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 On the numerical solution of pseudoparabolic equations Elischrift für Analysis

and the Anwendungen
 On the numerical solution of pseudoparabolic equations
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Don the numerical solution of pseudoparabolic equations

ROBERT P. GILBERT¹) and LEROY R. LUNDIN

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In der Arbeit werden diskrete Analoga der Fundamentalsingularitäten für die pseudoparabolische Differentialgleichung gefunden. Dabei dient eine' von COLTON und *GILBERT* für analytis'che Koeffizienten entwickelte Methode zur Motivation der. Annaherung. In der Tat, die Green'sche. Differenzenfunktion spielt hier die gleiche Rolle wie im analytischen Fall. Das Linienverfahren dient zur Behandlung der Differentiation nach der Zeit. Schließlich werden Fehlerabschatzungen erhalten, in dénen die stetigen Losüngen mit den Differenzenapproximationen vergliehen werden. In der Arbeit werden diskrete Analoga der Fundamentalsingulari
bolische Differentialgleichung gefunden. Dabei dient eine von Cota
lytische Koeffizienten entwickelte Methode zur Motivation der An
Green'sche. Differenzenfun

В работе найдены дискретные аналоги фундаментальных сингулярностей для псевдопараболических дифференциальных уравнений. При этом метод, разработанный Согтом и Gплект для аналитических коэффициентов, служит для мотивировки аппроксимации. Действительно, функция разности Грина играет здесь ту же самую роль как в аналитическом случае. Метод служит для обработки дифференцирования по времени. Наконец получены оценки погрешностей, в корорых сравниваются непрерывные решения 16 работе найдены дискретные аналоги фундаментальных сингулярностей для псевдо-
Параболических дифференциальных уравцений. При этом метод, разработанный Солтох
 16 нествительно, функция разности Грина играет здесь ту же

In this work the authors find discrete analogues of the fundamental singularities for pseudoparabolic equations. The method developed by COLTON and GILBERT for analytic coefficients is used to motivate the approach. Indeed, the finite difference Green's function is seen to play the \cdot same role here as in the analytic ease. The method of lines is employed to treat the time differentiation. Furthermore, error estimates are obtained which compare the continuous solutions to the finite difference approximations.

received much interest recently. In particular, the methods of functional analysis have been effectively brought to bear on these problems by SHOWALTER and Trng $[9-11, 13, 14]$. An alternate approach, which stresses the use of function theoretic methods has been developed by COLTON, GILBERT and HSIAO $[4, 5, 7, 8]$. Indeed, a fairly general function theoretic method now exists for investigating pseudo-parabolic equations in two space variables [7, 8]. These are equations of the form stigation of partial different
much interest recently. In
m effectively brought to be
3, 14]. An alternate appro-
has been developed by Co-
neral function theoretic mations in two space variable
 $\mathfrak{L}[u] := \underline{M}[u_t] - \underline{L}[u$

$$
\mathfrak{L}[u] := \mathfrak{M}[u_t] - \mathfrak{L}[u] = 0
$$

where ord $M>$ ord L and M is elliptic.

Pseudobarabolic equations arise in a variety of physical problems, such as the velocity of non-steady flows of viscous fluids $\bm{[1]}$, and the hydrostatic excess pressure occuring during the consolidation of clay [12].

/ 1) This work was supported in part by the National Science Foundation through Grant No. MCS 78-02452, and the 'Department of Energy through Grant No. DE-AC01-81 ER-1067.

The papers cited above by GrLBERT [7], and **GILBERT-HSIAO .** [8] generalize the approach used by COLTON [4, 5] for the case where ord $M = 2$ to order $2n$. In these works the approach depends on the analytical construction of fundamental singular, solutions to the adjoint equation **EXECUTE:** P. GILBERT and LEROY R. LUNDIN
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The papers cited above by GILBERT [7], and GILBERT-HSIAO [8] generaliz

approach used by COLTON [4, 5] for the case where ord $M = 2$ to order 2n. In

works the approach depends on

$$
\mathfrak{L}^*[v] := M^*[v_t] + L^*[v] = 0, \tag{1.2}
$$

where M^* , and L^* are the formal Lagrange adjoint operators for M and L respectively. It has been shown, moreover, that it is possible to develop the fundamental singularity in the form approach used by COLTON [4, 5] for the case where ord *M* works the approach depends on the analytical construction solutions to the adjoint equation
 $\mathcal{L}^* [v] := \mathcal{M}^* [v_t] + \mathcal{L}^* [v] = 0$,

where \mathcal{M}^* , and \mathcal

$$
S(P, t; Q, \tau) := A(P, t; Q, \tau) \ln \frac{1}{\tau} + B(P, t; Q, \tau),
$$

and

•
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e approach depends on t

to the adjoint equation
 $\mathcal{Q}^*[v] := \underline{M}^*[v_t] + \underline{L}^*[v]$

, and \underline{L}^ are the formal

s been shown, moreover,

in the form
 $S(P, t; Q, \tau) := A(P, t; Q)$
 $r := |P - Q|, \qquad P :=$
 $A(P, t; Q, \tau) := \sum_{n=1}^{\infty}$ $Q, \tau) \ln \frac{1}{r} + B(P, t; Q),$
 $(x, y), \qquad Q := (\xi, \eta),$ $\begin{aligned} S(P,t;Q,\tau) &:= A(P,t;Q,\tau) \ln \frac{\tau}{\eta} \ r &:= |P-Q|, \qquad P := (x,y), \ A(P,t;Q,\tau) &:= \sum\limits_{j=1}^{\infty} A_j(P,Q) \, \frac{(t-\tau)^2}{j} \end{aligned}$ $(\tau)^j$ 0,
 c ange adjoint operators for M and L respect

t it is possible to develop the fundamental
 $\ln \frac{1}{r} + B(P, t; Q, \tau),$
 $\theta := (\xi, \eta),$
 $\frac{(t-\tau)^j}{j!},$
 $\frac{(t-\tau)^j}{j!}$,

ach is that $A(P, Q)$ is the Riemann fund $\begin{aligned} H_1 &= |I_1 - Q|, & H_2 &= (x, y), \ H_3 &= |I_3 - Q|, & H_4(P, Q) \end{aligned}$
 $\begin{aligned} H_1(P, t; Q, \tau) &= \sum_{j=1}^{\infty} A_j(P, Q) \frac{(t - \tau)^j}{j} \end{aligned}$ is possible to develop the fundamental s
 $\frac{1}{r} + B(P, t; Q, \tau),$
 $Q := (\xi, \eta),$
 $\frac{1}{\eta!}$,
 $\frac{1}{\eta!}$,
 $\frac{1}{\eta!}$
 $\frac{1}{\eta!}$ is that $A(P, Q)$ is the Riemann function

if M, is written as a hyperbolic operator

The remarkable result of this approach is that $A(P, Q)$ is the Riemann function **/** associated with the operator M , that is, if M is written as a hyperbolic operator by formally mapping $(x, y) \rightarrow (z, z^*)$, $z = x + iy$, $z^* = x - iy$. The other coefficients A_j ($j \ge 2$), and the B_j ($j \ge 1$) may then be obtained by recursive schemes. The remarkable result of this approach is that $A(P, Q)$ is the Riemann function
associated with the operator M, that is, if M is written as a hyperbolic operator by
formally mapping $(x, y) \rightarrow (z, z^*)$, $z = x + iy$, $z^* = x - iy$. The

Two obvious disadvantages of the above method are that (1) the coefficients of M and *L* must be analytic in the space variables, and (2) it is very difficult, in general, to do the necessary analytical computations for the A_i ($i \geq 2$), B_i ((≥ 1) even when the Riemann function for *M* is already known. •

It is the purpose of the present paper to circumvent these difficulties by replacing the required analytical computations by numerical algorithms. Furthermore, we modify the **approach** cited above to include the case of nonanalytic coefficients. This permits circumvention of the necessary procedure of analytically continuing the coefficients into the *(z,* z*) space. For simplicity of exposition we discuss only the The remarkable result of this approach is that $A(P, Q)$ is the Riemann function
associated with the operator M , that is, if M is written as a hyperbolic operator by
formally mapping $(x, y) \rightarrow (z, z^*), z = x + iy, z^* = x - iy$. The ot Example 11 into the (z, z^*) space. For simplicity of
the into the (z, z^*) space. For simplicity of
 $M[v] := \bigtriangleup v - q(P) v$, $q(P) > 0$ for
 $L[v] := a(P) v$, $a(P) < 0$ for $P \in \Omega$,
we for purposes of numerical estimation we

and

$$
L[v] = a(P) v, \qquad a(P) < 0 \quad \text{for} \quad P \in \Omega,
$$

and where for purposes of numerical estimation we assume that the coefficients are in $C^{2,\alpha}(\overline{\Omega})$, [3], and in particular we consider the initial-boundary value problem

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$$
 for $P \in \Omega$,
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\n
$$
C^{2,a}(\overline{\Omega})
$$
, [3], and in particular we consider the initial-boundary value problem
\n
$$
\mathcal{L}[u] \cdot (P, t) = F(P, t), \qquad (P, t) \in \Omega \times [0, \infty)
$$
,
\n
$$
u(P, t) = f(P, t) \quad \text{for} \quad P \in \overline{\Omega}, \qquad t > 0
$$

\n
$$
u(P, 0) = 0, \qquad P \in \overline{\Omega}.
$$
\n

\nHere Ω is taken to be a simply-connected region, such that the boundary Ω , is
\nnooth enough for the various Green's identities to hold.

\nIn the expression which follows we shall treat first the problem of (1.4) with components of the equation.

Here Ω is taken to be a simply-connected region, such that the boundary $\dot{\Omega}$, is smooth enough for the various Green's identities to hold.

