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The Marcinkievicz Interpolation Theorem for Rearrangement-Invariant Function Spaces and Applications

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Der Interpolationssatz von J. MARCINKIEVICZ [17] besagt, daß jeder sublineare Operator T, der gleichzeitig vom schwachen Typ (p_1, q_1) und vom schwachen Typ (p_2, q_2) ist, notwendig auch ein beschränkter Operator des Lebesgueraumes $L_p(0, l)$, $0 < l \leqq \infty$, in sich ist, und zwar für alle p mit $p_2 < p < p_1$. Ziel der vorliegenden Arbeit ist, diesen Satz auf den Rahmen rearrangement-invarianter Banachscher Funktionenräume zu verallgemeinern und ihn damit einem sehr viel größeren Kreis von Anwendungen zugänglich zu machen.

Интерполяционная теорема И. МАРЦИНКЕВИЧА [17] утверждает, что каждый сублинейный оператор T, который одновременно имеет слабый (p_1, q_1)-тип и слабый (p_2, q_2)-тип, необходимо является ограниченным оператором в Лебеговом пространстве $L_p(0, l)$ $(0 < l \leqq \infty)$ для всех таких p, что $p_2 < p < p_1$. В настоящей работе эта теорема обобщается на случай перестановочно-инвариантных (симметричных) банаховых пространств функций. Тем самым теорема применима в значительно более широких рамках.

The interpolation theorem of J. MARCINKIEVICZ [17] states that any sublinear operator T which is simultaneously of weak types (p_1, q_1) and (p_2, q_2) is also a bounded operator from the Lebesgue space $L_p(0, l)$, $0 < l \leqq \infty$, into itself, provided $p_2 < p < p_1$. The aim of this paper is to generalize this theorem to the setting of rearrangement-invariant Banach function spaces, and thus to render the theorem available to a much larger range of applications.

1. Preliminaries

Let (Ω, Σ, μ) be a σ-finite, non-atomic measure space with $\mu(\Omega) =: l \leqq \infty$, $\mathcal{M}(\Omega)$ (resp. $\mathcal{P}(\Omega)$) the space of realvalued (resp. nonnegative), μ-measurable functions on Ω ϱ a *rearrangement-invariant* (= *r.i.*) function norm on $\mathcal{P}(\Omega)$, and $X \equiv X_\varrho(\Omega)$ the r.i. Banach function space generated by ϱ, in the sense of W. A. J. LUXEMBURG [14]. By $X' \equiv X_\varrho'(\Omega)$ we denote the associate r.i. Banach function space of X which is generated by the norm

$$\varrho'(g) := \sup \left\{ \int_\Omega fg \, d\mu : f \in \mathcal{P}(\Omega), \varrho(f) \leqq 1 \right\}.$$

Note that $\varrho'' = \varrho$. Finally, let $X_\lambda(\Omega^*)$ be the Luxemburg representation of the space $X_\varrho(\Omega)$, i.e., $\Omega^* := (0, l)$, $\mu = m =$ Lebesgue measure, and λ is a r.i. function norm on the set $\mathcal{P}(\Omega^*)$ of all nonnegative, Lebesgue measurable functions on Ω^* such that $\varrho(f) = \lambda(f^*)$ for all $f \in \mathcal{P}(\Omega)$, with f^* denoting the nonincreasing rearrangement of f. Explicitely, for $f \in \mathcal{P}(\Omega^*)$ the norm $\lambda(f)$ is given by

$$\lambda(f) = \sup \left\{ \int_0^l f^*(x)\, g^*(x)\, dx : g \in \mathcal{P}(\Omega), \varrho'(g) \leqq 1 \right\},$$

and the space $X_\lambda(\Omega^*)$ by

$$X_\lambda(\Omega^*) := \{f \in \mathcal{M}(\Omega^*) : \lambda(|f|) < \infty\},$$

where $\mathcal{M}(\Omega^*)$ is the set of all realvalued, Lebesgue measurable functions on Ω^*. This definition is meaningful, since supp $g^* \subset \Omega^*$ if $g \in \mathcal{P}(\Omega)$. In the sequel, the Luxemburg representation of $X_\varrho(\Omega)$ will systematically be used in order to reduce the problem to the situation where $\Omega = (0, l)$ is an interval and $\mu = m =$ Lebesgue measure, as treated in [8].

The first definition which involves the Luxemburg representation is the definition of the *Boyd indices* α_X and β_X of the space $X_\varrho(\Omega)$, namely

$$\alpha_X := \inf_{0<s<1} - \frac{\log \|E_s\|_{[X_\lambda(\Omega^*)]}}{\log s}, \quad \beta_X := \sup_{s>1} - \frac{\log \|E_s\|_{[X_\lambda(\Omega^*)]}}{\log s},$$

where E_s is the *dilation operator* on $\mathcal{M}(\Omega^*)$, given by

$$(E_s f)(t) := \begin{cases} f(st) & \text{if } \; st \in \Omega^* \\ 0 & \text{elsewhere}, \end{cases}$$

see [2]. If, in particular, $X_\varrho(\Omega) = L_p(\Omega)$, $1 \leqq p < \infty$, then $\|E_s\|_{[L_p(\Omega)]} = s^{-1/p}$ and $\alpha_{L_p(\Omega)} = \beta_{L_p(\Omega)} = 1/p$. Hence, these "Boyd indices" α_X and β_X generalize the number $1/p$ which characterizes the space L_p in the Lebesgue case. Generally, it can be shown that $0 \leqq \beta_X \leqq \alpha_X \leqq 1$ (just as $0 \leqq 1/p \leqq 1$), and $\alpha_{X'} = 1 - \beta_X$, $\beta_{X'} = 1 - \alpha_X$. For further properties of indices see [9, 10].

The second definition we need is that of an operator of weak type. As a substitute for the space weak-L_p in the original Marcinkiewicz theorem, we now use the rearrangement-invariant Lorentz spaces $\Lambda(X)$ and $\mathbf{M}(X)$ (see e.g. [19, 25]) which can be assigned to each r.i. Banach function space $X = X_\varrho(\Omega)$, namely

$$\Lambda(X) := \left\{f \in \mathcal{M}(\Omega) : \|f\|_{\Lambda(X)} := \int_0^l f^*(s)\, d\tau_X(s) < \infty\right\},$$

$$\mathbf{M}(X) := \left\{f \in \mathcal{M}(\Omega) : \|f\|_{\mathbf{M}(X)} := \sup_{t \in X^*} \frac{\tau_X(t)}{t} \int_0^t f^*(s)\, ds < \infty\right\},$$

where τ_X is the fundamental function of the space $X_\varrho(\Omega)$, i.e. $\tau_X(t) := \|\chi_{(0,\min(t,l))}\|_{X_\lambda(\Omega^*)}$ for $t > 0$. Without loss of generality, τ_X will be assumed to be concave, and $\tau_X(0+) = 0$. The spaces $\Lambda(X)$ and $\mathbf{M}(X)$, with $\|\cdot\|_{\Lambda(X)}$ and $\|\cdot\|_{\mathbf{M}(X)}$, respectively, as norms, are r.i. Banach function spaces such that $\Lambda(X) \subset X \subset \mathbf{M}(X)$ with continuous embeddings. Moreover, the space $\Lambda(X)$ (and $\mathbf{M}(X)$, resp.) is the smallest (largest) r.i. Banach function space contained in (containing) X with the same fundamental function, see [10: Corollary 3.2]. If $X = L_p$, $1 \leqq p < \infty$, then $\Lambda(L_p) = L_{p1}$ and $\mathbf{M}(L_p) = L_{p\infty}$ (L_{pq} denoting the Lorentz space).

