The Mareinkievicz Interpolation Theorem for Rearrangement-Invariant Function Spaces and Applications

F. **FEHR**

Der Interpolationssatz von J. MARCINKIEVICZ [17] besagt, daß jeder sublineare Operator T, der gleichzeitig vom schwachen Typ (p_1, q_1) und vom schwachen Typ (p_2, q_2) ist, notwendig auch ein beschränkter Operator des Lebesgueraumes' $L_p(0, l)$, $0 < l \leq \infty$, in sich ist, und zwar für alle p mit *P2 < p < p¹ .* Ziel der vorliegenden Arbeit ist, diesen *Satz* auf den Rahrnen rearrangement-invarianter Banachscher Funktionenräume zu verallgemeinern und ihn damit einem sehr viel größeren Kreis von Anwendungen zugänglich zu machen.

Интерполяционная теорема И. Марцинкввича [17] утверждает, что каждый сублинейный оператор *Т*, который одновременно имеет слабый (p_1, q_1) -тип и слабый (p_2, q_2) тип, необходимо является ограниченным опе́ратором в Лебеговом пространстве $L_p(0; l)$ $(0 < l \leq \infty)$ для всех таких р, что $p_2 < p < p_1$. В настоящей работе эта теорема обобщается на случай перестановочно-инвариантных (симметричных) банаховых проcтранств функций. Тем самым теорема применима в значительно более широких рамках. Интерполяционная теорема И. Магцинкевича [17] утверждает, что каждый сублиный оператор T , который одновременно имеет слабый (p_1, q_1) -тип и слабый (p_2, m_1) , тип, необходимо является ограниченным оператором в Лебегов

The interpolation theorem of J. MARCINKIEVICZ [17] states that any sublinear operator T which is simultaneously of weak types (p_1, q_1) and (p_2, q_2) is also a bounded operator from the Lebesgue space $L_p(0, l)$, $0 < l \leq \infty$, into itself, provided $p_2 < p < p_1$. The aim of this paper is to generalize this theorem to the setting of rearrangement-invariant Banach function spaces, and thus to render the theorem available to a much larger range of applications.

1. Preliminaries
Let (Ω, Σ, μ) be a σ -finite, non-atomic measure space with $\mu(\Omega) =: l \leq \infty$, *11(* Ω) (resp. $\mathcal{P}(\Omega)$) the space of realvalued (resp. nonnegative), μ -measurable functions on Ω ρ a *rearrangement-invariant* (= r.i.) function norm on $\mathscr{P}(\Omega)$, and $X \equiv X_{\rho}(\Omega)$ the r.i. Banach function space generated by ρ , in the sense of W, A. J. LUXEMBURG [14]. By $X' \equiv X_{\varrho'}(Q)$ we denote the associate r.i. Banach function space of X which is generated by the norm The interpolation theorem of J. MARCI:
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\varrho'(g) := \sup \left\{ \int_{\Omega} fg \, d\mu : f \in \mathscr{P}(\Omega), \varrho(f) \leq 1 \right\}.
$$

Note that $\varrho'' = \varrho$. Finally, let $X_{\lambda}(Q^*)$ be the Luxemburg representation of the space $\varrho'(g) := \sup \left\{ \int_{\Omega} fg \, d\mu : f \in \mathcal{P}(\Omega), \varrho(f) \leq 1 \right\}.$
Note that $\varrho'' = \varrho$. Finally, let $X_{\lambda}(\Omega^*)$ be the Luxemburg representation of the space $X_{\varrho}(\Omega)$, i.e., $\Omega^* := (0, l)$, $\mu = m =$ Lebesgue measure, and λ is a r $X_{\rho}(\Omega)$, i.e., $\Omega^* := (0, l)$, $\mu = m$ = Lebesgue measure, and λ is a r.i. function norm on $P(Q^*)$ of all nonnegative, Lebesgue measurable functions on Ω^* such that
 f^* for all $f \in \mathcal{P}(\Omega)$, with f^* denoting the nonincreasing rearrangement of f .
 $\lambda(f) = \sup \left\{ \int_0^1 f^*(x) g^*(x) dx : g \in \mathcal{P}(\Omega) \,; \, e'(g) \$ Explicitely, for $f \in \mathcal{P}(\Omega^*)$ the norm $\lambda(f)$ is given by $\begin{aligned} \n\mathbf{E}_g &= \mathbf{E}_g \mathbf{E}_g \mathbf{E}_g \mathbf{E}_g \mathbf{E}_g \mathbf{E}_g \mathbf{E}_g &= \frac{1}{2} \mathbf{E}_g \math$

$$
\lambda(f) = \sup \left\{ \int\limits_0^1 f^*(x) g^*(x) dx : g \in \mathcal{P}(\Omega), \varrho'(g) \leq 1 \right\},\
$$

$$
X_{\lambda}(\Omega^*) := \{f \in \mathscr{M}(\Omega^*) : \lambda(|f|) < \infty\},
$$

