p dr \right)^{1/p} \left( \int_{0}^{\infty} r^{2n+1} |u|^{2n+p} dr \right)^{(p-1)/p}.
$$
\nHence  
\n
$$
\left( \int_{0}^{\infty} r^{2n+1} |u(r)|^{2n+p} dr \right)^{1/p} \leq \frac{2n+p}{2n+2} \left( \int_{0}^{\infty} r^{2n+p+1} |u|^{2n} |u'|^p dr \right)^{1/p}
$$
\nas claimed.  
\nFrom the proof presented it follows that if the extremal function which eq  
\n(8) exists, then it must be of the form  $w(z) = zu(|z|)$ , because it should be such that  
\n $(f'(\xi))^{p/2} = a_0 = \text{const.}$  Thus, we are led to the following equalities

$$
\leq \frac{1}{2n+2} \left( \int_{0}^{\infty} r^{2n+p+1} |u|^{2n} |u'|^{p} dr \right) \left( \int_{0}^{\infty} r^{2n+1} |u|^{2n+p} dr \right)
$$
  

$$
\left( \int_{0}^{\infty} r^{2n+1} |u(r)|^{2n+p} dr \right)^{1/p} \leq \frac{2n+p}{2n+2} \left( \int_{0}^{\infty} r^{2n+p+1} |u|^{2n} |u'|^{p} dr \right)^{1/p}
$$
  
end.  
the proof presented it follows that if the extremal function which  
s, then it must be of the form  $w(z) = zu(|z|)$ , because it should be  
 $= a_0 = \text{const.}$  Thus, we are led to the following equalities

From the proof presented it follows that if the extremal function which equalizes (8) exists, then it must be of the form  $w(z) = zu(|z|)$ , because it should be such that

$$
\leq \frac{1}{2n+2} \left( \int_{0}^{\infty} r^{2n+p+1} |u|^{2n} |u'|^{p} dr \right) \left( \int_{0}^{\infty} r^{2n+1} |u|^{2n+p} dr \right)
$$
\n
$$
\left( \int_{0}^{\infty} r^{2n+1} |u(r)|^{2n+p} dr \right)^{1/p} \leq \frac{2n+p}{2n+2} \left( \int_{0}^{\infty} r^{2n+p+1} |u|^{2n} |u'|^{p} dr \right)^{1/p}
$$
\n
$$
\leq 0.
$$
\nthe proof presented it follows that if the extremal function which equalizes  $s$ , then it must be of the form  $w(z) = zw(|z|)$ , because it should be such that  $= a_0 = \text{const.}$ . Thus, we are led to the following equalities

\n
$$
\left( \int_{0}^{\infty} |u(r)|^{p} r dr \right)^{1/p} = \frac{p}{2} \left( \int_{0}^{\infty} |r u'(r)|^{p} r dr \right)^{1/p}, \quad \text{if } p \geq 2, \qquad (10)
$$
\n
$$
\left( \int_{0}^{\infty} |r^{-1}v(r)|^{p} r dr \right)^{1/p} = \frac{p}{2p-2} \left( \int_{0}^{\infty} |r^{-1}v'(r)|^{p} r dr \right)^{1/p}, \quad \text{if } 1 < p \leq 2.
$$
\n0

\n1

\n1

\n2

\n3

\n4

\n5

\n5

\n6

\n6

\n7

\n8

\n9

\n1

\n2

\n3

\n4

\n

and

$$
\left(\int_{0}^{\infty} |u(r)|^{p} r dr\right) = \frac{p}{2} \left(\int_{0}^{\infty} |r u'(r)|^{p} r dr\right) , \quad \text{if} \quad p \geq 2,
$$
\n
$$
\left(\int_{0}^{\infty} |r^{-1} v(r)|^{p} r dr\right)^{1/p} = \frac{p}{2p-2} \left(\int_{0}^{\infty} |r^{-1} v'(r)|^{p} r dr\right)^{1/p}, \quad \text{if} \quad 1 < p \leq 2.
$$

For simplicity we only explore the first one. Investigating again the proof of inequality (9), in the case  $n = 0$ , we easily find some necessary conditions for a function *u* which would satisfy (10). Such a, function must equalize all inequalities which appeared in the proof of (9). So, the first condition in question is duly explore the se  $n = 0$ , we explore the se  $n = 0$ , we explore the f of (9). So, the f of (9). So, the sequence of  $\frac{u(r)}{r}$ ,  $\frac{u(r)}{r}$ ,  $\frac{u(r)}{r}$ ,  $\frac{u(r)}{r}$ , able in the space For surfactor we only crystal and one necessary dominant or proof.<br>
Equality (9), in the case  $n = 0$ , we easily find some necessary conditions for a function  $u$  which would satisfy (10). Such a function must equalize all

Re 
$$
u'(r) \overline{u(r)} = -2 |u(r)| |u'(r)|
$$
.

The second one is a result of an application of the Holder inequality, giving

 $|u'(r)|^p r^{p+1} = \text{const.} \ r |u(r)|^p$ .

Both of these conditions imply a differential equation

$$
u'(r) = -c\,\frac{u(r)}{r}, \qquad c > 0
$$

which is neither solvable in the space  $C_0^1(\mathbf{R})$ , nor in the space  $W_p^1(\mathbf{R})$ . All the solu-Re  $u'(r) u(r) = -2 |u(r)| |u'(r)|$ <br> *u* d one is a result of an appl<br>  $|u'(r)|^p r^{p+1} = \text{const. } r |u(r)|^p$ <br>
these conditions imply a diff<br>  $u'(r) = -c \frac{u(r)}{r}$ ,  $c > 0$ <br>
neither solvable in the space<br>
his equation have the form<br>  $u(r) = \text{const. } r$ 

$$
u(r) = \text{const. } r^{-c}.
$$
 (11)

Thus the equality in (9) does not hold for any function *u*, which implies the same is ich is neither solvable in the space  $C_0^1(\mathbf{R})$ , nor in the space  $W_p^1(\mathbf{R})$ . All the solums of this equation have the form<br>  $u(r) = \text{const. } r^{-c}$ . (11)<br>
is the equality in (9) does not hold for any function *u*, which im

For  $u(r) u(r) =$ <br>
The second one is a res<br>  $|u'(r)|^p r^{p+1} =$ <br>
Both of these condition<br>  $u'(r) = -c \frac{u(r)}{r}$ <br>
which is neither solvab<br>
tions of this equation is<br>  $u(r) = \text{const.}$ <br>
Thus the equality in (9<br>
true for (8).<br>
The case  $c =$ grals in (10) are not convergent either at 0 or at  $\infty$ . By refining the function  $u(r) = r^{-2/p}$  near the points 0 and  $\infty$ , one can construct examples which show that the inequality (8) cannot be improved. This completes the proof of the theorem  $\blacksquare$ 

 $5'$ 

Proposition 1: Let  $R>1$ . Then we have

T. IWANIEC  
\nposition 1: Let 
$$
R > 1
$$
. Then we have  
\n
$$
\left(\int_{1}^{R} |u(r)|^{p} r dr\right)^{1/p} = A_{p} < \frac{p}{2}, \quad p \geq 2.
$$
\n
$$
u \in W_{p}(1,RI) \left(\int_{1}^{R} |r u'(r)|^{p} r dr\right)^{1/p} = A_{p} < \frac{p}{2}, \quad p \geq 2.
$$
\nof: The inequality  $A_{p} \leq p/2$  follows immediately from (10). On the  
\nthere exists an extremal function, say  $u_{0} \in W_{p}(1,R)$ , which realize  
\nnum. This statement is a consequence of the compactness of the mappi  
\n $r^{1+1/p} u'(r) \rightarrow r^{1/p} u(r)$   
\nfrom  $W_{p}(1,R)$  into  $L_{p}([1,R])$ . So, if the equality  $A_{p} = p/2$  were true  
\naction  $u_{0}$  would be extremal for (10). This contradicts the original as

**Proof:** The inequality  $A_p \leq p/2$  follows immediately from (10). On the other hand there exists an extremal function, say  $u_0 \in \mathring{W}_p^{-1}([1, R])$ , which realizes the supremum. This statement is a consequence of the compactness of the mapping

$$
r^{1+1/p}u'(r) \to r^{1/p}u(r)
$$

acting from  $\mathbf{W}_p^{-1}([1, R])$  into  $L_p([1, R])$ . So, if the equality  $A_p = p/2$  were true, then the function  $u_0$  would be extremal for (10). This contradicts the original assump**tioni**

For our further investigations it is important to extend the class  $\mathbf{W}_{p}^{-1}([1, R])$  in order to get the equality  $A_p = p/2$  for the extended class. For this purpose let us observe that inequality (9), with  $n = 0$ , and its proof remain true for functions  $u \in W_p^{-1}([1, R])$  which satisfy the boundary condition  $|u(R)| = R^{-2/p} |u(1)|$ .

