On the representation of Bergman-Vekua-operators for three-dimensional equations

E. LANCKATI

Die Integraloperatoren von S. BERGMAN und I. N. VEKUA transformieren holomorphe Funktionen einer komplexen Variablen in Lösungen linearer partieller Differentialgleichungen elliptischen Typs in der Ebene. Durch Verallgemeinerung dieser Methoden findet man Integraltransformationen für die Lösung partieller Differentialgleichungen verschiedenen Typs (z. B. parabolischen, elliptischen, pseudoparabolischen Typs) mit drei unabhängigen Veränderlichen. Die Transformationen verknüpfen holomorphe Funktionen von zwei Variablen und die erwähnten Lösungen miteinander. Für ein Beispiel geben wir eine explizite Darstellung des Kerns dieser Transformationen (durch eine Summe eines Duhamelprodukts und von Cauchyintegralen); dies verallgemeinert neuere Resultate von D. L. COLTON und R. P. GILBERT.

Митегральные операторы С. Бергмана и И. Н. Векул преобразуют голоморфные функции одного переменного в решения дифференциальных уравнений с частными производными эллиптического типа в плоскости. Обобщая этот метод можно найти интегральные преобразования для решения дифференциальных уравнений с частными производными разного (напр. параболического, эллиптического, псевдопараболического) типа в трехмерном случае. Эти преобразования связывают голоморфные функции двух переменных с выше названными решениями. Для примера дается явное представление ядра этих преобразований (как сумма произведения Духамель и интегралов Коши). Этим обобщены новые результаты Д. Л. Кольтона и Р. П. Гильберта.

The integral operators of S. BERGMAN and I. N. VERUA transform holomorphic functions of a complex variable into solutions of linear partial differential equations of elliptic type in the plane. Generalizing these methods we find integral transforms for the solution of partial differential equations of various type (e.g. parabolic, elliptic, pseudoparabolic) with three independent variables. The transforms associate holomorphic functions of two variables and the mentioned solution. We give (for an example) an explicit representation of the kernel of these transforms (by a sum of a Duhamel product and Cauchy integrals) which generalizes recent results of D. L. COLTON and R. P. GLBERT.

1. Introduction

Many efforts have been made in the study of linear partial differential equations with complex methods. The integral operators due to S. BERGMAN [1] and I. N. VEKUA $\lceil 9 \rceil$ allow to construct explicitly the solutions to these equations in the plane, associating them with holomorphic functions of a complex variable. In this way an extensive study of the function theoretic properties of the solutions to partial differential equations is possible.

Following some ideas of S. BERGMAN [1] recently D. L. COLTON (see [2]) and R. P. GILBERT (see [4]) enlarged considerably the investigations of generalizations of this method to higher dimensions. We also give a contribution to the construction of solutions to partial differential equations with three independent variables by

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integral transforms; Especially we attack the problem to find a suitable representation of this 'transforms, generalizing the statements of COLTON and GILBERT.

Let us consider a linear partial differential equation with the independent variables, x, y, λ . As usually in complex methods we perform the analytic continuation of its coefficients to complex values of x, y, \mathfrak{z} (in general "in the small") and introduce new independent variables $z = x + iy$, $\xi = x - iy$ ($\delta = \delta$). This, we study the differential equation everywhere in this complex version; mention especially $4\partial^2/\partial z \partial \xi$ $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. If the coefficients of the equation are real for real values of x, y, χ' we get a real solution from a complex one by putting $\xi = \overline{z}$ (the conjugate 2 E. LANCKAU

integral transforms. Especially we attack the problem to

tation of this transforms, generalizing the statements of (

Let us consider a linear partial differential equation with

x, y, λ . As usually in c ependent variables $z = x + iy$, $\xi = x$

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 $x^2 + \frac{\partial^2}{\partial y^2}$. If the coefficients of the

ve get a real solution from a complex

variable) and taking the real part.

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2. Construction of integral operators

Our concept to solve linear partial differential equations is as follows (see [7]; we give the method only for a special case)':

Let $u = u(z, \xi, \xi)$ with $z \in D$, $\xi \in D^*$, $\xi \in G$ $(D, D^*$, G bounded and simply connected, $0 \in D$, $0 \in D^*$) be a solution of Let $u = u(z, \xi, \delta)$ with

nected, $0 \in D, 0 \in D^*$ $\}$
 $\mathbf{L}u = \frac{\partial^2}{\partial z \partial \xi} u$
 \mathbf{H} ere **A** is a linear opera

We define a set **F** of

Definition 1: $f(\cdot, \cdot)$

(a) $f = f(z, \cdot)$ is holomo

(b) Constants $\alpha > 0$, C
 $|\mathbf$ et $u = u(z, \xi, \xi)$. with $z \in D$, $\xi \in D^*$, $\xi \in E$

ed, $0 \in D$, $0 \in D^*$) be a solution of
 $\mathbf{L}u = \frac{\partial^2}{\partial z \partial \xi} u + \mathbf{A}u = 0$.

e **A** is a linear operator, and let (1) be sel
 e define a set **F** of functions $f = f(z,$

$$
= u(z, \xi, \xi) \text{ with } z \in D, \xi \in D^*, \xi \in G \ (D, D^*, G \text{ bounded and simply con-}\n0 \in D, 0 \in D^*) be a solution of
$$
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$$
\mathbf{L}u = \frac{\partial^2}{\partial z \partial \xi} u + \mathbf{A}u = 0.
$$
\n
$$
\text{is a linear operator, and let (1) be self-adjoint with respect to } z, \xi.
$$
\n
$$
\text{if the a set } \mathbf{F} \text{ of functions } f = f(z, \xi) \text{ and a transform } \mathbf{R}[f] \text{ of these functions.}
$$
\n
$$
\text{a function } 1: f(\cdot, \cdot) \in \mathbf{F} \text{ if and only if}
$$
\n
$$
f(z, \cdot) \text{ is holomorphic with respect to } z \text{ in } D \text{ and continuous in } \overline{D}.
$$
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$$
\text{is a function } \mathbf{L} = \mathbf{0}, \mathbf{L} = \mathbf{0}, \mathbf{L} = \mathbf{0} \text{ and } \mathbf{L} = \mathbf{L} \mathbf{L} = \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} = \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} = \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} = \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} = \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} = \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} = \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} = \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} = \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} \mathbf{L} = \mathbf{L} \mathbf{L}
$$

Here A is a linear operator, and let (1) be self adjoint with respect to *z,* We define a set F of functions $f = f(z, \lambda)$ and a transform R[f] of these functions.

. Definition $1: f(\cdot, \cdot) \in \mathbf{F}$ if and only if .