Definition 1.1: Assume that $X \equiv X_\varrho(\Omega)$ is a r.i. Banach function space. A sublinear operator $T : \Lambda(X) \to \mathcal{M}(\Omega)$ is said to be of *weak type* (X, X), if and only if

$$\sup_{t \in \Omega^*} (Tf)^*(t)\, \tau_X(t) \leqq \text{const.}\, \|f\|_{\Lambda(X)} \qquad \big(f \in \Lambda(X)\big). \tag{1.1}$$

If, in addition, $\beta_{\Lambda(X)} > 0$, then the left side of (1.1) is equivalent to $\|Tf\|_{\mathbf{M}(X)}$. Indeed, $(Tf)^*(t) \leqq \left(\int_0^t (Tf)^*(s)\, ds\right)\Big/ t$ on account of the monotonicity of $(Tf)^*$, and, on the other hand,

$$(Tf)^*(t) \leqq \sup_{s\in\Omega^*} \{(Tf)^*(s)\,\tau_X(s)\}/\tau_X(t) \quad \text{for} \quad t > 0;$$

hence

$$\frac{1}{t}\int_0^t (Tf)^*(s)\,ds \leqq \frac{1}{t}\sup_{s\in\Omega^*}\{(Tf)^*(s)\,\tau_X(s)\}\int_0^t \frac{ds}{\tau_{X(s)}}$$

$$\leqq \sup_{s\in\Omega^*}\{(Tf)^*(s)\,\tau_X(s)\}\int_0^1 \|E_s\|_{[\Lambda(X_\lambda(\Omega^*))]}\,ds/\tau_{X'}(t). \tag{1.2}$$

Here we used the facts that

$$\tau_X(t)\,\tau_{X'}(t) = t \tag{1.3}$$

and

$$\int_0^t \frac{ds}{\tau_X(s)} \leqq \left(\int_0^1 \|E_s\|_{[\Lambda(X_\lambda(\Omega^*))]}\,ds\right)\tau_{X'}(t) \qquad (t \in \Omega^*), \tag{1.4}$$

see e.g. [20] and [10: (3.6)], respectively, noting that (1.4) is valid since $\beta_\Lambda(X) > 0$ by assumption. Multiplication of (1.2) by $\tau_X(t)$ and passing to the supremum over all $t \in \Omega^*$, yields

$$\|Tf\|_{\mathrm{M}(X)} \leqq \left(\int_0^1 \|E_s\|_{[\Lambda(X_\lambda(\Omega^*))]}\,ds\right)\sup_{s\in\Omega^*}\{Tf)^*(s)\,\tau_X(s)\}.$$

Hence we have (compare [20]) the following lemma.

Lemma 1.1: *If $\beta_{\Lambda(X)} > 0$, then a sublinear operator $T : \Lambda(X) \to \mathscr{M}(\Omega)$ is of weak type (X, X) if and only if T is a bounded operator from $\Lambda(X)$ into $\mathrm{M}(X)$, i.e.*

$$\|Tf\|_{\mathrm{M}(X)} \leqq \text{const}\cdot\|f\|_{\Lambda(X)} \qquad \big(f \in \Lambda(X)\big). \tag{1.4}$$

Finally we introduce the notations (Z) for the space of all bounded sublinear operators mapping a r.i. space Z into Z, and

$$W(X, Y) \equiv W\big(X(\Omega), Y(\Omega)\big) := \{\,T\colon \Lambda(X) + \Lambda(Y) \to \mathscr{M}(\Omega);\ T$$
$$\text{of weak types } (X, X) \text{ and } (Y, Y)\}. \tag{1.5}$$

2. Necessity of the Rearrangement-Invariant Property

If $X, Y \subset \mathscr{M}(\Omega)$ are any two Banach function spaces, and $T \subset (X + Y)$, we say that T is *admissable* (compare [5]), if the restriction $T|_X$ of T to the space X belongs to (X) and, simultaneously, $T|_Y \in (Y)$. By $ad(X, Y)$ we denote the set of all admissable operators with respect to the space X and Y. The *strong-type interpolation problem* consists in determining those spaces Z for which $ad(X, Y) \subset (Z)$, if X and Y are given. In the particular case that X, Y, Z are Lebesgue spaces, this problem was solved by the convexity theorem of M. Riesz/G. Thorin [18, 24]. In the frame work of r.i. spaces concrete methods of how to construct function spaces which solve the strong-type interpolation problem are studied in [6].

Since $ad(X, Y) \subset W(X, Y)$, a harder problem is the *weak-type interpolation problem* which consists in finding those spaces Z for which $W(X, Y) \subset (Z)$. This problem was solved by J. MARCINKIEVICZ [17] for Lebesgue spaces X, Y, Z, and by D. W. BOYD [2] in case that X, Y are Lebesgues spaces and Z is an arbitrary r.i. Banach function space. The purpose of this paper is to solve the weak-type interpolation problem for the case that X, Y are any abstract r.i. Banach function spaces and Z is any Banach function space. As a first step we now show that the space Z must necessarily also be rearrangement invariant.

Theorem 2.1: *Let X, $Y \subset \mathcal{M}(\Omega)$ be r.i. Banach function spaces such that* $0 < \min\{\beta_X, \beta_Y\} \leqq \max\{\alpha_X, \alpha_Y\} < 1$, *and $Z \subset \mathcal{M}(\Omega)$ be any Banach function space. If $W(X, Y) \subset (Z)$, then Z is rearrangement invariant.*

Proof: The idea of the proof consists in reducing the assertion to a particular result of A. P. CALDERÓN [5] for Lebesgue spaces by combining the interpolation theorems of M. Riesz/G. Thorin and of D. W. Boyd: By assumption, there exists a number $q \in (1, \infty)$ such that

$$0 < 1/q < \min\{\beta_X, \beta_Y\} \leqq \max\{\alpha_X, \alpha_Y\} < 1.$$

If we apply the interpolation theorem of [2] twice, namely to the spaces X and Y, respectively, we can conclude that $W(L_1, L_q) \subset (X) \cap (Y)$. Since $(X) \cap (Y) \subset W(X, Y)$, if follows that $W(L_1, L_q) \subset W(X, Y)$. On the other hand, the interpolation theorem of M. Riesz/G. Thorin yields that $ad(L_1, L_\infty) \subset [L_q] \subset (L_q)$. Since $ad(L_1, L_\infty) \subset (L_1)$ and, obviously, $(L_1) \cap (L_q) \subset W(L_1, L_q)$, we have $ad(L_1, L_\infty) \subset W(L_1, L_q)$, and therefore finally, $ad(L_1, L_\infty) \subset W(X, Y)$. If, by assumption, $W(X, Y) \subset (Z)$, then necessarily $ad(L_1, L_\infty) \subset (Z)$. So by a theorem of [5] this implies that the space Z is rearrangement invariant ∎

With the above theorem in mind, our next aim is to show that the property of rearrangement-invariance is also sufficient for a weak-type interpolation theorem to hold.

3. The Generalized Average Operators

The basic idea of the interpolation theorem to be established is to try to characterize those r.i. spaces which solve the weak-interpolation problem by conditions upon their Boyd indices. As a link between Boyd indices and operators of weak type we now briefly present two integral operators P_X and Q_X, as well as their basic properties, studied in detail in [8].

Definition 3.1: Let $X = X_\lambda(\Omega^*) \subset \mathcal{M}(\Omega^*)$ be a r.i. space. Then

a) $$(P_X f)(t) := \frac{1}{\tau_X(t)} \int_0^t f(s)\, d\tau_X(s) \qquad (f \in \mathcal{M}(\Omega^*), t \in \Omega^*);$$

b) $$(Q_X f)(t) := \frac{1}{\tau_X(t)} \int_t^\infty f(s)\, d\tau_X(s) \qquad (f \in \mathcal{M}(\Omega^*), t \in \Omega^*).$$

Note that $\tau_{X_\varrho(\Omega)}(\Omega) = \tau_{X_\lambda(\Omega^*)}$ if $X_\lambda(\Omega^*)$ is the Luxemburg representation of $X_\varrho(\Omega)$. The operator P_X, used by L. MALIGRANDA [15] in connection with Hardy's inequality, is a generalization of the average operator P_Θ of [2, 3]; the operator Q_X however is quite different to MALIGRANDA's [16] operator Q_φ. In the following lemmata we collect those properties of the operators P_X, Q_X which will be used in the sequel.