*Proposition 2: Let*  $u \in W_p^1([1, R])$ *,*  $|u(R)| = R^{-2/p} |u(1)|$ *. Then it follows that* 

\n If 
$$
f(x)
$$
 is a function of  $A_p = p/2$  for the extended class of  $A_p = p/2$  for the extended class of  $A_p = p/2$  for the extended class of  $A_p$  is a function of  $A_p$  is a function of  $A_p$  is a function of  $2$ :\n Let  $u \in W_p1([1, R])$ ,  $|u(R)| = R^{-2/p} |u(R)|$  is a function of  $|u(r)|^p r \, dr$ .\n

The functions which give equality have the form  $u(r) = \text{const.} r^{-2/p}$ .

## 2. Modified Hilbert transform

In this section we define a boundary value problem for elliptic systems which arose **•** Proposition 2: Let  $u \in W_p^1([1, R])$ ,  $|u(R)| = R^{-2/p} |u(1)|$ . Then it follows that<br>  $\left(\int_1^R |u(r)|^p r dr\right)^{1/p} \leq \frac{p}{2} \left(\int_1^R |r u'(r)|^p r dr\right)^{1/p}$ .<br>
The functions which give equality have the form  $u(r) = \text{const. } r^{-2/p}$ .<br>
2. Modifie simplest.case. **•** The functions which give equality have the form  $u(r) = \text{const. } r^{-2/p}$ .<br>
2. Modified Hilbert transform<br>
In this section we define a boundary value problem for elliptic sys<br>
in the study of the estimation of the operator S in the study of the estimation of the operator S in  $L_p$  spaces. We begin with the simplest case.<br>Let  $D_R = \{z: 1 < |z| < R\}$  and  $f \in L_p(D_R)$ . Consider the problem In this section we define a boundary value problem<br>in the study of the estimation of the operator S is<br>simplest case.<br>Let  $D_R = \{z: 1 < |z| < R\}$  and  $f \in L_p(D_R)$ . Considently,<br> $\{w_i = f\}$ <br> $w(Re^{i\theta}) = R^{1-2/p}w(e^{i\theta})$ ,  $\theta \in [0, 2\pi$ 

$$
\begin{cases} w_{\overline{z}} = f \\ w(Re^{i\theta}) = R^{1-2/p}w(e^{\theta i}), \quad \theta \in [0, 2\pi] \end{cases}
$$
\n(11)

where  $p \in (1, \infty)$ , and *w* is an unknown function from  $W_p^1(D_R)$ .

*of* (11). If  $p = 2$ , the solution is unique up to a constant.

Proposition 3: *For every*  $j \in L_p(D_R)$ ,  $p \neq 2$ , there exists exactly one solution of (11). *If*  $p = 2$ , *the solution is unique up to a constant*.<br>Proof: For the proof we shall show that an analytic function with boundar Proof: For the proof we shall show that an analytic function with boundary con-<br>dition (11) must be a constant, zero when  $p \neq 2$ . Let  $\sum a_n z^n$  be its Laurent's expan*ere p (2)*<br>Propo<br>(11). *If*<br>Proof:<br>ion (11<br>m. The<br>ree, by Ease.<br>  $\begin{aligned}\n&= |z:1 < |z| < R \} \text{ and } f \in L_p(D_R). \text{ Consider the pro}\n\begin{cases}\nw_z &= f \\
w(Re^{i\theta}) &= R^{1-2/p}w(e^{i\theta}), \quad \theta \in [0, 2\pi]\n\end{cases}.\n\end{aligned}$   $\begin{aligned}\n&(1, \infty), \text{ and } w \text{ is an unknown function from } W_p^{-1}(D_R) \\
&\text{sition 3: } For every } f \in L_p(D_R), \ p = 2, \text{ there exist } p = 2, \text{ the solution is unique up to a constant.}\n\end{aligned}$ For the proo  $\{w(Re^{i\theta}) = R^{1-2/p}w(e^{\theta t}), \quad \theta \in [0, 2\pi] \}$ <br>  $\in (1, \infty)$ , and w is an unknown function from  $W_p^1(D_R)$ .<br>
ssition 3: For every  $f \in L_p(D_R)$ ,  $p \neq 2$ , there exists exactly one so<br>  $\{p=2$ , the solution is unique up to a const

$$
\sum a_n R^{n} e^{in\theta} = \sum a_n R^{1-2/p} e^{in\theta}
$$

- Hence, by comparing the coefficients we get

1) must be a constant, zero when 
$$
p \neq 2
$$
. Let  $\sum a_n$   
the boundary condition becomes  

$$
\sum a_n R^n e^{in\theta} = \sum a_n R^{1-2/p} e^{in\theta}.
$$
by comparing the coefficients we get  

$$
a_n = 0 \text{ for } n = 0, \pm 1, \pm 2, ... \text{ if } p \neq 2,
$$

$$
a_n = 0 \text{ for } n = \pm 1, \pm 2, \pm 3, ... \text{ if } p = 2.
$$

 

Extremal inequalities and quasiconformal mappings

Exterminal inequalities and quasiconformal mappings

\n7

\nThe existence follows from the explicit formula

\n
$$
w(z) = (T_R f)(z) = \frac{-1}{\pi} \iint_{D_R} \left[ \frac{1}{t-z} + K(z, t) \right] f(t) d\sigma_t,
$$
\nwhere the kernel  $K(z, t)$  is given by\n
$$
K(z, t) = t^{-1} \sum_{n=0}^{\infty} \frac{z^{n}t^{-n}}{R^{n+2/p-1} - 1} + t^{-1} \sum_{-\infty}^{-1} \frac{z^{n}t^{-n}}{1 - R^{1-n-2/p}}.
$$
\nBoth series converge uniformly on compact subsets of  $D_R \times D_R$ , therefore  $K(z, t)$  is analytic in  $D_R \times D_R$ . We shall show that  $w(z)$ , as defined by (12), is in  $W_p^1(D_R)$ , and that it satisfies the boundary condition (11). First, we assume that  $\text{supp}(f) \subset D_R$ .

e kernel 
$$
K(z, t)
$$
 is given by  
\n
$$
K(z, t) = t^{-1} \sum_{n=0}^{\infty} \frac{z^{n}t^{-n}}{R^{n+2/p-1}-1} + t^{-1} \sum_{-\infty}^{-1} \frac{z^{n}t^{-n}}{1 - R^{1-n-2/p}}.
$$

Both series converge uniformly on compact subsets of  $D_R \times D_R$ , therefore  $K(z, t)$ is analytic in  $D_R \times D_R$ . We shall show that  $w(z)$ , as defined by (12), is in  $W_p^{-1}(D_R)$ and that it satisfies the boundary condition (11). First we assume that  $\text{supp}(f) \subset D_R$ . Hence we obviously have  $w \in W_p(1(D_R))$ , and  $w_{\bar{i}} = f$ , because the kernel  $K(z, t)$  is, analytic and  $-1/\pi(t-z)$  is a fundamental solution of the Cauchy-Riemann equation. Moreover

e Kernel 
$$
K(z, t)
$$
 is given by  
\n
$$
K(z, t) = t^{-1} \sum_{n=0}^{\infty} \frac{z^n t^{-n}}{R^{n+2/p-1} - 1} + t^{-1} \sum_{-\infty}^{-1} \frac{z^n t^{-n}}{1 - R^{1-n-2/p}}
$$
\nies converge uniformly on compact subsets of  $D_R \times D_R$ , therefore  $K(z, t)$  is in  $D_R \times D_R$ . We shall show that  $w(z)$ , as defined by (12), is in  $W_p(1)$  it satisfies the boundary condition (11). First we assume that  $\sup_p(f) \subset D$  be obviously have  $w \in W_p(1)R$ , and  $w_z = f$ , because the kernel  $K(z, t)$  and  $-1/\pi(t - z)$  is a fundamental solution of the Cauchy-Riemann equoover\n
$$
w(Re^{i\theta}) = \frac{-1}{\pi} \int_{D_R} \int \left[ (t - Re^{i\theta})^{-1} + K(Re^{i\theta}, t) \right] f(t) d\sigma_t
$$
\n
$$
= \frac{-R^{1-2/p}}{\pi} \int_{D_R} \int \left[ (t - e^{i\theta})^{-1} + K(e^{i\theta}, t) \right] f(t) d\sigma_t = R^{1-2/p} w(e^{i\theta}).
$$
\nequality is a consequence of the following property of the kernel  $K(z, t)$ 

$$
= \frac{-R^{1-2/p}}{\pi} \int_{D_R} \left[ (t - e^{i\theta})^{-1} + K(e^{i\theta}, t) \right] f(t) d\sigma_t = R^{1-2/p} w(e^{i\theta}).
$$
  