(a) $f = f(z, \cdot)$ is holomorphic with respect to *z* in *D* and continuous in \overline{D} . (b) Constants $\alpha > 0$, $C \geq 0$, $p \geq 0$ (integer) exist with t (1) be self-adjoint with re
 $= f(z, \delta)$ and a transform 1

only if

respect to z in D and conti

0 (integer) exist with

¹² for $m \ge m$.

$$
|\mathbf{A}^m f(z, \mathbf{A})| \leq \alpha C^m (m + p)!^2 \quad \text{for} \quad m \geq m_0
$$

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for all $z \in D$, $\zeta \in G_0 \subset G$.
A function $f \in F$ may be called an *associated function of equation* (1).

Definition 2: A *Riemann transform* $R[f] = R[f(t, \xi)]$ (z, ξ, t, τ, ξ) of equation (1) $|A^m/(z, \lambda)| \leq \alpha C^m (m + p)!^2$ for $m \geq m_0$ (2)

for all $z \in D, \lambda \in G_0 \subset G$.

A function $f \in F$ may be called an associated function of equation (1).

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(b) Constants $\alpha > 0$, $C \ge 0$, $p \ge 0$ (integer) exist with
 $|\mathbf{A}^m f(z, \delta)| \le \alpha C^m (m$

(i) $\partial/\partial z \mathbf{R}[f(t, \lambda)], \partial/\partial \xi \mathbf{R}[f(t, \lambda)], \partial^2/\partial z \partial \xi \mathbf{R}[f(t, \lambda)], \mathbf{A} \mathbf{R}[f(t, \lambda)]$ exist;

$$
\mathbf{R}[f(t)] = R(z, \xi, t, \tau) \cdot f(t).
$$

Theorem 1: For all associated functions $f \in \mathbf{F}$

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\text{Representation of Bergman-Vekua-operators} \qquad 3
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\text{rem 1: } For \ all \ associated \ functions \ f \in \mathbf{F}
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$$
u(z, \xi, \xi) = \int_0^z \mathbf{R}[f(t, \xi)] \, dt \qquad \qquad (7)
$$
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$$
\text{iemann transform} \ is \ a \ solution \ of \ (1) \ in \ D_0 \times D_0^* \times G_0.
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$$
\text{by-insertion of (7) into (1) using (3) (4)}
$$

(**R** the Riemann transform) is a solution of (1) in $D_0 \times D_0^* \times G_0$. *transform*) is a solution of (1) in $D_0 \times D_0^* \times G_0$.

ertion of (7) into (1) using (3), (4).
 . Let $G = G(z, \xi, \xi) \in F$ (for all $\xi \in D_0^*$).
 δ) = $\int \int$ **F** [$G(t, \tau, \xi)$] dt dt (7'

Proof by insertion of (7) into (1) using (3) , (4) .

Theorem 1': Let $G = G(z, \xi, \xi) \in F$ *(for all* $\xi \in D_0^*$ *).*

Representation of Bergman-Vekua-operators
\n
$$
\begin{array}{ll}\n\text{Theorem 1: For all associated functions } f \in \mathbf{F} \\
u(z, \xi, \delta) &= \int_{0}^{z} \mathbf{R}[f(t, \delta)] \, dt \\
(\mathbf{R}, \text{ the Riemann transform}) \text{ is a solution of (1) in } D_0 \times D_0^* \times G_0.\n\end{array}
$$
\n
$$
\begin{array}{ll}\n\text{Proof by-insertion of (7) into (1) using (3), (4).} \\
\text{Theorem 1':} \text{Let } G = G(z, \xi, \delta) \in F \text{ (for all } \xi \in D_0^*).\n\end{array}
$$
\n
$$
\begin{array}{ll}\n\cdot u_1(z, \xi, \delta) = \int_{0}^{z} \int_{0}^{z} \mathbf{R}[G(t, \tau, \delta)] \, dt \, dt \\
\text{is a solution of} \\
\mathbf{L}u_1 = G(z, \xi, \delta)\n\end{array}
$$
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$$
\begin{array}{ll}\n\cdot & \text{if } D_0 \times D_1^* \times G \text{ (B the Riemann transform)} \\
\cdot & \text{if } D_0 \times D_1^* \times G \text{ (B the Riemann transform)}\n\end{array}
$$
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$$
\begin{array}{ll}\n\cdot & \text{if } D_0 \times D_1^* \times G \text{ (B the Riemann transform)}\n\end{array}
$$

*Lu*₁ = $G(z, \xi, \lambda)$

in $D_0 \times D_0^* \times G_0$ (**R** the Riemann transform).

Proof by insertion of $(7')$ into $(1')$ using (3) , (4) , (5) , (6) .

Representation of Bergmani-Vekua-operators

Theorem 1: For all associated functions $f \in \mathbb{R}$
 $u(z, \xi, \xi) = \int \mathbf{R}[f(t, \xi)] dt$

(R the Riemann transform) is a solution of (1) in $D_0 \times D_0^* \times G_0$.

Proof by insertion of (7) There are two problems in this concept: the existence of the Riemann transform and its suitable representation. We give an answer only for a very special case.

Theorem 2: Let A be an operator not depending on z, ξ (this is, *if* $h = h(\delta)$ is a *function of f* alone, then $\bar{h} = Ah$ is also a function of *f* alone). The Riemann transform of equation (1) in $D_0 \times G_0$, where D_0 is a closed polydisc in **of Equation (1)** into (1) into (1) using (3), (4), (3), (6).

There are two problems in this concept: the existence of the Riemann tr

and its suitable representation. We give an answer only for a very special c

Theorem Lu₁ = $G(z, \xi, \delta)$
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function of λ alone, then $\tilde{h} = Ah$ is also a function of λ alone). The \tilde{h}
of equation (1) in $D_0 \times G_0$, where D_0 is a closed polydisc in

$$
D_z \times D_t^* \times D_t \times D_t^* \cap \{(z, \xi, t, \tau): |v| < 1/C\} \quad \text{with} \quad v = (z - t) \, (\tau - \xi)
$$

(8)

is given by

•

$$
\mathbf{R}[f(t, \lambda)] = \sum_{m=0}^{\infty} \frac{1}{m!^2} v^m \mathbf{A}^m[f(t, \lambda)]
$$

for all associated functions $f \in \mathbf{F}$ *.*

/ $Proof$ (and examples) see [7].

Resulting from the general nature of the operator A the above method is a very versatile one. But one of the main problems in the application of the method is'to. find a suitable representation of the transform R[f]. Following R. P. GILBERT *[4]* it is possible to express the transform R for elliptic and parabolic equations (1) by as given by
 $R[f(t, \delta)] = \sum_{m=0}^{\infty} \frac{1}{m!^2} v^m A^m[f(t, \delta)]$

for all associated functions $f \in F$.