Lemma 3.1: *Assume that* $X = X_{\lambda_1}(\Omega^*)$, $Y = Y_{\lambda_2}(\Omega^*)$, $Z = Z_{\lambda_2}(\Omega^*)$ *are r.i. spaces.*
a) *If* $0 < \beta_{\Lambda(X)} \leqq \alpha_{\Lambda(X)} < 1$, *then for every* $f \in Z$, $g \in Z'$

$$\frac{\beta_{\Lambda(X)}}{1-\beta_{\Lambda}(X)} \int_0^l f^*(t)\,(Q_{X'}\,g^*)\,(t)\,dt \leqq \int_0^l (P_X f^*)\,(t)\,g^*(t)\,dt$$

$$\leqq \frac{\alpha_{\Lambda(X)}}{1-\alpha_{\Lambda(X)}} \int_0^l f^*(t)\,(Q_{X'}\,g^*)\,(t)\,dt;$$

b) $(P_X f)^*(t) \leqq (P_X f^*)\,(t) \qquad \big(f \in \mathcal{M}(\Omega^*),\, t \in \Omega^*\big)$,
$(Q_X f)^*(t) \leqq (Q_X f^*)\,(t) \qquad \big(f \in \mathcal{M}(\Omega^*),\, t \in \Omega^*\big)$;
c) *the operator* P_X *is of weak type* (X, X); *if* $\beta_{\Lambda(Y)} > 0$ *and* τ_Y/τ_X *is a decreasing function, then* P_X *is also of weak type* (Y, Y);
d) *the operator* Q_Y *is of weak type* (Y, Y); *if* $\beta_{\Lambda(X)} > 0$ *and* τ_Y/τ_X *is decreasing, then* Q_Y *is also of weak type* (X, X);
e) $P_X + Q_Y = S$ *where* S *denotes the Calderón operator, defined by*

$$(Sf)\,(t) := \int_0^l f(s)\,d\left(\min\left\{\frac{\tau_X(s)}{\tau_X(t)}, \frac{\tau_Y(s)}{\tau_Y(t)}\right\}\right) \quad \big(f \in \mathcal{M}(\Omega^*),\, t \in \Omega^*\big).$$

For the proofs of these properties see [8]. In particular, the constant in the "duality" relation of a) can be evaluated by recalling that

$$\beta_{\Lambda(X)} \frac{\tau_X(t)}{t} \leqq \frac{d\tau_X(t)}{dt} \leqq \alpha_{\Lambda(X)} \frac{\tau_X(t)}{t}, \tag{3.1}$$

compare [10].

The Calderón operator S obtains its importance for interpolation theory from the facts that (see [20])

$$S \in W\big(X_{\lambda_1}(\Omega^*),\, Y_{\lambda_2}(\Omega^*)\big), \tag{3.2}$$

and, for each $t \in W\big(X_{\varrho_1}(\Omega),\, Y_{\varrho_2}(\Omega)\big)$,

$$(Tf)^* \leqq \text{const.}\; Sf^* \qquad \big(f \in \Lambda(X_{\varrho_1}(\Omega)) + \Lambda(X_{\varrho_2}(\Omega))\big), \tag{3.3}$$

If $X_{\lambda_1}(\Omega^*)$ and $Y_{\lambda_2}(\Omega^*)$ are the Luxemburg representations of $X_{\varrho_1}(\Omega)$ and $Y_{\varrho_2}(\Omega)$, respectively. On the other hand, the operators P_X and Q_Y are connected with the Boyd indices. In fact, for the case that $X, Z \subset \mathcal{M}(\Omega^*)$ are r.i. spaces of Lebesgue measurable functions on $\Omega^* = (0, l)$ such that $\beta_{\Lambda(X)} > 0$, the following holds; see [15], also [1, 3].

$$\text{If } \alpha_Z < \beta_{\Lambda(X)}, \textit{ then } P_X \in [Z]; \textit{ if } P_X \in [Z], \textit{ then } \alpha_Z \leqq \alpha_{\Lambda(X)}. \tag{3.4}$$

An analogous assertion for the operator Q_Y, which is not contained in [15], can be deduced from (3.4) by duality arguments, using Lemma 3.1a). Here we assume that $Y \subset \mathcal{M}(\Omega^*)$ is a r.i. space with $0 < \beta_{\Lambda(Y)} \leqq \alpha_{\Lambda(Y)} < 1$. Then:

$$\textit{If } \alpha_{\Lambda(Y)} < \beta_Z, \textit{ then } Q_Y \in [Z]; \textit{ if } Q_Y \in [Z], \textit{ then } \beta_{\Lambda(X)} < \beta_Z. \tag{3.5}$$

For the proof recall that $\alpha_{X'} = 1 - \beta_X$ and $\beta_{X'} = 1 - \alpha_X$ for any r.i. space X, note that $\Lambda(X)' = \mathbf{M}(Y')$, as well as $\tau_{\mathbf{M}(Y')} = \tau_{\Lambda(Y')}$; see [21, 10]. Hence $\beta_{\mathbf{M}(Y')} = \beta_{\Lambda(Y')}$, and we can argue as follows: If $\alpha_{A(Y)} < \beta_Z$, then $1 - \beta_{A(Y')} = 1 - \beta_{M(Y')} = 1 - \beta_{A(Y)'} = \alpha_{A(Y)} < \beta_Z = 1 - \alpha_Z$, i.e. $\alpha_{Z'} < \beta_{A(Y')}$. So by (3.4) we have $P_{Y'} \in [Z']$, this being equivalent to $Q_Y \in [Z]$ on account of Lemma 3.1a). The second part of (3.5) is proved similarly.

Concerning the operator norms of P_Y and Q_X we have the following facts.

Lemma 3.2:

a) *If* $\alpha_Z < \beta_{A(X)}$, *then* $\|P_X\|_{[Z]} \leqq \alpha_{A(X)} \int\limits_0^1 \|E_s\|_{[Z]}\, M(s, X) \frac{ds}{s} < \infty$;

b) *if* $\alpha_{A(Y)} < \beta_Z$, *then* $\|Q_Y\|_{[Z]} \leqq \alpha_{A(Y)} \int\limits_1^\infty \|E_s\|_{[Z]}\, M(s, Y) \frac{ds}{s} < \infty$.

Here $M(s, X) := \sup_{s,st \in \Omega^*} [\tau_X(st)/\tau_X(t)]$ and $M(s, Y)$ *analogously.*

Proof: A weaker version of part a) — without the factor $\alpha_{A(X)}$ — was proved in [16]. We confine ourselves to b), the proof of a) being similar. For this purpose, let $f \in Z$, $g \in Z'$ such that $g \geqq 0$, $\|g\|_{Z'} = 1$. Because of (3.1)

$$\int\limits_0^l (Q_Y|f|)(t)\, g(t)\, dt \leqq \alpha_{A(Y)} \int\limits_0^l \int\limits_t^l |f(s)| \frac{\tau_Y(s)}{\tau_Y(t)}\, g(t) \frac{ds}{s}\, dt$$

$$= \alpha_{A(Y)} \int\limits_0^l \int\limits_1^{l/t} |f(st)| \frac{\tau_Y(st)}{\tau_Y(t)}\, g(t) \frac{ds}{s}\, dt$$

$$\leqq \alpha_{A(Y)} \int\limits_0^l \left(\int\limits_1^\infty (E_s|f|)(t)\, M(s, Y) \frac{ds}{s} \right) g(t)\, dt,$$

yielding b), since $f \in Z$, $g \in Z'$ are arbitrary ∎

By means of the method of applying the Luxemburg representation, the assertions (3.4), (3.5), and Lemma 3.2 can be transferred to the more general situation of r.i. spaces in $\mathcal{M}(\Omega)$.

