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On the representation of Bergman-Vekua-operators for three-dimensional equations

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Die Integraloperatoren von S. BERGMAN und I. N. VEKUA transformieren holomorphe Funktionen einer komplexen Variablen in Lösungen linearer partieller Differentialgleichungen elliptischen Typs in der Ebene. Durch Verallgemeinerung dieser Methoden findet man Integraltransformationen für die Lösung partieller Differentialgleichungen verschiedenen Typs (z. B. parabolischen, elliptischen, pseudoparabolischen Typs) mit drei unabhängigen Veränderlichen. Die Transformationen verknüpfen holomorphe Funktionen von zwei Variablen und die erwähnten Lösungen miteinander. Für ein Beispiel geben wir eine explizite Darstellung des Kerns dieser Transformationen (durch eine Summe eines Duhamelprodukts und von Cauchyintegralen); dies verallgemeinert neuere Resultate von D. L. COLTON und R. P. GILBERT.

Интегральные операторы С. БЕРГМАНА и И. Н. ВЕКУА преобразуют голоморфные функции одного переменного в решения дифференциальных уравнений с частными производными эллиптического типа в плоскости. Обобщая этот метод можно найти интегральные преобразования для решения дифференциальных уравнений с частными производными разного (напр. параболического, эллиптического, псевдопараболического) типа в трехмерном случае. Эти преобразования связывают голоморфные функции двух переменных с выше названными решениями. Для примера дается явное представление ядра этих преобразований (как сумма произведения Духамель и интегралов Коши). Этим обобщены новые результаты Д. Л. Кольтона и Р. П. Гильберта.

The integral operators of S. BERGMAN and I. N. VEKUA transform holomorphic functions of a complex variable into solutions of linear partial differential equations of elliptic type in the plane. Generalizing these methods we find integral transforms for the solution of partial differential equations of various type (e.g. parabolic, elliptic, pseudoparabolic) with three independent variables. The transforms associate holomorphic functions of two variables and the mentioned solution. We give (for an example) an explicit representation of the kernel of these transforms (by a sum of a Duhamel product and Cauchy integrals) which generalizes recent results of D. L. COLTON and R. P. GILBERT.

1. Introduction

Many efforts have been made in the study of linear partial differential equations with complex methods. The integral operators due to S. BERGMAN [1] and I. N. VEKUA [9] allow to construct explicitly the solutions to these equations in the plane, associating them with holomorphic functions of a complex variable. In this way an extensive study of the function theoretic properties of the solutions to partial differential equations is possible.

Following some ideas of S. BERGMAN [1] recently D. L. COLTON (see [2]) and R. P. GILBERT (see [4]) enlarged considerably the investigations of generalizations of this method to higher dimensions. We also give a contribution to the construction of solutions to partial differential equations with three independent variables by integral transforms. Especially we attack the problem to find a suitable representation of this transforms, generalizing the statements of COLTON and GILBERT.

Let us consider a linear partial differential equation with the independent variables, $x, y, \mathfrak{z}$. As usually in complex methods we perform the analytic continuation of its coefficients to complex values of $x, y, \mathfrak{z}$ (in general "in the small") and introduce new independent variables $z = x + iy$, $\xi = x - iy$ ($\mathfrak{z} = \mathfrak{z}$). This, we study the differential equation everywhere in this complex version; mention especially $4\partial^2/\partial z\,\partial\xi = \partial^2/\partial x^2 + \partial^2/\partial y^2$. If the coefficients of the equation are real for real values of $x, y, \mathfrak{z}$, we get a real solution from a complex one by putting $\xi = \bar{z}$ (the conjugate complex variable) and taking the real part.

2. Construction of integral operators

Our concept to solve linear partial differential equations is as follows (see [7]; we give the method only for a special case):

Let $u = u(z, \xi, \mathfrak{z})$ with $z \in D$, $\xi \in D^*$, $\mathfrak{z} \in G$ (D, D^*, G bounded and simply connected, $0 \in D$, $0 \in D^*$) be a solution of

$$\mathbf{L}u = \frac{\partial^2}{\partial z\,\partial\xi} u + \mathbf{A}u = 0. \tag{1}$$

Here $\mathbf{A}$ is a linear operator, and let (1) be self-adjoint with respect to z, ξ.