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3. Representation of the Riemann transform

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 E transform **R** for ellipt

$$
\mathbf{R}[f(t, \delta)] = \frac{1}{2\pi i} \oint_{\mathbf{R}} H(z, \xi, t, \tau, \delta, s) f(t, s) \frac{ds}{s - \delta}, \qquad (9)
$$

here K is a circle in G_0 surrounding $s = s$. For pseudoparabolic equations (1) (of composed type) D.L. Corron [2] has given a representation of *R[f]* through a *Duhamel product* ible to express the integral
 R[*f*(*t*, 3)] = $\frac{1}{2\pi i}$
 R_[*f*(*t*, 3)] = $\frac{1}{2\pi i}$
 R[*j*(*t*, 3)] = $\frac{\partial}{\partial \delta}$ \int_0^1

$$
\mathbf{R}[f(t, \delta)] = \frac{\partial}{\partial \delta} \int\limits_0^t H(z, \xi, t, \tau, s - \delta) f(t, s) ds.
$$

For these cases both it is possible to construct the kernel H with the aid of differential equations found from (1) . For explicit examples see $[6, 8]$.

But for more general equations these representations of the transform $R[f]$ do not hold. We show by an example of an equation of composed type that the transform $R[f]$ is to be represented by a sum of Cauchy integrals and a Duhamel product. The representations of R. P. GILBERT and D. L. COLTON are special cases of it. Let therefore

$$
\mathbf{A} = \alpha D^p + \beta I + \gamma S^p,
$$

where α , β , γ are constants, D is the differentiation with respect to $\delta: D = \partial/\partial \delta$, S is the integration

$$
S=\int\limits_{0}^{3}\cdot ds,
$$

I is the identity, and let $p = 1$ or $p = 2$. Thus we consider the equation

$$
\frac{\partial^{2+p'}}{\partial z \partial \xi \partial \zeta^{p}} u + \alpha \frac{\partial^{2p}}{\partial \zeta^{2p}} u + \beta \frac{\partial^{p}}{\partial \zeta^{p}} u + \gamma u = 0.
$$
 (11)

Lemma 1: For $n=0$ one has

$$
A^{n}[f] = A_{I}^{[n]}[f] - \alpha \gamma \sum_{m=0}^{n-2} A^{n-2-m}[z_{m}(f)] \qquad (12)
$$

with

$$
A_{I}^{[n]}[f] = \sum_{k+l+m=n} \frac{n!}{k!l!m!} \alpha^{k} \beta^{l} \gamma^{m} L^{k-m}, \quad where \quad L^{j} = \begin{cases} D^{pj}, & j \geq 0, \\ S^{-pj}, & j < 0, \end{cases}
$$
(13)

and with

$$
z_m = \sum_{\substack{k+l+q=m+1 \ k \geq q}} \frac{(m+1)!}{(k+1)! \ l! \ q!} \alpha^k \beta^l \gamma^q (D^{p(k-q)} - S^p D^{p(k-q+1)}) [f].
$$

Remark: We give some explanations of these expressions. We have $DS = I$, $SD = I - N$ with $Nf(t, \lambda) = f(t, 0)$. From this we find

$$
D^{pk} - S^{p}D^{pk+p} = \begin{cases} ND^{k} & \text{for } p = 1, \\ ND^{2k} + SND^{2k+1} & \text{for } p = 2. \end{cases}
$$

Thus z_m is a constant with respect to δ (for $p=1$) or linearly depending on δ (for $p = 2$:

 $z_m = A_{m0} + A_{m10}$,

and the coefficients A_{m} $(j = 0, 1; m = 0, 1, 2, ...)$ are linearly depending on $f(t, 0)$ and the derivatives of $f(t, \lambda)$ with respect to λ in $\lambda = 0$:

$$
A_{mj} = \sum_{\substack{k+l+q=m\\k\geq q}} \frac{(m+1)!}{(k+1)! \, l! q!} \, \alpha^k \beta^l \gamma^q \, \frac{\partial^{pk-pq+j}}{\partial \delta^{pk-pq+j}} \, f(t, \, \delta)|_{\delta=0} \tag{14}
$$

for $j = 0, 1$, if $p = 2$, for $j = 0$ if $p = 1$, and

$$
A_{m1}=0 \quad \text{for} \quad p=1.
$$

Thus (12) reduces the problem to find $Aⁿ[f]$ for an arbitrary f to the problem to find it for a linear f. Furtherly: The definition (13) is made in such a way that we would have $\mathbf{A}^n = \mathbf{A}_I^{(n)}$ if $SD = DS = I$.

Representation of Bergman-Vekua-operators 5
Proof of Lemma 1 by induction: Obviously (12) is true for $n = 0$ and $n = 1$.

\n Representation of Bergman-Vekua-operators 5
\n Proof of Lemma 1 by induction: Obviously (12) is true for
$$
n = 0
$$
 and $n = 1$.
\n From the validity for *n* follows for $n + 1$ 4
\n
$$
A^{n+1}[f] = AA_I^{[n]}[f] - \alpha \gamma \sum_{m=0}^{n-2} A^{n-1-m}[z_m(f)].
$$
\n\n Consider $A^{[n+1]}[f] - AA_I^{[n]}[f]$. There is a contribution to this difference only from the (left) multiplication of the sum (13) by γS^p , and this contribution is\n

Consider $A^{[n+1]}[f] - AA^{[n]}[f]$. There is a contribution to this difference only from the (left) multiplication of the sum (13) by γS^p , and this contribution is

\n Representation of Bergman-Vekua-operators\n
$$
\text{Proof of Lemma 1 by induction: Obviously (12) is true for } n = 0 \text{ and } n
$$
\n From the validity for n follows for $n + 1$ \n

\n\n
$$
A^{n+1}[f] = AA^{[n]}[f] - \alpha \gamma \sum_{m=0}^{n-2} A^{n-1-m}[z_m(f)].
$$
\n

\n\n Consider
$$
A^{[n+1]}[f] = AA^{[n]}[f].
$$
 There is a contribution to this difference only the (left) multiplication of the sum (13) by γS^p , and this contribution is\n
$$
A^{[n+1]}[f] - AA^{[n]}[f] = \alpha \gamma \sum_{k+m+l=n} \frac{n!}{k! \, m! \, l!} \alpha^{k-1} \beta^l \gamma^m L^{k-m} = \alpha \gamma z_{n-1}.
$$
\n

\n\n Inserting this into (15) we have (12) holding for $n + 1$ instead of n .\n

\n\n Now we give a set of associated functions for the equation (11).\n
$$
\text{Lemma 2: } f \in F \text{ if } f = f(\cdot, \cdot, \cdot) \text{ is holomorphic with respect to } \cdot \cdot \cdot \text{ in } G \text{ and contain } \alpha, \beta, G_0 \subset G \text{ is given by}
$$
\n
$$
|s - s| \geq \delta > 0 \quad (\delta < 1)
$$
\n

\n\n for all $s \in \partial G, \cdot s \in G_0.$ Property (a) of Definition 1 may be fulfilled.\n
$$
\text{Proof: We prove the validity of condition (2) by the use of (12) in Lemma 2, i.e., } \alpha \text{ is a nontrivial.}
$$
\n

Inserting this into (15) we have (12) holding for $n + 1$ instead of n

Now we give a set of associated functions for the equation (11).