In the exposition which follows we shall treat first the problem of (1.4) with continuous coefficients, and develop a representation formula for its solution. Having done this we shall turn our attention to various, discretized forms of this problem and $\rm{obtain\ error\ estimates\ comparing\ the\ solution\ of\ (1.4)\ with\ the\ discretized\ solutions}$

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 Continuous Case

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2. The Continuous Case

Following the idea of [4, 5] and [7, 8] we attempt to cons

lution. By a "Green's function" for (1.4), we shall mean Following the idea of $[4, 5]$ and $[7, 8]$ we attempt to construct a fundamental solution. By a "Green's function" for (1.4), we shall mean a function *G* defined on $\overline{Q} \times \overline{Q} \times (0, \infty)$ which satisfies **Example 12: Z C**
Z: The Continuous Case
**Following the idea of [4, 5] and [7, 8] we attempt

lution. By a "Green's function" for (1.4), we shall
** $\overline{Q} \times \overline{Q} \times [0, \infty)$ **which satisfies
** $\Omega_P G(P, Q; t) = \delta(P - Q)$ **f** $\frac{1}{2}$

Numerical solution of pseudoparabolic equations 27
\n2. The Continuous Case
\nFollowing the idea of [4, 5] and [7, 8] we attempt to construct a fundamental solution. By a "Green's function" for (1.4), we shall mean a function G defined on
\n
$$
\overline{Q} \times \overline{Q} \times [0, \infty)
$$
 which satisfies
\n $2_P G(P, Q; t) = \delta(P - Q)$ for $P \in \Omega$, $t \ge 0$;
\n $G(P, Q; t) = 0$ for $P \in C$, $t > 0$;
\n $G(P, Q; t) = 0$ for $P, Q \in \overline{\Omega}$.
\nTheorem 2.1: There is a unique function G which is analytic in t and satisfies (2.1).
\nFurthermore G may be represented in the form
\n $\begin{aligned}\n&G(P, Q; t) = \sum_{n=0}^{\infty} G_n(P, Q) \frac{t^{n+1}}{(n+1)!} \\
&G(P, Q; t) = \sum_{n=0}^{\infty} G_n(P, Q) \frac{t^{n+1}}{(n+1)!}\n\end{aligned}$ \n(2.2)
\nwhere G_0 is the Grèen's function associated with the problem $Mw = f$, $w|_C = 0$. The coefficients G_{n+1} may be uniquely determined as solutions of the recursive system
\n $M_P G_{n+1}(P, Q) = L_P G_n(P, Q)$ for $P \in \Omega$,
\n $G_{n+1}(P, Q) = 0$ for $P \in C$.
\nProof: For the general scheme used to construct fundamental solutions see [4, 5,
\n7, 8]. The result then follows by noting G satisfies definition (2.1). It can be seen that G
\nconverges as it is a special fundamental solution of the form investigated already in

Theorem 2.1: *There is a unique function G-which is analytic in t and satisfies (2.1). ^V* ^V **Furthermore G may be represented in the form**

$$
G(P, Q; t) = \sum_{n=0}^{\infty} G_n(P, Q) \frac{t^{n+1}}{(n+1)!}
$$
 (2.2)

where G_0 *is the Green's function associated with the problem* $Mw = f$ *,* $w|_C = 0$ *. The* $\mathfrak{n} = 0$
 G_0 *is the Green's function associated with the pricents* G_{n+1} *may be uniquely determined as solutions*
 $M_P G_{n+1}(P, Q) = L_P G_n(P, Q)$ for $P \in \Omega$,

is the Green's function associated with the problem
$$
Mw = f
$$
, $w|_C = 0$. The *u* is the G_{n+1} may be uniquely determined as solutions of the recursive system $M_P G_{n+1}(P,Q) = L_P G_n(P,Q)$ for $P \in \Omega$, $G_{n+1}(P,Q) = 0$ for $P \in C$. $\{2.3\}$

For the general scheme used to construct fundamental solutions see [4, 5].

For the genral scheme used to construct fundamental solutions see [4, 5].

Proof: For the general scheme used to construct fundamental solutions $M_P G_{n+1}(P,Q) = L_P G_n(P,Q)$ for $P \in \Omega$,
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Proof: For the general scheme used to construct fundamental solutions see [4, 5, 7, 8]. The result then follows by noting *G* satisfies definition (2.1). It $[4, 5, 7, 8]$

In what follows, G will always be taken to mean the fundamental solution (2.2 $-$ 2.3). An important property of Green's functions for elliptic equations is their reproducing
property. This is also the case for our G. Let v be defined on $\overline{Q} \times [0, \infty)$ such that it is 7, 8]. The result then follows by noting *G* satisfies definition (2.1). It can be seen that *G*
converges as it is a special fundamental solution of the form investigated already in
[4, 5, 7, 8] **I**
In what follows, *G* simultaneously C^2 in the space-variables and C^1 in the time variable. We designate this space as Σ . Starting with the integral identity *V, W_PG_{n+1}(P, Q)* = $L_PG_n(P, Q)$ for $P \in \Omega$,
 $G_{n+1}(P, Q) = 0$ for $P \in C$. $\}$ (2.3)
 \therefore For the general scheme used to construct fundamental solutions see [4, 5, result then follows by noting G satisfies definition

simultaneously
$$
C^2
$$
 in the space-variables and C^1 in the time variable. We define this space as \sum . Starting with the integral identity

\n
$$
v(P, t) = \int_0^t d\tau \int_{\Omega} \frac{\partial}{\partial \tau} v(Q, \tau) \, \Omega_0[G(Q, P; t - \tau)] \, dA(Q_0)
$$
\n
$$
= -\int_0^t d\tau \int_{\Omega} \frac{\partial}{\partial \tau} v(Q, \tau) \left\{ [\triangle_Q - q(Q)] \frac{\partial}{\partial \tau} [G(Q, P; t - \tau)] + a(Q) G(Q, P; t - \tau) \right\} dA(Q_0), \quad dA := dx \, dy,
$$
\nand applying Green's third identity one obtains

\n
$$
v(P, t) = -\int_0^t d\tau \int_{\Omega} \left\{ M \left[\frac{\partial}{\partial \tau} v \right](Q, \tau) \cdot \frac{\partial}{\partial \tau} [G(Q, P; t - \tau)] \right\} dA(Q_0)
$$

$$
\int_{0}^{1} \int_{0}^{1} \frac{\partial}{\partial t} v(Q, \tau) \left\{ [\Delta_{0} - q(Q)] \frac{\partial}{\partial \tau} [G(Q, P; t - \tau)] \right\}
$$

\n
$$
+ a(Q) G(Q, P; t - \tau) \Big\} dA(\Omega_{0}), \quad dA := dx dy,
$$

\n
$$
v(P, t) = - \int_{0}^{t} d\tau \int_{\Omega} \left\{ M \left[\frac{\partial}{\partial \tau} v \right] (Q, \tau) \cdot \frac{\partial}{\partial \tau} [G(Q, P; t - \tau)] \right\}
$$

\n
$$
+ a(Q) \frac{\partial}{\partial \tau} v(Q, \tau) G(Q, P; t - \tau) \Big\} dA(\Omega_{0})
$$

\n
$$
- \int_{0}^{t} d\tau \int_{C} \frac{\partial}{\partial \tau} v(Q, \tau) \frac{\partial^{2}}{\partial n_{0} \partial \tau} [G(Q, P; t - \tau)] d\sigma_{0}.
$$

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Here $\frac{v}{2n}$ denotes differentiation in the direction of the outward normal, and $d\sigma_Q$ denotes the are length differential along *C.* Integrating by parts the *a(Q)* term and **28** • ROBERT P. GILBERT and LEROY R. LUNDIN

Here $\frac{\partial}{\partial n_Q}$ denotes differentiation in the direction of the outward normal,

denotes the arc length differential along C. Integrating by parts the $a(Q)$ t
 $\int d\tau \int a(Q) \frac$ **a** differentiation in the direction of
length differential along C. Integration
conditions for v and G, yields
 $a(Q) \frac{\partial}{\partial \tau} v(Q, \tau) G(Q, P; t - \tau) dA(\Omega_Q)$

e initial conditions for v and G, yields
\n
$$
\int_{0}^{t} d\tau \int_{\Omega} a(Q) \frac{\partial}{\partial \tau} v(Q, \tau) G(Q, P; t - \tau) dA(\Omega_0)
$$
\n
$$
= - \int_{0}^{t} a(Q) v(Q, \tau) \frac{\partial}{\partial \tau} [G(Q, P; t - \tau)] dA(\Omega_0).
$$
\n
$$
= \int_{0}^{t} a(Q) v(Q, \tau) \frac{\partial}{\partial \tau} [G(Q, P; t - \tau)] dA(\Omega_0).
$$
\n
$$
= \int_{0}^{t} a(Q) v(Q, \tau) \frac{\partial}{\partial \tau} [G(Q, P; t - \tau)] dA(\Omega_0).
$$

We obtain in this way the integral representation given in the following theorem.

Theorem 2.2: Let $v \in \sum$ and satisfy the homogeneous initial condition $v(P, 0) = 0$ *for* $P \in \Omega$. Then for $(P, t) \in \overline{\Omega} \times (0, t)$, v has the representation

We obtain in this way the integral representation given in the follow
\n
$$
\begin{aligned}\n\text{We obtain in this way the integral representation given in the follow.} \\
\text{Therefore, } P \in \Omega. \text{ Then, for } (P, t) \in \overline{\Omega} \times [0, t), v \text{ has the representation} \\
v(P, t) &= -\int_0^t d\tau \int \frac{\partial}{\partial \tau} G(Q, P; t - \tau) \, \Omega v(Q, \tau) \, dA(\Omega_Q) \\
&\quad - \int_0^t d\tau \int_0^t \frac{\partial}{\partial \tau} v(Q, \tau) \frac{\partial}{\partial n_Q} \frac{\partial}{\partial \tau} G(Q, P; t - \tau) \, d\sigma_Q.\n\end{aligned}
$$
\n3. **Diskretization in the Space Variables**\nA natural method of approximate the fundamental solution $G(P, Q)$ approximate the coefficients of the powers of t. For simplicity in the

A natural method of approximate the fundamental solution *G(P, Q; t)* of (2.2) is *to* approximate the coefficients of the powers of *t.* For simplicity in exposition we first assume that *Q* is a rectangle such that we may place a rectangular grid of equal spacing over Ω and thereby discretize the space variables. In the usual way we designate certain grid points as interior points and their set as Q_h . The boundary points are the intersections of the mesh with C , and this set is designated by C_h . We set $\Omega_h := \Omega_h \cup C_h$. Our particular choice for Ω does not require that the neighborhood of the boundary be treated in a special way at this point. The discrete version of \mathfrak{L} is obtained by replacing \triangle by its centered-difference approximation \triangle . This results 3. Diskretization in the Space Varia

4. A natural method of approximate to

approximate the coefficients of the

assume that Ω is a rectangle such

spacing over Ω and thereby discre-

designate certain grid points $\begin{array}{l} \text{powers}\ \text{that we} \ \text{time}\ \text{the}\ \text{erior}\ \text{pos} \ \text{with}\ \text{size}\ \text{for}\ \text{if}\ \text{in}\ \text{way} \ \text{tered-di}\ \text{(}P,t)\in \mathbb{R} \end{array}$ in the problem sume that Ω is a rectangle such that we may place a rectangular grid of equal

bacing over Ω and thereby discretize the space variables. In the usual way we

ssignate certain grid points as interior points and their A matural method of approximate the jumdamental solution $G(F, Q; t)$
approximate the coefficients of the powers of *t*. For simplicity in expose
assume that *Q* is a rectangle such that we may place a rectangular *p*
spacin approximate the coencients of the powers of *t*. For sin
assume that Ω is a rectangle such that we may place
spacing over Ω and thereby discretize the space variation
designate certain grid points as interior points *Q*_{*h*} *Qh X* and interest in the usual way the same variables. In the usual way the designate certain grid points as interior points and their set as Ω_h . The both points are the intersections of the mesh with *C 2h_a i Q_t Q_t Q_t Q_t Q*_t *Q*_t *Q*_t *Q*_t *Q*_t *Q*_t *Q*_t *Q*_t *Qt Q* *Qte Qte d Qter d Qter d dt neghborhood <i>A*_{*n*} *P*_{*N*} *Qter A*_{*n*} *P*_{*N*} *P*_{*P*} *D*

of the boundary be treated in a special way at this point. The discrete version of
$$
\mathfrak{L}
$$
 is obtained by replacing \triangle by its centered-difference approximation \triangle_h . This results in the problem\n
$$
\mathfrak{L}_h u^h(P,t) = F(P,t) \quad \text{for} \quad (P,t) \in \Omega_h \times [0,\infty),
$$
\n
$$
u^h(P,t) = f(P,t) \quad \text{for} \quad P \in C_h, \quad t > 0,
$$
\n
$$
u^h(P,0) = 0 \quad \text{for} \quad P \in \bar{\Omega}_h,
$$
\nwhere $\mathfrak{L}_h v := M_h v_t - Lv, M_h := \triangle_h - qE$, and where u^h is defined on $\Omega_h \times [0,\infty)$.

