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On a Boundary Value Problem for a Special Class of First Order Semilinear Elliptic Systems in the Plane

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Mit Hilfe bekannter Resultate von Brézis und Browder über Hammersteinsche Integralgleichungen werden Existenzsätze für eine Klasse von Riemann-Hilbert-Problemen bei Vekuaschen Differentialgleichungssystemen mit potenzartiger Nichtlinearität hergeleitet.

С помощью известных результатов Брезиса и Браудера об интегральных уравнениях типа Гаммерстейна доказываются теоремы существования для класса задач Римана-Гильберта у систем уравнений с частными производными типа И. Н. Векуа со степенной нелинейностью.

By means of known results of Brézis and Browder on integral equations of Hammerstein type existence theorems for a class of Riemann-Hilbert problems for Vekua's differential equation systems with power-like nonlinearity are proved.

Introduction

In the recent papers of the writer [6, 7] methods of monotone operator theory are applied to some classes of boundary value problems for first order semilinear elliptic systems in the plane. Transforming the boundary value problems to Hammerstein integral equations on the domain or the boundary of the domain, we obtained existence assertions for superlinear nonlinearities of a special kind up to growth order three. In this note we deal with linear boundary value problems for a typical special class of such nonlinearities. Utilizing results of H. Brézis and F. E. Browder [1–4] on integral equations of Hammerstein type, we show on the one hand that for this special class of problems for nonlinearities up to growth order three the usual monotonicity and coercivity assumptions on the nonlinearity can be omitted and on the other hand that under these assumptions also nonlinearities with growth order greater than three can be handled.

Statement of problem

Let G be the unit disk in the complex z plane with boundary $\Gamma = \{t = e^{is}: -\pi \leqq s \leqq \pi\}$. The *problem* is to find a solution $w(z) = u + iv$ to the differential equation

$$\partial w/\partial \bar{z} - a(z)\,\overline{w} = H(z, w) + F(z) \quad \text{in } G, \tag{1}$$

the boundary condition

$$u(s) - \mu(s)\,v(s) = f(s) \quad \text{on } \Gamma, \tag{2}$$

and the additional condition

$$v(0) - \nu u(0) = c \quad \text{in } z = 0. \tag{3}$$

The data in (1)—(3) fulfil the following *assumptions* (i)—(v):

$$a(z) \in L_p(G), \qquad p > 2, \quad \text{with} \tag{i}$$

$$\operatorname{Re} \frac{a(z)}{z} \geqq 0 \quad \text{a.e. in } G. \tag{4}$$

$$H(z, w) = z\overline{w}h(|w|, z), \tag{ii}$$

where the non-negative real-valued function $h(\xi, z)$, $0 \leqq \xi < \infty$ is continuous with respect to ξ in $0 < \xi < \infty$ for almost all $z \in G$, measurable with respect to z in G for all $0 \leqq \xi < \infty$, and the function $g(\xi, z) = \xi h(\xi, z)$ satisfies the inequality

$$g(\xi, z) \leqq E(z) + D\xi^{r-1}, \qquad 2 \leqq r < \infty, \tag{5}$$

with a real-valued function $E(z) \in L_s(G)$, s the exponent conjugate to r, and a positive constant D; and the condition $g(0, z) = 0$ for almost all z in G.

$$F(z) \in L_s(G) \quad \text{with} \quad \frac{1}{z} F(z) \in L_1(G). \tag{iii}$$

$\mu(s) \in H_\alpha(\Gamma)$, $\quad 0 < \alpha \leqq 1$, is a real Hölder continuous function and ν is a real constant, where (iv)

$$|\mu(s)| \leqq 1, \qquad -\pi \leqq s \leqq \pi, \quad \text{and} \quad |\nu| \leqq 1, \tag{6}$$

and in addition to inequality (4) either this inequality holds in strict form in a subset of G with positive measure or $\nu\mu(s) \not\equiv 1$, i.e., the cases $\mu(s) \equiv 1, \nu = 1$ and $\mu(s) \equiv -1$, $\nu = -1$ are excluded, and for $\nu\mu(s) \equiv -1$ the additional inequalities

$$\operatorname{Im} \frac{a(z)}{z} \leqq 0 \quad \text{and} \quad \operatorname{Im} \frac{a(z)}{z} \geqq 0 \quad \text{a.e. in } G \tag{7}$$

in the case $\mu(s) \equiv 1$, $\nu = -1$ and $\mu(s) \equiv -1$, $\nu = 1$, respectively, are fulfilled.

$$\begin{aligned} &f(s) \in L_\delta(\Gamma) \quad \text{with} \quad r/2 \leqq \delta < \infty \quad \text{if} \quad r > 2 \\ &\text{and} \quad 1 < \delta < \infty \quad \text{if} \quad r = 2 \end{aligned} \tag{v}$$

is a real summable function and c is a real constant.

We ask for *generalized solutions* $w(z) \in L_r(G)$ of (1)—(3) which possess generalized derivatives in Sobolev sense (cf. [5]) $\partial w/\partial \bar{z} \in L_s(G)$ and boundary values $w(t) \in L_\gamma(\Gamma)$ with $\gamma = \min [\delta, s/(2-s)]$ and which are continuous for $z = 0$.