We define a set $\mathbf{F}$ of functions $f = f(z, \mathfrak{z})$ and a transform $\mathbf{R}[f]$ of these functions.

Definition 1: $f(\cdot,\cdot) \in \mathbf{F}$ if and only if

(a) $f = f(z, \cdot)$ is holomorphic with respect to z in D and continuous in $\overline{D}$.

(b) Constants $\alpha > 0$, $C \geqq 0$, $p \geqq 0$ (integer) exist with

$$|\mathbf{A}^m f(z, \mathfrak{z})| \leqq \alpha C^m (m + p)!^2 \quad \text{for} \quad m \geqq m_0 \tag{2}$$

for all $z \in D$, $\mathfrak{z} \in G_0 \subset G$.

A function $f \in \mathbf{F}$ may be called an *associated function of equation* (1).

Definition 2: A *Riemann transform* $\mathbf{R}[f] = \mathbf{R}[f(t, \mathfrak{z})](z, \xi, t, \tau, \mathfrak{z})$ *of equation* (1) is a transform with the properties: For $z \in D_0 \subset D$, $\xi \in D_0^* \subset D^*$, $t \in D_0$, $\tau \in D_0^*$ ($0 \in D_0$, $0 \in D_0^*$), $\mathfrak{z} \in G_0$:

(i) $\partial/\partial z\mathbf{R}[f(t, \mathfrak{z})]$, $\partial/\partial\xi\mathbf{R}[f(t, \mathfrak{z})]$, $\partial^2/\partial z\,\partial\xi\mathbf{R}[f(t, \mathfrak{z})]$, $\mathbf{A}\mathbf{R}[f(t, \mathfrak{z})]$ exist;

(ii) $\mathbf{R}[f(t, \mathfrak{z})]$ is integrable with respect to t in D_0;

(iii)
$$\mathbf{L}\mathbf{R}[f(t, \mathfrak{z})] = 0, \tag{3}$$

$$\partial/\partial\xi\mathbf{R}[f(t, \mathfrak{z})] = 0 \quad \text{for} \quad t = z, \tag{4}$$

$$\partial/\partial z\mathbf{R}[f(t, \mathfrak{z})] = 0 \quad \text{for} \quad \tau = \xi, \tag{5}$$

$$\mathbf{R}[f(t, \mathfrak{z})] = f(t, \mathfrak{z}) \quad \text{for} \quad t = z, \quad \tau = \xi. \tag{6}$$

Remark: In the theory of I. N. VEKUA [9] the Riemann transform is the multiplication with the complex Riemann function:

$$\mathbf{R}[f(t)] = R(z, \xi, t, \tau) \cdot f(t).$$

Theorem 1: *For all associated functions* $f \in \mathbf{F}$

$$u(z, \xi, \mathfrak{z}) = \int_0^z \mathbf{R}[f(t, \mathfrak{z})]\, dt \tag{7}$$

($\mathbf{R}$ *the Riemann transform*) *is a solution of* (1) *in* $D_0 \times D_0^* \times G_0$.

Proof by insertion of (7) into (1) using (3), (4).

Theorem 1': *Let* $G = G(z, \xi, \mathfrak{z}) \in F$ *(for all* $\xi \in D_0^*$*).*

$$u_1(z, \xi, \mathfrak{z}) = \int_0^z \int_0^\xi \mathbf{R}[G(t, \tau, \mathfrak{z})]\, dt\, d\tau \tag{7'}$$

is a solution of

$$\mathbf{L}u_1 = G(z, \xi, \mathfrak{z}) \tag{1'}$$

in $D_0 \times D_0^* \times G_0$ ($\mathbf{R}$ *the Riemann transform*).

Proof by insertion of (7') into (1') using (3), (4), (5), (6).

There are two problems in this concept: the existence of the Riemann transform and its suitable representation. We give an answer only for a very special case.