Theorem 3.3: *Assume that* $X = X_{\varrho_1}(\Omega)$, $Y = Y_{\varrho_2}(\Omega)$, $Z = Z_{\varrho_3}(\Omega) \subset \mathcal{M}(\Omega)$ *are r.i. spaces with* $\beta_{A(X)} > 0$ *and* $0 < \beta_{A(Y)} \leqq \alpha_{A(Y)} < 1$. *Let* $X_{\lambda_1}(\Omega^*)$, $Y_{\lambda_2}(\Omega^*)$, *and* $Z_{\lambda_3}(\Omega^*)$ *denote the Luxemburg representations of* X, Y, *and* Z *respectively.*

a) *If* $\alpha_Z < \beta_{A(X)}$, *then for every* $f \in Z_{\varrho_3}(\Omega)$

$$\|P_X f^*\|_{Z_{\lambda_3}(\Omega^*)} \quad \left(\alpha_{A(X)} \int\limits_0^1 \|E_s\|_{[Z_{\lambda_3}(\Omega^*)]}\, M(s, X) \frac{ds}{s} \right) \|f^*\|_{Z_{\lambda_3}(\Omega^*)}; \tag{3.6}$$

conversely, if (3.6) *holds for every* $f \in Z_{\varrho_3}(\Omega)$, *then* $\alpha_Z < \alpha_{A(X)}$.

b) *If* $\alpha_{A(Y)} < \beta_Z$, *then for every* $f \in Z_{\varrho_3}(\Omega)$

$$\|Q_Y f^*\|_{Z_{\lambda_3}(\Omega^*)} \leqq \left(\alpha_{A(Y)} \int\limits_1^\infty \|E_s\|_{[Z_{\lambda_3}(\Omega^*)]}\, M(s, Y) \frac{ds}{s} \right) \|f^*\|_{Z_{\lambda_3}(\Omega^*)}; \tag{3.7}$$

conversely, if (3.7) *holds for every* $f \in Z_{\varrho_3}(\Omega)$, *then* $\beta_{A(Y)} < \beta_Z$.

Proof: Note that the indices of a r.i. space coincide with the indices of its Luxemburg representation and, moreover, that inequality (3.6) for all $f \in Z_{\varrho_3}(\Omega)$ is equivalent to $P_X \in [Z_{\lambda_3}(\Omega^*)]$, in view of Lemma 3.1 b); similarly inequality (3.7) for all $f \in Z_{\lambda_3}(\Omega)$ is equivalent to $Q_Y \in [Z_{\lambda_3}(\Omega^*)]$ ∎

4. The Interpolation Theorem for Rearrangement-Invariant Spaces

Now we are ready to prove the following weak-type interpolation theorem.

Theorem 4.1: *Assume that $X \equiv X_{\varrho_1}(\Omega)$, $Y \equiv Y_{\varrho_2}(\Omega)$, $Z \equiv Z_{\varrho_3}(\Omega) \subset \mathcal{M}(\Omega)$ are r.i. spaces such that $Z \subset \Lambda(X) + \Lambda(Y)$, $\beta_{\Lambda(X)} > 0$, $0 < \beta_{\Lambda(Y)} \leqq \alpha_{\Lambda(Y)} < 1$, and τ_Y/τ_X is decreasing.*

a) *If $\alpha_{\Lambda(Y)} < \beta_Z \leqq \alpha_Z < \beta_{\Lambda(X)}$, then $W(X, Y) \subset (Z)$;*
b) *If $W(X, Y) \subset (Z)$, then $\beta_{\Lambda(Y)} < \beta_Z \leqq \alpha_Z < \alpha_{\Lambda(X)}$.*

Proof: a) Let $T \in W(X, Y)$ and $f \in Z$ be given. As above denote by $X_{\lambda_1}(\Omega^*)$, $Y_{\lambda_2}(\Omega^*)$, $Z_{\lambda_3}(\Omega^*)$ the Luxemburg representations of X, Y, Z, respectively. Since $f \in Z$, we have $f^* \in Z_{\lambda_3}(\Omega^*) \subset \mathcal{M}(\Omega^*)$ and, by Lemma 3.1 e), $Sf^* = P_X f^* + Q_Y f^*$. If the index condition of a) holds, we can apply Lemma 3.2 to conclude that

$$\|Sf^*\|_{Z_{\lambda_3}(\Omega^*)} \leqq \left(\|P_X\|_{[Z_{\lambda_3}(\Omega^*)]} + \|Q_Y\|_{[Z_{\lambda_3}(\Omega^*)]}\right) \|f^*\|_{Z_{\lambda_3}(\Omega^*)} < \infty.$$

This means, in particular, that $Sf^* \in Z_{\lambda_3}(\Omega^*)$, and hence $(Sf^*)(t) < \infty$ almost everywhere, since λ_3 is a r.i. norm. This implies that $f^* \in \Lambda(X_{\lambda_1}(\Omega^*)) + \Lambda(Y_{\lambda_2}(\Omega^*))$. In order to show that $f \in \Lambda(X_{\varrho_1}(\Omega)) + \Lambda(Y_{\varrho_2}(\Omega))$, we consider the norm of f^* in $\Lambda(X_{\lambda_1}(\Omega^*)) + \Lambda(Y_{\lambda_2}(\Omega^*))$. By definition of the norm of a sum of Banach spaces,

$$\begin{aligned}\|f^*\|_{\Lambda(X_{\lambda_1}(\Omega^*))+\Lambda(Y_{\lambda_2}(\Omega^*))} = \inf\big\{&\|g_1\|_{\Lambda(X_{\lambda_1}(\Omega^*))} + \|g_2\|_{\Lambda(Y_{\lambda_2}(\Omega^*))} :\\ &f^* = g_1 + g_2,\, g_1 \in \Lambda((X_{\lambda_1}\Omega^*)),\, g_2 \in \Lambda(Y_{\lambda_2}(\Omega^*)),\\ &g_1 \geqq 0,\, g_2 \geqq 0\big\}.\end{aligned}$$

According to [5] there exists a measure preserving transformation $f^* \to f$ from $\mathcal{M}(\Omega^*)$ to $\mathcal{M}(\Omega)$ such that for each decomposition $f^* = g_1 + g_2$ with g_1, g_2 as above, there exist functions $f_1 \in \Lambda(X_{\varrho_1}(\Omega))$, $f_2 \in \Lambda(Y_{\varrho_2}(\Omega))$ with $f_1^* = g_1$, $f_2^* = g_2$ and $f = f_1 + f_2$. Hence

$$\begin{aligned}&\|f^*\|_{\Lambda(X_{\lambda_1}(\Omega^*))+\Lambda(Y_{\lambda_2}(\Omega^*))}\\ &\quad= \inf\big\{\|f_1^*\|_{\Lambda(X_{\lambda_1}(\Omega^*))} + \|f_2^*\|_{\Lambda(Y_{\lambda_2}(\Omega^*))}:\\ &f = f_1 + f_2,\, f_1 \in \Lambda(X_{\varrho_1}(\Omega)),\, f_2 \in \Lambda(Y_{\varrho_2}(\Omega))\big\}\\ &\quad= \|f\|_{\Lambda(X_{\varrho_1}(\Omega))+\Lambda(Y_{\varrho_2}(\Omega))}.\end{aligned}$$

