On a Boundary Value Problem for a Special Class of First Order Semilinear Elliptic Systems in the Plane

L. V. WOLFERSDORF

Mit Hilfe bekannter Resultate von Brézis und Browper über Hammersteinsche Integralgleichungen werden Existenzsätze für eine Klasse von Riemann-Hilbert-Problemen bei Vekuaschen Differentialgleichungssystemen mit potenzartiger Nichtlinearität hergeleitet.

С помощью известных результатов Брезиса и Браудера об интегральных уравнениях типа Гаммерстейна доказываются теоремы существования для класса задач Римана-Гильберта у систем уравнений с частными производными типа И. Н. Векуа со степенной нелинейностью.

By means of known results of Brézis and Browper on integral equations of Hammerstein type existence theorems for a class of Riemann-Hilbert problems for Vekua's differential equation systems with power-like nonlinearity are proved.

Introduction

In the recent papers of the writer $[6, 7]$ methods of monotone operator theory are applied to some classes of boundary value problems for first order semilinear elliptic systems in the plane. Transforming the boundary value problems to Hammerstein integral equations on the domain or the boundary of the domain, we obtained existence assertions for superlinear nonlinearities of a special kind up to growth order three. In this note we deal with linear boundary value problems for a typical special class of such nonlinearities. Utilizing results of H. BRÉZIS and F. E. BROWDER $[1-4]$ on integral equations of Hammerstein type, we show on the one hand that for this special class of problems for nonlinearities up to growth order three the usual monotonicity and coercivity assumptions on the nonlinearity can be omitted and on the other hand that under these assumptions also nonlinearities with growth order greater than three can be handled.

Statement of problem

Let G be the unit disk in the complex z plane with boundary $\Gamma = \{t = e^{is} : -\pi \leq s\}$ $\leq \pi$. The problem is to find a solution $w(z) = u + iv$ to the differential equation

$$
\frac{\partial w}{\partial \bar{z}} - a(z) \bar{w} = H(z, w) + F(z) \quad \text{in } G.
$$

the boundary condition

$$
u(s) - \mu(s) v(s) = f(s) \text{ on } \Gamma.
$$

and the additional condition

$$
v(0) - \nu u(0) = c
$$
 in $z = 0$.

 (3)

 (1)

 (2)

The data in (1) - (3) fulfil the following *assumptions* (i) - (v) : L. v. WOLFERSDORF

ta in (1)-(3) fulfil the following assumptions (i)-(v):
 $a(z) \in L_p(G), \qquad p > 2$, with

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 $a(z) \in L_p(G)$, $p > 2$, with
 $\operatorname{Re} \frac{a(z)}{z} \ge 0$ a.e. in G. (4)
 $H(z, w) = z\overline{w}h(|w|, z)$, (ii)

e non-negative real-valued function $h(\xi, z)$, $0 \$ **188** *L. v. WOLFERSDORF*
 Find data in (1)-(3) fulfil the following assumptions (i)-(v):
 $a(z) \in L_p(G), \quad p > 2$, with (i)
 $Re \frac{a(z)}{z} \ge 0$ a.e. in *G*. (4)
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where the non-negative real-value $a(z) \in L_p(G)$, $p > 2$, with

Re $\frac{a(z)}{z} \ge 0$ a.e. in G.
 $H(z, w) = z\overline{w}h(|w|, z)$, (4)

where the non-negative real-valued function $h(\xi, z)$, $0 \le \xi < \infty$ is continuous with

respect to ξ in $0 < \xi < \infty$ for almost all z The data in (1)-(3) fulfil the following assumptions (i)-(v):
 $a(z) \in L_p(G)$, $p > 2$, with
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 $\text{Re } \frac{a(z)}{z} \ge 0$ a.e. in *G*.
 $H(z, w) = z\overline{w}h(|w|, z),$ (ii)
 $e \text{ non-negative real-valued function } h(\xi, z), 0 \le \xi < \infty$ is continuous with
 $\log \xi$ in $0 < \xi < \infty$ for almost all $z \in G$, measurable with respect to z in 38 L. v. Wortzwaspone