The last equality is a consequence of the following property of the Kernel  $K(z, t)$ :  

$$
R^{1-2/p} K(e^{i\theta}, t) - K(Re^{i\theta}, t)
$$

$$
= \frac{R^{1-2/p}}{t} \left[ \sum_{n=0}^{\infty} \frac{t^{-n} e^{in\theta}}{R^{n+2/p-1} - 1} + \sum_{-\infty}^{-1} \frac{t^{-n} e^{in\theta}}{1 - R^{1-n-2/p}} \right]
$$

$$
= \frac{1}{t} \left[ \sum_{n=0}^{\infty} \frac{R^{n} t^{-n} e^{in\theta}}{R^{n+2/p-1} - 1} + \sum_{-\infty}^{-1} \frac{R^{n} t^{-n} e^{in\theta}}{1 - R^{1-n-2/p}} \right]
$$

$$
= \frac{-1}{t} \sum_{n=0}^{\infty} \frac{(R^n - R^{1-2/p}) t^{-n} e^{in\theta}}{R^{n+2/p-1} - 1} - \frac{1}{t} \sum_{-\infty}^{-1} \frac{(R^n - R^{1-2/p}) t^{-n} e^{in\theta}}{1 - R^{1-n-2/p}}
$$

$$
= \frac{-R^{1-2/p}}{t - e^{i\theta}} - \frac{1}{t - Re^{i\theta}}.
$$
Therefore  $w(z)$  is the solution of the problem if  $\text{supp}(f) \subset D_R$ . Now we define the following modified Hilbert transform  
 $(S_n f)(z) = (T_n f)(z) = \frac{-1}{t} \int_{-\infty}^{t} f((t - z)^{-2} + K(z, t)) f(t) d\sigma_t$ .

Therefore  $w(z)$  is the solution of the problem if supp  $(f) \subset D_R$ . Now we define the

$$
= \frac{1}{t - e^{i\theta}} - \frac{1}{t - Re^{i\theta}}.
$$
  
re  $w(z)$  is the solution of the problem if  $\text{supp}(f) \subset D_R$ . Now we define the  
lg modified Hilbert transform  

$$
(S_R f)(z) = (T_R f)_z (z) = \frac{-1}{\pi} \iint_{D_R} [(t - z)^{-2} + K_z(z, t)] f(t) d\sigma_t.
$$
  
lines to prove that  $S_R$  is a bounded operator in  $L_p(D_R)$ .  
orem 2: We have  

$$
\sup_{f \in L_p(D_R)} \frac{||S_R f||_p}{||f||_p} = ||S||_p
$$
 (1)

It remains to prove that  $S_R$  is a bounded operator in  $L_p(D_R)$ .

Theorem 2: We have

$$
\sup_{f \in L_p(D_R)} \frac{\|S_R f\|_p}{\|f\|_p} = \|S\|_p
$$

(13)

and

$$
\inf_{f \in L_p(D_R)} \frac{\|S_R f\|_p}{\|f\|_p} = \frac{1}{\|S\|_p}.
$$

Proof: Letting  $M_p{}^1(D_R) = \{w; w \in W_p{}^1(D_R), w(Re^{i\theta}) = R^{1-2/p}w(e^{i\theta})\}$  gives

$$
||S||_p = \sup_{w \in W_p \cap (\mathcal{C})} \frac{||w_z||_p}{||w_z||_p} = \sup_{w \in \mathcal{W}_p \cap (D_R)} \frac{||w_z||_p}{||w_z||_p} \leq \sup_{w \in M_p \cap (D_R)} \frac{||w_z||_p}{||w_z||_p} = \sup_{f \in L_p(D_R)} \frac{||S_Rf||_p}{||f||_p}
$$

 $(14)$ 

and

$$
\frac{1}{\|S\|_p} = \inf_{w \in W_p^{-1}(C)} \frac{\|w_z\|_p}{\|w_z\|_p} = \inf_{w \in \hat{W}_p^{-1}(D_R)} \frac{\|w_z\|_p}{\|w_z\|_p} \ge \inf_{M_p^{-1}(D_R)} \frac{\|w_z\|_p}{\|w_z\|_p} = \inf_{f \in L_p(D_R)} \frac{\|S_R f\|_p}{\|f\|_p}
$$

Now we shall prove the opposite inequalities. For every positive integer  $n$  we construct an extension of any  $w \in M_p(1/p_R)$  as follows:

$$
\tilde{v}(z) = \begin{cases}\n0 & \text{for } |z| \leq 1, \\
\frac{|z| - 1}{R - 1} w(z) & \text{for } 1 \leq |z| \leq R, \\
R^{j(1-2/p)}w(zR^{-j}) & \text{for } R^j \leq |z| \leq R^{j+1}, \quad j = 1, 2, ..., n, \\
\frac{R^{n+2} - |z|}{R^{n+2} - R^{n+1}} w(zR^{-n-1}) & \text{for } R^{n+1} \leq |z| \leq R^{n+2}, \\
0 & \text{for } R^{n+2} \leq |z|,\n\end{cases}
$$

From the definition of  $M_p(1D_R)$ , it follows that  $w \in W_p(1(C))$ . Moreover

$$
\tilde{w}_t(z) = \begin{cases}\n0 & \text{for} \quad |z| \leq 1, \\
\frac{|z| - 1}{R - 1} w_t(z) + \frac{\overline{z}w(z)}{z |z| (R - 1)} & \text{for} \quad 1 < |z| \leq R, \\
\frac{|R^{-2j/p}w_t(R^{-j}z)}{B^{-2j/p}w_t(R^{-j}z)} & \text{for} \quad R^j < |z| \leq R^{j+1}, \quad j = 1, 2, ..., n \\
\frac{(R^{n+2} - |z|) w_t(R^{-n-1}z) - |z| z^{-2} R^{n+1} w(R^{-n-1}z)}{(R - 1) R^{(1+2/p)(n+1)}} & \text{for} \quad R^{n+1} < |z| \leq R^{n+2}, \\
0 & \text{for} \quad R^{n+2} < |z|.\n\end{cases}
$$

We have to estimate the integral  $\int \int |\tilde{w}_t(z)|^p d\sigma_z$ . Therefore we first transform it into an integral over the ring  $D_R$ , and then we decompose this integral into a finite sumof integrals over the rings in which  $\tilde{w}(z)$  is defined. By using a natural substitution we reduce each of these integrals to an integral over  $D_R$ . Accordingly we write

$$
\iiint_{D_R} |\tilde{w}_z(z)|^p d\sigma_z
$$
\n
$$
= n \iint_{D_R} |w_z(z)|^p d\sigma_z + \iint_{D_R} \left| \frac{|z| - 1}{R - 1} w_z + \frac{\overline{z}w}{|z| |(R - 1)} \right|^p d\sigma_z
$$
\n
$$
+ \iint_{D_R} \left| \frac{R - |z|}{R - 1} w_z - \frac{\overline{z}w}{|z| |(R - 1)} \right|^p d\sigma_z
$$
\n
$$
= n \iint_{D_R} |w_z(z)|^p d\sigma_z + \mathcal{O}(1).
$$

8

# Extremal inequalities and quasiconformal mappings<br>
at the set of the

Similarly we can prove that

$$
\int \int |\tilde{w}_{\tilde{z}}(z)|^p\,d\sigma_z=n\int\limits_{D_R} |w_{\tilde{z}}(z)|^p\,d\sigma_z+\mathcal{O}(1).
$$

Letting *n* approach infinity we get

*ff jW(Z) <sup>P</sup> da*  lint IüJ(z)I *da. + (--) DR If* Iw(z)' *da2 - n\_.00ff I(z <sup>P</sup> do + 0 (.1)*  • sup = *IJSIJP, wEJV'(C) IIWp*  • inf JJw IJ P 1 *• - ;* = *• WEW9'(C)* JW: Il<sup>p</sup> ' 118I1p' *JELP(DR) fELp(DR)* LfI

Since  $w$  was chosen arbitrarily, we can write

$$
\left| \geq \inf_{w \in W_p^{-1}(\mathbb{C})} \frac{\sup_{\|w_z\|_p p}{\|w_z\|_p p} = \frac{1}{\|S\|_p p}}{\|S\|_p p}.
$$
  
was chosen arbitrarily, we can write  

$$
\sup_{\{L_p(D_R)\}} \frac{\|S_R f\|_p}{\|f\|_p} \leq \|\mathcal{S}\|_p \text{ and } \inf_{f \in L_p(D_R)} \frac{\|\mathcal{S}_R f\|_p}{\|f\|_p} \geq \frac{1}{\|\mathcal{S}\|_p}.
$$

The proof of the theorem is complete

Our theorem implies that  $||S_R||_p = ||S||_p$ . There are many reasons why it has a practical value. It is much easier to study the modified Hubert transform than the original one. For example,  $\frac{2}{5} S_R$  has a very simple spectral decomposition.