and is analytic in its second variable.
As a space discretized-Green's function we shall require a function G^h defined on

$$
\mathfrak{L}_{h}u^{u}(P, t) = F(P, t) \text{ for } (P, t) \in \mathfrak{L}_{h} \times [0, \infty),
$$

\n
$$
u^{h}(P, t) = f(P, t) \text{ for } P \in C_{h}, t > 0,
$$

\n
$$
u^{h}(P, 0) = 0 \text{ for } P \in \overline{\Omega}_{h},
$$

\n
$$
\mathfrak{L}_{h}v := M_{h}v_{t} - Lv, M_{h} := \bigtriangleup_{h} - qE, \text{ and where } u^{h} \text{ is defined on } \Omega_{h} \times [0, \infty)
$$

\nanalytic in its second variable.
\nspace discretized-Green's function we shall require a function G^{h} defined on
\n
$$
\times [0, \infty) \text{ which is analytic in } t \text{ and satisfies}
$$

\n
$$
\mathfrak{L}_{h,P}G^{h}(P, Q; t) = -h^{-2}\delta(P, Q) \text{ for } P, Q \in \Omega_{h}, t \geq 0,
$$

\n
$$
G^{h}(P, Q; t) = \delta(P, Q) t \text{ for } P \in C_{h}, t > 0,
$$

\n
$$
G^{h}(P, Q; 0) = 0 \text{ for } P, Q \in \overline{\Omega}_{h},
$$

\n
$$
(3.2)
$$

5

 $\label{eq:2.1} \frac{1}{\left|\mathcal{L}_{\text{max}}\right|}\leq \frac{1}{\sqrt{2}}\sum_{i=1}^{N}\frac{1}{\left|\mathcal{L}_{\text{max}}\right|}$

where

$$
\delta(P,Q):=\begin{cases}1 & \text{if} \quad P=Q\\ 0 & \text{if} \quad P\neq 0.\end{cases}.
$$

-

For a review of the discrete generalizations of the classical fundamental singular where
 $\delta(P,Q) := \begin{cases} 1, & \text{if } P = Q \\ 0 & \text{if } P \neq 0. \end{cases}$

For a review of the discrete generalizations of the classical funds

solutions the reader is referred to WENDLAND [16: Chapter 7].

Theorem 3.1: Problems (3.1) and (3

Theorem 3.1: *Problems* (3.1) and (3.2) have unique solutions, designated by u^h and 6k *respectively. Moreover, they have expansions of the forms*

Numerical solution of pseudoparabolic equations
\nwhere
\n
$$
\delta(P,Q) := \begin{cases}\n1. & \text{if } P = Q \\
0 & \text{if } P \neq 0.\n\end{cases}
$$
\nFor a review of the discrete generalizations of the classical fundamental singular solutions the reader is referred to WENDLAND [16: Chapter 7].
\nTheorem 3.1: Problems (3.1) and (3.2) have unique solutions, designated by u^h and G^h respectively. Moreover, they have expansions of the forms
\n
$$
u^h(P,Q;t) = \sum_{n=0}^{\infty} u_n^h(P,Q) \frac{t^{n+1}}{(n+1)!},
$$
\n
$$
G^h(P,Q;t) = \sum_{n=0}^{\infty} G_n^h(P,Q) \frac{t^{n+1}}{(n+1)!}.
$$
\n(3.2a)
\nHere $G_0^h(P,Q)$ is the discrete Green's function associated with M_h , and the G_{n+1}^h are determined recursively. (3.2b)

$$
G^{h}(P,Q;t) = \sum_{n=0}^{\infty} G_{n}^{h}(P,Q) \frac{t^{n+1}}{(n+1)!}.
$$
 (3.2b)

 $\delta(P,Q) := \begin{cases} 0 & \text{if } P \neq 0. \end{cases}$
 *For a review of the discrete generalizations of the classical fundameisolutions the reader is referred to WENDLAND [16: Chapter 7].

Theorem 3.1: Problems (3.1) and (3.2) have unique so* Proof: Direct substitution of (3.26) into the differential equation of (3.2) and aomparing powers of *t* shows that G_0^h must be the discrete Green's function as defined Example $G_0(T, Q)$ is the discrete Green's function associated with M_h , and the G_{n+1}^* are determined recursively.

Proof: Direct substitution of (3.2b) into the differential equation of (3.2) and aomparing powers of

above. Furthermore, we must have for
$$
n \geq 0
$$
 that G_{n+1}^h are the unique solutions of $M_{h,P}[G_{n+1}^h(P,Q)] = L_P[G_n^h(P,Q)]$ $(P \in \Omega_h)$, $G_{n+1}^h(P,Q) = 0$ for $P \in C_h$, (3.3)

\nwhich yields

\n
$$
G_{n+1}^h(P,Q) = -h^2 \sum_{T \in \Omega_h} G_0^h(P,T) G_n^h(T,Q) a(T).
$$
\nTo show that the infinite series (3.2b) exists it is sufficient to demonstrate that the finite sum of positive terms?

\n
$$
\sum_{\sigma \in \Omega_h} G^h(P,Q,t) \qquad (t > 0)
$$

which yields **•**

$$
G_{n+1}^h(P,Q) = -h^2 \sum_{T \in \mathcal{Q}_h} G_0^h(P,T) G_n^h(T,Q) a(T).
$$

To show that the infinite series (3.2b) exists it is sufficient to demonstrate that the

 $G_1(P, Q) = -h^2 \sum_{T \in Q_h} G_0$
 t the infinite series (3)
 *G*ⁿ(*P*, *Q*, *t*) (*t* > 0) $\begin{aligned}\n &\frac{d}{dt} \sum_{T \in \mathcal{Q}_h} G_0^h(P, T) G_n^h(T, Q) a(T).\n \end{aligned}$

Series $(3.2b)$ exists it is sufficient to demonstrate the inequality
 $\begin{aligned}\n &\frac{k}{h^2} & (k > 0)\n \end{aligned}$

Consequently

Consequently

Consequently

$$
\sum_{Q\in\mathcal{Q}_h}G^h(P,Q,t)\qquad (t>0)
$$

Is bounded. To this end, we note that from Lemma 5.2 we have the inequality
 $\sum G_0^h(P,Q) \leq \frac{k}{h^2}$ $(k > 0)$

$$
r\epsilon O_h
$$

that the infinite series (3.2b) exists in
of positive terms²)

$$
\sum_{Q\in\mathcal{Q}_h} G^h(P, Q, t) \qquad (t > 0)
$$

ed. To this end, we note that from Le

$$
\sum_{Q\in\mathcal{Q}_h} G_0^h(P, Q) \leq \frac{k}{h^2} \qquad (k > 0)
$$

=
$$
[\min_{Q(Q)}]^{-1}
$$
. Consequently

where $k := \left[\min_{Q \in \mathcal{Q}_h} q(Q)\right]^{-1}$. Consequently

is bounded. To this end, we note that from Lemma 5.2 we have the inequ
\n
$$
\sum_{Q \in \mathcal{Q}_h} G_0^h(P,Q) \leq \frac{k}{h^2} \qquad (k > 0)
$$
\nwhere $k := \left[\min_{Q \in \mathcal{Q}_h} q(Q)\right]^{-1}$. Consequently
\n
$$
h^2 \sum_{Q \in \mathcal{Q}_h} h^2 \sum_{T \in \mathcal{Q}_h} G_0^h(P,T) G_0^h(T,Q) \leq kh^2 \sum_{T \in \mathcal{Q}_h} G_0^h(P,T) \leq k^2.
$$
\nHence,
\n
$$
h^2 \sum_{Q \in \mathcal{Q}_h} h^2 \sum_{T \in \mathcal{Q}_h} G_0^h(P,T) G_n^h(T,Q) a(T) \leq A^{n+1}[k]^{n+2},
$$
\nwhere $A := \max |a(T)|$ and $\sum |G|^h(P,Q,t) \leq \frac{t}{L^2} k \exp |Atk| \quad (h > 0)$.

Hence,

$$
h^2 \sum_{Q \in \mathcal{Q}_h} h^2 \sum_{T \in \mathcal{Q}_h} G_0^{h}(P, T) G_n^{h}(T, Q) a(T) \leq A^{n+1}[k]^{n+2},
$$

where $k := \left[\min_{Q \in \mathcal{Q}_h} q(Q)\right]^{-1}$. Consequently
 $h^2 \sum_{Q \in \mathcal{Q}_h} h^2 \sum_{T \in \mathcal{Q}_h} G_0^h(P, T) G_0^h(T, Q) \leq kh^2 \sum_{T \in \mathcal{Q}_h} G_0^h(P, T) \leq k^2$.

Hence,
 $h^2 \sum_{Q \in \mathcal{Q}_h} h^2 \sum_{T \in \mathcal{Q}_h} G_0^h(P, T) G_n^h(T, Q) a(T) \leq A^{n+1}[k]^{n+2}$ $\frac{t}{h^2}$ *k* exp [*Atk*] $(h > 0)$. Uniquene of the solution to either problem follows by considering the case of homogeneous $\sum_{q \in D_h} G_0^h(P, Q) \leq \frac{k}{h^2}$ $(k > 0)$

where $k := [\min_{Q \in D_h} q(Q)]^{-1}$. Consequently
 $h^2 \sum_{Q \in D_h} h^2 \sum_{T \in D_h} G_0^h(P, T) G_0^h(T, Q) \leq k h^2 \sum_{T \in D_h} G_0^h(P, T) \leq k^2$.

Hence,
 $h^2 \sum_{Q \in D_h} h^2 \sum_{T \in D_h} G_0^h(P, T) G_n^h(T, Q) a(T) \leq A^{n+1$ $\begin{bmatrix} Atk \\ th \end{bmatrix}$ ($h > 0$). Unique
the case of homogen

 $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

5;

Existence of the solution (3.2a) follows from its unique representation in terms of the Green's function, which we list below.