The data in (1)-(3) fulfil the following assumptions (f)-(v):
 $a(z) \in L_p(G)$, $p > 2$, with
 $\text{Re} \frac{a(z)}{z} \ge 0$, a.e. in O , (4)
 $H(z, w) = z\mathbb{R}A|w|, z$),

Where the non-negative real-valued function

respect to ξ in $0 < \xi < \infty$ for almost all $z \in G$, measurable with respect to z in G for all $0 \le \xi < \infty$, and the function $g(\xi, z) = \xi h(\xi, z)$ satisfies the inequality
 $g(\xi, z) \le E(z) + D\xi^{r-1}$, $2 \le r < \infty$, (5) $\begin{align*} H(z,w) &= z\overline{w}h(|w|,z), \ \text{(ii)}\ \text{e non-negative real-valued function } h(\xi,z),\ 0\leq\xi<\infty \text{ is continuous with } \\ s\text{ in } 0<\xi<\infty \text{ for almost all } z\in G\text{, measurable with respect to } z\text{ in }G\text{ for } z\infty\text{, and the function } g(\xi,z) = \xi h(\xi,z) \text{ satisfies the inequality} \ g(\xi,z)\leq E(z)+D\xi^{r-1},\qquad 2\leq r<\infty,\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\$ (s, w) = zw $n(|w|, z)$,

(i)

non-negative real-valued function $h(\xi, z)$, $0 \le \xi < \infty$ is continuous with

in $0 < \xi < \infty$ for almost all $z \in G$, measurable with respect to z in G for
 $(z) \in E(z) + D\xi^{r-1}$, $2 \le r < \infty$, (5)

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$$
g(\xi, z) \leq E(z) + D\xi^{r-1}, \qquad 2 \leq r < \infty,
$$
\n⁽⁵⁾

with a real-valued function $E(z) \in L_{\alpha}(G)$, *s* the exponent conjugate to *r*, and a positive constant. D; and the condition $g(0, z) = 0$ for almost all z in G.

$$
F(z) \in L_s(G) \quad \text{with} \quad \frac{1}{z} \ F(z) \in L_1(G). \tag{iii}
$$

 $\mu(s) \in H_a(\Gamma)$, $0 < \alpha \leq 1$, is a real Hölder continuous function and *v* is a real constant, where (iv)

$$
|\mu(s)| \leq 1, \quad -\pi \leq s \leq \pi, \text{ and } |\nu| \leq 1, \tag{6}
$$

and in addition to inequality (4) either this inequality holds in strict form in a subset $F(z) \in L_s(G)$ with $\frac{1}{z} F(z) \in L_1(G)$. (iii)
 $\mu(s) \in H_a(\Gamma)$, $0 < \alpha \leq 1$, is a real Hölder continuous function and ν is a

real constant, where
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and in addition to inequa $|\mu(s)| \leq 1$, $-\pi \leq s \leq \pi$, and $|\nu| \leq 1$,
and in addition to inequality (4) either this inequality holds in strict for *G* with positive measure or $\nu\mu(s) \equiv 1$, i.e., the cases $\mu(s) \equiv 1$, $\nu = 1$, $\nu = -1$. are exclude and Im $E(z) \in L_s(G)$, *s* the exponent conjugate to *r*, and a positive
notition $g(0, z) = 0$ for almost all *z* in *G*.
(iii)
 $0 < \alpha \leq 1$, is a real Hölder continuous function and *v* is a
where
 $-\pi \leq s \leq \pi$, and $|v| \leq$