Theorem 3: The operator  $\frac{z}{\bar{z}} S_R$  has a point spectrum only. The numbers

$$
\sup_{f\in L_p(D_R)} \frac{||S_Rf||_p}{||f||_p} \leq ||S||_p \text{ and } \inf_{f\in L_p(D_R)} \frac{||S_Rf||_p}{||f||_p} \geq \frac{1}{||S||_p}.
$$
  
of of the theorem is complete  
theorem implies that  $||S_R||_p = ||S||_p$ . There are many reasons' why it has a  
value. It is much easier to study the modified Hilbert transform than the  
one. For example,  $\frac{z}{z} S_R$  has a very simple spectral decomposition.  
rem 3: The operator  $\frac{z}{z} S_R$  has a point spectrum only. The numbers  

$$
\beta_{m,n} = \frac{1 - \frac{2}{p} + m + \frac{2\pi in}{\ln R}}{1 - \frac{2}{p} - m + \frac{2\pi in}{\ln R}} \quad (m, n = 0, \pm 1, \pm 2, ...)
$$
 (15)  
genvalues and the corresponding eigenfunctions  

$$
f_{m,n} = z^{m+1}(|z|)^{\frac{-2}{p} - m + \frac{2\pi in}{\ln R} - 1} \quad (m, n = 0, \pm 1, \pm 2, ...)
$$
 (16)  
mplete system in  $L_p(D_R)$ ,  $p \geq 2$ .  
We have to find the solutions of the boundary value problem  
 $zw_z(z) = \beta \bar{z}w_z(z)$ ,  $1 \leq |z| \leq R$ ,  
 $w(Re^{i\theta}) = R^{1-2/p}w(e^{i\theta})$ ,  $0 \leq \theta \leq 2\pi$ .  

$$
w = \frac{1}{\sqrt{2\pi \epsilon}} \left( \frac{1}{\sqrt{2\pi \epsilon}} \right)
$$

*are its eigenvalues and the corresponding eigen/unctions* 

$$
f_{m,n} = z^{m+1}(|z|)^{\frac{-2}{p} - m + \frac{2\pi \ln n}{\ln R} - 1} \quad (m, n = 0, \pm 1, \pm 2, \ldots)
$$
 (16)

*form a complete system in*  $L_p(D_R)$ ,  $p \geq 2$ .

Proof: We have to find the solutions of the boundary value problem

complete system in 
$$
L_p(D_R)
$$
,  $p \geq 2$ .

\nf: We have to find the solutions of the boundary value problem

\n $zw_z(z) = \beta \bar{z}w_z(z)$ ,  $1 < |z| < R$ ,

\n $w(Re^{i\theta}) = R^{1-2/p}w(e^{i\theta})$ ,  $0 \leq \theta \leq 2\pi$ .

\n(17)

It is convenient to work with polar coordinates  $(r, \theta)$ :

$$
\beta_{m,n} = \frac{P}{1 - \frac{2}{p} - m + \frac{2\pi in}{\ln R}} \quad (m, n = 0, \pm 1, \pm 2, ...)
$$
(15)  
\nare its eigenvalues and the corresponding eigenfunctions  
\n
$$
f_{m,n} = z^{m+1}(|z|)^{\frac{-2}{p} - m + \frac{2\pi in}{\ln R} - 1} \quad (m, n = 0, \pm 1, \pm 2, ...)
$$
(16)  
\nform a complete system in  $L_p(D_R)$ ,  $p \ge 2$ .  
\nProof: We have to find the solutions of the boundary value problem  
\n $zw_z(z) = \beta \bar{z}w_z(z), \qquad 1 < |z| < R$ ,  
\n $w(Re^{i\theta}) = R^{1-2/p}w(e^{i\theta}), \qquad 0 \le \theta \le 2\pi$ .  
\nIt is convenient to work with polar coordinates  $(r, \theta)$ :  
\n
$$
\begin{pmatrix} w_r - \frac{i}{r} & w_\theta \end{pmatrix} = \beta \left( w_r + \frac{i}{r} w_\theta \right),
$$
\n $w(R, \theta) = R^{1-2/p}w(1, \theta).$ \n(18)  
\n $w(R, \theta) = R^{1-2/p}w(1, \theta).$ \n(18)  
\nand  $(R - 1)$ -periodic in r. Therefore we can expand  $w(r, \theta)$  into the Fourier series

The boundary condition implies that the function  $r^{-1+2/p}w(r, \theta)$  is  $2\pi$ -periodic in  $\theta$ and  $(R - 1)$ -periodic in *r*. Therefore we can expand  $w(r, \theta)$  into the Fourier series

$$
w(r, \theta) = \sum \sum a_{mn} w_{mn}(r, \theta),
$$

where.

 

S<br>S<br>S

T. IWANIRO  
\n
$$
w_{mn}(r, \theta) = r^{-\frac{2}{p} + 1 + \frac{2\pi in}{\ln R}} e^{im\theta} = z^m |z|^{1 - \frac{2}{p} - m + \frac{2\pi in}{\ln R}}.
$$
\ng this into (18) we get

Inserting this into (18) we get

10. T. IWANIEO  
\nwhere  
\n
$$
w_{mn}(r, \theta) = r^{-\frac{2}{p}+1+\frac{2\pi in}{\ln R}} e^{im\theta} = z^m |z|^{1-\frac{2}{p}-m+\frac{2\pi in}{\ln R}}.
$$
\nInserting this into (18) we get  
\n
$$
\sum_{m,n} a_{mn} \left(1 - \frac{2}{p} + m + \frac{2\pi in}{\ln R}\right) r^{\frac{2\pi in}{\ln R} - \frac{2}{p}} e^{im\theta}
$$
\n
$$
= \beta \sum_{m,n} a_{mn} \left(1 - \frac{2}{p} - m + \frac{2\pi in}{\ln R}\right) r^{\frac{2\pi in}{\ln R} - \frac{2}{p}} e^{im\theta}.
$$
\nBy comparing the coefficients of these series we immediately obtain (15) and (16) with  $f_{m,n} = (w_{mn})z$  -const.

with  $f_{m,n} = (w_{mn})_{\bar{s}}$  const.  $\blacksquare$ <br>The eigenvalue of  $\frac{z}{z} S_R$  with the greatest modul is  $\beta_{1,0} = 1 - p$ , which corresponds to  $w_{1,0}(z) = z |z|^{-2/p}$ , a quasiformal solution of (17). That with the smallest modul is  $w_{0}(z) = z |z|^{-2/p}$ , a quasiformal solution of (17). That with the smallest modul is<br>=  $(1 - p)^{-1}$  which corresponds to  $w_{-1,0}(z) = w_{1,0}(z) = \overline{z} |z|^{-2/p}$ . We do not By comparing the coefficients of these series we immediately obtain (15) and (16)<br>with  $f_{m,n} = (w_{mn})_{\bar{z}}$  const.  $\blacksquare$ <br>The eigenvalue of  $\frac{z}{\bar{z}} S_R$  with the greatest modul is  $\beta_{1,0} = 1 - p$ , which correspond<br>to  $w_{1$ know whether  $|\beta_{1,0}| = p - 1$  is the norm of  $S_R$  in  $L_p(D_R)$ , but there is strong evidence for expecting it. Namely, if one writes the Euler equations  $\sum_{m,n} a_{mn} \left(1 - \frac{2}{p} + m\right) + \frac{2\pi i n}{\ln R} \int_{R}^{\frac{2\pi i n}{10R} - \frac{2}{p}} e^{im\theta}$ <br>  $= \beta \sum_{m,n} a_{mn} \left(1 - \frac{2}{p} - m + \frac{2\pi i n}{\ln R}\right) + \frac{2\pi i n}{\ln R} - \frac{2}{p} e^{im\theta}$ <br>
aring the coefficients of these series we immediately ob<br>  $= (w_{mn})_z$ The eigenvalue of  $\frac{z}{z} S_R$  with the greatest modul is  $\beta_{1,0} = 1 - p$ , which corresponds<br>to  $w_{1,0}(z) = z |z|^{-2/p}$ , a quasiformal solution of (17). That with the smallest modul<br> $\beta_{-1,0} = (1 - p)^{-1}$  which corresponds to  $w_{-$ 