Theorem 3.2: Let $v \in \sum$ and satisfy the homogeneous initial data $v^h(P, 0) = 0$ for $P \in \Omega_h$ Then for each $P \in \Omega_h$,

e Green's function, which we list below.
\nTheorem 3.2: Let
$$
v \,\epsilon \sum_{n=1}^{\infty} a_n d
$$
 satisfy the homogeneous initial data $v^h(P, 0) = 0$ for $\epsilon \Omega_h$. Then for each $P \epsilon \Omega_h$,
\n
$$
v(P, t) = h^2 \int_0^t d\tau \sum_{Q \epsilon \Omega_h} \frac{d}{dt} G^h(P, Q; t - \tau) \Omega_h v(Q, \tau)
$$
\n
$$
+ \int_0^t d\tau \sum_{Q \epsilon \Omega_h} G^h(P, Q; t - \tau) \frac{dv}{d\tau} (Q, \tau).
$$
\nProof: Let $w(P, t)$ denote the right-hand side of (3.4). Clearly, $w(P, 0) = 0$ for

$$
\int_{0}^{b} d\tau \sum_{Q \in \mathcal{Q}_{h}} G^{h}(P, Q; t - \tau) \frac{dv}{d\tau} (Q, \tau).
$$

Proof: Let $w(P, t)$ denote the right-hand side of (3.4). Clearly, w
all P. If $P \in C_{h}$ and $t > 0$, $\frac{d}{dt} G(P, Q; t - \tau) = \delta(P, Q)$ and so

$$
w(P, t) = \int_{0}^{t} d\tau \sum_{Q \in C_{h}} \delta(P, Q) \frac{d}{d\tau} v(Q, \tau) = \int_{0}^{t} \frac{dv}{d\tau} (P/\tau) d\tau
$$

$$
= v(P, t) - v(P, 0) = v(P, t).
$$

Integrating (3.4) by parts yields

$$
w(P, t) = -h^{2} \sum_{Q \in \mathcal{Q}_{h}} G^{h}(P, Q; t) \ \mathfrak{L}_{h} v(Q, 0) - \sum_{Q \in C_{h}} G^{h}(P, Q; t) \frac{dv}{dt}
$$

Integrating (3.4) by parts yields

$$
w(P, t) = -h^2 \sum_{Q \in \mathcal{Q}_h} G^h(P, Q; t) \mathfrak{L}_h v(Q, 0) - \sum_{Q \in \mathcal{C}_h} G^h(P, Q; t) \frac{dv}{dt} (Q, 0)
$$

$$
- h^2 \int_{0}^{t} d\tau \sum_{Q \in \mathcal{Q}_h} G^h(P, Q; t - \tau) \frac{d}{d\tau} [\mathfrak{L}_h v(Q, \tau)]
$$

$$
- \int_{0}^{t} d\tau \sum_{Q \in \mathcal{C}_h} G^h(P, Q; t - \tau) \frac{d^2v}{d\tau^2} (Q, \tau).
$$
Thus, for $P \in \mathcal{Q}_h$ and $t \ge 0$,
$$
\mathfrak{L}_h w(P, t) = \sum_{Q \in \mathcal{C}_h} \delta(P, Q) \mathfrak{L}_h v(Q, 0) + h^{-2} \sum_{Q \in \mathcal{C}_h} \delta(P, Q) \frac{dv}{dt} (Q, 0)
$$

$$
-\int_{0}^{1} d\tau \sum_{Q \in C_{h}} G^{h}(P, Q; t - \tau) \frac{d^{2}v}{d\tau^{2}} (Q, \tau).
$$

\nThus, for $P \in \Omega_{h}$ and $t \ge 0$,
\n
$$
\Omega_{h}w(P, t) = \sum_{Q \in \Omega_{h}} \delta(P, Q) \cdot \Omega_{h}v(Q, 0) + h^{-2} \sum_{Q \in C_{h}} \delta(P, Q) \frac{dv}{dt} (Q, 0)
$$
\n
$$
+ \int_{0}^{t} d\tau \sum_{Q \in \Omega_{h}} \delta(P, Q) \frac{d}{d\tau} [\Omega_{h}v(Q, \tau)]
$$
\n
$$
+ h^{-2} \int_{0}^{t} d\tau \sum_{Q \in C_{h}} \delta(P, Q) \frac{d^{2}v}{d\tau^{2}} (Q, \tau)
$$
\n
$$
\approx \Omega_{h}v(P, 0) + \int_{0}^{t} \frac{d}{d\tau} [\Omega_{h}v(Q, \tau)] d\tau = \Omega_{h}v(P, t).
$$
\nFrom the uniqueness result of Theorem 3.1 follows that $w \equiv v$, and so the proof complete **E**

From the uniqueness result of Theorem 3.1 follows that $w \equiv v$, and so the proof is

.
【注意】

EXECUTE:
 A. The Method of Lines

One method of removing the infinite sure

singularity is to truncate the series, anot

dures is to discretize the time regished 15 One method of **removing** he infinite sum in the representation of the fundamental singularity is to truncate the series, another and perhaps better for numerical procedures is to discretize the time variable [15]. We do this now, using equal-spacing with step-size *k,* and use a forward-difference scheme for which the discrete operator is **4.** The Method of Lines

One method of removing the infinite sum in

singularity is to truncate the series, another idures is to discretize the time variable [15]. W

step-size k, and use a forward-difference scher
 \mathcal Solution of pseudoparabolic equations

4. The Method of Lines

One method of convoring the infinite sum in the representation of the fundamental

singularity is to truncate the series, another and perhaps better for numer

$$
\Omega_k^{\,F}v_i^{\,i} := k^{-1}M[v_{i+1} - v_i] - Lv_i.
$$

The adjoint backward-difference operator is

$$
\Omega_k^{*,B}v_{i+1}=k^{-1}M[v_{i+1}-v_i]+Lv_{i+1}.
$$

Let N denote the set of natural numbers and $N^+ = N - \{0\}$. We seek G^k defined on $\bar{Q} \times \bar{\Omega} \times {\rm N}$ satisfying Let $\frac{N}{\Omega} \times \bar{\Omega}$

One method of removing the infinite sum in the representation of the fundamental
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ures is to discrete the time variable [15]. We do this now, using equal-spacing with
tep-size k, and use a forward-difference scheme for which the discrete operator is

$$
\Omega_k^F v_i := k^{-1} M[v_{i+1} - v_i] - Lv_i
$$
.
The adjoint backward-difference operator is
 $\Omega_k^{*,b} v_{i+1} = k^{-1} M[v_{i+1} - v_i] + Lv_{i+1}$.
Let N denote the set of natural numbers and $N^+ = N - \{0\}$. We seek G^k defined on
 $2 \times \overline{\Omega} \times N$ satisfying
 $\Omega_k^F v_i^G(P, Q; i) = \delta(P - Q)$ for $P \in \Omega$, $i \in N$,
 $G^k(P, Q; i) = 0$ for $P \in C$, $i \in N^+$,
 $G^k(P, Q; i) = 0$ for $P, Q \in \Omega$.
Theorem 4.1: G^k exists and is unique. Furthermore, $G^k(P, Q; 0) = 0$ and for $i \in N^+$,
 $G^k(P, Q; i) = \sum_{j=1}^l {i \choose j} k^j G_{j-1}(P, Q)$ (4.2)
where $\{G_j\}_{j=0}^{\infty}$ is as in (2.3).
Proof: Existence may be verified directly using formula (4.2) and substituting
into (4.1). Uniqueness may be shown using formula (4.3) which is given below.
Lemma 4.2: Let *i* be a (fixed) positive integer and let $Q \in \overline{\Omega}$. Then for $1 \leq j \leq i$,
 $\{\Omega_k^k B_j G^k(P, Q; i - j) = -\delta(P - Q)\}$ for $P \in \Omega$,
 $(G^k(P, Q; i - j) = 0$ for $P \in \Omega$,
 $(G^k(P, Q; i - j) = 0$ for $P \in \Omega$.
Theorem 4.3: Let $\{v_i\}_{i \in N}$ be defined on $\overline{\Omega}$ with $v_0 = 0$. Then for each $i \in N$ and for

Theorem 4.1; G^k exists and is unique. Furthermore, $G^k(P, Q; 0) \equiv 0$ and for i

$$
G^{k}(P, Q; 0) = 0
$$
 for $P, Q \in \Omega$.
\n
$$
G^{k}(P, Q; i) = \sum_{j=1}^{i} {i \choose j} k^{j} G_{j-1}(P, Q)
$$
 (4.2)
\n
$$
\sum_{j=0}^{n} i^{j} g_{j} = \sum_{j=1}^{i} {i \choose j} k^{j} G_{j-1}(P, Q)
$$
 (4.2)
\n
$$
\sum_{j=0}^{n} i^{j} g_{j} = 0
$$
 is as in (2.3).
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y^{j} = 0
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 is as in (2.3

where ${G_i}_{i=0}^{\infty}$ *is as in* (2.3).

-

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Proof: Existence may be verified directly using formula (4.2) and substituting into (4.1) . Uniqueness may be shown using formula (4.3) which is given below.

Lemma 4.2: Let *i* be a (fixed) positive integer and let $Q \in \overline{\Omega}$. Then for $1 \leq j \leq i$
 $G^k(P, Q; i - j)$ *satisfies*

$$
\begin{cases}\n i-j) \ satisfies \\
 \begin{cases}\n \Omega_k^*, B_j G^k(P, Q; i-j) = -\delta(P-Q) & \text{for} \quad P \in \Omega, \\
 G^k(P, Q; i-j) = 0 & \text{for} \quad P \in C.\n\end{cases}
$$

Theorem 4.3: Let $\{v_i\}_{i\in \mathbb{N}}$ be defined on $\overline{\Omega}$ with $v_0\equiv 0$. Then for each $i\in \mathbb{N}$ and for *each P E Q we have the representation formula*

Lemma 4.2: Let *i* be a (fixed) positive integer and let
$$
Q \in \Omega
$$
. Then for $1 \leq j \leq i$,
\n
$$
G^{k}(P,Q; i - j) \text{ satisfies}
$$
\n
$$
\begin{cases}\n\mathfrak{L}_{k,P,j}^{*,B}G^{k}(P,Q; i - j) = -\delta(P - Q) & \text{for } P \in \Omega, \\
G^{k}(P,Q; i - j) = 0 & \text{for } P \in C.\n\end{cases}
$$
\n
$$
\begin{cases}\n\mathfrak{L}_{k,P,j}^{*,B}G^{k}(P,Q; i - j) = -\delta(P - Q) & \text{for } P \in \Omega, \\
G^{k}(P,Q; i - j) = 0 & \text{for } P \in C.\n\end{cases}
$$
\n
$$
\begin{cases}\n\text{Theorem 4.3: Let } \{v_i\}_{i \in \mathbb{N}} \text{ be defined on } \overline{\Omega} \text{ with } v_0 = 0. \text{ Then for each } i \in \mathbb{N} \text{ and for each } P \in \Omega \text{ we have the representation formula} \\
v_{i+1}(P) = \sum_{j=0}^{i} \int_{\Omega} [G^{k}(Q, P; i - j + 1) - G^{k}(Q, P; i - j)] \mathfrak{L}_{k}^{*}v_{j}(Q) dA(\Omega_{Q}) \\
+ k^{-1} \sum_{j=0}^{i} \int_{C} [v_{j+1}(Q) - v_{j}(Q)] \frac{\partial}{\partial n_{Q}} [G^{k}(Q, P; i - j + 1) - G^{k}(Q, P; i - j + 1)] \\
- G^{k}(Q, P; i - j)] d\sigma_{Q}.\n\end{cases}
$$
\n
$$
\begin{cases}\n\text{Proof: We begin by employing Lemma 4.2, a telescoping series and the definition of } v_{0}. \text{ Setting } \Gamma_{j}(P, Q) = G^{k}(P, Q; i + 1 - j) \text{ we have}\n\end{cases}
$$

$$
\lim_{j=0} \frac{1}{2} \sum_{j=0}^{n} \int_{C} [v_{j+1}(Q) - v_{j}(Q)] \frac{\partial}{\partial n_{Q}} [G^{k}(Q, P; i - j + 1)]
$$

\n
$$
- G^{k}(Q, P; i - j)] d\sigma_{Q}.
$$

\nProof: We begin by employing Lemma 4.2, a telescoping series and the definition
\nof v_{Q} . Setting $\Gamma_{j}(P, Q) = G^{k}(P, Q; i + 1 - j)$ we have
\n
$$
v_{i+1}(P) = -\sum_{j=0}^{i} \int_{Q} [v_{j+1}(Q) - v_{j}(Q)] \mathfrak{L}_{k,Q}^{*B} \Gamma_{j+1}(Q, P) dA(\Omega_{Q})
$$