Theorem 2: *Let* $\mathbf{A}$ *be an operator not depending on* z, ξ *(this is, if* $h = h(\mathfrak{z})$ *is a function of* $\mathfrak{z}$ *alone, then* $\bar{h} = \mathbf{A}h$ *is also a function of* $\mathfrak{z}$ *alone). The Riemann transform of equation* (1) *in* $D_0 \times G_0$*, where* D_0 *is a closed polydisc in*

$$D_z \times D_\xi^* \times D_t \times D_\tau^* \cap \{(z, \xi, t, \tau) : |v| < 1/C\} \quad \text{with} \quad v = (z - t)(\tau - \xi)$$

is given by

$$\mathbf{R}[f(t, \mathfrak{z})] = \sum_{m=0}^{\infty} \frac{1}{m!^2} v^m \mathbf{A}^m[f(t, \mathfrak{z})] \tag{8}$$

for all associated functions $f \in \mathbf{F}$.

Proof (and examples) see [7].

3. Representation of the Riemann transform

Resulting from the general nature of the operator $\mathbf{A}$ the above method is a very versatile one. But one of the main problems in the application of the method is to find a suitable representation of the transform $\mathbf{R}[f]$. Following R. P. Gilbert [4] it is possible to express the transform $\mathbf{R}$ for elliptic and parabolic equations (1) by a Cauchy integral

$$\mathbf{R}[f(t, \mathfrak{z})] = \frac{1}{2\pi i} \oint_K H(z, \xi, t, \tau, \mathfrak{z}, s)\, f(t, s) \frac{ds}{s - \mathfrak{z}}, \tag{9}$$

here K is a circle in G_0 surrounding $s = \mathfrak{z}$. For pseudoparabolic equations (1) (of composed type) D. L. Colton [2] has given a representation of $\mathbf{R}[f]$ through a *Duhamel product*

$$\mathbf{R}[f(t, \mathfrak{z})] = \frac{\partial}{\partial \mathfrak{z}} \int_0^{\mathfrak{z}} H(z, \xi, t, \tau, s - \mathfrak{z})\, f(t, s)\, ds. \tag{10}$$

For these cases both it is possible to construct the kernel H with the aid of differential equations found from (1). For explicit examples see [6, 8].

But for more general equations these representations of the transform $\mathbf{R}[f]$ do not hold. We show by an example of an equation of composed type that the transform $\mathbf{R}[f]$ is to be represented by a sum of Cauchy integrals and a Duhamel product. The representations of R. P. GILBERT and D. L. COLTON are special cases of it. Let therefore

$$\mathbf{A} = \alpha D^p + \beta I + \gamma S^p,$$

where α, β, γ are constants, D is the differentiation with respect to $\mathfrak{z}$: $D = \partial/\partial\mathfrak{z}$, S is the integration

$$S = \int_0^{\mathfrak{z}} \cdot\, ds,$$

I is the identity, and let $p = 1$ or $p = 2$. Thus we consider the equation

$$\frac{\partial^{2+p}}{\partial z\, \partial\xi\, \partial\mathfrak{z}^p} u + \alpha \frac{\partial^{2p}}{\partial\mathfrak{z}^{2p}} u + \beta \frac{\partial^p}{\partial\mathfrak{z}^p} u + \gamma u = 0. \tag{11}$$

Lemma 1: *For $n = 0$ one has*

$$\mathbf{A}^n[f] = \mathbf{A}_I^{[n]}[f] - \alpha\gamma \sum_{m=0}^{n-2} \mathbf{A}^{n-2-m}[z_m(f)] \tag{12}$$

with

$$\mathbf{A}_I^{[n]}[f] = \sum_{k+l+m=n} \frac{n!}{k!\,l!\,m!} \alpha^k\beta^l\gamma^m L^{k-m}, \quad \textit{where} \quad L^j = \begin{cases} D^{pj}, & j \geqq 0, \\ S^{-pj}, & j < 0, \end{cases} \tag{13}$$

and with

$$z_m = \sum_{\substack{k+l+q=m+1\\ k\geqq q}} \frac{(m+1)!}{(k+1)!\,l!\,q!} \alpha^k\beta^l\gamma^q (D^{p(k-q)} - S^p D^{p(k-q+1)})\,[f].$$