$$
\begin{cases}\n(\vert w_z\vert^{p-2} \overline{w}_z)_z = \beta^p (\vert w_{\overline{z}}\vert^{p-2} \overline{w}_{\overline{z}})_z \\
w(Re^{i\theta}) = R^{1-2/p} w(e^{i\theta}),\n\end{cases}
$$

for the variational problem'

$$
\beta = \sup_{w \in M_p^{-1}(D_R)} \frac{\|w_z\|_p}{\|w_{\bar{z}}\|_p},
$$

This then it turns out that the  $w_{m,n}$  are the solutions corresponding to  $\beta = |\beta_{m,n}|$ . This again suggests Conjecture 1 and provides another reason to study the following  $\int w(Re^{i\theta})$ <br>for the variationa<br> $\beta = \sup_{w \in M_p^{-i}}$ <br>then it turns out<br>again suggests Corpoblem.<br>*Generalized eige .*  $\beta = \sup_{w \in M_p^{-1}(D_R)} \frac{||w_x||_p}{||w_x||_p}$ ,<br>  $\beta = \sup_{w \in M_p^{-1}(D_R)} \frac{||w_x||_p}{||w_x||_p}$ ,<br>
en it turns out that the  $w_{m,n}$  are the solutions corresponding to  $\beta = |\beta|$ <br>
ain suggests Conjecture 1 and provides another reason to stud arms out that the  $w_{m,n}$  are the solutions corresponding to  $\beta = |\beta_{m,n}|$ . This ggests Conjecture 1 and provides another reason to study the following lized eigenvalues problem: Let  $p > 2$ . Find  $w \in W_p^{-1}(D_R)$  such that  $\begin$  $w(Re^{i\theta}) = R^{1-2/p}w(e^{i\theta}),$ <br>
iational problem<br>  $= \sup_{w \in M_p^{-1}(D_R)} \frac{||w_z||_p}{||w_z||_p},$ <br>
This out that the  $w_{m,n}$  are the soless Conjecture 1 and provide<br>  $\begin{aligned} &\text{and }~\text{by} &\text{and }~\text{by} &\text{and }~\text{by} &\text{and} &\text{and} &\text{by} \ &\text{and }~\text{by}$  $\begin{aligned} \n\left\{ \begin{array}{l} w(Re^w) = R^{1-2/p}w(e^{i\theta}), \end{array} \right. \ \ \beta & = \sup_{w\in M_p^{-1}(D_R)} \frac{\|w_\varepsilon\|_p}{\|w_\varepsilon\|_p}, \ \text{then it turns out that the } w_{m,n} \text{ are the solutions corresponding to } \beta = |\beta_m \text{ again suggests Conjecture 1 and provides another reason to study the from from the image.} \end{aligned} \end{aligned}$ Ionis out that<br>
Integests Conject<br>
Integral consider the  $\begin{cases} w_z(z) = \beta(z) \\ w(Re^{i\theta}) = R^i \end{cases}$ <br>
(b) is a measure<br>  $|\beta(z)| \ge \frac{1}{\beta}$  > that this provider hand we

$$
\begin{cases} w_z(z) = \beta(z) w_{\overline{z}}(z), & 1 < |z| < R, \\ w(Re^{i\theta}) = R^{1-2/p} w(e^{i\theta}) \end{cases}
$$

where  $\beta(z)$  is a measurable function which satisfies the ellipticity condition

$$
|\beta(z)| \geqq \frac{1}{\beta} > 1 \quad \text{or} \quad |\beta(z)| \leqq \beta < 1.
$$

It is clear that this problem admits only the trivial solution if  $\beta ||S||_p = \beta ||S_R||_p < 1.$ On the other hand we shall prove the following statement.. problem.<br> *Generalized eigenvalues problem:* Let  $p > 2$ .<br>  $\begin{cases} w_z(z) = \beta(z) w_{\overline{z}}(z), & 1 < |z| < R, \\ w(Re^{i\theta}) = R^{1-2/p}w(e^{i\theta}) \end{cases}$ <br>
where  $\beta(z)$  is a measurable function which sat<br>  $|\beta(z)| \geq \frac{1}{\beta} > 1$  or  $|\beta(z)| \leq \beta < 1$ <br>
It is

*Theorem 4: A necessary condition for the existence of a non-trivial solution. of (19)* 

In the proof we use the following:

Definition *2:* By a *ring D* we mean a domain the boundary of which consists of two Jordan curves  $\mathscr{C}_1$ ,  $\mathscr{C}_2$ . The *modulus M(D)* of the ring *D* is

$$
M(D) = \inf_{q} \iint_{\mathbf{C}} q^2(z) d\sigma_z
$$

where the infimum is taken over all Borel measurable functions  $q(z) \geq 0$  such that

e infimum is tak  

$$
\int_{l} q(z) |dz| \geq 1
$$

for every locally rectifiable' curve  $l \subset D$  which joins the boundary components  $\mathscr{C}_1$  and  $\mathscr{C}_2$ . where  $\frac{1}{2}$ <br>**for every** and  $\mathscr{C}_2$ .<br>**For t** Extremal inequalities and quasiconformal mappings<br>where the infimum is taken over all Borel measurable functions  $q(z) \ge 0$  such the  $\int_{l} q(z) |dz| \ge 1$ .<br>for every locally rectifiable curve  $l \subset D$  which joins the boundary co *• '2={az:z€'1), a>1.*  for every locally rectifiable<br>
and  $\mathscr{C}_2$ .<br> *For the spherical ring D<sub>i</sub>*<br> *Lemma 1: Let G be a ri*<br>  $\mathscr{C}_2 = \{az : z \in \mathscr{C}_1\}$ ,<br> *Then*  $M(G) \ge \frac{2\pi}{\ln a}$ .<br>
Proof: Let *g* be an arbi

For the spherical ring  $D_R = \{z: 1 < |z| < R\}$  we have  $M(D_R) = \frac{2\pi}{\ln R}$ 

$$
\mathscr{C}_2 = \{az : z \in \mathscr{C}_1\}, \qquad a > 1.
$$

 $\int q(z) |dz| \geq 1$ <br>  $\int q(z) |dz| \geq 1$ <br>  $\int \text{locally rectifiable}$ <br>  $\Rightarrow$  spherical ring  $D_R$ <br>  $\Rightarrow$  a 1: Let G be a ring<br>  $\mathscr{C}_2 = \{az : z \in \mathscr{C}_1\},$ <br>  $\Rightarrow$   $\sum_{n=1}^{\infty} \frac{2\pi}{\ln a}$ <br>  $\therefore$  Let q be an arbitrical point  $\int_a^b t_a = \{z : z = re^{i\theta}, n\}$ Proof: Let *q* be an arbitrary admissible function. For every angle  $\theta \in [0, 2\pi]$  we denote by  $l_{\theta} = \{z: z = re^{i\theta}, r_1 \le r \le r_2\}$  the radial segment contained in G which joins the boundary components  $\mathscr{C}_1$  and  $\mathscr{C}_2$ . Lemma 1: Let G be a ring whose bounda<br>  $\mathscr{C}_2 = \{az : z \in \mathscr{C}_1\}, \quad a > 1.$ <br>
Then  $M(G) \ge \frac{2\pi}{\ln a}$ .<br>
Proof: Let q be an arbitrary admissible<br>
denote by  $l_{\theta} = \{z : z = re^{i\theta}, r_1 \le r \le r_2\}$  the joins the boundary components  $\$ 



Abb.1.