\n
$$
= -\sum_{j=0}^{i} \int_{Q} [v_{j+1}(Q) - v_{j}(Q)] \{k^{-1}M_{Q}[\Gamma_{j+1}(Q, P) - \Gamma_{j}(Q, P)] + a(Q) \Gamma_{j+1}(Q, P)\} dA(\Omega_{Q})
$$

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\n
$$
= -\sum_{j=0}^{i} \int_{\Omega} [\Gamma_{j+1}(Q, P) - \Gamma_{j}(Q, P)] \ \mathfrak{L}_{k}^{F} v_{j}(Q) \ da(\Omega_{Q})
$$
\n
$$
- k^{-1} \sum_{j=0}^{i} \int_{C} [v_{j+1}(Q) - v_{j}(Q)] \frac{\partial}{\partial n_{Q}} [\Gamma_{j+1}(Q, P) - \Gamma_{j}(Q, P)] \ d\sigma_{Q}
$$
\n
$$
- \sum_{j=1}^{i} \int_{Q} \{a(Q) v_{j}(Q) [\Gamma_{j+1}(Q, P) - \Gamma_{j}(Q, P)]
$$
\n
$$
+ a(Q) \ \Gamma_{j+1}(Q, P) [v_{j+1}(Q) - v_{j}(Q)] \} \ dA(\Omega_{Q}).
$$
\nt sum vanishes due to obvious cancellations; hence, the proof i placing the Γ_{j} -terms by the corresponding G^{k} -terms

The last sum vanishes due to obvious cancellations; hence, the proof is complete upon replacing the Γ_i -terms by the corresponding G^k -terms **I**

At this point we investigate the simultaneous discretization of the space and time variables. To this end we define $\overline{\Omega}_h := \Omega_h \cup C_h$ as in Section 3, and we consider the operator

$$
\mathbb{E}_{k,h}^F v_i^h(P) := M_h[v_{i+1}^h(P) - v_i^h(P)] - Lv_i^h(P)
$$

where $\{v_i^h\}_{i\in \mathbb{N}}$ is a sequence of functions defined on $\bar{\Omega}_h$. We seek $\{u_i^{\ k,h}\}_{i\in \mathbb{N}}$ satisfying

$$
-\sum_{j=1}^{n} \sum_{j} \{a(Q) v_j(Q)\{I_{j+1}(Q, P) - I_j(Q, P)\}\} dA(Q_0).
$$
\nThe last sum vanishes due to obvious cancellations; hence, the proof is complete upon replacing the Γ_j -terms by the corresponding G^k -terms.
\nAt this point we investigate the simultaneous discretization of the space and time variables. To this end we define $\overline{Q}_h := \mathcal{Q}_h \cup C_h$ as in Section 3, and we consider the operators
\noperator
\n
$$
\Omega_{k,h}^F v_i^h(P) := M_h[v_{i+1}^h(P) - v_i^h(P)] - L v_i^h(P)
$$
\nwhere $\{v_i^h\}_{i\in\mathbb{N}}$ is a sequence of functions defined on \overline{Q}_h . We seek $\{u_i^{k,h}\}_{i\in\mathbb{N}}$ satisfying
\n
$$
\Omega_{k,h}^F u_i^{k,h}(P) = F(P, ik)
$$
 for $P \in \Omega_h$, $i \in \mathbb{N}$,
\n
$$
u_i^{k,h}(P) = f(P, ik)
$$
 for $P \in \Omega_h$, $i \in \mathbb{N}^*$,
\n
$$
u_0^{k,h}(P) = 0
$$
 for $P \in \Omega_h$, $i \in \mathbb{N}^*$,
\n
$$
u_0^{k,h}(P) = 0
$$
 for $P \in \Omega_h$.
\nTheorem 4.4: There exists a unique solution $\{u_i^{k,h}\}_{i\in\mathbb{N}}$ to problem (4.4).
\nProof: This follows by the fact that for each fixed *i* we may uniquely solve
\n
$$
M_h[u_{i+1}^h(P)] = F(P, ik) + M_h[u_i^h(P)] - Lu_i^h(P)
$$
 for $u_{i+1}^{k,h}$ in terms of a known u_i^h .
\nIndeed
\n
$$
u_{i+1}^{k,h}(P) = -h^2 \sum G_0^h(P, Q) \{F(P, ik) + M_h[u_i^h(Q)] - Lu_i^h(Q)\}
$$

Theorem 4.4: There exists a unique solution ${u_i}^{k,h}$ _{ieN} to problem (4.4).

Proof: This follows by the fact that for each fixed *i* we may uniquely solve $M_h[u_{i+1}^h(P)] = F(P, ik) + M_h[u_i^h(P)] - Lu_i^h(P)$ for $u_{i+1}^{k,h}$ in terms of a known u_i^h .
Indeed

$$
\mathfrak{L}_{k,h}^{P_{k,h}(P)} = F(P, ik) \quad \text{for} \quad P \in \Omega_{h}, \quad i \in \mathbb{N},
$$
\n
$$
u_{i}^{k,h}(P) = f(P, ik) \quad \text{for} \quad P \in C_{h}, \quad i \in \mathbb{N}^{+},
$$
\n
$$
u_{0}^{k,h}(P) = 0 \quad \text{for} \quad P \in \Omega_{h}, \quad i \in \mathbb{N}^{+},
$$
\n
$$
u_{0}^{k,h}(P) = 0 \quad \text{for} \quad P \in \Omega_{h}.
$$
\n
$$
\text{from 4.4: There exists a unique solution } \{u_{i}^{k,h}\} \text{ is to problem (4.4)}.
$$
\n
$$
\therefore \text{ This follows by the fact that for each fixed } i \text{ we may uniquely solve}
$$
\n
$$
P) = F(P, ik) + M_{h}[u_{i}^{h}(P)] - Lu_{i}^{h}(P) \text{ for } u_{i+1}^{k,h} \text{ in terms of a known } u_{i}^{h}.
$$
\n
$$
u_{i+1}^{k,h}(P) = -h^{2} \sum_{Q \in \Omega_{h}} G_{0}^{h}(P,Q) \{F(P, ik) + M_{h}[u_{i}^{h}(Q)] - Lu_{i}^{h}(Q)\}
$$
\n
$$
+ \sum_{Q \in \Omega_{h}} G_{0}^{h}(P,Q) \{Q, ik\}, \quad i \geq 0,
$$
\n
$$
\text{(4.5)}
$$
\n
$$
\text{(4.6)}
$$
\n
$$
P) := 0 \quad \blacksquare
$$
\n
$$
\text{ated with the problem (4.4) we define a discrete Green's function } G^{k,h} \text{ de-}
$$
\n
$$
\overline{\Omega}_{h} \times \overline{\Omega}_{h} \times \overline{\Omega}_{h} \times \mathbb{N} \text{ as follows:}
$$
\n
$$
G^{k,h}(P,Q;0) := 0,
$$
\n
$$
G^{k,h}(P,Q;i) := \sum_{n=0}^{i} {i \choose n} k^{n} G_{n-1}^{h}(P,Q) \quad \text{for} \quad i > 0,
$$
\n
$$
\text{and} \quad G^{k,h}(P,k) = 0 \quad \text{and} \quad G^{k,h}(P,k) = 0 \quad
$$

and $u_0^{k,h}(P) := 0$ **I**

•

\ Associated with the problem (4.4) we define a discrete Green's function $G^{k,h}$ de-Associated with the problem (4.4) we define a discrete Green
fined on $\overline{Q}_h \times \overline{Q}_h \times N$ as follows:

$$
+\sum_{Q \in C_h} G_0^h(P, Q) f(Q, ik), \qquad i \geq 0,
$$
\nand $u_0^{k,h}(P) := 0$
\nAssociated with the problem (4.4) we define a discrete Green's function $G^{k,h}$ defined on $\overline{\Omega}_h \times \overline{\Omega}_h \times \overline{\Omega}_h \times \overline{\Omega}_h$ as follows:
\n $G^{k,h}(P, Q; 0) := 0$,
\n $G^{k,h}(P, Q; i) := \sum_{n=0}^r {i \choose n} k^n G_{n-1}^h(P, Q)$ for $i > 0$,
\nwhere the G_n^h , $n \geq 0$, are defined by (3.3). It is easy to show that $G^{k,h}$ acts as a re-
\nproducing singularity. To this end we observe the following result, whose proof is
\ndirect.
\nTheorem 4.5: $G^{k,h}$ satisfies
\n $\Omega_{k,h}^F(G^{k,h}(P, Q; i) = -h^{-2}\delta(P, Q)$ for $P \in \Omega_h$, $i \in \mathbb{N}$,
\n $G^{k,h}(P, Q; i) = ik\delta(P, Q)$ for $P \in \overline{\mathcal{O}}_h$, $i \in \mathbb{N}^+$,
\n $G^{k,h}(P, Q; 0) = 0$ for $P, Q \in \overline{\Omega}_h$.
\n(4.7)
\n $G^{k,h}(P, Q; 0) = 0$

where the G_n^h , $n \ge 0$, are defined by (3.3). It is easy to show that $G^{k,h}$ acts as a re-
producing singularity. To this end we observe the following result, whose proof is
direct.
Theorem 4.5: $G^{k,h}$ *satisfies*
 $\$ producing singularity. To this end we observe the following result, whose proof is
 i direct.
 12F_{*zh*}*G*^{*k*}*h*^{*R*}*fsatisfies*
 $\mathfrak{L}_{k,h}^{F} G^{k,h}(P, Q; i) = -h^{-2}\delta(P, Q)$ for $P \in \mathcal{Q}_h$, $i \in \mathbb{N}$,
 $G^{k,h}(P, Q$

Theorem $4.5: G^{k,n}$ *satisfies*

$$
\mathfrak{L}_{k,h}^{F}G^{k,h}(P,Q;i) = -h^{-2}\delta(P,Q) \quad \text{for} \quad P \in \mathcal{Q}_h, \quad i \in \mathbb{N},
$$
\n
$$
G^{k,h}(P,Q;i) = ik\delta(P,Q) \quad \text{for} \quad P \in \bar{\mathcal{C}}_h, \quad i \in \mathbb{N}^+,
$$
\n
$$
G^{k,h}(P,Q;0) = 0 \quad \text{for} \quad P, Q \in \bar{\mathcal{Q}}_h.
$$
\n(4.7)

The following representation theorem may be used to solve the nonhomogeneous initial-boundary value problem..