Lemma 2: $f \in \mathbf{F}$ if $f = f(\cdot, \mathfrak{z})$ is holomorphic with respect to \mathfrak{z} in G and continuous *in* \overline{G} , $G_0 \subset G$ *is given by* Now we give a set of associ

Lemma 2: $f \in \mathbf{F}$ if $f = f(\cdot)$
 $\vec{G}, G_0 \subseteq G$ is given by
 $|s - \mathbf{a}| \geq \delta > 0$

all $s \in \partial G$, $s \in G_0$, Property

$$
|s - \mathfrak{z}| \geq \delta > 0 \quad (\delta < 1)
$$

Consider $A^{[n+1]}[f] - AA_f^{[n]}[f]$. There is a contribution to this difference on
the (left) multiplication of the sum (13) by γS^p , and this contribution is
 $A_f^{[n+1]}[f] - AA_f^{[n]}[f] = \alpha \gamma \sum_{k+m} \frac{n!}{k! \, m! \, l!} \alpha^{k-1} \beta^l \gamma^$ Proof: We prove the validity of condition (2) by the use of (12) in Lemma 1. for all $s \in \partial G$, $\delta \in G_0$. Property (a) of Definition 1 may be fulfilled.
Proof: We prove the validity of condition (2) by the use of (12) in Lemma 1.
All constants, not depending on *n*, we denote by $c_1, c_2, ...$ withou their relations. A constant depending on 6 we denote by Ck(). Let a = niax-(jI, I, $|s - \lambda| \ge \delta > 0$
for all $s \in \partial G$, $\lambda \in G_0$. Pro
Proof: We prove the
All constants, not depend
their relations. A constantly, 1). Using $\frac{n!}{m!k!l!} \le \frac{1}{n!k!l!}$ $k+m+i=n$ *w.m.o.*
 $k>m$
 $k>m$
 $k>m$
 $k+m$
 $k+m$
 $k+m$
 $\frac{1}{k}m$
 Lemma $2: f \in \mathbf{F}$ if $f = f(\cdot, \delta)$ is holomorphic with respect to δ in G and continuous \overline{G} , $G_0 \subset G$ is given by
 $|s - \delta| \ge \delta > 0$ ($\delta < 1$)

all $s \in \partial G$, $\delta \in G_0$. Property (a) of Definition 1 may be fulfille *i.e.* \log_{10} *ii* $\int_{R_0}^{\infty} f(x) dx = \int_{\infty}^{\infty} f(x) dx$ *ii* $\int_{\infty}^{\infty} f(x) dx = \int_{\infty}^{\infty} f(x) dx$ *is* $\int_{\infty}^{\infty} f(x) dx = \int_{\infty}^{\infty} f(x) dx$ $\int_{\infty}^{\infty} f(x) dx = \int_{\infty}^{\infty} f(x) dx$ *is* $\int_{\infty}^{\infty} f(x) dx = \int_{\infty}^{\infty} f(x) dx$ *\int_{\in* $\lim_{m \to \infty} \frac{n!}{m! k! l!} \leq 3^n$ we have from (13)
 $\mathbf{A}_I^{[n]}[f] \leq (3a)^n \sum |L^{k-m}[f]|.$
 δ) $|< M$ in $D \times \overline{G}$ we have for $0 \leq k - L^{k-m}[f]| = |D^{p(k-m)}[f]| \leq M \frac{(pn)!}{\delta^{pn}}.$ **Example 12 Condition (2)** by the use of (12) in 1.
 E denote by c_1 , c_2 , ... without the description of the denote by $c_k(\delta)$. Let $a = \text{max}$
 For $0 \le k - m \le n$ from Cauchy's if $\frac{(pn)!}{\delta^{pn}}$.
 $\frac{n}{\delta^{pn}}$ *we h*

•

With $|f(z, \lambda)| < M$ in $D \times \overline{G}$ we have for $0 \le k - m \le n$ from Cauchy's inequality.

$$
|L^{k-m}[f]| = |D^{p(k-m)}[f]| \leq M \frac{(pn)!}{\delta^{pn}}.
$$

On the other hand, for $0 \leq m - k \leq n$ we have from

With
$$
|f(z, \delta)| < M
$$
 in $D \times G$ we have for $0 \le k - m \le n$ from
\n
$$
|L^{k-m}[f]| = |D^{p(k-m)}[f]| \le M \frac{(pn)!}{\delta^{pn}}.
$$
\nOn the other hand, for $0 \le m - k \le n$ we have from
\n
$$
S^{pj}[f] = \frac{1}{\Gamma(pj)} \int_{0}^{\delta} f(\ldots, s) (s - s)^{pj-1} ds.
$$
\n
$$
|L^{k-m}[f]| = |S^{p(k-m)}[f]| \le \frac{1}{\Gamma(pj)} M_{\delta}^{*pn} \le M_{\delta}^{*pn}.
$$
\nHere $s^* = \max(1, \text{diam } G)$. Thus for all k, m
\n $|L^{k-m}[f]| \le M \cdot c_1^{n}(\delta) \cdot (pn)!$.
\nThe sum (13) has $(n + 1)^2 \le 4^n$ terms, therefore we have

$$
|L^{k-m}[f]| = |S^{p(k-m)}[f]| \leq \frac{1}{\Gamma(pj)} M_{\delta}^{*pn} \leq M_{\delta}^{*pn}.
$$

Here $\delta^* = \max(1, \text{diam } G)$. Thus for all k, m
 $|L^{k-m}[f]| \leq M \cdot c_n(n(\delta) \cdot (pn)!)$.

The sum (13) has $(n + 1)^2 \leq 4^n$ terms, therefore we have
 $|A_I^{[n]}[f]| \leq M \cdot c_2^n(\delta) \cdot (pn)!$.

$$
|A_I^{[n]}[f]| \leq M \cdot c_2^{n}(\delta) \cdot (pn)!.
$$
 (16)

 $|L^{k-m}[f]| = |S^{p(k-m)}[f]| \leq \frac{1}{\Gamma(pj)} M_{\delta}^{*pn} \leq M_{\delta}^{*pn}.$

Here $\delta^* = \max(1, \text{diam } G)$. Thus for all k, m
 $|L^{k-m}[f]| \leq M \cdot c_1^n(\delta) \cdot (pn)!$.

The sum (13) has $(n + 1)^2 \leq 4^n$ terms, therefore we have
 $|A_I^{[n]}[f]| \leq M \cdot c_2^n(\delta) \cdot (pn)!$ Here $\delta^* = \max(1,$
 $|L^{k-m}[f]|$

The sum (13) has (
 $|A_I^{[n]}[f]|$ is

Now we estimate

of terms
 $S^*D^{\mu}[z_m]$ $\begin{aligned} [m] &\leq \frac{1}{\Gamma(p)} M \delta^{+p} \leq M \delta^{+p}. \end{aligned}$
 $\begin{aligned} [m] &\leq \frac{1}{\Gamma(p)} M \delta^{+p}. \end{aligned}$
 $\begin{aligned} [m] &\geq \frac{1}{\Gamma(p)} \end{aligned}$. Thus for all k, m
 $\begin{aligned} [m] &\leq \frac{1}{\Gamma(p)} M \delta^{+p}. \end{aligned}$
 $\begin{aligned} [m] &\geq \frac{1}{\Gamma(p)} M \delta^{+p}. \end{aligned}$
 $\begin{aligned} [m] &\geq \$ *Denoted II + Amoc (1, diam G).* Thus for all k, m
 $k-m[f] \leq M \cdot c_1^n(\delta) \cdot (pn)!$.