$$
\nu = -1 \text{ are excluded, and for } \nu \mu(s) = -1 \text{ the additional inequalities}
$$
\n
$$
\text{Im} \frac{a(z)}{z} \leq 0 \quad \text{and} \quad \text{Im} \frac{a(z)}{z} \geq 0 \quad \text{a.e. in } G
$$
\n
$$
\text{in the case } \mu(s) = 1, \nu = -1 \text{ and } \mu(s) = -1, \nu = 1, \text{ respectively, are fulfilled.}
$$
\n(7)

$$
f(s) \in L_{\delta}(T) \quad \text{with} \quad r/2 \leq \delta < \infty \quad \text{if} \quad r > 2 \tag{v}
$$
\n
$$
\text{and} \quad 1 < \delta < \infty \quad \text{if} \quad r = 2 \tag{v}
$$

is a real summable function and *c* is a real constant.

We ask for *generalized solutions* $w(z) \in L_r(G)$ of $(1) - (3)$ which possess generalized derivatives in Sobolev sense (cf. [5]) $\partial w/\partial \overline{z} \in L_s(G)$ and boundary values $w(t) \in L_s(\Gamma)$ where the positive measure of $\nu\mu(s) = 1$, i.e., the cases $\mu(s) = 1$, $\nu = 1$ and $\mu(s) = -1$.
 $\nu = -1$ are excluded, and for $\nu\mu(s) = -1$ the additional inequalities
 $\nu = -1$ are excluded, and for $\nu\mu(s) = -1$ the additional and $1 < \delta < \infty$ if $r =$
is a real summable function and c i
We ask for *generalized solutions* a
derivatives in Sobolev sense (cf. [5]
with $\gamma = \min [\delta, s/(2-s)]$ and which
Existence theorems
Under the above assumptions the with $\gamma = \min \{ \delta, s/(2-s) \}$ and which are continuous for $z = 0$. $f(s) \in L_b(\Gamma)$ with $r/2 \le \delta < \infty$ if $r > 2$

and $1 < \delta < \infty$ if $r = 2$

is a real summable function and c is a real constant.

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es in Sobolev sense (cf. [5]) $\partial w/\partial \overline{z} \in L_s(G)$ and boundary value:

min [δ , $s/(2-s)$] and which are continuous for $z = 0$.

 theorems
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Under the above assumptions the boundary value problem (1) -(3) is equivalent to

$$
w = MNw = \varphi \quad \text{in } L_r(G), \tag{8}
$$

where $\varphi \in L_r(G)$ is a known function determined by the data, $N: L_r(G) \to L_s(G)$ is the Nemytski operator. Under the
the follow
where $\varphi \in$
Nemytsk
 \therefore

$$
= \min \left[\delta, s/(2-s) \right] \text{ and which are continuous for } z = 0.
$$
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$$
e \text{ theorems}
$$
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$$
n e \text{ above assumptions. the boundary value problem (1)–(3) is equivalent to
$$
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$$
w = M N w = \varphi \quad \text{in } L_r(G),
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$$
\in L_r(G) \text{ is a known function determined by the data, } N: L_r(G) \to L_s(G) \text{ is the}
$$
\n
$$
N w = \frac{1}{z} \overline{H(z, w)},
$$
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$$
\sum_{i=1}^{n} \overline{H(z, w)},
$$
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$$
\sum_{i=
$$

First order semilinear elliptic systems in the plane $\frac{39}{2}$

and M is a linear integral operator of the form

First order semilinear elliptic systems in the plane
\nand M is a linear integral operator of the form
\n
$$
(M\psi)(z) = \iint_G [M_1(z, \zeta) \psi(\zeta) + M_2(z, \zeta) \overline{\psi(\zeta)}] d\zeta d\eta
$$
\nwith kernels $M_j(z, \zeta)$, $j = 1, 2$, continuous for $\zeta + z$ and satisfying the estimates