Bob. 1

\nSo, for every 
$$
\theta \in [0, 2\pi]
$$
 we have

\n
$$
1 \leq \int_{t_{\theta}} q(z) |dz| \leq \int_{0}^{\infty} q(re^{i\theta}) dr.
$$
\nHermiting over all  $\theta$  and applying the

Integrating over all  $\theta$  and applying the Hölder inequality one can obtain

$$
2\pi \leq \int_{0}^{2\pi} \int_{0}^{\infty} q(re^{i\theta}) dr d\theta \leq \left(\int_{0}^{2\pi} \int_{0}^{\infty} q^2(re^{i\theta}) r dr d\theta\right)^{1/2} \left(\int_{0}^{2\pi} \int_{0}^{\infty} \frac{\chi_{G}(re^{i\theta})}{r} dr d\theta\right)^{1/2}
$$

where  $\chi_G$  is the characteristic function of G. The integral  $\int_{0}^{\infty} \frac{\chi_G(re^{i\theta})}{r} dr$  does not depend on the angle  $\theta$  and is equal to  $\ln a$  (we omit the proof of this). Hence we get Integrating over all  $\theta$  and applying the Hölder inequality one can obtain.<br>
Integrating over all  $\theta$  and applying the Hölder inequality one can obtain.<br>  $2\pi \leq \int_0^{2\pi} \int_0^{\infty} q(re^{i\theta}) dr d\theta \leq \left(\int_0^{2\pi} \int_0^{\infty} q^2(re^{i$  $2\pi \leq \int_{0}^{2\pi} \int_{0}^{\infty} q(re^{i\theta}) dr d\theta \leq \left(\int_{0}^{2\pi} \int_{0}^{\infty} q^2(re^{i\theta}) dr\right)$ <br>is the characteristic function of *G*. Then the angle  $\theta$  and is equal to ln *a* (we om  $2\pi \leq (\int \int q^2(z) d\sigma_z)^{1/2} (2\pi \ln a)^{1/2}$ , ly  $M(G) = \inf_{q$ 

$$
M(G) = \inf_{q} \int \int q^2(z) \, d\sigma_z \geq \frac{2\pi}{\ln a}
$$

T. IWANIEC<br>
Proof of Theorem 4: For simplicity we examine the case  $|\beta(z)| \geq \frac{1}{\beta} > 1$  only.<br>
Pery non-trivial solution of (19) is a  $\beta$ -quasiregular map. Suppose that w is  $\beta$ -quasi-<br>
nformal. It transforms the spheri Every non-trivial solution of (19) is a  $\beta$ -quasiregular map. Suppose that *w* is  $\beta$ -quasiconformal. It transforms the spherical ring  $D_R$  onto a domain G which satisfies the assumptions of Lemma 1 with  $a = R^{1-2/p}$ . It is a fundamental result of the theory of quasiconformal mappings that **T. IWANIEC**<br>
of Theorem 4: For simplicity we examine the in-trivial solution of (19) is a  $\beta$ -quasiregular ms<br> *M*. It transforms the spherical ring  $D_R$  onto a<br>
ons of Lemma 1 with  $a = R^{1-2/p}$ . It is a funconformal map 12 T. IWANIEC<br>
Proof of Theorem 4: For simplicity w<br>
Every non-trivial solution of (19) is a  $\beta$ -qua<br>
conformal. It transforms the spherical ring<br>
assumptions of Lemma 1 with  $a = R^{1-2/p}$ .<br>
of quasiconformal mappings that roof of Theorem 4: For<br>
ry non-trivial solution of (1<br>
ormal. It transforms the sp<br>
mptions of Lemma 1 with<br>
uasiconformal mappings th<br>  $M(G) \leq KM(D_R) = \frac{1}{1}$ <br>
ce, we have<br>  $\frac{2\pi}{\ln R^{1-2/p}} \leq \frac{1+\beta}{1-\beta} \frac{2}{\ln R}$ <br>
hat

onformal mappings that

\n
$$
M(G) \leq KM(D_R) = \frac{1+\beta}{1-\beta} M(D_R).
$$
\ne have

\n
$$
\frac{2\pi}{\ln R^{1-2/p}} \leq \frac{1+\beta}{1-\beta} \frac{2\pi}{\ln R}.
$$

*(20)* 

$$
\frac{2\pi}{\ln R^{1-2/p}} \leqq \frac{1+\beta}{1-\beta} \frac{2\pi}{\ln R},
$$

so that

$$
\beta\geq \frac{1}{p-1}.
$$

Hence, we have<br>  $\frac{2\pi}{\ln R^{1-2/p}} \leq \frac{1+\beta}{1-\beta} \frac{2\pi}{\ln R}$ ,<br>
so that<br>  $\beta \geq \frac{1}{p-1}$ .<br>
The inequality (20) remains true, in its proper formulation, for any quasiregular mapping (see [7]). This permits us to prove the theorem completely  $\blacksquare$ 

As a result of this theorem we obtain, in particular, a solution of the following extremal problem:

*Problem:* In the class of functions  $w \in M_p^{-1}(D_R)$  find a quasiconformal mapping with minimal dilatation.

tremal problem:<br> *Problem*: In the class of functions  $w \in M_p^{-1}(D_R)$ <br>
th minimal dilatation.<br>
Actually, the minimal dilatation  $\beta$  is at least eular,<br>
find<br>  $\frac{1}{p-1}$ <br>
mini  $\frac{1}{1}$  (because of Theorem 4). On the other hand, the map  $w(z) = z(|z|)^{-2/p}$  has the minimal dilatation  $\beta = 1/(p-1)$ ,  $\beta(z) = \frac{w_{\overline{z}}}{w_z} = \frac{-1}{p-1} \frac{z^2}{|z|^2}$ . this checked we<br>
n:<br>
he class of funct<br>
atation,<br>
minimal dilatatie<br>
d, the map  $w(z) =$ <br>  $\frac{w_z}{w_z} = \frac{-1}{p-1} \frac{z^2}{|z|^2}$ mai dilatation.<br>
y, the minimal dilatation  $\beta$  is at least  $\frac{1}{p-1}$  (because of Theorem 4).<br>
her hand, the map  $w(z) = z(|z|)^{-2/p}$  has the minimal dilatation  $\beta = 1/(p-1)$ ,<br>  $g(z) = \frac{w_z}{w_z} = \frac{-1}{p-1} \frac{z^2}{|z|^2}$ .<br>
is extre

$$
\beta(z) = \frac{w_{\bar{z}}}{w_z} = \frac{-1}{p-1} \frac{z^2}{|z|^2}.
$$

Thus  $w(z)$  is extremal; it is Teichmüller's quasiconformal map.

## **3. Asymptotic behaviour of**  $\|\mathbf{S}\|_p$

The operator S is of the type  $(1,1)$ ; i.e.

$$
\text{meas } \{z \colon |(Sf)(z)| > \alpha\} \leq \frac{A}{\alpha} \, \|f\|_1, \quad \text{for every } \alpha > 0,\tag{21}
$$

where *A* is a constant which does not depend on  $f \in L_1(\mathbb{C})$ . The smallest of such constants defines a norm of the operator  $S$ . We will denote this also by  $A$ . Using the method presented in the book of E. STEIN [8], it can be proved that Thus  $w(z)$  is extremal; it is Teichmüller's quasiconformal map.<br>
3. Asymptotic behaviour of  $||S||_p$ <br>
The operator S is of the type  $(1,1)$ ; i.e.<br>
meas  $\{z: |(Sf)(z)| > \alpha\} \leq \frac{A}{\alpha} ||f||_1$ , for every  $\alpha > 0$ ,<br>
where A is a c

$$
(\ln 2)^{-1} < A < 30.
$$

Lemma *2: Assume that S is an arbitrary linear operator which satisfies (21) and is bounded in some*  $L_{p_0}$ *,*  $p_0 > 1$ *. Then for every*  $p \in (1, p_0)$  *the operator S is bounded in*  $L_p$  and Ine operator  $S$  is of the  $C_1$ <br>
meas  $\{z: |(Sf)(z)|$ <br>
where A is a constant which<br>
stants defines a norm of  $\{ln 2\}^{-1} < A < 30$ <br>
We shall use the following<br>
Lemma 2: Assume that<br>
is bounded in some  $L_{p_o}$ , p<br>
in  $L_p$  and<br>

$$
\limsup_{p\to 1} (p-1) ||S||_p \leq A.
$$

This supplements the well known lemma of interpolation theory for operators in  $L_p$ spaces. The proof only requires a slight modification. A careful examination of it yields that pplement<br>he proof<br> $\|S\|_p p \leq 1$ Extremal iner<br>
s the well known lemm<br>
only requires a slight<br>  $\frac{pA}{(p-1)t} + \frac{p||S}{(p_0-p)(t)}$ <br>
The adjoint Hilbert Extremal inequalities and quasiconform<br>
This supplements the well known lemma of interpolation theor<br>
spaces. The proof only requires a slight modification. A carefu<br>
yields that<br>  $||S||_p^p \leq \frac{pA}{(p-1)t} + \frac{p ||S||_{p_t}^{p_t}}{(p$ This supplements the well kn<br>
spaces. The proof only require<br>
yields that<br>  $||S||_p^p \le \frac{pA}{(p-1)t} + \frac{1}{(p-1)t}$ <br>
whence the lemma. The adjoint<br>  $\frac{1}{(S^*f)(z)} = \frac{1}{(S^*f)(z)} = -\frac{1}{(S^*f)(z)} = -\frac{1}{(S^*f)(z)} = -\frac{1}{(S^*f)(z)} = -\frac{$ 