This shows that $f \in \Lambda(X_{\varrho_1}(\Omega)) + \Lambda(X_{\varrho_2}(\Omega))$, and therefore (3.3) can be applied to f, yielding that

$$\begin{aligned}\|Tf\|_{Z_{\varrho_3}(\Omega)} &= \|(Tf)_0^*\|_{Z_{\lambda_3}(\Omega^*)} \leqq \text{const.}\ \|Sf^*\|_{Z_{\lambda_3}(\Omega^*)}\\ &\leqq \text{const.}\left(\|P_X\|_{[Z_{\lambda_3}(\Omega^*)]} + \|Q_Y\|_{[Z_{\lambda_3}(\Omega^*)]}\right)\|f^*\|_{Z_{\lambda_3}(\Omega^*)}\\ &= \text{const.}\ \|f\|_{Z_{\varrho_3}(\Omega)} < \infty.\end{aligned}$$

Therefore we finally have $Tf \in Z_{\varrho_3}(\Omega)$ and $\|T\|_{[Z_{\varrho_3}(\Omega)]} < \infty$.

b) Since the σ-finite measure space (Ω, Σ, μ) was assumed to be nonatomic, there exists a measure preserving transformation $\pi : \Omega \to \Omega^*$. By means of this transformation π the operators P_X and Q_Y on $\mathcal{M}(\Omega^*)$ can be transferred into operators, say $\mathbf{P}_X$ and $\mathbf{Q}_Y$, on $\mathcal{M}(\Omega)$. For $v \in \Omega$ we introduce two kernels k_1 and k_2 by

$$k_1(v) := \begin{cases} \tau_X(s)/s & \text{if } \ v \in \pi^{-1}(\{s\}) \\ 0 & \text{elsewhere}, \end{cases} \qquad k_2(v) := \begin{cases} \tau_Y(s)/s & \text{if } \ v \in \pi^{-1}(\{s\}) \\ 0 & \text{elsewhere}, \end{cases}$$

and then define the operators $\mathbf{P}_X$ and $\mathbf{Q}_Y$ for $f \in \mathcal{M}(\Omega)$, $v \in \Omega$ by

$$(\mathbf{P}_X f)(v) := \begin{cases} \dfrac{1}{\tau_X(t)} \displaystyle\int_{\pi^{-1}(0,t]} f k_1 \, d\mu & \text{if } \ v \in \pi^{-1}(\{t\}) \\ 0 & \text{elsewhere}, \end{cases}$$

$$(\mathbf{Q}_Y f)(v) := \begin{cases} \dfrac{1}{\tau_Y(t)} \displaystyle\int_{\pi^{-1}[t,l)} f k_2 \, d\mu & \text{if } \ v \in \pi^{-1}(\{t\}) \\ 0 & \text{elsewhere}. \end{cases}$$

In case $\Omega = \Omega^*$, the operators $\mathbf{P}_X$ and $\mathbf{Q}_Y$ are equivalent to P_X and Q_X respectively. In the general case

$$(\mathbf{P}_X \bar{f})^*(t) \geqq \frac{1}{\alpha_{A(X)}} (P_X f^*)(t) \qquad \big(f \in \mathcal{M}(\Omega), t \in \Omega^*\big), \tag{4.1}$$

$$(\mathbf{Q}_Y \bar{f})^*(t) \geqq \frac{1}{\alpha_{A(Y)}} (Q_Y f^*)(t) \qquad \big(f \in \mathcal{M}(\Omega), t \in \Omega^*\big), \tag{4.2}$$

where $\bar{f}(v) := f^*(s)$ if $v \in \pi^{-1}(\{s\})$, and $\bar{f}(v) = 0$ elsewhere. Indeed since $(\bar{f})^* = f^*$ because of the measure preserving property of π, we can estimate $(\mathbf{P}_X f)^*$ from below by

$$(\mathbf{P}_X \bar{f})^*(t) = \left\{ \frac{1}{\tau_X(\cdot)} \int_0^{\cdot} f^*(s) \frac{\tau_X(s)}{s} \, ds \right\}^* (t) \geqq \left\{ \frac{1}{\alpha_{A(X)}} (P_X f^*) \right\}^* (t)$$
$$= \frac{1}{\alpha_{A(X)}} (P_X f^*)(t).$$

Here we used (3.1) and the fact that $P_X f^*$ is decreasing (see [8]). Inequality (4.2) is proved analogously. Note that $\alpha_{A(X)} > 0$ and $\alpha_{A(Y)} > 0$ by assumption.

Conversely, it can similarly be shown that

$$(\mathbf{P}_X f)^*(t) \leqq \frac{1}{\beta_{A(X)}} (P_X g)^*(t) \qquad \big(f \in \mathcal{M}(\Omega), t \in \Omega^*\big), \tag{4.3}$$

$$(\mathbf{Q}_Y f)^*(t) \leqq \frac{1}{\beta_{A(Y)}} (Q_Y g)^*(t) \qquad \big(f \in \mathcal{M}(\Omega), t \in \Omega^*\big), \tag{4.4}$$

with $g(t) := f(v)$ if $v \in \pi^{-1}(\{t\})$, and $g(t) = 0$ elsewhere.

Next we benefit from the fact that P_X, as stated in Lemma 3.1 c) is of weak type $\big(X_{\lambda_1}(\Omega^*), X_{\lambda_1}(\Omega^*)\big)$, in order to show that the new operator $\mathbf{P}_X$ on $\mathcal{M}(\Omega)$ is of weak type $\big(X_{\varrho_1}(\Omega), X_{\varrho_1}(\Omega)\big)$. In fact, multiplying (4.3) by $\tau_X(t)$ and passing to the supremum, we have by Lemma 3.1 b) and c) that

$$\sup_{t\in\Omega^*} (\mathbf{P}_X f)^*(t)\,\tau_X(t) \leqq \frac{1}{\beta_{A(X)}} \sup_{t\in\Omega^*} (P_X g)^*(t)\,\tau_X(t)$$

$$\leqq \frac{1}{\beta_{A(X)}} \sup_{t\in\Omega^*} (P_X' g^*)(t)\,\tau_X(t)$$

$$\leqq \frac{\text{const.}}{\beta_{A(X)}} \|g^*\|_{A(X_{\lambda_1}(\Omega^*))} = \frac{\text{const.}}{\beta_{A(X)}} \|f\|_{A(X_{\varrho_1}(\Omega))},$$

since $g^* = f^*$, and $X_{\lambda_1}(\Omega^*)$ is the Luxemburg representation of $X_{\varrho_1}(\Omega)$. If we multiply (4.3) by $\tau_Y(t)$ instead of $\tau_X(t)$, an analogous calculation leads to

$$\sup_{t\in\Omega^*} (\mathbf{P}_X f)^*(t)\,\tau_Y(t) \leqq \frac{\text{const.}}{\beta_{A(X)}} \|f\|_{A(Y_{\varrho_2}(\Omega))}.$$

Starting with (4.4) instead of (4.3), one can similarly show that

$$\sup_{t\in\Omega^*} (\mathbf{Q}_Y f)^*(t)\,\tau_X(t) \leqq \frac{\text{const.}}{\beta_{A(Y)}} \|f\|_{A(X_{\varrho_1}(\Omega))},$$

$$\sup_{t\in\Omega^*} (\mathbf{Q}_Y f)^*(t)\,\tau_Y(t) \leqq \frac{\text{const.}}{\beta_{A(Y)}} \|f\|_{A(Y_{\varrho_2}(\Omega))}.$$

Collecting all these estimates we have that $\mathbf{P}_X, \mathbf{Q}_Y \in W\big(X_{\varrho_1}(\Omega), Y_{\varrho_2}(\Omega)\big)$, and hence $\mathbf{P}_X, \mathbf{Q}_Y \in (Z)$, on account of the assumption upon Z.