Existence theorems

Under the above assumptions the boundary value problem (1)—(3) is equivalent to the following *Hammerstein equation* (cf. [6])

$$w = MNw = \varphi \quad \text{in } L_r(G), \tag{8}$$

where $\varphi \in L_r(G)$ is a known function determined by the data, $N: L_r(G) \to L_s(G)$ is the Nemytski operator

$$(Nw)(z) = \frac{1}{\bar{z}} \overline{H(z, w)}, \tag{9}$$

and M is a linear integral operator of the form

$$(M\psi)(z) = \iint_G [M_1(z, \zeta)\, \psi(\zeta) + M_2(z, \zeta)\, \overline{\psi(\zeta)}]\, d\xi\, d\eta \tag{10}$$

with kernels $M_j(z, \zeta)$, $j = 1, 2$, continuous for $\zeta \neq z$ and satisfying the estimates

$$|M_j(z, \zeta)| \leqq \frac{C_1 |\zeta|}{|z - \zeta|} + C_2 \leqq \frac{C_0}{|z - \zeta|} \tag{11}$$

with certain positive constants C_k, $k = 0, 1, 2$. Moreover, the operator M fulfils the inequality

$$\operatorname{Re} \iint_G \psi(z)\, \overline{M\psi}\, dx\, dy \geqq 0 \tag{12}$$

for any $\psi \in L_\varrho(G)$, $\varrho \geqq 4/3$.

Firstly, let be $2 \leqq r < 4$. Then the operator $M\colon L_s(G) \to L_s(G)$ is compact and monotone. Further, there holds the relation

$$\operatorname{Re}\left[w \frac{1}{z} H(z; w)\right] = |w|^2 h(|w|, z) = |w| \left|\frac{1}{z} H(z, w)\right|. \tag{13}$$

Therefore, from Theorem 4 of [2] (cf. also [7]) we obtain

Theorem 1: *Under the above assumptions* (i)—(v) *the problem* (1)—(3) *has a generalized solution* $w(z) \in L_r(G)$ *if* $2 \leqq r < 4$.

Remark: It suffices to assume the function $h(\xi, z)$ as real-valued and nonnegative for $\xi > R$ with some $R > 0$, only. (The inequality (5) has to hold for the absolute value of $g(\xi, z)$ then.)

In the case $r = 4$ the operator $M\colon L_s(G) \to L_r(G)$ is merely bounded and monotone. We additionally suppose that the function $g(\xi, z)$ is *non-decreasing* in ξ for almost all z in G. Then $H(z, w) = z[\partial\Phi/\partial w]$ with the continuous convex function in w for almost all z in G

$$\Phi(z, w) = \int_0^{|w|^2} h\left(\sqrt{\zeta}, z\right) d\zeta. \tag{14}$$

Therefore, the Nemytski operator $N\colon L_r(G) \to L_s(G)$ is cyclically trimonotone and Theorems 1—3 of [1] (cf. also [7]) yield

Theorem 2: *Under the above assumptions* (i)—(v) *the problem* (1)—(3) *has a unique generalized solution* $w(z) \in L_r(G)$ *if* $2 \leqq r \leqq 4$ *and additionally the function* $g(\xi, z) = \xi h(\xi, z)$ *is non-decreasing in* ξ *for almost all* z *in* G. *Moreover, this solution depends continuously upon the data* $\varphi \in L_r(G)$.

If $r > 4$ the operator M from $L_s(G)$ into $L_r(G)$ is not defined on the whole space $L_s(G)$ and the Hammerstein equation (8) is singular in the sense of Browder. We rely on the results of Brézis and Browder [3, 4] for the singular case. At first, the operator M is a bounded linear map of $L_1(G)$ into $L_1(G)$ satisfying the relation (12) for all ψ in $L_\infty(G)$. Further, we again make the additional assumption that the function $g(\xi, z)$ is non-decreasing in ξ for almost all z in G. Thus, the continuous Nemytski operator N: $L_r(G) \to L_s(G)$ is cyclically trimonotone. Finally, we require the *coercivity condition*

$$g(\xi, z) \geqq C\xi^{r-1} - G_0(z), \tag{15}$$

with a positive constant C and a real-valued function $G_0(z) \in L_s(G)$. Then the Nemytski operator N maps $L_r(G)$ onto $L_s(G)$ because for any $\zeta \in L_s(G)$ there exists a $w \in L_r(G)$ of the form $w = \zeta k(|\zeta|, z)$ where $l(\varrho, z) = \varrho k(\varrho, z)$ is an inverse mapping to $g(\xi, z)$ with respect to ξ for almost all z in G. By Theorem 3 of [3] or [4] there follows

Theorem 3: *In the case $r > 4$ the problem* (1)—(3) *has a generalized solution $w(z) \in L_r(G)$ if the assumptions* (i)—(v) *and the coercivity condition* (15) *are fulfilled and the function $g(\xi, z)$ is non-decreasing in ξ for almost all z in G.*

Remark: The solution $w(z) \in L_r(G)$ of (1)—(3) is uniquely determined if $g(\xi, z)$ is strictly increasing in ξ for almost all z in G.

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