3) has $(n + 1)^2 \leq 4^n$ terms, therefore we have
 $I^{[n]}[f] \leq M \cdot c_2^n(\delta) \cdot (pn)!$.

timate the polynomial $\sum_{m=0}^n \mathbf{A}^{n-m}[z_m(f)]$. Each summand

$$
S^{\kappa}D^{\mu}[z_m] = A_{m0}S^{\kappa}D^{\mu}[1] + A_{m1}S^{\kappa}D^{\mu}[3] \qquad (\kappa, \mu \leq n-m \leq n),
$$

 $\begin{array}{l} \mathbf{r}^{\mathbf{r}}_{\mathbf{r}}(\mathbf{r})=\mathbf{r}^{\mathbf{r}}_{\mathbf{r}}(\mathbf{r})\ \mathbf{r}^{\mathbf{r}}_{\mathbf{r}}(\mathbf{r})=\mathbf{r}^{\mathbf{r}}_{\mathbf{r}}(\mathbf{r})\ \mathbf{r}^{\mathbf{r}}_{\mathbf{r}}(\mathbf{r})=\mathbf{r}^{\mathbf{r}}_{\mathbf{r}}(\mathbf{r})\ \mathbf{r}^{\mathbf{r}}_{\mathbf{r}}(\mathbf{r})=\mathbf{r}^{\mathbf{r}}_{\mathbf{r}}(\mathbf{r})\ \$

i.e. of powers of δ with the highest degree pn or pn + 1; thus

$$
|S^*D^{\mu}[z_m]| \leq \lambda^{*pn+1}(|A_{m0}| + |A_{m1}|).
$$

 A^{n-m} has $3^{n-m} \leq 3^n$ such terms, multiplied by $\alpha^k \beta^l \gamma^q$ with $k+l+q \leq n-m \leq n$, therefore

$$
|A^{n-m}[z_m(f)]| \leq c_3 n_3^* \cdot 2 \max_{j=0,1} |A_{mj}|
$$

and

$$
\left|\sum_{m=0}^n \mathbf{A}^{n-m}[z_m(f)]\right| = (n+1) c_3^n \mathfrak{F}^* 2 \max_{\substack{j=0,1\\ m=0,1,\ldots,n}} |A_{mj}|.
$$

From (14) we find, again using Cauchy's inequality, as above

$$
|A_{mj}| \leq (m+1)^2 3^{m+1} a^m M \frac{(pm+j)!}{\delta^{pm+j}},
$$

$$
\max |A_{mj}| \leq \begin{cases} (n+2)! 3^{n+1} a^n M \delta^{-n} & \text{for } p = 1, \\ (2n+3)! 3^{n+1} a^n M \delta^{-2n-1} & \text{for } p = 2. \end{cases}
$$

From this we have

$$
\left|\sum_{m=0}^n \Lambda^{n-m}[z_m(f)]\right| \leqq \begin{cases} (n+3)!\,c_4{}^n(\delta)\cdot 2M\delta^* & \text{for } p=1, \\ (2n+4)!\,c_5{}^n(\delta)\cdot 2M\,\frac{\delta^*}{\delta} & \text{for } p=2, \end{cases}
$$

and together with (12) and (16) we have (using $(2n)! \leq 4^n \cdot n!^2$)

$$
|\mathbf{A}^n[f]| \leq \begin{cases} 2M_{\delta}^* \cdot c_6 n(\delta) \cdot (n+1)! & \text{for } p = 1, \\ 2M \cdot \frac{\delta^*}{\delta} \cdot c_7 n(\delta) \cdot 16(n+2)!^2 & \text{for } p = 2. \end{cases}
$$
(17)

This is (2)

Remark: For $p = 1$ we have an essential fact. From the convergence of the series $\sum d^m/m!$ we have $d^n < (n + 1)!$ for every d and sufficiently large $n > n_0(d)$. Thus.

$$
|A^n[f]| \leq 2M\delta^* \left(\frac{c_5(\delta)}{d}\right)^n (n+1)!^2
$$

 $R[f] = R_1[f] - \alpha \gamma \cdot R_2[f]$

for arbitrary $d > 0$, if $n > n_0(d)$. This means, (2) holds for all $C > 0$.

Now we may construct the Riemann transform. From (8) and (12) we have

$$
\dot{\mathbf{w}}^{\mathbf{ith}}
$$

$$
\mathbf{R}_1 = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} v^n \mathbf{A}_I^{[n]}, \qquad \mathbf{R}_2[f] = \sum_{n=2}^{\infty} \frac{v^n}{n!^2} \sum_{m=0}^{n-2} \mathbf{A}^{n-2-m}[z_m(f)].
$$

Firstly we consider $R_1(f)$: By simple calculations we find from (13) the symmetric expression

$$
\mathbf{R}_{1} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n!m!k!(n+m+k)!} \alpha^{n} \beta^{m} \gamma^{k} v^{n+m+k} L^{n-k}
$$

$$
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n!k!} \left(\alpha \sqrt{\frac{v}{\beta}} \right)^{n} \left(\gamma \sqrt{\frac{v}{\beta}} \right)^{k} I_{n+k} \left(2 \sqrt{v \beta} \right) L^{n-k}
$$

 $\frac{1}{6}$

7

 (18)

with the modified Bessel
function I_{n+k} . From this we have

$$
\mathbf{R}_1 = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(n+k)! \, k!} \left(\sqrt{\frac{v}{\beta}} \right)^n \left(\alpha \gamma \sqrt{\frac{v}{\beta}} \right)^k I_{n+2k} (2 \sqrt{v \beta}) \left(\alpha^n D^{pn} + \gamma^n S^{pn} - \delta_{n0} \cdot I \right)
$$

with the Kronecker symbol δ_{n0} . Representing the differentiations D^{pn} by Cauchy integrals, the integrations Spn by Duhamel products

$$
S^{pn}[f(\ldots,\delta)]=\frac{1}{(pn)!}\frac{\partial}{\partial\delta}\int_{0}^{\delta}(\delta-s)^{pn}f(\ldots,s)\,ds
$$

we find.

$$
\mathbf{R}_1[f] = \frac{1}{2\pi i} \oint\limits_k H_1 f(t,s) \frac{ds}{s-s} + \frac{\partial}{\partial s} \int\limits_0^{\cdot} H_{-1} \cdot f(t,s) \, ds \, - H_1(0) \cdot f(t,s)
$$

with -

$$
H_1 = H_1(\alpha(s-\delta)^{-p}), \qquad H_{-1} = H_{-1}(\gamma(\delta - \delta)^p)
$$