The following representation theorem may be used to solve the nonhomogeneous itial-boundary value problem.
Theorem 4.6: Let ${v_i}^h_{i\in N}$ be defined on $\overline{\Omega}_h^{\cdot}$ where $v_0^h \equiv 0$. Then for $i \in N$ and for $i \in \overline{\Omega}_h$ **• The following representional probability of** $P \in \overline{\Omega}_h$ **,
** $v_{i+1}(P) = -h$

Numerical solution of pseudoparabolic equations 33
\nllowing representation theorem may be used to solve the nonhomogeneous
\nundary value problem.
\n
$$
e_m 4.6: Let \{v_i^h\}_{i\in\mathbb{N}} be defined on \bar{\Omega}_h^{\prec} where v_0^h \equiv 0. Then for i \in \mathbb{N} and for\n
$$
v_{i+1}(P) = -h^2 \sum_{j=0}^{i} \sum_{Q \in \mathcal{Q}_h} [G^{k,h}(P, Q; i+1-j) - G^{k,h}(P, Q; i-j)] \mathfrak{L}_{k,h}^F v_{j+1}(Q)
$$
\n
$$
+ k^{-1} \sum_{j=0}^{i} \sum_{Q \in \mathcal{C}_h} [G^{k,h}(P, Q; i+1-j) - G^{k,h}(P, Q; i-j)]
$$
\n
$$
\times [v_{j+1}(Q) - v_j(Q)]. \qquad (4.8)
$$
\n
$$
e_m \sum_{Q \in \mathcal{C}_h} [G^{k,h}(P, Q; i+1-j) - G^{k,h}(P, Q; i-j)]
$$
\n
$$
= 0.4
$$
\n
$$
e_m \sum_{Q \in \mathcal{C}_h} [G^{k,h}(Q, Q; i+1-j)] \mathfrak{L}_{k,h}^F v_{j+1}(Q)
$$
\n
$$
= 0.4
$$
\n
$$
e_m \sum_{Q \in \mathcal{C}_h} [G^{k,h}(Q, Q; i+1-j)] \mathfrak{L}_{k,h}^F v_{j+1}(Q)
$$
\n
$$
= 0.4
$$
\n
$$
e_m \sum_{Q \in \mathcal{C}_h} [G^{k,h}(Q; i+1-j)] \mathfrak{L}_{k,h}^F v_{j+1}(Q)
$$
\n
$$
= 0.4
$$
\n
$$
e_m \sum_{Q \in \mathcal{C}_h} [G^{k,h}(Q; i+1-j)] \mathfrak{L}_{k,h}^F v_{j+1}(Q)
$$
\n
$$
= 0.4
$$
\n
$$
e_m \sum_{Q \in \mathcal{C}_h} [G^{k,h}(Q; i+1-j)] \mathfrak{L}_{k,h}^F v_{j+1}(Q)
$$
\n
$$
= 0.4
$$
\n
$$
e_m \sum_{Q \in \mathcal{C}_h} [G
$$
$$

Proof: Let $w(P)$ denote the right-hand side of (4.8). For $P \in \overline{\Omega}_h$, we have

$$
i=0 \text{ Ge } a_{h}
$$
\n
$$
+ k^{-1} \sum_{j=0}^{i} \sum_{Q \in C_{h}} [G^{k,h}(P, Q; i+1-j) - G^{k,h}(P, Q; i-j)]
$$
\n
$$
\times [v_{j+1}(Q) - v_{j}(Q)].
$$
\nProof: Let $w(P)$ denote the right-hand side of (4.8). For $P \in \overline{\Omega}_{h}$, we have\n
$$
\Omega_{k,h}^{F}w(P) = -h^{2} \sum_{j=0}^{i-1} \sum_{Q \in \Omega_{h}} [-h^{-2}\delta(P, Q) + h^{-2}\delta(P, Q)] \Omega_{k,h}^{F}v_{j+1}(Q)
$$
\n
$$
- h^{2} \sum_{Q \in \Omega_{h}} [-h^{-2}\delta(P, Q) - 0] \Omega_{k,h}^{F}v_{i+1}(Q).
$$
\n
$$
+ k^{-1} \sum_{j=0}^{i-1} \sum_{Q \in C_{h}} [-h^{-2}\delta(P, Q) + h^{-2}\delta(P, Q)] [v_{j+1}(Q) - v_{j}(Q)].
$$
\n
$$
+ k^{-1} \sum_{Q \in C_{h}} [-h^{-2}\delta(P, Q) - 0] [v_{i+1}(Q) - v_{i}(Q)].
$$
\nOnly the second term above fails to vanish, and it equals $\Omega_{k,h}^{F}v_{j+1}(P)$.
\nIf $P \in C_{h}$,
\n
$$
w(P) = -h^{2} \sum_{j=0}^{i} \sum_{Q \in \Omega_{h}} [(i-j+1) k\delta(P, Q) - (i-j) k\delta(P, Q)] \Omega_{k,h}^{F}v_{j+1}(Q).
$$

•

$$
\int_{0}^{2} \sum_{i=0}^{n} \sum_{q \in C_{h}} [-h^{-2}\delta(P, Q) + h^{-2}\delta(P, Q)] [v_{i+1}(Q) - v_{i}(Q)]
$$
\n
$$
+ k^{-1} \sum_{j=0}^{n-1} \sum_{q \in C_{h}} [-h^{-2}\delta(P, Q) + h^{-2}\delta(P, Q)] [v_{i+1}(Q) - v_{i}(Q)]
$$
\n
$$
+ k^{-1} \sum_{q \in C_{h}} [-h^{-2}\delta(P, Q) - 0] [v_{i+1}(Q) - v_{i}(Q)]
$$
\nOnly the second term above fails to vanish, and it equals $\Omega_{k,h}^{F}v_{i+1}(P)$.

\nIf $P \in C_h$,

\n
$$
w(P) = -h^{2} \sum_{j=0}^{n} \sum_{q \in C_{h}} [(i-j+1) k\delta(P, Q) - (i-j) k\delta(P, Q)] \Omega_{k,h}^{F}v_{i+1}(Q) + k^{-1} \sum_{j=0}^{n} \sum_{q \in C_{h}} [(i-j+1) k\delta(P, Q)]
$$
\n
$$
- (i-j) k\delta(P, Q)] [v_{i+1}(Q) - v_{i}(Q)]
$$
\n
$$
= \sum_{j=0}^{i} (v_{i+1}(P) - v_{j}(P)] = v_{i+1}(P) - v_{0}(P) = v_{i+1}(P).
$$
\nThe proof is completed by invoking Theorem 4.4

\nFormula (4.8) is simplified somewhat by using the identity

\n
$$
G^{k,h}(P, Q; i+1-j) - G^{k,h}(P, Q; i-j) = \sum_{n=0}^{i-j} {i-j \choose n} k^{n+1}G_{n}^{h}(P, Q).
$$
\nDefining

\n
$$
\Gamma_{i+1,j}^{k,h}(P, Q) := \sum_{n=0}^{i-j} {i-j \choose n} k^{n}G_{n}^{h}(P, Q).
$$
\nWe may rewrite (4.8) in the form

\n
$$
v_{i+1}(P) = -h^{2}k \sum_{j=0}^{i} \sum_{q \in R_{h}^{F}} \Gamma_{i+1,j}^{k,h}(P, Q) \Omega_{k,h}^{F}v_{j+1}(Q).
$$

The proof is completed-by invoking Theorem 4.4 **^I**

f is completed by invoking Theorem 4.4
\nla
\n
$$
G^{k,h}(P,Q;i+1-j) = G^{k,h}(P,Q;i-j) = \sum_{n=0}^{i-j} {i-j \choose n} k^{n+1} G_n^h(P,Q).
$$
\n(4.9)
\n
$$
\Gamma_{i+1,j}^{k,h}(P,Q) := \sum_{n=0}^{i-j} {i-j \choose n} k^n G_n^h(P,Q).
$$

V •

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$$
\int_{\Gamma_{i+1,j}}^{\mu,h}(P,Q) := \sum_{n=0}^{i-j} {i-j \choose n} k^n G_n^h(P,Q)
$$
\n
$$
\text{rewrite (4.8) in the form} \tag{4.10}
$$

we may rewrite (4.8) in the form

Formula (4.8) is simplified somewhat by using the identity
\n
$$
G^{k,h}(P, Q; i+1-j) - G^{k,h}(P, Q; i-j) = \sum_{n=0}^{i-j} {i-j \choose n} k^{n+1} G_n^h(P, Q).
$$
\nDefining
\n
$$
\Gamma_{i+1,j}^{k,h}(P, Q) := \sum_{n=0}^{i-j} {i-j \choose n} k^n G_n^h(P, Q).
$$
\nwe may rewrite (4.8) in the form
\n
$$
v_{i+1}(P) = -h^2 k \sum_{j=0}^{i} \sum_{Q \in G_n} \Gamma_{i+1,j}^{k,h}(P, Q) \Omega_{k,h}^P v_{j+1}(Q)
$$
\n
$$
+ \sum_{j=0}^{i} \sum_{Q \in C_h} \Gamma_{i+1,j}^{k,h}(P, Q) [v_{j+1}(Q) - v_j(Q)].
$$
\n(4.11)
\n3 Analysis Bd. 2, Hett 1 (1983)

5. Error Bounds

In this section we incorporate the usual error estimates which have been developed for elliptic partial differential equations, solutions represented in terms of fundamental singularities $[2, 3, 16]$. In order to extend these estimates to the case of pseudo-**34** ROBERT P. GILBERT and LEROY R. LUNDIN

5. Error Bounds

In this section we incorporate the usual error estimates which have been developed

for elliptic partial differential equations, solutions represented in terms **34** ROBERT P. GILBERT and LEROY R. LUNDIN
 5. Error Bounds
 **In this section we incorporate the usual error estimates which have

for elliptic partial differential equations, solutions represented in term

singularit Bounds**

Getion we incorporate the usual error estimates which have been

c partial differential equations, solutions represented in terms of fu

ies [2, 3, 16]. In order to extend these estimates to the case

equations In this section we incorporate

for elliptic partial differential ec

singularities [2, 3, 16]. In ord

parabolic equations we must fi

Lemma 5.1: For each $i \ge 0$

in (4.10).

Proof: We know $G_0^h \ge 0$ by
 $G_{n+1}^h(P,Q) =$

in (4.10).
Proof: We know $G_0^h \ge 0$ by the Collatz maximum principal. Further, for $n \ge 0$,

$$
G_{n+1}^h(P,Q) = -h^2 \sum_{T \in \mathcal{Q}_h} G_0^h(P,T) a(T) G_n^h(T,Q),
$$

Proof: We know $G_0^h \ge 0$ by the Collatz maximum principal. Further, for $n \ge 0$,
 $G_{n+1}^h(P,Q) = -h^2 \sum_{T \in \mathcal{Q}_h} G_0^h(P,T) a(T) G_n^h(T,Q)$,

and since $a \le 0$, it follows by induction that $G_n^h \ge 0$ for all *n*. Hence, the d Lemma 5.1. For each $i \geq 0$, and for $0 \leq j \leq i$, $i_{i+1,j} \leq 0$, and $i_{i+1,j} \leq 0$, $\lim_{i \to 1} (4.10)$.