First order semilinear elliptic systems in the plane
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$$
(M\psi)(z) = \iint_{G} [M_1(z, \zeta) \psi(\zeta) + M_2(z, \zeta) \overline{\psi(\zeta)}] d\zeta d\eta
$$
\nwith kernels $M_j(z, \zeta), j = 1, 2$, continuous for $\zeta \neq z$ and satisfying the estimates
\n
$$
|M_j(z, \zeta)| \leq \frac{C_1 |\zeta|}{|z - \zeta|} + C_2 \leq \frac{C_0}{|z - \zeta|}
$$
\nwith certain positive constants $C_k, k = 0, 1, 2$. Moreover, the operator M fulfils the inequality
\n
$$
\text{Re} \iint_{G} \psi(z) \overline{M\psi} dz dy \geq 0
$$
\nfor any $\psi \in L_e(G), \varrho \geq 4/3$.
\nFirstly, let be $2 \leq r < 4$. Then the operator M: $L_s(G) \to L_s(G)$ is compact and monotone. Further, there holds the relation
\n
$$
\text{Re} \left[w \frac{1}{z} H(z, w) \right] = |w|^2 h(|w|, z) = |w| \left| \frac{1}{z} H(z, w) \right|.
$$
\nTherefore, from Theorem 4 of [2] (cf. also [7]). we obtain
\nTheorem 1: Under the above assumptions (i) = (v) the problem (1) = (3) has a

with certain positive constants C_k , $k = 0, 1, 2$. Moreover, the operator M fulfils the

$$
\operatorname{Re} \iint_{G} \psi(z) \overline{M\psi} \, dx \, dy \ge 0 \tag{12}
$$
\n
$$
\psi \in L_{\varrho}(G), \, \varrho \ge 4/3.
$$

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for any $\psi \in L_{\rho}(G)$, $\rho \geq 4/3$.
Firstly, let be $2 \leq r < 4$. Then the operator'M: $L_{s}(G) \to L_{s}(G)$ is compact and monotone. Further, there holds the relation

$$
\operatorname{Re}\left[w\,\frac{1}{z}\,H(z;\,w)\right] = |w|^2\,\dot{h}(|w|,z) = |w|\,\left|\,\frac{1}{z}\,H(z,\,w)\,\right|.\tag{13}
$$

Therefore, from Theorem 4 of [2] (cf. also [7]). we obtain

Theorem 1: Under the above assumptions $(i)-(v)$ the problem $(1)-(3)$ has a Therefore, from Theorem 4 of [2] (cf. also [7]
Theorem 1: *Under the above assumption generalized solution w*(z) \in $L_r(G)$ *if* $2 \le r < 4$

Remark: It suffices to assume the function $h(\xi, z)$ as real-valued and nonnegativefor $\xi > R$ with some $R > 0$, only. (The inequality (5) has to hold for the absolute Friedmann 1: Und
generalized solution w
Remark: It suffice
for $\xi > R$ with some
value of $g(\xi, z)$ then.)
In the case $r = 4$ then.

In the case $r = 4$ the operator $M: L_s(G) \to L_r(G)$ is merely bounded and monotone. We additionally suppose that the function $g(\xi, z)$ is *non-decreasing* in ξ for almost all z in G. Then $H(z, w) = z[\partial \Phi/\partial w]$ with the continuous convex function in w for almost all z in G *c₂*, from Theorem 4 of [2] (cf. also [
 cm 1: Under the above assumptid solution $w(z) \in L_r(G)$ *if* $2 \le r < k$ *: It suffices to assume the function* $q(\xi, z)$ *then.)

case* $r = 4$ *the operator* $M: L_s(G) - \text{ionally suppose that the function } g$ *

in H(z, w) = z[* we additionally suppose that the full of
in G. Then $H(z, w) = z[\partial \Phi/\partial w]$ with the
all z in G
 $\Phi(z, w) = \int_0^{|w|^4} h(\sqrt{\zeta}, z) dz$.
Theorems 1-3 of [1] (cf. also [7]) yield
Theorem 2: Under the above assump

$$
\Phi(z, w) = \int\limits_{0}^{|w|^{2}} h(\sqrt{\zeta}, z) d\zeta.
$$

Therefore, the Nemytski operator $N: L_r(G) \to L_s(G)$ is cyclically trimonotone and