112 F. FERER

and the space $X_{\lambda}(\Omega^*)$ by
 $X_{\lambda}(\Omega^*) := \{f \in \mathcal{M}(\Omega^*) : \lambda(|f|) \}$

where $\mathcal{M}(\Omega^*)$ is the set of all real valued where $\mathcal{M}(\Omega^*)$ is the set of all realvalued, Lebesgue measurable functions on Ω^* . This definition is meaningful, since supp $g^* \subset \Omega^*$ if $g \in \mathscr{P}(\Omega)$. In the sequel, the Luxemburg representation of $X_e(\Omega)$ will systematically be used in order to reduce the problem to the situation where $\Omega = (0, l)$ is an interval and $\mu = m =$ Lebesgue measure, as treated in [8]. $\mathcal{M}(\Omega^*) : \lambda(|f|) < \infty$,

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involves the Luxemburg represent
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The first definition which involves the Luxemburg representation is the definition of the *Boyd indices* α_X and β_X of the space $X_{\rho}(Q)$, namely

\n first definition which involves the Luxemburg representation is the *Soyd indices*
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$$
 and β_X of the space $X_e(\Omega)$, namely:\n
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\alpha_X := \inf_{0 < s < 1} \frac{\log ||E_s||_{(X_x(\Omega^*))}}{\log s}, \quad \beta_X := \sup_{s > 1} \frac{\log ||E_s||_{(X_x(\Omega^*))}}{\log s},
$$
\n is the *dilation operator* on $A(\Omega^*)$, given by\n

where E_s is the *dilation operator* on $\mathcal{M}(\Omega^*)$, given by

 (E_sf) (t) := $\begin{cases} f(st) & \text{if } st \in \Omega^* \\ 0 & \text{if } s \end{cases}$ $\frac{\log ||E_s||_{(X_{\lambda}(\Omega^{\bullet}))}}{\log s}$

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where E_s is the dilation operator on $\mathcal{M}(\Omega^*)$, given by
 $(E_s f)(t) := \begin{cases} f(st) & \text{if } st \in \Omega^* \\ 0 & \text{elsewhere} \end{cases}$

see [2]. If, in particular, $X_e(\Omega) = L_p(\Omega)$, $1 \leq p < \infty$, then $||E_s||_{L_p(\Omega)} = s^{-1/p}$ and
 $\alpha_{L_s(\Omega)} = \beta_{L_s(\Omega)} = 1/p$. H $\alpha_{L_p(Q)} = \beta_{L_p(Q)} = 1/p$. Hence, these "Boyd indices" α_X and β_X generalize the number $1/p$ which characterizes the 'space L_p in the Lebesgue case. Generally, it can be shown that $0 \le \beta_X \le \alpha_X \le 1$ (just as $0 \le 1/p \le 1$), and $\alpha_{X'} = 1 - \beta_X, \beta_{X'} = 1 - \alpha_X$. The first definition w

of the *Boyd indices* α_X
 $\alpha_X := \inf_{0 \le s \le 1}$

where E_s is the dilation
 $(E_s f) (t) := \begin{cases} f \\ (t) \\ (t) \end{cases}$

see [2]. If, in particula
 $\alpha_{L_p(0)} = \beta_{L_p(0)} = 1/p$.

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 x, $X_e(\Omega) = L_p(\Omega)$, $1 \leq p < \infty$, then $||E_s||_{[L_p(\Omega)]} = s^{-1/2}$

Hence, these "Boyd indices" α_X and β_X generalize the number

is the `space L_p in the Lebesgue case. Generally, it c For further properties of indices see [9, 10].