Remark: We give some explanations of these expressions. We have $DS = I$, $SD = I - N$ with $Nf(t, \mathfrak{z}) = f(t, 0)$. From this we find

$$D^{pk} - S^p D^{pk+p} = \begin{cases} ND^k & \text{for} \quad p = 1, \\ ND^{2k} + SND^{2k+1} & \text{for} \quad p = 2. \end{cases}$$

Thus z_m is a constant with respect to $\mathfrak{z}$ (for $p = 1$) or linearly depending on $\mathfrak{z}$ (for $p = 2$):

$$z_m = A_{m0} + A_{m1}\mathfrak{z},$$

and the coefficients A_{mj} ($j = 0, 1$; $m = 0, 1, 2, \ldots$) are linearly depending on $f(t, 0)$ and the derivatives of $f(t, \mathfrak{z})$ with respect to $\mathfrak{z}$ in $\mathfrak{z} = 0$:

$$A_{mj} = \sum_{\substack{k+l+q=m\\ k\geqq q}} \frac{(m+1)!}{(k+1)!\,l!\,q!} \alpha^k\beta^l\gamma^q \frac{\partial^{pk-pq+j}}{\partial\mathfrak{z}^{pk-pq+j}} f(t, \mathfrak{z})\big|_{\mathfrak{z}=0} \tag{14}$$

for $j = 0, 1$, if $p = 2$, for $j = 0$ if $p = 1$, and

$$A_{m1} = 0 \quad \text{for} \quad p = 1.$$

Thus (12) reduces the problem to find $\mathbf{A}^n[f]$ for an arbitrary f to the problem to find it for a linear f. Furtherly: The definition (13) is made in such a way that we would have $\mathbf{A}^n = \mathbf{A}_I^{[n]}$ if $SD = DS = I$.

Proof of Lemma 1 by induction: Obviously (12) is true for $n = 0$ and $n = 1$. From the validity for n follows for $n + 1$

$$\mathbf{A}^{n+1}[f] = \mathbf{A}\mathbf{A}_I^{[n]}[f] - \alpha\gamma \sum_{m=0}^{n-2} \mathbf{A}^{n-1-m}[z_m(f)]. \tag{15}$$

Consider $\mathbf{A}^{[n+1]}[f] - \mathbf{A}\mathbf{A}_I^{[n]}[f]$. There is a contribution to this difference only from the (left) multiplication of the sum (13) by γS^p, and this contribution is

$$\mathbf{A}_I^{[n+1]}[f] - \mathbf{A}\mathbf{A}_I^{[n]}[f] = \alpha\gamma \sum_{\substack{k+m+l=n\\ k>m}} \frac{n!}{k!\,m!\,l!}\, \alpha^{k-1}\beta^l\gamma^m L^{k-m} = \alpha\gamma z_{n-1}.$$

Inserting this into (15) we have (12) holding for $n + 1$ instead of n ∎

Now we give a set of associated functions for the equation (11).

Lemma 2: *$f \in \mathbf{F}$ if $f = f(\cdot, \mathfrak{z})$ is holomorphic with respect to $\mathfrak{z}$ in G and continuous in $\bar{G}$, $G_0 \subset G$ is given by*

$$|s - \mathfrak{z}| \geqq \delta > 0 \qquad (\delta < 1)$$

for all $s \in \partial G$, $\mathfrak{z} \in G_0$. Property (a) *of Definition* 1 *may be fulfilled.*