 \cdot Theorem 2: Under the above assumptions (i)-(v) the problem (1)-(3) has a unique Therefore, the Nemytski operator $N: L_r(G) \to L_s(G)$ is cyclically trimonotone and

Theorems 1-3 of [1] (cf. also [7]) yield

Theorem 2: Under the above assumptions (i)-(v) the problem (1)-(3) has a unique

"generalized soluti *generalized solution* $w(z) \in L_r(G)$ *if* $2 \leq r \leq 4$ and additionally the function $g(\xi, z) = \xi h(\xi, z)$ is non-decreasing in ξ for almost all z in G. Moreover, this solution depends In the case $r = 4$ the operator $M : L_s(G) \to L_r(G)$ is merely bounded
We additionally suppose that the function $g(\xi, z)$ is non-decreasing in i
in G. Then $H(z, w) = z[\partial \Phi/\partial w]$ with the continuous convex function
all z in G
 $\Phi(z, w$

continuously upon the data $\varphi \in L_r(G)$.
If $r > 4$ the operator *M* from $L_s(G)$ into $L_r(G)$ is not defined on the whole space $L_s(G)$ and the Hammerstein equation (8) is singular in the sense of Browder. We rely. on the results of BRÉZIS and BROWDER $[3, 4]$ for the singular case. At first, the operator M/is a bounded linear map of $L_1(G)$ into $L_1(G)$ satisfying the relation (12) for all ψ in $L_{\infty}(G)$. Further, we again make the additional assumption that the function $g(\xi, z)$ is non-decreasing in ξ for almost all z in G. Thus, the continuous Nemytski operator N : $L_r(G) \to L_s(G)$ is cyclically trimonotone. Finally, we require the *coercivity condition g*, the Nemytski
 g(1-3 of [1] (cf. *i*
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from $L_s(G)$ into $L_r(G)$

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trimonotone. Finally, v
 $G_0(z)$, nuous Ner
quire the
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$$
g(\xi, z) \geq C\xi^{r-1} - G_0(z),
$$

(14)

(15)

with a positive constant *C* and a real-valued function $G_0(z) \in L_s(G)$. Then the Nemytski operator *N* maps $L_r(G)$ onto $L_s(G)$ because for any $\zeta \in L_s(G)$ there exists a $w \in L_r(G)$ of the form $w = \zeta k(|\zeta|, z)$ where $l(\varrho, z) = \varrho k(\varrho, z)$ is an inverse mapping to $g(\xi, z)$ with respect to ξ for almost all z in G . By Theorem 3 of [3] or [4] there follows

Theorem 3: In the case $r>4$ the problem $(1)-(3)$ has a generalized solution $w(z)$ \in $L_r(G)$ if the assumptions (i)–(v) and the coercivity condition (15) are fulfilled and the *function* $g(\xi, z)$ *is non-decreasing in* ξ *for almost all z in G.* L_t(G) of the form $w = \zeta k(|\zeta|, z)$ where $l(\varrho, z) = \varrho k(\varrho, z)$ is a $\zeta(\xi, z)$ with respect to ξ for almost all z in G. By Theorem 3 of [3]

Theorem 3: In the case $r > 4$ the problem $(1) - (3)$ has a general $\zeta_r(G)$ i

Remark: The solution $w(z) \in L_r(G)$ of $(1) - (3)$ is uniquely determined if $g(\xi, z)$ is strictly increasing in ξ for almost all z in G . Remark: The solution $w(z) \in L_r(G)$ of $(1) - (3)$ is uniquely det
trictly increasing in ξ for almost all z in G .
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