The second definition we need is that 6f an operator of weak type. As a substitute for the space weak- L_p in the original Marcinkievicz theorem, we now use the rearrangement-invariant Lorentz spaces $A(X)$ and $M(X)$ (see e.g. [19, 25]) which can be assigned to each r.i. Banach function space $X = X_e(\Omega)$, namely

11, in particular,
$$
X_e(S) = L_p(S)
$$
, $l \leq p < \infty$, then $||E_s||_{L_p(l)} = \beta_{L_p(Q)} = 1/p$. Hence, these "Boyd indices" α_X and β_X general, the characteristic of the space L_p in the Lebesgue case. Generals of $\leq \beta_X \leq \alpha_X \leq 1$ (just as $0 \leq 1/p \leq 1$), and $\alpha_{X'} = 1 - \beta_X$, the properties of indices see [9, 10].
second definition we need is that of an operator of weak type, space weak L_p in the original Marcinkiewicz theorem, we need to each r.i. Banach function space $X = X_e(Q)$, namely $\mathcal{A}(X) := \left\{ f \in \mathcal{M}(\Omega) : ||f||_{\mathcal{A}(X)} := \int_0^t f^*(s) \, d\tau_X(s) < \infty \right\}$, and $M(X) := \left\{ f \in \mathcal{M}(\Omega) : ||f||_{\mathcal{M}(X)} := \sup_{l \in X^*} \frac{\tau_X(l)}{l} \int_0^l f^*(s) \, ds < \infty \right\}$, is the fundamental function of the space $X_e(Q)$, i.e. $\tau_X(l) := \emptyset$.
b. Without loss of generality, τ_X will be assumed to be conca.

where τ_X is the fundamental function of the space $X_e(\Omega)$, i.e. $\tau_X(t) := ||\chi_{(0,\min\{t,0\})}||_{X_\lambda(\Omega^*)}$ for $t>0$. Without loss of generality, τ_X will be assumed to be concave, and $\tau_X(0+1)$ see [2]. If, in particular, $X_c(2) = L_p(2)$, $1 \le p < \infty$, then $||E_x||_{L_p(2)}$
 $\alpha_{L_p(A)} = \beta_{L_p(A)} = 1/p$. Hence, these 'Boyd indices'' α_X and β_X generality
 $1/p$ which characterizes the 'space L_p in the Lebesgue case. Gene $= 0$. The spaces $A(X)$ and $M(X)$, with $\|\cdot\|_{A(X)}$ and $\|\cdot\|_{M(X)}$, respectively, as norms, are r.i. Banach function spaces such that $A(X) \subset X \subset M(X)$ with continuous embeddings. Moreover, the space $A(X)$ (and $M(X)$, resp.) is the smallest (largest) r.i. Banach function space contained in (containing) X with the same fundamental function, see [10: Corollary 3.2]. If $X = L_p$, $1 \leq p < \infty$, then $A(L_p) = L_{p1}$ and $M(L_p) = L_{p\infty}$ (L_{pq}) denoting the Letron space such that $A(X) \subset X \subset M(X)$ with continuous embeddings. Moreover, the space $A(X)$ (and $M(X)$, resp.) is the denoting the Lorentz space). *sup 3.2].* If $X = L_p$, $1 \leq p < \infty$, the the Lorentz space).

it ion 1.1: Assume that $X \equiv X_e(\Omega)$ is

erator $T : A(X) \to \mathcal{M}(\Omega)$ is said to be

sup $(Tf)^*(t) \tau_X(t) \leq \text{const.} ||f||_{\mathcal{A}(X)}$

ition $\mathcal{A} \to 0$ than the left side of

Definition 1.1: Assume that $X \equiv X_{\varrho}(\varOmega)$ is a r.i. Banach function space. A sublinear operator $T : A(X) \to \mathcal{M}(\Omega)$ is said to be of *weak type* (X, X) , if and only if

$$
\sup_{t\in\Omega^*} (Tf)^* (t) \tau_X(t) \leq \text{const. } ||f||_{\mathcal{A}(X)} \qquad \big(f\in\mathcal{A}(X)\big). \quad \text{for} \quad (1.1)
$$

If, in addition, $\beta_{A(X)} > 0$, then the left side of (1.1) is equivalent to $||Tf||_{M(X)}$. Indeed, $\sup_{t \in \Omega^*} (Tf)^* (t) \tau_X(t) \leq \text{const.} ||f||_{A(X)} \qquad (f \in A(X)).$ *(1.1)*
 If, in addition, $\beta_{A(X)} > 0$ *, then the left side of (1.1) is equivalent to* $||Tf||_{M(X)}$ *. Indeed,
* $(Tf)^* (t) \leq \left(\int_0^t (Tf)^* (s) ds \right) / t$ *on account of the monotoni*