Proof: We prove the validity of condition (2) by the use of (12) in Lemma 1. All constants, not depending on n, we denote by $c_1, c_2, \ldots$ without the description of their relations. A constant depending on δ we denote by $c_k(\delta)$. Let $a = \max\{|\alpha|, |\beta|, |\gamma|, 1\}$. Using $\frac{n!}{m!\,k!\,l!} \leqq 3^n$ we have from (13)

$$\mathbf{A}_I^{[n]}[f] \leqq (3a)^n \sum |L^{k-m}[f]|.$$

With $|f(z, \mathfrak{z})| < M$ in $D \times \bar{G}$ we have for $0 \leqq k - m \leqq n$ from Cauchy's inequality

$$|L^{k-m}[f]| = |D^{p(k-m)}[f]| \leqq M \frac{(pn)!}{\delta^{pn}}.$$

On the other hand, for $0 \leqq m - k \leqq n$ we have from

$$S^{pj}[f] = \frac{1}{\Gamma(pj)} \int_0^{\mathfrak{z}} f(\ldots, s)\,(\mathfrak{z} - s)^{pj-1}\, ds:$$

$$|L^{k-m}[f]| = |S^{p(k-m)}[f]| \leqq \frac{1}{\Gamma(pj)} M\mathfrak{z}^{*pn} \leqq M\mathfrak{z}^{*pn}.$$

Here $\mathfrak{z}^* = \max(1, \operatorname{diam} G)$. Thus for all k, m

$$|L^{k-m}[f]| \leqq M \cdot c_1^n(\delta) \cdot (pn)!.$$

The sum (13) has $(n + 1)^2 \leqq 4^n$ terms, therefore we have

$$|A_I^{[n]}[f]| \leqq M \cdot c_2^n(\delta) \cdot (pn)!. \tag{16}$$

Now we estimate the polynomial $\sum_{m=0}^{n} \mathbf{A}^{n-m}[z_m(f)]$. Each summand $\mathbf{A}^{n-m}[z_m]$ consists of terms

$$S^\varkappa D^\mu[z_m] = A_{m0} S^\varkappa D^\mu[1] + A_{m1} S^\varkappa D^\mu[\mathfrak{z}] \qquad (\varkappa, \mu \leqq n - m \leqq n),$$

i.e. of powers of $\mathfrak{z}$ with the highest degree pn, or $pn + 1$; thus

$$|S^{\varkappa}D^{\mu}[z_m]| \leqq \mathfrak{z}^{*pn+1}(|A_{m0}| + |A_{m1}|).$$

A^{n-m} has $3^{n-m} \leqq 3^n$ such terms, multiplied by $\alpha^k\beta^l\gamma^q$ with $k + l + q \leqq n - m \leqq n$, therefore

$$|\mathbf{A}^{n-m}[z_m(f)]| \leqq c_3{}^n\mathfrak{z}^* \, 2 \max_{j=0,1} |A_{mj}|$$

and

$$\left| \sum_{m=0}^{n} \mathbf{A}^{n-m}[z_m(f)] \right| = (n + 1)\, c_3{}^n\mathfrak{z}^* \, 2 \max_{\substack{j=0,1 \\ m=0,1,\ldots,n}} |A_{mj}|.$$

From (14) we find, again using Cauchy's inequality, as above

$$|A_{mj}| \leqq (m + 1)^2\, 3^{m+1}a^mM \frac{(pm + j)!}{\delta^{pm+j}},$$

$$\max |A_{mj}| \leqq \begin{cases} (n + 2)!\, 3^{n+1}a^nM\delta^{-n} & \text{for } p = 1, \\ (2n + 3)!\, 3^{n+1}a^nM\delta^{-2n-1} & \text{for } p = 2. \end{cases}$$

From this we have

$$\left| \sum_{m=0}^{n} \mathbf{A}^{n-m}[z_m(f)] \right| \leqq \begin{cases} (n + 3)!\, c_4{}^n(\delta) \cdot 2M\mathfrak{z}^* & \text{for } p = 1, \\ (2n + 4)!\, c_5{}^n(\delta) \cdot 2M \dfrac{\mathfrak{z}^*}{\delta} & \text{for } p = 2, \end{cases}$$

and together with (12) and (16) we have (using $(2n)! \leqq 4^n \cdot n!^2$)

$$|\mathbf{A}^n[f]| \leqq \begin{cases} 2M\mathfrak{z}^* \cdot c_6{}^n(\delta) \cdot (n + 1)! & \text{for } p = 1, \\ 2M \dfrac{\mathfrak{z}^*}{\delta} \cdot c_7{}^n(\delta) \cdot 16(n + 2)!^2 & \text{for } p = 2. \end{cases} \tag{17}$$