Now, let $f \in Z_{\varrho_2}(\Omega)$ be given and construct $\bar f$ as above. Since $\mathbf{P}_X \in (Z)$ and $(\bar f)^* = f^*$, we have

$$\|\mathbf{P}_X \bar f\|_{Z_{\varrho_2}(\Omega)} \leqq \text{const.}\ \|\bar f\|_{Z_{\varrho_2}(\Omega)} = \text{const.}\ \|f^*\|_{Z_{\lambda_2}(\Omega^*)}.$$

Passing to f and applying (4.1), we therefore have the estimate

$$\|P_X f^*\|_{Z_{\lambda_2}(\Omega^*)} \leqq \alpha_{A(X)} \|\mathbf{P}_X \bar f\|_{Z_{\varrho_2}(\Omega)} \leqq \alpha_{A(X)}\ \text{const.}\ \|f^*\|_{Z_{\lambda_2}(\Omega)}.$$

This implies $\alpha_Z < \alpha_{A(X)}$ by Theorem 3.3 a), as maintained.

Concerning the first part of the index condition asserted, it follows from (4.2) and $\mathbf{Q}_Y \in (Z)$ that

$$\|\mathbf{Q}_Y f^*\|_{Z_{\lambda_2}(\Omega^*)} \leqq \alpha_{A(Y)}\ \text{const.}\ \|f^*\|_{Z_{\lambda_2}(\Omega^*)},$$

and hence $\beta_{A(Y)} < \beta_Z$ by Theorem 3.3 b). This concludes the proof of Theorem 4.1 ∎

If the spaces X and Y are of fundamental type (see [10] and § 5) then $\alpha_{A(X)} = \alpha_X$, $\beta_{A(X)} = \beta_X$, and $\alpha_{A(Y)} = \alpha_Y$, $\beta_{A(Y)} = \beta_Y$, and we have in addition

Corollary 4.2: *In addition to the assumptions of Theorem 4.1 let X and Y be of fundamental type such that $\alpha_X = \beta_X$ and $\alpha_Y = \beta_Y$. Then $W(X, Y) \subset (Z)$ if and only if $\beta_Y < \beta_Z \leqq \alpha_Z < \alpha_X$.*

Note that most of the known r.i. spaces such as Lebesgue spaces, Lorentz spaces, and Orlicz spaces are of fundamental type, see [7]. In particular, Corollary 4.2 contains the interpolation theorems of D. W. Boyd [2] (for $X = L_p$, $Y = L_q$, Z arbitrary, $1 \leqq p < q < \infty$), of A. P. Calderón [5] (for $X = L_p$, $Y = L_q$, $Z = L_r$, $1 \leqq p < r < q < \infty$) and its weaker version of J. E. Marcinkievicz [17], as well as the theorem of M. Riesz/G. Thorin, since each bounded operator on L_p is a fortiori of weak type

(L_p, L_p). Moreover, part b) of Theorem 4.1 might be regarded as an answer to Conjecture 5.4 in [25].

Explicitely, Theorem 4.1 a) can be reformulated as follows: If T is of weak type (X, X), we denote by $\|T\|_{W(X)}$ the lowest positive constant such that (1.1) holds; similarly we define $\|T\|_{W(Y)}$.

Corollary 4.3: *Under the assumptions of Theorem 4.1 a) we have*

$$\|T\|_{[Z_{\varrho_2}(\Omega)]} \leqq \|E_{1/2}\|_{Z_{\lambda_2}(\Omega^*)} \max\{\|T\|_{W(X)}, \|T\|_{W(Y)}\} \times \left\{\alpha_{\Lambda(X)} \int_0^1 \|E_s\|_{[Z_{\lambda_2}(\Omega^*)]} M(s, X) \frac{ds}{s} + \alpha_{\Lambda(Y)} \int_1^\infty \|E_s\|_{[Z_{\lambda_2}(\Omega^*)]} M(s, Y) \frac{ds}{s}\right\}.$$

Here we used Lemma 3.2, Lemma 3.1 e), and (3.3) with the constant being equal to $\|E_{1/2}\|_{Z_{\lambda_2}(\Omega^*)} \max\{\|T\|_{W(X)}, \|T\|_{W(Y)}\}$ as a slight modification of Sharpley's [20] argument shows. Concerning the distribution function $D_{Tf}(\sigma) := \mu\{x \in \Omega : |Tf|(x) > \sigma\}$ of Tf we have the estimate

$$(D_{Tf})(\sigma) \leqq \tau_X^{-1}\left(\frac{2\,\|T\|_{W(X)}}{\sigma}\|g_1\|_{\Lambda(X)}\right) + \tau_Y^{-1}\left(\frac{2\,\|T\|_{W(Y)}}{\sigma}\|g_2\|_{\Lambda(Y)}\right) \tag{4.6}$$

for any representation $f = g_1 + g_2$ with $g_1 \in \Lambda(X)$, $g_2 \in \Lambda(Y)$. Indeed, on one hand it is known that $D_{Tf}(\sigma) \leqq D_{Tg_1}(\sigma/2) + D_{Tg_2}(\sigma/2)$ for $\sigma > 0$. On the other hand, if T is of weak type (X, X), then

$$\sup_{0<\sigma<\infty} \sigma\tau_X\big(D_{Tg_1}(\sigma)\big) = \sup_{0<t<\infty} (Tg_1)^*(t)\,\tau_X(t) \leqq \|T\|_{W(X)}\|g_1\|_{\Lambda(X)}.$$

If T is of weak type (Y, Y) then analogously it follows that $\sup\limits_{0<\sigma<\infty} \sigma\tau_Y\big(D_{Tg_2}(\sigma)\big) \leqq \|T\|_{W(Y)} \times \|g_2\|_{\Lambda(Y)}$. Therefore

$$(D_{Tg_1})(\sigma/2) \leqq \tau_X^{-1}\frac{\|T\|_{W(X)}\|g_1\|_{\Lambda(X)}}{\sigma/2},$$

$$(D_{Tg_2})(\sigma/2) \leqq \tau_Y^{-1}\left(\frac{\|T\|_{W(X)}\|g_2\|_{\Lambda(X)}}{\sigma/2}\right)$$

if $T \in W(X, Y)$, yielding (4.6). Let us remark that in case of Lebesgue spaces L_p, L_q, and L_r with $1/r = \theta/q + (1-\theta)/p$, $0 < \theta < 1$, this leads to the well known constant $\|T\|_{W(L_q)}^{\theta}\,\|T\|_{W(L_p)}^{1-\theta}$ in Marcinkievicz's theorem.

5. Applications

5.1. Applications to particular spaces

Of particular interest with respect to applications is the case when the space $Z = Z_{\varrho_2}(\Omega)$ is of fundamental type, i.e., when $\|E_s\|_{[Z_{\lambda_2}(\Omega^*)]} = M(1/s, Z)$. In this case, it follows that $\|E_s\|_{[Z_{\lambda_2}(\Omega^*)]} = M\big(1/s, \Lambda(Z)\big)$, since $\tau_{\Lambda(Z)} = \tau_Z$ is valid even for any r.i. space Z. From this equality it then can easily be deduced that $\alpha_Z = \alpha_{\Lambda(Z)}$ and $\beta_Z = \beta_{\Lambda(Z)}$. For a more detailed discussion of spaces of fundamental type see [10].