A Marcinkiewicz Interpolation
\nother hand,
\n
$$
(Tf)^{*}(t) \le \sup_{s \in \Omega^{*}} \{(Tf)^{*}(s) \tau_{X}(s)\} / \tau_{X}(t) \text{ for } t > 0;
$$
\nhence
\n
$$
\frac{1}{t} \int_{0}^{t} (Tf)^{*}(s) ds \le \frac{1}{t} \sup_{s \in \Omega^{*}} \{(Tf)^{*}(s) \tau_{X}(s)\} \int_{0}^{t} \frac{ds}{t \tau_{X}(s)}
$$

other hand,
\n
$$
(Tf)^*(t) \le \sup_{s \in \Omega^*} \{(Tf)^*(s) \tau_X(s)\}/\tau_X(t) \quad \text{for} \quad t > 0;
$$
\nhence
\n
$$
\frac{1}{t} \int_0^t (Tf)^*(s) \, ds \le \frac{1}{t} \sup_{s \in \Omega^*} \{(Tf)^*(s) \tau_X(s)\} \int_0^t \frac{ds}{\tau_{X(s)}}
$$
\n
$$
\le \sup_{s \in \Omega^*} \{(Tf)^*(s) \tau_X(s)\} \int_0^1 ||E_s||_{[A(X_1(\Omega^*))]} ds / \tau_X(t).
$$
\n(1.2)
\nHere we used the facts that
\n
$$
\tau_X(t) \tau_X(t) = t
$$
\nand
\n
$$
\int \frac{ds}{\tau_X(s)} \le \left(\int_0^1 ||E_s||[\lambda(x_1(\Omega^*))] \, ds\right) \tau_X(t) \qquad (t \in \Omega^*),
$$
\n(1.4)

$$
\leq \sup_{s\in\Omega^*} \left\{ (Tf)^* \ (s) \ \tau_X(s) \right\} \int\limits_0^{\cdot} {\left\|E_s\right\|_{\left[A(X_\lambda(\Omega^*)) \right]} ds}/{\tau_X(t)} . \qquad (1.2)
$$

$$
\tau_X(t) \tau_{X'}(t) = t \tag{1.3}
$$

$$
\leq \sup_{s \in \Omega^*} \{(Tf)^* (s) \tau_X(s)\} \int_0^1 \|E_s\|_{[A(X_1(\Omega^*))]} ds/\tau_X(t). \tag{1.2}
$$
\nHere we used the facts that\n
$$
\tau_X(t) \tau_{X'}(t) = t \tag{1.3}
$$
\nand\n
$$
\int_0^t \frac{ds}{\tau_X(s)} \leq \left(\int_0^1 \|E_s\|_{[A(X_1(\Omega^*))]} ds\right) \tau_{X'}(t) \quad (t \in \Omega^*), \tag{1.4}
$$
\nsee e.g. [20] and [10: (3.6)], respectively, noting that (1.4) is valid since $\beta_A(X) > 0$ by assumption. Multiplication of (1.2) by $\tau_X(t)$ and passing to the supremum over all $t \in \Omega^*$, yields\n
$$
\|Tf\|_{\mathbf{M}(X)} \leq \left(\int_0^1 \|E_s\|_{[A(X_1(\Omega^*))]} ds\right) \sup_{s \in \Omega^*} \{Tf\}^*(s) \tau_X(s) \}.
$$
\nHence we have (compare [20]) the following lemma.\nLemma 1.1: If $\beta_{A(X)} > 0$, then a sublinear operator $T : A(X) \to A'(\Omega)$ is of weak type (X, X) if and only if T is a bounded operator from $A(X)$ into $\mathbf{M}(X)$, i.e.