This is (2) ∎

Remark: For $p = 1$ we have an essential fact. From the convergence of the series $\sum d^m/m!$ we have $d^n < (n + 1)!$ for every d and sufficiently large $n > n_0(d)$. Thus

$$|\mathbf{A}^n[f]| \leqq 2M\mathfrak{z}^* \left(\frac{c_5(\delta)}{d}\right)^n (n + 1)!^2$$

for arbitrary $d > 0$, if $n > n_0(d)$. This means, (2) holds for all $C > 0$.

Now we may construct the Riemann transform. From (8) and (12) we have

$$\mathbf{R}[f] = \mathbf{R}_1[f] - \alpha\gamma \cdot \mathbf{R}_2[f]$$

with

$$\mathbf{R}_1 = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\, v^n \mathbf{A}_I{}^{[n]}, \qquad \mathbf{R}_2[f] = \sum_{n=2}^{\infty} \frac{v^n}{n!^2} \sum_{m=0}^{n-2} \mathbf{A}^{n-2-m}[z_m(f)].$$

Firstly we consider $\mathbf{R}_1[f]$: By simple calculations we find from (13) the symmetric expression

$$\mathbf{R}_1 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n!m!k!(n + m + k)!}\, \alpha^n\beta^m\gamma^k v^{n+m+k} L^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n!k!} \left(\alpha \sqrt{\frac{v}{\beta}}\right)^n \left(\gamma \sqrt{\frac{v}{\beta}}\right)^k I_{n+k}(2\sqrt{v\beta})\, L^{n-k}$$

with the modified Besselfunction I_{n+k}. From this we have

$$\mathbf{R}_1 = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(n+k)!\,k!} \left(\sqrt{\frac{v}{\beta}}\right)^n \left(\alpha\gamma \sqrt{\frac{v}{\beta}}\right)^k I_{n+2k}\left(2\sqrt{v\beta}\right) \left(\alpha^n D^{pn} + \gamma^n S^{pn} - \delta_{n0} \cdot I\right)$$

with the Kronecker symbol δ_{n0}. Representing the differentiations D^{pn} by Cauchy integrals, the integrations S^{pn} by Duhamel products

$$S^{pn}[f(\ldots, \mathfrak{z})] = \frac{1}{(pn)!} \frac{\partial}{\partial \mathfrak{z}} \int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{pn} f(\ldots, s)\, ds$$

we find

$$\mathbf{R}_1[f] = \frac{1}{2\pi i} \oint_k H_1 f(t, s) \frac{ds}{s - \mathfrak{z}} + \frac{\partial}{\partial \mathfrak{z}} \int_0^{\mathfrak{z}} H_{-1} \cdot f(t, s)\, ds - H_1(0) \cdot f(t, \mathfrak{z})$$

with

$$H_1 = H_1\left(\alpha(s - \mathfrak{z})^{-p}\right), \qquad H_{-1} = H_{-1}\left(\gamma(\mathfrak{z} - s)^p\right),$$

and

$$H_\delta(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(pn)!}{(n+k)!\,k!} \left(\frac{\alpha\gamma v}{\beta}\right)^k \left(\sqrt{\frac{v}{\beta}}\right)^n I_{n+2k}\left(2\sqrt{v\beta}\right) x^n. \tag{18}$$

By a simple way we prove the convergence of this series. Firstly we use

$$I_{n+2k}\left(2\sqrt{|v\beta|}\right) \leq |v\beta|^{k+n/2} \frac{1}{(n+2k)!} I_0\left(2\sqrt{|v\beta|}\right),$$

and from this immediately follows

$$|H_\delta| \leqq I_0\left(2\left|\sqrt{v\beta}\right|\right) \cdot \sum_{k=0}^{\infty} \frac{1}{(2k)!\,k!^2} |\alpha\gamma v^2| \cdot \sum_{n=0}^{\infty} \frac{(pn)!^\delta}{(n!)^2} \cdot |vx|^n.$$