Concerning the weak-interpolation problem we now show that for this important class of r.i. spaces (which contains the Lebesgue-, Lorentz-, and Orlicz spaces) there exist two further conditions which are equivalent to the index condition of Corollary 4.2:

Theorem 5.1: *As in Theorem 4.1 let $X, Y, Z \subset \mathcal{M}(\Omega)$ be r.i. spaces such that $\beta_{\Lambda(X)} > 0$, $0 < \beta_{\Lambda(Y)} \leqq \alpha_{\Lambda(Y)} < 1$ and τ_Y/τ_X decreasing. Further assume that $\alpha_{\Lambda(X)} = \beta_{\Lambda(X)}$, $\alpha_{\Lambda(Y)} = \beta_{\Lambda(Y)}$, and Z is of fundamental type. Then the following statements are equivalent*:

(a) $W(X, Y) \subset Z)$;

(b) $W(X, Y) \subset (\Lambda(Z))$;

(c) $\beta_{\Lambda(Y)} < \beta_Z \leqq \alpha_Z < \alpha_{\Lambda(X)}$;

(d) *there exists a finite number $A > 0$ such that*

$$\int_0^t F(s, t)\, d\tau_Z(t) \leqq A \tag{5.1}$$

uniformly in $s \in \Omega^*$, the function $F : \Omega^* \times \Omega^* \to \mathbf{R}$ being defined by

$$F(s, t) := \min \left\{ \frac{\tau_X(s)}{\tau_X(t)}, \frac{\tau_Y(s)}{\tau_Y(t)} \right\} \Big/ \tau_Z(s) \qquad (s, t \in \Omega^*). \tag{5.2}$$

The proof of this theorem now follows readily. The equivalence of (a) with (c) is essentially Corollary 4.2. Since Z is assumed to be of fundamental type, (c) can be rewritten as $\beta_{\Lambda(Y)} < \beta_{\Lambda(Z)} \leqq \alpha_{\Lambda(Z)} < \alpha_{\Lambda(X)}$. Therefore, the equivalence of (b) and (c) is again Corollary 4.2, but now applied to $\Lambda(Z)$ instead of Z. Finally, the equivalence of (b) and (d) was proved in [20]. Note that in addition to the theorem of [20] we now also have the equivalence of (d) with (a) ∎

As a more concrete example we next consider the case when X and Y are Lebesgue spaces, and Z is arbitrary.

Corollary 5.2: *If Z is of fundamental type and $1 \leqq p < q < \infty$, then the following statements are equivalent*:

(a) $W(L_p, L_q) \subset (Z)$;

(b) $W(L_p, L_q) \subset (\Lambda(Z))$;

(c) $1/q < \beta_Z \leqq \alpha_Z < 1/p$;

(d) *there exists a finite number $A > 0$ such that uniformly in $s \in \Omega^*$,*

$$\frac{s^{1/q}}{\tau_Z(s)} \int_0^s t^{-1/q}\, d\tau_Z(t) + \frac{s^{1/p}}{\tau_Z(s)} \int_s^l t^{-1/p}\, d\tau_Z(t) \leqq A.$$

Indeed, since $\tau_{L_q}(t) = t^{1/q}$ and $\tau_{L_p}(t) = t^{1/p}$, the function τ_{L_p}/τ_{L_q} is decreasing if and only if $p < q$. Moreover, $\beta_{\Lambda(L_p)} = \beta_{L_p} = 1/p > 0$ and $0 < \alpha_{\Lambda(L_q)} = \alpha_{L_q} = 1/q < 1$. Finally, observe that in this case

$$\begin{aligned} F(s, t) &= \min \{(s/t)^{1/p}, \ (s/t)^{1/q}/\tau_Z(s)\} \\ &= \begin{cases} (s/t)^{1/q}/\tau_Z(s) & \text{if } 0 < t < s \\ (s/t)^{1/p}/\tau_Z(s) & \text{if } t \geqq s, \end{cases} \end{aligned}$$

and therefore Theorem 5.1 furnishes the above Corollary.

Remark 5.1: For the equivalence of (a) and (c) it is not necessary that Z be of fundamental type. Then this equivalence is the theorem of Boyd.

Corollary 5.3: *If $1 \leqq p < q < \infty$, $1 < r < \infty$, and Z is one of the spaces L_r, L_{rs}, $L^r(\log^+ L)$ or the Marcinkievicz space $M_{1-1/r}$, then the following statements are equivalent:*

(a) $W(L_p, L_q) \subset (Z)$;

(b) $W(L_p, L_q) \subset (L_{r_1})$;

(c) $1/q < 1/r < 1/p$.

Condition (d) is trivial in this case. This corollary follows from the preceeding corollary since in any case $\tau_Z(t) = t^{1/r}$, and hence $\Lambda(Z) = L_{r1}$.

Remark 5.2: The implication (c) $\Rightarrow$ (a) contains the theorems of Riesz/Thorin and of Marcinkievicz, whereas (c) $\Rightarrow$ (b) is the theorem of Calderón.

Let us conclude this paragraph with the relation

$$\int_0^t F(s, t)\, d\tau_Z(t) = \tau_Y(s)\, [P_Z(1/\tau_Y)] + \tau_X(s)\, [Q_Z(1/\tau_X)]\, (s), \tag{5.3}$$

between the integral of (5.1) and the average operators of Section 3.

5.2. Applications to Particular Operators

As a first example we consider the Hardy-Littlewood maximal operator θ on $\mathbf{R}_n$, $n \geqq 1$, in its spherical form, given for $f \in \mathbf{R}_n$ by

$$(\theta f)\,(v) := \sup_{B(v)} \frac{1}{m\big((B(v)\big)} \int_{B(v)} |f(u)|\, du. \tag{5.4}$$

Here the supremum has to be taken over all balls $B(v)$ with positive radius and center v. If $Q(v)$ is the circumscribed cube of $B(v)$ with its sides parallel to the coördinate axes, then there exists a constant $A_n > 0$, depending only on n (with $A_1 = 1$) such that $m\big(Q(v)\big) \leqq A_n m\big(B(v)\big)$, see [23], As an application of Corollaries 4.2 and 4.3 we obtain the following mapping theorem for the maximal operator.

Theorem 5.4: *If $Z = Z_\varrho(\mathbf{R}_n)$, $n \geqq 1$, has indices strictly between 0 and 1, i.e., $0 < \beta_Z \leqq \alpha_Z < 1$, then $\theta \in (Z)$ and*

$$\|\theta\|_{(Z)} \leqq 2^n A_n\, \|E_{1/2}\|_{[Z_\lambda(\Omega^*)]} \int_0^1 \|E_s\|_{[Z_\lambda(\Omega^*)]}\, ds, \tag{5.5}$$

where $Z_\lambda(\Omega^*)$ is the Luxemburg representation of Z.

Proof: In order to derive this theorem from Corollary 4.2, choose $X := L_1(\mathbf{R}_n)$ and $Y := L_q(\mathbf{R})$ with $1 < q < \infty$. It is well known that the maximal operator θ is of weak type (X, X) and a bounded operator on Y. More precisely, if $D_{\theta f}(\sigma) := \mu\{x \in \mathbf{R}_n\colon |\theta(f)\,(x)| > \sigma\}$ is the distribution function of θf, then (see [23])

$$D_{\theta f}(\sigma) \leqq \frac{2^n A_n}{\sigma}\, \|f\|_X \qquad (f \in X), \tag{5.6}$$

$$\|\theta\|_{(Y)} \leqq 2\left(\frac{q 2^n A_n}{q-1}\right)^{1/q}. \tag{5.7}$$

From (5.6) it follows that $(\theta f)^*(t) \leqq (2^n A_n/t)\,\|f\|_X = (2^n A_n/t)\,\|f\|_{11}$ for $f \in X$. Recalling that $\tau_X(t) = t$ and $\Lambda(X) = L_{11}$, we can conclude that θ is of weak type in the more general sense of (1.1) with the weak norm

$$\|\theta\|_{W(X)} \leqq 2^n A_n. \tag{5.8}$$

On the other hand, $\sigma^q D_{\theta f}(\sigma) \leqq \|\theta f\|_Y{}^q$ for $f \in Y$, see [23]; hence $(\theta f)^*(t) \leqq (\|\theta\|_{(Y)}/t^{1/q}) \times \|f\|_Y \leqq (\|\theta\|_{(Y)}/t^{1/q})\,\|f\|_{q1}$. On account of $\tau_Y(t) = t^{1/q}$ and $\Lambda(Y) = L_{q1}$, this — together with (5.7) — means that θ is of weak type (Y, Y) with the weak-norm