Proof: We know $G_0^h \geq 0$ by the Collatz maximum principal. Further, for $n \geq 0$,
 $G_{n+1}^h(P,Q) = -h^2 \sum_{T \in Q_h} G_$ $G_{n+1}^h(P,Q) = -\hbar^2 \sum_{f \in G_h} G_0^h(P,T) a(T) G_n^h(T,Q)$,

and since $a \le 0$, it follows by induction that $G_n^h \ge 0$ for all *n*. Hence, the c

result is immediate from (4.10) \blacksquare

Lemma 5.2: For $P \in \Omega_h$, $\hbar^2 \sum_{Q \in \Omega_h} G_0^$

and since
$$
a \le 0
$$
, it follows by induction that $G_n^a \ge 0$ for all n. I result is immediate from (4.10) \blacksquare
\nLemma 5.2: For $P \in \Omega_h$, $h^2 \sum_{Q \in \Omega_h} G_0^h(P, Q) \le K := \left[\min_{Q \in \Omega_h} q(Q)\right]^{-1}$.
\nProof: As usual, let C_h^* denote all points of Ω_h which are neither C_h , and for $Q \in \Omega_h$ let $N_h(Q)$ denote the set of neighbors of Q. There
\n
$$
1 = -h^2 \sum_{Q \in \Omega_h} G_0^h(P, Q) \left[-q(Q)\right] + h^2 \sum_{Q \in C} G_0^h(P, Q) \cdot \text{card}(N) \cdot \text{diag}(Q) \cdot \text{diag}(N) \cdot \text{diag}(N)
$$

Definition 3.2. For
$$
I \in \mathbb{Z}_{h}
$$
, $h \leqslant \log_{h} 0$ $(I, \mathcal{L}) \leqslant I$. The inequality $G_{\epsilon_0} \circ \mathcal{L}_h$ is a real product of Q_h which are neighbors of Q_h , and for $Q \in \mathcal{Q}_h$ let $N_h(Q)$ denote the set of neighbors of Q . Then for $P \in \mathcal{Q}_h$, $1 = -h^2 \sum_{Q \in \mathcal{Q}_h} G_0^h(P, Q) \left[-q(Q) \right] + h^2 \sum_{Q \in \mathcal{C}^*} G_0^h(P, Q) \cdot \text{card}(N_h(Q) \cap C_h)$. Since $G_0^h \geq 0$, we conclude

Since $G_0^h \geq 0$, we conclude *- -* Since G_0^h
from which the main the main \mathbf{F}

$$
1 \geq h^2 K^{-1} \sum_{Q \in \mathcal{Q}_h} G_0^h(P,Q),
$$

 $1 \geq h^2 K^{-1} \sum_{Q \in \Omega_h} G_0^h(P, Q);$
 Lemma 5.3: Let i ≥ 0 and *let* $0 \leq j \leq i$. Then for $P \in \Omega_h$, Ω_h is Γ_h with Ω_h $Q_h \leq K$ is h leads K line Ω_h .

From which the desired result easily follows

\nLemma 5.3: Let
$$
i \geq 0
$$
 and let $0 \leq j \leq i$. Then for

\n
$$
h^2 \sum_{Q \in \mathcal{Q}_h} \Gamma_{i+1,j}^{k,h}(P,Q) \leq K[i+k ||a||_{\infty} K]^{i-j}.
$$
\nProof: For $n \geq 0$ and $P \in \mathcal{Q}_h$,

\n
$$
h^2 \sum_{Q \in \mathcal{Q}_h} \Gamma_{i+1,Q}^{k,h}(P,Q) = h^2 \sum_{Q \in \mathcal{Q}_h} \Gamma_{i+1,Q}^{k,h}(P,Q).
$$

$$
C_h
$$
, and for $Q \in \Omega_h$ let $N_h(Q)$ denote the set of neighbors of Q . Then for $P \in \Omega_h$.
\n
$$
1 = -h^2 \sum_{Q \in \Omega_h} G_0{}^h(P,Q) [-q(Q)] + h^2 \sum_{Q \in C} G_0{}^h(P,Q) \cdot \text{card}(N_h(Q) \cap C_h).
$$
\nSince $G_0{}^h \geq 0$, we conclude
\n
$$
1 \geq h^2 K^{-1} \sum_{Q \in \Omega_h} G_0{}^h(P,Q),
$$
\nfrom which the desired result easily follows
\nLemma 5.3: Let $i \geq 0$ and let $0 \leq j \leq i$. Then for $P \in \Omega_h$,
\n
$$
h^2 \sum_{Q \in \Omega_h} \Gamma_{i+1,j}^{k+1}(P,Q) \leq K[i+k ||a||_{\infty} K]^{i-j}.
$$
\nProof: For $n \geq 0$ and $P \in \Omega_h$.
\n
$$
h^2 \sum_{Q \in \Omega_h} G_{n+1}^h(P,Q) = h^2 \sum_{Q \in \Omega_h} [-h^2 \sum_{T \in \Omega_h} G_0{}^h(P,T) a(T) G_n{}^h(T,Q)]
$$
\n
$$
= h^2 \sum_{T \in \Omega_h} [-a(T)] G_0{}^h(P,T) h^2 \left[\sum_{Q \in \Omega_h} G_n{}^h(T,Q) \right]
$$
\n
$$
\leq ||a||_{\infty} \left[\max_{T \in \Omega_h} h^2 \sum_{Q \in \Omega_h} G_n{}^h(T,Q) \right] \left[\sum_{T \in \Omega_h} G_0{}^h(P;T) \right].
$$
\nBy induction, it follows that for $n \geq 0$ and $P \in \Omega_h$,
\n
$$
h^2 \sum_{Q \in \Omega_h} G_n{}^h(P,Q) \leq ||a||_{\infty}^h K^{n+1}.
$$

By induction, it follows that for $n \geq 0$ and $P \in \Omega_h$,

$$
h^2 \sum_{Q \in \Omega_h} G_n^{\ h}(P,Q) \leq ||a||_{\infty}^n K^{n+1}
$$

Inequality (5.1) easily follows now from (4.10)

 $\begin{split} \mathbf{f} &\approx \frac{1}{2} \left(\begin{array}{ccc} \mathbf{f} & \mathbf{f} & \mathbf{f} \\ \mathbf{f} & \mathbf{f} & \mathbf{f} \end{array} \right) \mathbf{f} \end{split}$

 $h^2 \sum_{Q \in \mathcal{Q}_h} G_n^h(P, Q) \leq ||a||_{\infty}^n K^{n+1}$.
equality (5.1) easily follows now from (4.10) \blacksquare
Finally, we bound the error $\varepsilon_i^{k,h}(P) := u(P, ik) -$
(4.4). We have $\varepsilon_0^{k,h} \equiv 0$ and $\varepsilon_i^{k,h}(P) = 0$ for P \in $h^2 \sum_{Q \in \mathcal{Q}_h} G_n^h(P, Q) \leq ||a||_{\infty}^h K^{n+1}.$
Inequality (5.1) easily follows now from (4.10) \blacksquare
Finally, we bound the error $\varepsilon_i^{k,h}(P) := u(P, ik) - u_i^{k,h}(P)$, where $u_i^{k,h}$ is a solution
of (4.4). We have $\varepsilon_0^{k,h} \equiv 0$ apply (4.11) to obtain **r** in
of (4.
apply
.

$$
\mathcal{L}(\mathcal{E}D_n)
$$
\n
$$
\leq ||a||_{\infty} \left[\max_{T \in \Omega_h} h^2 \sum_{\Omega} G_n h(T, Q) \right] \left[h^2 \sum_{T \in \Omega_h} G_0 h(P, T) \right].
$$
\nBy induction, it follows that for $n \geq 0$ and $P \in \Omega_h$,
\n
$$
h^2 \sum_{Q \in \Omega_h} G_n h(P, Q) \leq ||a||_{\infty}^n K^{n+1}.
$$
\n
$$
\text{Inequality (5.1) easily follows now from (4.10)} \blacksquare
$$
\nFinally, we bound the error $\varepsilon_i^{k,h}(P) := u(P, ik) - u_i^{k,h}(P)$, where $u_i^{k,h}$ is a solution
\nof (4.4). We have $\varepsilon_0^{k,h} = 0$ and $\varepsilon_i^{k,h}(P) = 0$ for $P \in C_n$. For $i \geq 0$ and $P \in \Omega_h$, we
\napply (4.11) to obtain
\n
$$
\varepsilon_i^{k,h}(P) = -h^2 k \sum_{j=0}^i \sum_{Q \in \Omega_h} \Gamma_{i+1,j}^{k,h}(P, Q) \Omega_{k,h}^F \varepsilon_{j+1}(Q)
$$
\n
$$
= -h^2 k \sum_{j=0}^i \sum_{Q \in \Omega_h} \Gamma_{i+1,j}^{k,h}(P, Q) \{M_h[u(Q, (j+1) k) - u(Q, jk)] - Mu(Q, jk)\}.
$$

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Numerical solution of pseudoparabolic equations 35
\nSo applying (5.1) we find
\n
$$
|\varepsilon_{i+1}^{k,h}(P)| \leq \varkappa_{i+1}Kk \sum_{j=0}^{l} [1+k ||a||_{\infty} K]^{i-j} \leq \varkappa_{i+1}K(i+1) k[1+k ||a||_{\infty} K]^{i},
$$
\nwhere
\n
$$
\varkappa_{i+1} := \max_{0 \leq j \leq i} \max_{0 \leq \rho \leq \rho} |M_h[u(Q, (j+1)k) - u(Q, jk)] - Mu(Q, jk)|.
$$
\nOur error bound now follows from (5.2) under typical continuity assumptions on u.
\nTheorem 5.4: Let u be simultaneously C^4 on Q and C^2 on $[0, T]$. Then there exists a constant \varkappa independent of k and h, such that for $P \in \Omega_h$ and $0 \leq i \leq m + 1 \leq [T/k]$,
\n
$$
|\varepsilon_i^{k,h}(P)| \leq \varkappa KT[1+k ||a||_{\infty} K]^m kh^2.
$$
\n(5.3)

$$
Z_{i+1} := \max_{0 \leq j \leq i} \max_{Q \in \mathcal{Q}_k} |M_h[u(Q, (j+1) k) - u(Q, jk)] - M u(Q, jk)|.
$$

Our error bound now follows from (5.2) under typical continuity assumptions on u .

Theorem 5.4: Let u be simultaneously C^4 on Ω and C^2 on $[0, T]$. Then there exists a Our error bound now follows from (5.2) under typical continuity assumptions on u.

Theorem 5.4: Let u be simultaneously C^4 on Ω and C^2 on $[0, T]$. Then there existe

constant x independent of k and h, such that f value of $|\varepsilon_{i+1}^{k,h}(P)| \leq \varkappa_{i+1}Kk \sum_{j=0}^{i} [1 + k ||a||_{\infty} K]^{i-j} \leq \varkappa_{i+1}$
where
 $\varkappa_{i+1} := \max_{0 \leq j \leq i} \max_{0 \leq h} |M_h[u(Q, (j + 1) k) - u(Q, jk)|$
Our error bound now follows from (5.2) under typical con
Theorem 5.4: Let u be *:'*

$$
|\varepsilon_i^{k,h}(P)| \leq \varepsilon K T[1+k \, ||a||_{\infty} K]^m \, kh^2. \tag{5.3}
$$

6. Concluding Remarks

It is not necessary that Ω be a rectangle; indeed, we may consider irregular regions Ω . providing that $\dot{\Omega}$ is sufficiently smooth. We ask that $\dot{\Omega}$ be such that our Green's identities hold. As before, we place a rectangular grid of equal spacing over Ω . The boundary points, which we shall designate as points of C_h , are the intersections of $\dot{\Omega}$ with the grid lines. Next we designate an "inner boundary" C_h^* , as the set of points which are nearest neighbors to points of C_h . The remaining points of Ω which coincide. with grid points are "interior points" and their collection is designated by Ω_h . The It is not necessary that Ω
providing that Ω is sufficial
identities hold. As before,
boundary points, which w
with the grid lines. Next which are nearest neighbolow
with grid points are "inte
closure of Ω_h is \over closure of Ω_h is $\Omega_h := \Omega_h + C_h + C_h^*$. boundary points, which we shall designate as points of C_h , are the intersections of Ω with the grid lines. Next we designate an "inner boundary" C_h^* , as the set of points which are nearest neighbors to points of *s*, which we shall designate an

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 $\overline{\Omega}_h := \Omega_h + C_h + C_h^*$.