 $and \, \cdot$

 \mathcal{C}^{∞} .

$$
H_{\delta}(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(pn)!}{(n+k)! k!} \left(\frac{\alpha \gamma v}{\beta}\right)^{k} \left(\sqrt{\frac{v}{\beta}}\right)^{n} \cdot I_{n+2k}\left(2 \sqrt{v \beta}\right) x^{n}.
$$

By a simple way we prove the convergence of this series. Firstly we use

$$
I_{n+2k}(2 \sqrt{|v\beta|}) \leq |v\beta|^{k+n/2} \cdot \frac{1}{(n+2k)!} I_0 \left(2 \sqrt{|v\beta|}\right)
$$

and from this immediately follows

$$
|H_{\delta}| \leq I_0\left(2\left|\sqrt{v\beta}\right|\right) \sum_{k=0}^{\infty} \frac{1}{(2k)! |k|^2} |\alpha \gamma v^2| \sum_{n=0}^{\infty} \frac{(pn)!^{\delta}}{(n!)^2} \cdot |vx|^n.
$$

The generalized hypergeometric series

$$
\sum_{k=0}^{\infty} \frac{1}{(2k) \, |\, (k!)^2} \, |\alpha \gamma v^2| = {}_0F_3(1, 1, 1/2; 4 \, |\alpha \gamma v^2|)
$$

converges everywhere, the series

$$
\sum_{n=0}^{\infty} \frac{(pn)!^{\delta}}{(n!)^2} |vx|^n = \begin{cases} e^{|vx|} & \text{for } p = 1, \delta = 1, \\ 0^F_2(1, 1; |vx|) & \text{for } p = 1, \delta = -1 \\ \text{(generalized hypergeometric series)} \\ \frac{1}{\sqrt{1 - 4 |vx|}} & \text{for } p = 2, \delta = 1, \\ 0^F_3(1, 1, 1/2, 4 |vx|) & \text{for } p = 2, \delta = -1; \end{cases}
$$

converges for $p = 1$ and for $p = 2$, $\delta = -1$ everywhere, for $p = 2$, $\delta = 1$ for $|vx| < 1/4$. (We remark, that these assertions coincide with the remarks on the choice of C in (2) , made in the proof of Lemma 2.)

Secondly we consider the transform $R_2[f]$: We write it symmetrically

$$
\mathbf{R}_{2}[f] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{v^{n+m+2}}{((n+m+2)!)^2} \mathbf{A}^{n}[z_{m}(f)].
$$

8 E. LANCKAU

Z_m(*f*) depends linearly on */(t, 0)* and the derivatives of *f*(*t,* δ) at the point $\delta = 0$ up to.

the order $p(m + 1) - 1$, see (14). We express all these derivatives by Cauchy in-

tegrals, thus we have
 $z_m[f(t, \delta$ the order $p(m + 1) - 1$, see (14). We express all these derivatives by Cauchy integrals, thus we have on $f(t, 0)$ and the derivatives of $f(t, \delta)$ at the point 1, see (14). We express all these derivatives
 $\frac{1}{\pi i} \oint_{K_0} z_m \left[\frac{1}{s - \delta} \right] f(t, s) \frac{ds}{s},$ **E.** LANCRAU
 $n(t)$ depends linearly on $f(t, 0)$ and the derivatives of $f(t, \delta)$ at the pierroof $p(m + 1) - 1$, see (14). We express all these derivatives

ggrals, thus we have
 $z_m[f(t, \delta)] = \frac{1}{2\pi i} \oint_{K_0} z_m \left[\frac{1}{s - \delta} \right]$ B. E. LANCKAU
 $z_m(t)$ depends linearly on $f(t, 0)$ and the derivatives of $f(t,$

the order $p(m + 1) - 1$, see (14). We express all these

tegrals, thus we have
 $z_m[f(t, \delta)] = \frac{1}{2\pi i} \oint_{K_0} z_m \left[\frac{1}{s - \delta} \right] f(t, s) \frac{ds}{s},$
 K

E. LASCKAU
ends linearly on
$$
f(t, 0)
$$
 and the derivatives
or $p(m + 1) - 1$, see (14). We express all
thus we have

$$
z_m[f(t, \delta)] = \frac{1}{2\pi i} \oint_{K_0} z_m \left[\frac{1}{s - \delta} \right] f(t, s) \frac{ds}{s},
$$

 K_0 a circle in G_0 , surrounding $s = 0$. Finally we have

$$
\mathbf{R}_2[f] = \frac{1}{2\pi i} \oint \mathbf{H}_0 \cdot f(t, s) \frac{ds}{s}
$$

V

V

 \cdot ^V

V

 $\frac{1}{\sqrt{2}}$

N

$$
K_0
$$
 a circle in G_0 , surrounding $s = 0$. Finally we have
\n
$$
\mathbf{R}_2[f] = \frac{1}{2\pi i} \oint_{K_0} H_0 \cdot f(t, s) \frac{ds}{s}
$$
\nwith the series
\n
$$
H_0 = H_0(z, \xi, t, \tau, \delta, s) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{v^{n+m+2}}{((n+m+2)!)^2} \mathbf{A}^n \left[z_m \left(\frac{1}{s - \delta} \right) \right].
$$
\n(19)
\nWe omit here the detailed proof of the convergence of this series. It is based on the

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We omit here the detailed proof of the convergence of this series. It is based on the fact that $z_m\left[\frac{1}{s-1}\right] = A_{m0} + A_{m10}$, where the coefficients A_{mj} are found explicitly by (14) : series
 $H_0 = H_0(z, \xi, t, \tau, \lambda, s) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{v^{n+m+2}}{((n+m+2)!)^2} \mathbf{A}^n \left[z_m \left(\frac{1}{s-\lambda} \right) \right]$.

here the detailed proof of the convergence of this series. It is based o
 $z_m \left[\frac{1}{s-\lambda} \right] = A_{m0} + A_{m1\lambda}$, *k*₆
 $k, \xi, t, \tau, \lambda, s$ = $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{v^{n+m+2}}{((n+m+2)!)^2} \mathbf{A}^n \Big[z_m \Big(\frac{1}{s-\lambda} \Big) \Big]$

letailed proof of the convergence of this series. It is ba
 $\left[\frac{1}{s} - A_{m0} + A_{m1\lambda} \right]$, where the coefficients $\mathbf{R}_2[f] = \frac{1}{2\pi i} \oint_{\kappa_2} H_0 \cdot f(t, s) \frac{ds}{s}$

with the series
 $H_0 = H_0(z, \xi, t, \tau, \lambda, s) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{v^{n+m}}{((n+m+1)!)}$