$$\|\theta\|_{W(Y)} \leqq 2\left(\frac{q 2^n A_n}{q-1}\right)^{1/q}. \tag{5.9}$$

Summarizing we can say that $\theta \in W(X, Y)$ for any $q \in (1, \infty)$. The spaces X and Y are of fundamental type with indices $\alpha_X = \beta_X = 1$ and $\alpha_Y = \beta_Y = 1/q$. Further, τ_Y/τ_X is decreasing since $q > 1$, so that all the assumptions of Corollary 4.2 are satisfied. For any r.i. space Z with $1/q < \beta_Z \leqq \alpha_Z < 1$ we therefore have by Corollary 4.2 that $\theta \in (Z)$, and by Corollary 4.3 that

$$\|\theta\|_{(Z)} \leqq \|E_{1/2}\|_{[Z_\lambda(\Omega^*)]} \max\left\{2^n A_n,\, 2\left(\frac{q 2^n A_n}{q-1}\right)^{1/q}\right\}$$
$$\times \left\{\int_0^1 \|E_s\|_{[Z_\lambda(\Omega^*)]}\, ds + \frac{1}{q}\int_1^\infty \|E_s\|_{[Z_\lambda(\Omega^*)]}\, s^{1/q}\, ds/s\right\}.$$

Here we used that $M(s, X) = s$ and $M(s, Y) = s^{1/q}$. Note that the latter two integrals are finite on account of the index condition assumed, see [3]. Letting q tend to infinity we obtain (5.5), observing that $A_n \geqq 1$, and the second integral is decreasing in q ∎

Remark 5.3: A result similar to Theorem 5.4 could be deduced for the cubic maximal operator by the same methods. The only change is that the factor A_n in (5.5), ..., (5.9) is omitted.

For concrete spaces Z the norm estimate of (5.5) can be evaluated explicitely. Indeed, we have

Corollary 5.5: a) *If $Z = \Lambda(\phi, p)$, $p > 1$, is a uniformly convex Lorentz space, then $(\theta) \in (Z)$ and*

$$\|\theta\|_{(Z)} \leqq 2^n A_n N(1/2) \int_0^1 N(s)^{1/p}\, ds \leqq 2^{n+1/p} A_n \frac{p}{p-1} \tag{5.10}$$

with $N(s) := \sup_{t \in \Omega^*} \left[\int_0^t \phi(u)\, du \Big/ \int_0^{st} \phi(u)\, du\right]$.

b) *If $Z = L_{M\Psi}$ is an Orlicz space with strictly increasing Young function Ψ such that $L_{M\Psi}$ is reflexive, then*

$$\|\theta\|_{(Z)} \leqq \frac{2^n A_n}{K_\Psi{}^{-1}(1/2)} \int_0^1 ds/K_\Psi{}^{-1}(s), \tag{5.11}$$

$K_\Psi{}^{-1}(s)$ denoting the right-continuous inverse of $K_\Psi(s) := \sup_{t \in \Omega^*} [\Psi(st)/\Psi(t)]$.

This corollary follows from Theorem 5.4 by inserting the norm $\|E_s\|_{[Z_\lambda(\Omega^*)]}$ of the respective space. If $Z = \Lambda(\phi, p)$, then $\|E_s\|_{[Z]} = N(s)^{1/p}$, and for $Z = L_{M\Psi}$ one has $\|E_s\|_{[Z]} = 1/K_\Psi{}^{-1}(s)$. Concerning the index conditions note that for Lorentz spaces

$Z = \Lambda(\phi, p)$ one has $0 < \beta_Z \leqq \alpha_Z < 1$ if and only if Z is uniformly convex, whereas in case of Orlicz spaces $Z = L_{M\Psi}$ this condition is equivalent to the reflexivity of $L_{M\Psi}$. The second estimate in (5.10) follows from $N(1/2) \leqq 2^{1/p}$ and $\int_0^1 N(s)^{1/p}\, ds \leqq \int_0^1 s^{-1/p}\, ds = p/(p-1)$ for $p > 1$. For the one-dimensional case $A_1 = 1$, and (5.10) therefore improves the estimate $\|\theta\|_{(Z)} \leqq 2 \cdot 2^{1/p} p/(p-1)$ as stated in [13].

Remark 5.4: If $Z = L_r$, $r > 1$, one should expect (5.7), but Theorem 5.4 leads in this case to

$$\|\theta\|_{(L_r)} \leqq 2^n A_n 2^{1/r} \frac{r}{r-1},$$

which, for $n = 1$, differs from the classical constant of Hardy-Littlewood by the methodic factor $2^{1/r}$. For arbitrary n, the constant given in [22], namely $2^{2n} r/(r-1)$, is larger than the above constant by the factor $2^{n-1/r}/A_n$.

For $n = 1$ and $\Omega = (0, l)$, $0 < l < \infty$, $\mu = m =$ Lebesgue measure it is well known [12] that $(\theta f^*)(t) = (1/t) \int_0^t f^*(s)\, ds$. By means of this representation we now can give an application to ergodic theory. For this purpose, let $\mathscr{E}$ be a measure space with finite measure l, and let G be an ergodic group of one to one measure preserving transformations g of $\mathscr{E}$. Further assume that for each measurable function f on $\mathscr{E}$ and each $g \in G$ the product fg is measurable on $G \times \mathscr{E}$. Then the expression

$$(p_a f)(v) := \frac{1}{|N_a|} \int_{N_a} f(gv)\, dg \qquad (v \in \mathscr{E}, a > 0)$$

exists for any nonnegative function $f \in L_1(\mathscr{E})$. Here $\{N_a : a > 0\}$ denotes a family of compact, symmetric neighbourhoods of the identity of G such that $N_a N_b \subset N_{a \cdot b}$, and $|N_{2a}| \leqq K\, |N_a|$ where $|N_a|$ is the left invariant measure of N_a. Concerning the operator

$$(Pf)(v) := \operatorname*{ess\,sup}_{a>0} p_a(|f|)(v) \qquad \left(f \in L_1(\mathscr{E}), v \in \mathscr{E}\right)$$

it is shown in [4] that

$$(Pf)^*(t) \leqq \frac{K^2}{t} \int_0^t |f|^*(s)\, ds \qquad \left(t \in (0, l)\right).$$

Hence, P is of weak type $\left(L_1(\mathscr{E}), L_1(\mathscr{E})\right)$ and $(Pf)^*(t) \leqq K^2 (\theta f^*)(t)$. Since θ is bounded on $L_q(\mathscr{E})$ for $1 < q < \infty$, the latter implies that P is also bounded and, a fortiori, of weak type $\left(L_q(\mathscr{E}), L_q(\mathscr{E})\right)$. By Theorem 5.4 we therefore have

Corollary 5.6: *If $\mu(\mathscr{E}) = l < \infty$ and $Z(\mathscr{E})$ is a r.i. space of μ-measurable functions on $\mathscr{E}$ such that $0 < \beta_Z \leqq \alpha_Z < 1$, then $P \in (Z)$.*

Remark 5.5: Similar considerations could be applied e.g. to the Hilbert transform, to the conjugate operator, the Poisson operator or, more generally, to kernel operators with a kernel which is homogeneous of degree -1. In these cases Theorem 5.4 would furnish mapping properties of these operators on Lorentz — or Orlicz spaces, such as Hardy- and Hardy-Schur inequalities. In particular by means of the conjugate operator a theorem [11] about norm convergence of Fourier series on r.i. spaces could be reestablished of.

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