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which we shall designate as points of C_h are the intersections of Ω
s. Next we d

For the most part, we may proceed as before by replacing in our definitions and the Laplacian on points of C_h^* . For this case, we choose an interpolated Laplacian [3] $\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}})$

From the most part, we may proceed as before by replacing in our definitions and formulae
$$
\Omega_h
$$
 is $\overline{\Omega}_h = \Omega_h + C_h + C_h + \overline{C_h^*}$.

\nFor the most part, we may proceed as before by replacing in our definitions and formulae Ω_h by $\Omega_h + C_h^*$. Some concern must be taken, however, when dealing with the Laplacian on points of C_h^* . For this case, we choose an interpolated Laplacian [3]

\n
$$
\triangle_h u := h^{-2} \left\{ \frac{u(x + \alpha_1 h, y)}{(\alpha_1 + \alpha_2) \alpha_1} + \frac{u(x - \alpha_2 h, y)}{(\alpha_1 + \alpha_2) \alpha_2} + \frac{u(x, y + \beta_1 h)}{(\beta_1 + \beta_2) \beta_1} + \frac{u(x, y - \beta_1 h)}{(\beta_1 + \beta_2) \beta_2} - \left(\frac{1}{\alpha_1 \alpha_2} + \frac{1}{\beta_1 \beta_2} \right) u(x, y) \right\},
$$
\n(6.1)

\n(where $0 < \alpha_1, \alpha_2, \beta_1, \beta_2 \leq 1$. Not all of these parameters will be strictly less than one, since the distances $\alpha_i h$, $\beta_i h$ ($i = 1, 2$) where α_i , or $\beta_i < 1$ are measured from a point on C_h^* to a neighbour or C_h , and h is chosen small enough so that Ω_h is simply connected. To verify Lemma (5.2) for the present case, we introduce the function $Z(P) := \begin{cases} 1, P \in \Omega_h + C_h^*, \\ 0, P \in C_h. \end{cases}$.

\nProceeding as before we employ a discrete Green's identity [2, 3, 16] associated with the finite-difference operator.

 $+\frac{u(x, y - \beta_1 h)}{(\beta_1 + \beta_2)\beta_2} - \left(\frac{1}{\alpha_1 \alpha_2} + \frac{1}{\beta_1 \beta_2}\right)u(x, y)\right\},$ (6.1)
where $0 < \alpha_1, \alpha_2, \beta_1, \beta_2 \leq 1$. Not all of these parameters will be strictly less than one,
since the distances $\alpha_i h, \beta_i h$ ($i = 1, 2$) w on C_h^* to a neighbour or C_h , and h is chosen small enough so that Ω_h is simply connected. + $\frac{u(x, y - \beta_1 h)}{(\beta_1 + \beta_2) \beta_2}$ - $\left(\frac{1}{\alpha_1 \alpha_2} + \frac{1}{\beta_1 \beta_2}\right) u(x, y)$,

where $0 < \alpha_1, \alpha_2, \beta_1, \beta_2 \le 1$. Not all of these parameters will be strictly less that

since the distances $\alpha_i h, \beta_i h$ ($i = 1, 2$) where where $\sigma \le \alpha_1, \alpha_2, \mu_1, \mu_2 \le 1$. Not all of these parameters will be strictly less than one,
since the distances $\alpha_i h, \beta_i h$ ($i = 1, 2$) where α_i , or $\beta_i < 1$ are measured from a point
on C_h^* to a neighbour or $+\frac{(\beta_1 + \beta_2) \beta_2}{(\beta_1 + \beta_2) \beta_2} - \left(\frac{\alpha_1 \alpha_2}{\alpha_1 \alpha_2} + \frac{\beta_1 \beta_2}{\beta_1 \beta_2}\right) u(x, y)$,

where $0 < \alpha_1, \alpha_2, \beta_1, \beta_2 \leq 1$. Not all of these parameters will be strictly

since the distances $\alpha_i h, \beta_i h$ ($i = 1, 2$) where

$$
Z(P) := \begin{cases} 1, \ P \in \Omega_h + C_h^* \\ 0, \ P \in C_h. \end{cases}
$$

Proceeding as before we employ a discrete Green's identity [2, 3, 16] associated with the finite-difference operator

$$
M_h[U]:=(\triangle_h-q)[U].
$$

•

Proceeding as before we employ a discrete Green's identity [2, 3, 10] associated to the finite-difference operator

\n
$$
M_h[U] := (\triangle_h - q) U.
$$
\nHence, for irregular regions we have

\n
$$
U(P) = -h^2 \sum_{Q \in \mathcal{Q}_h + \mathcal{C}_h} M_h[U](Q) G_0(P,Q) + \sum_{Q \in \mathcal{C}_h} G_0(P,Q) U(Q).
$$

• '.

Now inserting $Z(P)$ for $U(P)$ in (6.3) yields

$$
1 = h^2 \sum_{Q \in \mathcal{Q}_h + C_h^*} q(Q) G_0(P,Q) - h^2 \sum_{Q \in \mathcal{C}_h^*} (\bigtriangleup_h Z) (Q) G_0(P,Q) \text{ for } P \in \mathcal{Q}_h + C_h^*.
$$

A short computation with the interpolated Laplacian (6.1) verifies that ($\triangle_{h}Z$) (Q) < 0 for $Q \in C_h^*$; hence, as before, we have the inequality of Lemma (5.2), namely rting $Z(P)$ for $U(P)$ in (6.3) yields
 $1 = h^2 \sum_{Q \in \mathcal{Q}_h + C_h^*} q(Q) G_0(P,Q) - h^2 \sum_{Q \in \mathcal{C}_h}$

pomputation with the interpolated I
 \ast^* ; hence, as before, we have the i
 $1 \ge h^2 \sum_{Q \in \mathcal{Q}_h + C_h^*} q(Q) G_0(P,Q)$. A short computation with the interpolated Laplactor $Q \in C_h^*$; hence, as before, we have the inequality of $1 \ge h^2 \sum_{Q \in \mathcal{Q}_h + C_h^*} q(Q) G_0(P, Q)$.

It is now clear that the formula for the error est present case when we de utation with the interpolated Laplacian (6.
hence, as before, we have the inequality of
 $\iota_k^2 \sum_{Q \in \mathcal{Q}_h + C_h^*} q(Q) G_0(P, Q)$.
ar that the formula for the error estimate
when we define
 $= \max_{Q \in \mathcal{Q}_h + C_h^*} [q(Q)]^{-1}$ and $||$

It is now clear that the formula for the error estimate (5.3) may be extended to the 36

Now ins
 $\frac{1}{2}$
 $\frac{1}{2}$
 • Network that the formula for the error estimate (

ent case when we define
 $K := \max_{Q \in \Omega_h + C_h^*} [q(Q)]^{-1}$ and $||a||_{\infty} := \sup_{Q \in \Omega_h + C_h^*} [a]$

FERENCES

BARENBLAT, G., ZHELTOV, I., and I. KOCHIVA: Basic conce

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$$
K := \max_{Q \in \mathcal{Q}_h + C_{\bullet}^*} [q(Q)]^{-1} \quad \text{and} \quad ||a||_{\infty} := \sup_{Q \in \mathcal{Q}_h + C_{\bullet}^*} [a(Q)].
$$

REFERENCES

- [1] BARENBLAT, G., ZHELTOV, I., and I. KOCHIVA: Basic concepts in the theory of seepage of homogeneous fluids in fissured rock. J. Appl. Math. Mech. 24 (1960), 1286-1303.
- [2] BRAMBLE, I., and B. HUBBARD: A priori bounds on the discretization error in the numerical solution of the Dirichlet problem. Contrib. Diff. Equat. 1 (1963), 229-252.
- **[3] COLLATZ,** L.: The Numerical Treatment of Differential Equations'(3rd ëd.). Springer-Ver-
- **[4] COLTON,** D. L.: Integral operators and the first initial boundary value problem for pseudoparabolic equations with analytic coefficients. J. Diff. Equat. 13 (1973), 506-522.
- **[5] COLTON,** D. L.: On the analytic theory of pseudoparabolie equations. Quart. J. Math. 23 * $(1972), 179 - 192.$
- **[6] COURANT,** D. L., and D. **HILBERT: .** Methods of Mathematical Physics (Vol. 11). Wiley: New York 1962..
- **[7] GILBERT,** & P.: A Lewy-Type reflection principle for pseudoparabolic equations. Journal of Diff. Equat. 37 (1980), 261-284.
- **[8] GILBERT,** R. P., and G: C. **HSIA0:** Constructive function theoretic methods for higher order pseudoparalolic equations. In: Lecture Notes in Math. &61. Springer: Berlin 1976.
- [9] SHOWALTER, R. E.: Well-posed problems for a partial differential equation of order $2m + 1$. SIAM J. Math. Anal. 1 (1970), 214-231.
- 31 (1969), 789-794.
- [11] SHOWALTER, R. E., and T. W. Tryo: Pseudoparabolic partial differential equations. SIAM/ J. Math. Anal. 1 (1970), $1-26$. Japan R. P.: S. (1969), 440-453.

[19] COURANT, D. L., and D. HILBERT: Methods of Mathematical Physics (Vol. I1). Wiley:

[7] GILBERT, R. P.: A Lewy-Type reflection principle for pseudoparabolic equations. Journal

of Diff
- **[12] TAYLOR,** D. W.: Research on Consolidation of Clays. MIT Press: Cambridge 1942.
- 2m + 1. SIAM J. Math. Anal. 1 (1970), 214-231.

[10] SHOWALTER, R. E.: Partial differential equations of Sobolev-Calpern type. Pacific J. Math.

31 (1969), 789-794.

[11] SHOWALTER, R. E., and T. W. Tixo: Pseudoparabolic p
	- [14] TING, T. W.: Certain non-steady flows of second order fluids. Arch. Rat. Mech. Anal. 14 (1963), $1-26$.
- [15] WALTER, W.: Differential and Integral Inequalities. Ergeb. Math. Grenzgeb. Bd. *55. .* Springer-Verlag: Berlin 1970. [10] SHOWALTER, R. E.: Partial differentia

31 (1969), 789-794.

[11] SHOWALTER, R. E., and T. W. Tixo:

J. Math. Anal. 1 (1970), $1-26$.

[12] TAYLOR, D. W.: Research on Consoli

[13] TING, T. W.: Parabolic and pseudop

	- [16] WENDLAND, W.: Elliptic Systems in the Plane. Pitman-Verlag: London 1979.

Manuskripteingang: 26. 2. 1982
VERFASSER:

- .

Prof. Dr. **ROBERT** P. **GILBERT** and Dr. **LEROY** R. **LUNDIN** OR, D. W.: Research on Consolutation of Clays. MIT Fress: Campinge 1942.

T. W.: Parabolic and pseudoparabolic partial differential equations. J. Math. So

1. W.: Certain non-steady flows of second order fluids. Arch. Rat. 1. W.: Farabone and Beeddoparabone pa

1. 21 (1969), 440–453.

T. W.: Certain non-steady flows of second

1. 1. 26.

EER, W.: Differential and Integral Inequal

ger-Verlag: Berlin 1970.

Manuskripteingang: 26. 2. 1982

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