$$
\|\mathcal{S}\|_{p}^{p} \leq \frac{pA}{(p-1)t} + \frac{p \|\mathcal{S}\|_{p_{0}}^{p_{0}}}{(p_{0}-p) (1-t)^{p_{0}}}, \text{ for every } t \in (0, 1),
$$

whence the lemma. The adjoint Hilbert operator  $S^*$  has the conjugate kernel

$$
||S||_p^p \leqq \frac{1}{(p-1)t} + \frac{1}{(p_0 - p)(1-t)^{p_0}}
$$
  
whence the lemma. The adjoint Hilbert operator  

$$
\langle S^*f \rangle(z) = \overline{(S_f^f)(z)} = \frac{-1}{\pi} \int \int \frac{f(t) d\sigma_t}{(\overline{z}-t)^2},
$$
from which it follows that  

$$
||S||_p = ||S^*||_p = ||S|| \frac{p}{p-1}.
$$
  
Therefore,  

$$
\lim_{p \to 1} \sup_{p \to 1} (p-1) ||S||_p = \lim_{p \to 1} \sup_{p \to 1} \frac{p-1}{p} ||S||
$$
  
Let the  

$$
\lim_{p \to \infty} \inf \frac{||S||_p}{p}
$$

from which it follows that

$$
||S||_p = ||S^*||_p = ||S||_{\frac{p}{p-1}}.
$$

Therefore,

at  
\n
$$
||S||_{p}^{p} \leq \frac{pA}{(p-1)t} + \frac{p ||S||_{p_{\bullet}}^{p_{\bullet}}}{(p_{0}-p) (1-t)^{p_{\bullet}}}, \text{ for every } t \in (0, 1),
$$
\nhe lemma. The adjoint Hilbert operator  $S^*$  has the conjugate  $\text{kernel}$   
\n
$$
(S^*f)(z) = \overline{(S\overline{f})(z)} = \frac{-1}{\pi} \int \int \frac{f(t) d\sigma_{t}}{(\overline{z}-t)^{2}},
$$
\nch it follows that  
\n
$$
||S||_{p} = ||S^*||_{p} = ||S||_{p} \frac{p}{p-1}.
$$
\n
$$
P^2
$$
\n
$$
P^3
$$
\n
$$
\limsup_{p \to 1} (p-1) ||S||_{p} = \limsup_{p \to 1} \frac{p-1}{p} ||S||_{p} \frac{p}{p-1} = \limsup_{q \to \infty} \frac{||S||_{q}}{q}.
$$
\n
$$
\liminf_{p \to \infty} \frac{||S||_{p}}{p}
$$
\n
$$
\text{and by } a. \text{ We have proved that } 1 \leq a \leq A < 30. \text{ There has been conjecture.}
$$

$$
\liminf_{p\to\infty}\frac{\|S\|_p}{p}
$$

be denoted by a. We have proved that  $1 \le a \le A < 30$ . There has been conjectured from which it follows the<br>  $||S||_p = ||S^*||_p =$ <br>
Therefore,<br>  $\limsup_{p\to 1} (p-1)$ <br>
Let the<br>  $\liminf_{p\to \infty} \frac{||S||_p}{p}$ <br>
be denoted by a. We have that  $a = 1$ .<br>
Theorem 5: The extr<br>  $1 + \beta$ 

Theorem 5: The extremal exponent function  $p = p(\beta)$  satisfies<sup>2</sup>)

$$
\limsup_{p\to 1} (p-1) \|S\|_p = \limsup_{p\to 1} \frac{p}{p} = \limsup_{q\to \infty} \frac{p}{q}.
$$
\n
$$
\liminf_{p\to \infty} \frac{\|S\|_p}{p}
$$
\n
$$
\text{and so } \liminf_{p\to \infty} \frac{\|S\|_p}{p}
$$
\n
$$
\text{and so } P
$$
\n
$$
\
$$

Proof: The proof is based on the following inequality

$$
p\rightarrow\infty
$$
 P  
\nred by a. We have proved that  $1 \le a \le A < 30$ . There has been conjectured  
\n= 1.  
\nrem 5: The extremal exponent function  $p = p(\beta)$  satisfies<sup>2</sup>)  
\n
$$
\frac{1+\beta}{\beta} \ge p(\beta) \ge \frac{2(1+\beta)^{o}}{(1+\beta)^{o}-(1-\beta)^{o}} \ge \frac{1+\beta}{\beta} \cdot 2^{1-o}.
$$
\n[: The proof is based on the following inequality  
\n
$$
p\left(\frac{\beta'+\beta''}{1+\beta'\beta''}\right) \ge \frac{p(\beta')p(\beta'')}{p(\beta')+p(\beta'')-2} \text{ for } 0 \le \beta', \beta'' < 1.
$$
\n(23)  
\nwe we first prove this inequality.

Therefore we first proved that  $1 \le a \le A < 30$ . There has been conjectured<br>
that  $a = 1$ .<br>
Theorem 5: The extremal exponent function  $p = p(\beta)$  satisfies<sup>2</sup>)<br>  $\frac{1+\beta}{\beta} \ge p(\beta) \ge \frac{2(1+\beta)^a}{(1+\beta)^a - (1-\beta)^a} \ge \frac{1+\beta}{\beta} \cdot 2^{1-a}$ .<br> ave proved that  $1 \le a \le A < 30$ .<br>
tremal exponent function  $p = p(\beta)$  s<br>  $\ge \frac{2(1+\beta)^a}{(1+\beta)^a - (1-\beta)^a} \ge \frac{1+\beta}{\beta}$ <br>
based on the following inequality<br>  $\ge \frac{p(\beta')p(\beta'')}{p(\beta') + p(\beta'') - 2}$  for  $0 \le \beta'$ <br>
re this inequality.<br>  $\frac{\beta' +$ be an arbitrary exponent strictly, less than  $\frac{p(\beta') \cdot p(\beta'')}{p(\beta') + p(\beta'') - 2}$ . Our aim is to prove that  $w \in W_{p,loc}^1(\Omega)$ . By applying the existence theorems for the Beltrami system, the following decompbsition property can be proved: Therefore we first prove this inequality.<br>
Let  $w = w(z)$  be a  $\frac{\beta' + \beta''}{1 + \beta'\beta'}$  quasiconformal map in the domain  $\Omega \subset \mathbb{C}$  and<br>
be an arbitrary exponent strictly less than  $\frac{p(\beta') \cdot p(\beta'')}{p(\beta'') - 2}$ . Our aim is to prov

$$
w=g\stackrel{\centerdot}{\circ}h
$$

where *h* is a  $\beta$ "-quasiconformal map in  $\Omega$  and g is a  $\beta$ '-quasiconformal map in the domain  $h(\Omega)$ . According to the definition of the extremal exponents, we have  $h \in W^1_{p'',loc}(\Omega)$  and  $g \in W^1_{p',loc}(h(\Omega))$  for every  $p' < p(\beta')$  and  $p'' < p(\beta'')$ , respectively. Without loss of generality we may omit the symbol "loc" by eventually considering

2) A similar result has also been proved in [5].

Let  $E$  be an arbitrary measurable subset of  $\Omega$ . We estimate the measure of the set  $h(E)$  as follows

$$
\begin{aligned}\n\text{meas } h(E) &= \iint\limits_E \left( |h_z(z)|^2 - |h_{\bar{z}}(z)|^2 \right) d\sigma_z \\
&\leq (\text{meas } E)^{1-2/p} \left( \iint\limits_E |h_z(z)|^{p''} d\sigma_z \right)^{2/p''} \leq \text{const. } (\text{meas } E)^{1-2/p''}.\n\end{aligned}
$$

As above we also deduce that

$$
\text{meas } w(E) = \text{meas } g(h(E)) \leq \text{const. } (\text{meas } h(E))^{1-2/p'}.
$$

Takes together these yield the inequality

meas  $w(E) \leq$  Const. (meas  $E)^{(1-2/p')(1-2/p')}$ .