We omit here the detailed proof of the convergence

fact that $z_m \left[\frac{1}{s-\lambda} \right] = A_{m0} +$

$$
A_{mj}=s^{-j-1}\sum_{\substack{k+l+q=m\\k\geq q}}\frac{(m+1)!(pk-pq+1)!}{(k+1)!\,l!q!}\left(\frac{\alpha}{s^p}\right)^k\beta^l(\gamma\cdot s^p)^q.
$$

Theorem 3: The Riemann transform of equation (11) is given for $f \in F$ by

$$
H_0 = H_0(z, \xi, t, \tau, \delta, s) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{v^{n+m}}{((n+m+2)!)^2} A^n \Big[z_m \Big(\frac{1}{s-\delta} \Big) \Big].
$$
 (19)
We omit here the detailed proof of the convergence of this series. It is based on the
fact that $z_m \Big[\frac{1}{s-\delta} \Big] = A_{m0} + A_{m1\delta}$, where the coefficients A_{mj} are found explic-
itly by (14):

$$
A_{mj} = s^{-j-1} \sum_{\substack{k+l=n \\ k \geq q}} \frac{(m+1)! (pk - pq + 1)!}{(k+1)! l! q!} \Big(\frac{x}{s^p} \Big)^k \beta^l (\gamma \cdot s^p)^q.
$$
By these considerations we have found:
Theorem 3: The Riemann transform of equation (11) is given for $f \in \mathbf{F}$ by

$$
\sum_{k=0}^{\infty} \mathbf{R}[f(t, \delta)] = \frac{1}{2\pi i} \Big[\oint_{\kappa_{\delta}} H_1 \cdot f(t, s) \frac{ds}{s-\delta} - \alpha \gamma \oint_{\kappa_{\delta}} H_0 \cdot f(t, s) \frac{ds}{s} \Big] + \frac{\partial}{\partial \delta} \int_{\delta}^{s} H_{-1} \cdot f(t, s) ds - H_1(0) f(t, \delta),
$$

here K_{δ} is a circle in G_0 surrounding $s = \delta$, the kernels $H_{\pm 1}$, H_0 are given by (18), (19)
Remark: In the equation (11) are some interesting special cases. (Noticing that
 $H_1(0) = H_{-1}(0)$.) For $\alpha = 0$ we have a pseudoparabolic equation

here K $_{3}$ *is a circle in G*₀ surrounding $s =$ $_{3}$, the kernels H _{±1}, H ₀ are given by (18), (19). Remark: In the equation (11) are some interesting special cases. (Noticing that $H_1(0) = H_{-1}(0)$.) For $\alpha = 0$ we have a pseudoparabolic equation $+\frac{\partial}{\partial \delta} \int_{0}^{s} H_{-1} \cdot f(t, s) ds - H_{1}(0) f(t, \delta),$
 s a circle in G_{0} surrounding $s = \delta$, the kernels $H_{\pm 1}$, H_{0} are given by
 rk: In the equation (11) are some interesting special cases. (No
 $H_{-1}(0)$.) For $+\frac{\partial}{\partial \delta} \int H_{-1} \cdot f(t, s) ds - H_1(0) f(t, s))$
 in G_0 surrounding $s = \delta$, the kernels H

le equation (11) are some interesting
 For $\alpha = 0$ we have a pseudoparabolic
 $u + \beta \frac{\partial^p}{\partial \delta^p} u + \gamma u = 0;$

sform is a single Duh $\frac{1}{2}$, $\frac{1}{2}$, H_0 are gives

special cases.

equation

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 $\frac{1}{2}$
 $\frac{1}{2}$ studied
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 $\frac{1}{2}$ studied

$$
\frac{\partial^{2+p}}{\partial z \partial \xi \partial \zeta^{p}} u + \beta \frac{\partial^{p}}{\partial \zeta^{p}} u + \gamma u = 0; \qquad (11')
$$

the Riemann transform is a single Duhamelproduct (without Cauchy integral), and a solution of equation (11') is

$$
H_{-1}(0), F \circ F \circ \alpha \equiv 0 \text{ we have a pseudo-\n\frac{\partial^2 F}{\partial z \partial \xi \partial \delta^p} u + \beta \frac{\partial^p}{\partial \delta^p} u + \gamma u = 0;
$$

\n
$$
\text{mann transform is a single Duhamel}
$$

\n
$$
\text{on of equation (11') is}
$$

\n
$$
u(z, \xi, \xi) = \int_0^z \frac{\partial}{\partial \xi} \int_0^{\xi} H_{-1}/(t, s) \, ds \, dt.
$$

Using expressions of this type (for $p = 1$) D. L. COLTON [2] studied the properties of the solutions of pseudoparabolic equations. In our special case we have *1 ds dt.*

= 1) D. L. CoLTC

uations. In our sp
 $\sqrt{\frac{v}{\beta}}^n \cdot x^n$.

$$
H_{-1} = \sum_{n=0}^{\infty} \frac{1}{n!(pn)!} I_n(2\sqrt{p}v) \sqrt{\frac{v}{\beta}}^n \cdot x
$$

Representation of Bergman-Vekua-
\nFor
$$
\alpha = \beta = 0
$$
 this kernel is a generalized hypergeometric function
\n
$$
H_{-1} = \begin{cases} \sigma F_2(1, 1; \gamma(\delta - s) v) & \text{for } p = 1, \\ \sigma F_3(1, 1, 1/2; 4\gamma(\delta - s) v) & \text{for } p = 2. \end{cases}
$$
\nFor $\gamma = 0$, $p = 1$ we have a parabolic' equation
\n
$$
\frac{\partial^2}{\partial z \partial \xi} u + \alpha \frac{\partial}{\partial \delta} u + \beta u = 0,
$$

For $\gamma = 0$, $p = 1$ we have a parabolic' equation

Representation of Bergman-Vekua-operators
\n
$$
\beta = 0
$$
 this kernel is a generalized hypergeometric function
\n
$$
H_{-1} = \begin{cases} {}_{0}F_{2}(1, 1; \gamma(\delta - s) v) & \text{for } p = 1, \\ {}_{0}F_{3}(1, 1, 1/2; 4\gamma(\delta - s) v) & \text{for } p = 2. \end{cases}
$$
\n
$$
= 0, p = 1 \text{ we have a parabolic' equation}
$$
\n
$$
\frac{\partial^{2}}{\partial z \partial \xi} u + \alpha \frac{\partial}{\partial \delta} u + \beta u = 0,
$$
\n(11'')
\n
$$
\text{sform R}[f] \text{ is a single Cauchy integral (without Duhamel product), and a of equation (11'') is}
$$

 $\begin{array}{cc}\n & \text{For } t \\
 & \text{otherwise}\n \end{array}$ the transform $R[f]$ is a single Cauchy integral (without Duhamel product), and a solution of equation (11") is