The generalized hypergeometric series

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!\,(k!)^2} |\alpha\gamma v^2| = {}_0F_3(1, 1, 1/2; 4\,|\alpha\gamma v^2|)$$

converges everywhere, the series

$$\sum_{n=0}^{\infty} \frac{(pn)!^\delta}{(n!)^2} |vx|^n = \begin{cases} e^{|vx|} & \text{for} \quad p = 1, \delta = 1, \\ {}_0F_2(1, 1; |vx|) & \text{for} \quad p = 1, \delta = -1 \\ \text{(generalized hypergeometric series)} & \\ \dfrac{1}{\sqrt{1 - 4\,|vx|}} & \text{for} \quad p = 2, \ \delta = 1, \\ {}_0F_3(1, 1, 1/2, 4\,|vx|) & \text{for} \quad p = 2, \delta = -1, \end{cases}$$

converges for $p = 1$ and for $p = 2$, $\delta = -1$ everywhere, for $p = 2$, $\delta = 1$ for $|vx| < 1/4$. (We remark, that these assertions coincide with the remarks on the choice of C in (2), made in the proof of Lemma 2.)

Secondly we consider the transform $\mathbf{R}_2[f]$: We write it symmetrically

$$\mathbf{R}_2[f] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{v^{n+m+2}}{((n+m+2)!)^2} \mathbf{A}^n[z_m(f)].$$

$z_m(f)$ depends linearly on $f(t, 0)$ and the derivatives of $f(t, \mathfrak{z})$ at the point $\mathfrak{z} = 0$ up to the order $p(m + 1) - 1$, see (14). We express all these derivatives by Cauchy integrals, thus we have

$$z_m[f(t, \mathfrak{z})] = \frac{1}{2\pi i} \oint_{K_0} z_m \left[\frac{1}{s - \mathfrak{z}}\right] f(t, s) \frac{ds}{s},$$

K_0 a circle in G_0, surrounding $s = 0$. Finally we have

$$\mathbf{R}_2[f] = \frac{1}{2\pi i} \oint_{K_0} H_0 \cdot f(t, s) \frac{ds}{s}$$

with the series

$$H_0 = H_0(z; \xi, t, \tau, \mathfrak{z}, s) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{v^{n+m+2}}{((n + m + 2)!)^2} \mathbf{A}^n \left[z_m \left(\frac{1}{s - \mathfrak{z}}\right)\right]. \tag{19}$$

We omit here the detailed proof of the convergence of this series. It is based on the fact that $z_m \left[\frac{1}{s - \mathfrak{z}}\right] = A_{m0} + A_{m1}\mathfrak{z}$, where the coefficients A_{mj} are found explicitly by (14):

$$A_{mj} = s^{-j-1} \sum_{\substack{k+l+q=m \\ k \geq q}} \frac{(m + 1)!\,(pk - pq + 1)!}{(k + 1)!\,l!\,q!} \left(\frac{\alpha}{s^p}\right)^k \beta^l (\gamma \cdot s^p)^q.$$

By these considerations we have found:

Theorem 3: *The Riemann transform of equation* (11) *is given for $f \in \mathbf{F}$ by*

$$\mathbf{R}[f(t, \mathfrak{z})] = \frac{1}{2\pi i} \left[\oint_{K_\mathfrak{z}} H_1 \cdot f(t, s) \frac{ds}{s - \mathfrak{z}} - \alpha\gamma \oint_{K_0} H_0 \cdot f(t, s) \frac{ds}{s} \right]$$

$$+ \frac{\partial}{\partial \mathfrak{z}} \int_0^{\mathfrak{z}} H_{-1} \cdot f(t, s)\, ds - H_1(0)\, f(t, \mathfrak{z}),$$

here $K_\mathfrak{z}$ is a circle in G_0 surrounding $s = \mathfrak{z}$, the kernels $H_{\pm 1}$, H_0 are given by (18), (19).