The left-hand side of this inequality is nothing other than the integral over  $E$  of the Jacobian  $J(z) = |w_z(z)|^2 - |w_{\overline{z}}(z)|^2$  of the map w. Thus it reads

$$
\iint\limits_E J(z) \, d\sigma_z \leq \text{const.} \, (\text{meas } E)^{(1-2/p')(1-2/p')}.
$$

We utilize this inequality for estimating the measure of the set

 $E_T = \{z \in \Omega : J(z) > T\},\$ 

where  $T$  is an arbitrary positive parameter.

meas 
$$
(E_T)
$$
 =  $\int\int\limits_{E_T} d\sigma_z \leq \frac{1}{T} \int\limits_{E_T} \int J(z) d\sigma_z \leq \frac{C}{T} (\text{meas } E_T)^{(1-2/p')(1-2/p')}$ 

Hence

$$
\text{meas}\, E_T \leq \text{const.}\, T^{\frac{r}{2(p'+p''-2)}}
$$

On the other hand it is well known that

$$
\iint_{0}^{1} J(z)^{p/2} d\sigma_{z} = \frac{p}{2} \int_{0}^{\infty} T^{p/2 - 1} \text{ meas } E_{T} dT
$$
  

$$
\leq \frac{p \text{ meas } (\Omega)}{2} \int_{0}^{1} T^{(p-2)/2} dT
$$
  
+ const. 
$$
\int_{1}^{\infty} T^{(p-2)/2} T^{2(p'+p''-2)} dT <
$$

This last follows from the assumption  $p < \frac{p'p''}{p' + p'' - 2}$ . So, we have that  $J(z) \in L_{p/2}(\Omega)$ . By the quasiconformality of w we finally get  $w \in W_p(\Omega)$ . This completes the proof of (23).

Remark: We have shown that it always holds that  $p(\beta) \leq \frac{1+\beta}{\beta}$ . Notice that equality occurs in (23) when  $p(\beta) = \frac{1+\beta}{\beta}$ .

•Extremal inequalities and quasiconformal mappings 15

(24)

Now, we shall prove-(22). We begin by. defining the preparatory function

Extremal inequalities and quasiconformal mappings

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\n
$$
F(s) = -\ln\left(1 - \frac{2}{p(\beta)}\right), \text{ for } s = \ln\frac{1+\beta}{1-\beta} \ge 0.
$$
\nFrom  $p(0) = \infty$  it follows that  $F(0) = 0$ . Now, inequality (23) becomes

\n
$$
F(s_1 + s_2) \le F(s_1) + F(s_2).
$$
\nIndeed,

\n
$$
F(s_1 + s_2) = F(s_1) + F(s_2).
$$
\n
$$
F(s_2 + s_1) = \frac{1+\beta'}{1+\beta'} - \frac{1+\beta'}{1+\beta'}.
$$

• . . . 

 

Extremal inequalities and quasiconformal mappings 1:  
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$$
\n(24)  
\nIndeed,  
\n
$$
F(s(\beta') + s(\beta'')) = F\left(\ln\frac{1+\beta'}{1-\beta'} + \ln\frac{1+\beta''}{1-\beta'}\right) = F\left(\ln\frac{1+\frac{\beta'+\beta''}{1+\beta'\beta''}}{1-\frac{\beta'+\beta''}{1+\beta'\beta''}}\right)
$$
\n
$$
= F\left(s\left(\frac{\beta'+\beta''}{1+\beta'\beta''}\right)\right) = -\ln\left(1 - \frac{2}{p\left(\frac{\beta'+\beta''}{1+\beta'\beta''}\right)}\right)
$$
\n
$$
\le -\ln\left(1 - \frac{2p(\beta') + 2p(\beta'') - 4}{p(\beta')p(\beta'')} \right)
$$
\n
$$
= -\ln\left(1 - \frac{2}{p(\beta')}\right)\left(1 - \frac{2}{p(\beta'')}\right) = F(s(\beta')) + F(s(\beta'')).
$$
\nInequality (24) implies that the function  $\frac{F(s)}{s}$  is decreasing in  $s \in (0, \infty)$ . In particular we get  
\n
$$
\frac{F(r)}{r} \le \liminf_{s \to 0} \frac{F(s)}{s} = \liminf_{s \to 0} \frac{e^s + 1}{e^s - 1} \frac{e^{F(s)} - 1}{2e^{F(s)}} = \liminf_{s \to 0} \frac{1}{\beta p(\beta)}.
$$
\nBut from (5) it follows that  $\frac{1}{p(\beta)} = \frac{S_{p(\beta)}}{p(\beta)}$ . Hence,  
\nBy the definition of  $F$  we immediately get  
\n
$$
1 - \frac{2}{p(\beta)} = e^{-f\left(\ln\frac{1+\beta}{1-\beta}\right)} \ge e^{-\rho\ln\frac{1+\beta}{1-\beta}} = \left(\frac{1-\beta}{1+\beta}\right).
$$

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$$
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$$
  
from (5) it follows that  $\frac{1}{\beta p(\beta)} \leq \frac{S_{p(\beta)}}{p(\beta)}$ . Hence,  

$$
\frac{F(r)}{r} \leq \liminf_{\beta \to 0} \frac{S_{p(\beta)}}{p(\beta)} = \liminf_{p \to \infty} \frac{||S||_p}{p} = a.
$$
  
the definition of F we immediately get

$$
\frac{F(r)}{r} \leq \liminf_{\beta \to 0} \frac{S_{p(\beta)}}{p(\beta)} = \liminf_{p \to \infty} \frac{||S||_p}{p} = a
$$

By the definition of  $F$  we immediately get

$$
1-\frac{2}{p(\beta)}=e^{-F\left(\ln\frac{1+\beta}{1-\beta}\right)}\geq e^{-\beta\ln\frac{1+\beta}{1-\beta}}=\left(\frac{1-\beta}{1+\beta}\right)^{\alpha}.
$$

4) implies that<br>  $\leq \liminf_{s\to 0} \frac{F(s)}{s}$ <br>
it follows that<br>  $\leq \liminf_{\beta\to 0} \frac{S_{p(\beta)}}{p(\beta)}$ <br>
ion of F we im<br>  $\frac{2}{p(\beta)} = e^{-F\left(\ln\frac{1}{1}\right)}$ <br>
obtain (22). We<br>
ken together wi function  $\frac{F(s)}{s}$  is decreasing in  $s \in (0, \infty)$ . In<br>  $\min_{\epsilon \to 0} \frac{e^{\epsilon} + 1}{e^{\epsilon} - 1} \frac{e^{F(\epsilon)} - 1}{2e^{F(\epsilon)}} = \lim_{\beta \to 0} \inf \frac{1}{\beta p(\beta)}$ .<br>  $\frac{S}{p} \leq \frac{S_{p(\beta)}}{p(\beta)}$ . Hence,<br>  $\lim_{p \to \infty} \frac{\inf \frac{||S||_p}{p}}{p} = a$ .<br>  $\lim_{p \to \infty} \frac{$ But from (5) it follows that  $\frac{1}{\beta p(\beta)} \leq \frac{S_{p(\beta)}}{p(\beta)}$ . Hence,<br>  $\frac{F(r)}{r} \leq \liminf_{\beta \to 0} \frac{S_{p(\beta)}}{p(\beta)} = \liminf_{p \to \infty} \frac{\|S\|_p}{p} = a$ .<br>
By the definition of F we immediately get<br>  $1 - \frac{2}{p(\beta)} = e^{-f\left(\ln \frac{1+\beta}{1-\beta}\right)} \geq e^{-a \ln$ ( $\beta$ )  $\rightarrow$   $\frac{1}{\sqrt{1 + \beta}}$ <br>btain (22). We remark that the equality From this we obtain (22). We remark that the equality  $\liminf_{p \to \infty} \frac{\|S\|_p}{p} = 1$  conviously, taken together with (22), implies  $p(\beta) = 1 + 1/\beta$  **s** But from (5) it follows that  $\frac{1}{\beta p(\beta)} \leq \frac{S_{p(\beta)}}{p(\beta)}$ . Hence,<br>  $\frac{F(r)}{r} \leq \liminf_{\beta \to 0} \frac{S_{p(\beta)}}{p(\beta)} = \liminf_{p \to \infty} \frac{||S||_p}{p} = a$ .<br>
By the definition of F we immediately get<br>  $1 - \frac{2}{p(\beta)} = e^{-F\left(\ln \frac{1+\beta}{1-\beta}\right)} \geq e^{-a \ln \$ om this we obtain (22). We remark that the equality lim inf  $\frac{||S||_p}{p} = 1$  conjectionsly, taken together with (22), implies  $p(\beta) = 1 + 1/\beta$  **P**<br> **EFERENCES**<br>
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