Representation of Bergman-Vekua-operators
\nFor
$$
\alpha = \beta = 0
$$
 this kernet is a generalized hypergeometric function
\n
$$
H_{-1} = \begin{cases} 0^{\Gamma_2}(1, 1; \gamma(3 - s) v) & \text{for } p = 1, \\ 0^{\Gamma_3}(1, 1, 1/2; 4\gamma(3 - s) v) & \text{for } p = 2. \end{cases}
$$
\nFor $\gamma = 0, p = 1$ we have a parabolic' equation
\n
$$
\frac{\partial^2}{\partial z \partial \xi} u + \alpha \frac{\partial}{\partial \xi} u + \beta u = 0,
$$
\nthe transform $\mathbf{R}[f]$ is a single Cauchy integral (without Duhamel product), and a solution of equation (11'') is
\n
$$
u(z, \xi, \xi) = \frac{1}{2\pi i} \int_{0}^{\xi} \oint H_1 \cdot f(t, s) \frac{ds}{s - \xi} dt.
$$
\n(20)
\nHere we have

Here we have

$$
H_1 = \Phi_3 \left(1, 1; \frac{\alpha v}{s - \beta}, \beta v \right)
$$

-- this is a hypergeometric function of two variables $[3]$ -, for $\beta = \gamma = 0$ we have simply

$$
H_1=\exp\frac{\alpha v}{s-\frac{1}{6}}.
$$

The latter result is due to C. D. HILL [5].

For $\gamma = 0$, $p = 2$ we have an elliptic equation

$$
H_1 = \exp \frac{\alpha v}{s - \delta}.
$$

For result is due to C. D. HILL [8]

$$
= 0, p = 2
$$
 we have an elliptic

$$
\frac{\partial^2}{\partial z \partial \xi} u + \alpha \frac{\partial^2}{\partial \delta^2} u + \beta u = 0;
$$

 $u(z, \xi, \delta) = \frac{1}{2\pi i} \int_{0}^{1} \int_{K_{\delta}} H_1 \cdot f(t, \delta) \frac{1}{s - \delta} dt$

We have
 $H_1 = \Phi_3 \left(1, 1; \frac{\Delta v}{s - \delta}, \beta v \right)$

is a hypergeometric function of two varia
 $H_1 = \exp \frac{\Delta v}{s - \delta}$.
 $u = \exp \frac{\Delta v}{s - \delta}$.
 $v = 0, p = 2$ we have an elli again the transform $R[f]$ is a single Cauchy integral, and the solution u is given by (20). Using the equivalent Bergman operators together with Cauchy integrals D. L $\frac{\partial^2}{\partial z \partial \xi} u + \alpha \frac{\partial^2}{\partial \xi^2} u + \beta u = 0;$
again the transform **R**[*f*] is a single Cauchy integral, and the solution *u* is given by
(20). Using the equivalent Bergman operators together with Cauchy integrals D. L.
COL dimensional" elliptic equations and of parabolic equations with two space variables.
Here the kernel is - this is a hypergeometric function of two variables $[3]$ -, for $\beta = \gamma = 0$ we have

simply
 $H_1 = \exp \frac{\alpha v}{s - \delta}$.

The latter result is due to C.D. HILL [5].

For $\gamma = 0$, $p = 2$ we have an elliptic equation
 $\frac{\partial^2}{\partial z$ • tter result is due to C. D. HILL [5]
 $\gamma = 0$, $p = 2$ we have an elliptic ϵ
 $\frac{\partial^2}{\partial z \partial \xi} u + \alpha \frac{\partial^2}{\partial \xi^2} u + \beta u = 0$;

the transform **R**[*f*] is a single Caud

Jsing the equivalent Bergman oper

N [2] and R. P. GLE $s = \frac{3}{2}$

The latter result is due to C. D.
 For $\gamma = 0$, $p = 2$ we have ar
 $\frac{\partial^2}{\partial z \partial \xi} u + \alpha \frac{\partial^2}{\partial \xi^2} u + \beta$

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(20). Using the equivalent Berg

COLTON [2] and R. P. GILBER The latter result is due to C.

For $\gamma = 0$, $p = 2$ we have
 $\frac{\partial^2}{\partial z \partial \xi} u + \alpha \frac{\partial^2}{\partial \xi^2} u$

again the transform R[f] is :

(20). Using the equivalent B

COLTON [2] and R. P. GLBE

dimensional" elliptic equation

H $\frac{\partial^2}{\partial z \partial \xi} u + \alpha \frac{\partial^2}{\partial \xi^2} u +$
 21 i transform **R**[*f*] is a
 ng the equivalent Be

21 and R. P. GILBER

21 and R. P. GILBER
 *R*₁ = Φ_2 (1/2, 1, 1;
 *A*₁ = Φ_2 (1/2, 1, 1;
 *A*₁ = $\frac{s-3}{\sqrt{(s-3)^$ igle Cauchy integral, and the s

man operators together with C_é

4] constructed in this way the

and of parabolic equations with
 $\frac{\alpha v}{-\lambda}$, βv
 $\left(\frac{\alpha v}{\lambda}\right)$
 $\left(\frac{\alpha v}{\lambda}\right)$
 $\left(\frac{\alpha v}{\lambda}\right)$
 $\left(\frac{\alpha v}{\lambda}\right)$

$$
H_1 = \Phi_2 \left(1/2, 1, 1; \frac{4 \alpha v}{(s - \lambda)^2}, \beta v \right)
$$

 $\frac{1}{2}$ or $\frac{1}{2}$ or

$$
H_{1i} = \frac{s - \delta}{\sqrt{(s - \delta)^2 - 4\alpha v}}.
$$

- **again a hypergeometric function of two variables** [3] -, for $\beta = \gamma = 0$ we have simply [6]
*H*₁ = $\frac{s - \delta}{\sqrt{(s - \delta)^2 - 4\alpha v}}$.
Finally, for $\alpha = \gamma = 0$ we have a two-dimensional equation; the Riemann transform is th dimensional" elliptic equations and of parabolic equations with two space variables.

Here the kernel is
 $H_1 = \Phi_2 \left(1/2, 1, 1; \frac{4\alpha v}{(s-3)^2}, \beta v \right)$
 $-$ again a hypergeometric function of two variables [3] $-$, for $\$ Finally, for $\alpha = \gamma = 0$ we have a two-dimensional equation; the Riemann transform is the multiplication with the Riemann function (due to I. N. VERUA)
 $\mathbf{R}[f] = H_1(0) \cdot f(t, \mathfrak{z}) = I_0(2 \sqrt{\beta v}) f(t, \mathfrak{z}).$

(For $\alpha = \beta = \gamma =$ - again a hypergeometric function of two variables [3] -, for $\beta = \gamma = 0$ we have
simply [6]
 $H_L = \frac{s - \delta}{\sqrt{(s - \delta)^2 - 4\alpha v}}.$
Finally, for $\alpha = \gamma = 0$ we have a two-dimensional equation; the Riemann transform is the multiplica

$$
\mathbf{R}[f] = H_1(0) \cdot f(t, \mathbf{\delta}) = I_0\left(2 \sqrt{\beta v}\right) f(t, \mathbf{\delta}).
$$

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