Remark: In the equation (11) are some interesting special cases. (Noticing that $H_1(0) = H_{-1}(0)$.) *For $\alpha = 0$* we have a pseudoparabolic equation

$$\frac{\partial^{2+p}}{\partial z\, \partial \xi\, \partial \mathfrak{z}^p} u + \beta \frac{\partial^p}{\partial \mathfrak{z}^p} u + \gamma u = 0; \tag{11'}$$

the Riemann transform is a single Duhamelproduct (without Cauchy integral), and a solution of equation (11′) is

$$u(z, \xi, \mathfrak{z}) = \int_0^z \frac{\partial}{\partial \mathfrak{z}} \int_0^{\mathfrak{z}} H_{-1} f(t, s)\, ds\, dt.$$

Using expressions of this type (for $p = 1$) D. L. COLTON [2] studied the properties of the solutions of pseudoparabolic equations. In our special case we have

$$H_{-1} = \sum_{n=0}^{\infty} \frac{1}{n!\,(pn)!} I_n(2\sqrt{\beta v}) \sqrt{\frac{v}{\beta}}^{\,n} \cdot x^n.$$

For $\alpha = \beta = 0$ this kernel is a generalized hypergeometric function

$$H_{-1} = \begin{cases} {}_0F_2\big(1, 1; \gamma(\mathfrak{z} - s)\, v\big) & \text{for} \quad p = 1; \\ {}_0F_3\big(1, 1, 1/2; 4\gamma(\mathfrak{z} - s)\, v\big) & \text{for} \quad p = 2. \end{cases}$$

For $\gamma = 0$, $p = 1$ we have a parabolic equation

$$\frac{\partial^2}{\partial z\, \partial \xi} u + \alpha \frac{\partial}{\partial \mathfrak{z}} u + \beta u = 0, \tag{11''}$$

the transform $\mathbf{R}[f]$ is a single Cauchy integral (without Duhamel product), and a solution of equation (11″) is

$$u(z, \xi, \mathfrak{z}) = \frac{1}{2\pi i} \int_0^z \oint_{K_\mathfrak{z}} H_1 \cdot f(t, s) \frac{ds}{s - \mathfrak{z}}\, dt. \tag{20}$$

Here we have

$$H_1 = \Phi_3 \left(1, 1; \frac{\alpha v}{s - \mathfrak{z}}, \beta v\right)$$

— this is a hypergeometric function of two variables [3] —, for $\beta = \gamma = 0$ we have simply

$$H_1 = \exp \frac{\alpha v}{s - \mathfrak{z}}.$$

The latter result is due to C. D. Hill [5].

For $\gamma = 0$, $p = 2$ we have an elliptic equation

$$\frac{\partial^2}{\partial z\, \partial \xi} u + \alpha \frac{\partial^2}{\partial \mathfrak{z}^2} u + \beta u = 0;$$

again the transform $\mathbf{R}[f]$ is a single Cauchy integral, and the solution u is given by (20). Using the equivalent Bergman operators together with Cauchy integrals D. L. Colton [2] and R. P. Gilbert [4] constructed in this way the solutions of "three-dimensional" elliptic equations and of parabolic equations with two space variables. Here the kernel is

$$H_1 = \Phi_2 \left(1/2, 1, 1; \frac{4\alpha v}{(s - \mathfrak{z})^2}, \beta v\right)$$

— again a hypergeometric function of two variables [3] —, for $\beta = \gamma = 0$ we have simply [6]

$$H_1 = \frac{s - \mathfrak{z}}{\sqrt{(s - \mathfrak{z})^2 - 4\alpha v}}.$$

Finally, for $\alpha = \gamma = 0$ we have a two-dimensional equation; the Riemann transform is the multiplication with the Riemann function (due to I. N. Vekua)

$$\mathbf{R}[f] = H_1(0) \cdot f(t, \mathfrak{z}) = I_0\big(2\sqrt{\beta v}\big)\, f(t, \mathfrak{z}).$$

(For $\alpha = \beta = \gamma = 0$ the Riemann transform is the identity, $\mathbf{R}[f] = f$.)

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