see e.g. [20] and [10: (3.6)], respectively, noting that (1.4) is valid since $\beta_A(X) > 0$ by see e.g. [20] and [10: (3.6)], respectively, noting that (1.4) is valid since $\beta_A(X) > 0$ by
assumption. Multiplication of (1.2) by $\tau_X(t)$ and passing to the supremum over all
 $t \in \Omega^*$, yields
 $||Tf||_{\mathbf{M}(X)} \leqq \left(\int_0^1 ||E$ $\begin{aligned} \text{see e.g.}\ \text{as} \text{sum}\ t \in \mathcal{Q}^\clubsuit, \end{aligned}$ UJ and [10: (3.6)], respectively, noting that (1.4) is valid since $\beta_A(X) > 0$ by

on: Multiplication of (1.2) by $\tau_X(t)$ and passing to the supremum over all

elds
 $|Tf||_{\mathbf{M}(X)} \leqq \int_0^1 ||E_s||_{[A(X_1(\Omega^*))]} ds \sup_{s \in \Omega^*} \{Tf\}$ see e.g. [20] and [10: (3.6)], respectively, noting that (1.4

assumption. Multiplication of (1.2) by $\tau_X(t)$ and passin
 $t \in \Omega^*$, yields
 $||Tf||_{\mathbf{M}(X)} \leqq \left(\int_0^1 ||E_s||_{[A(X_k(\Omega^*))]} ds\right) \sup_{s \in \Omega^*} {Tf}^* (s) \tau$

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||Tf||_{\mathbf{M}(X)} \leq \left(\int_{0}^{1} ||E_{s}||_{[A(X_{\lambda}(Q^*))]} ds\right) \sup_{s \in \Omega^*} {Tf)^* (s) \tau_X(s)}.
$$
\ne have (compare [20]) the following lemma.
\nna 1.1: If $\beta_{A(X)} > 0$, then a sublinear operator $T : A(X)$
\nX) if and only if T is a bounded operator from $A(X)$ into
\n $||Tf||_{\mathbf{M}(X)} \leq \text{const} \cdot ||f||_{A(X)} \qquad (f \in A(X)).$
\n \forall we introduce the notations (Z) for the space of a
\ns mapping a r.i. space Z into Z, and
\n $W(X, Y) \equiv W(X(\Omega), Y(\Omega)) := \{ T : A(X) + A(Y) \rightarrow$
\nof weak types (X, X) and (Y, Y) .
\n $\text{with } Y \in \mathbb{R}$ is the **refinement-
\n** Invariant Property

Lemma 1.1: *If* $\beta_{A(X)} > 0$, then a sublinear operator $T : A(X) \rightarrow \mathcal{M}(\Omega)$ is of weak *type* (X, X) *if and only if* T *is a bounded operator from* $A(X)$ *into* $M(X)$ *, i.e.* $\begin{array}{c} \begin{array}{c} \text{\Large \mid} \end{array} \ \text{\Large \mid} \ \begin{array}{c} \text{\Large \mid} \ \text{\Large \mid} \end{array} \ \text{\Large \mid} \$ (2) is of ϵ .
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• - ' Finally we introduce the notations *(Z)* for the space of all bounded sublinear *We* introduce the notations $(f \in A(X))$.
 We introduce the notations (Z) for the space of all bound
 W(*X*, *Y*) = *W*(*X*(*Q*), *Y*(*Q*)) := { *T*: *A*(*X*) + *A*(*Y*) → *M*(*Q*); *T*
 M(*X*, *Y*) = *W*(*X*(*Q*), *Y* 12. $||Tf||_{M(X)} \leq \text{const} \cdot ||f||_{A(X)} \qquad (f \in A(X)).$ (1.4)

Finally we introduce the notations (Z) for the space of all bounded sublinear

operators mapping a r.i. space Z into Z, and
 $W(X, Y) \equiv W(X(\Omega), Y(\Omega)) := \{ T : A(X) + A(Y) \rightarrow \mathcal{M}(\Omega) ; T \}$

$$
W(X, Y) \equiv W(X(\Omega), Y(\Omega)) := \{ T \colon A(X) + A(Y) \to \mathscr{M}(\Omega) ; T \}
$$

If *X*, $Y \subset \mathcal{M}(\Omega)$ are any two Banach function spaces, and $T \subset (X + Y)$; we say that *T* is *admissable* (compare [5]), if the restriction $T|_X$ of *T* to the space *X* belongs to (*X*) 2. Necessity of the Rearrangement-Invariant Property
 If X, *Y* \subset *M*(Ω) are any two Banach function spaces, and *T* \subset (*X* + *Y*); we say that
 T is *admissable* (compare [5]), if the restriction *T*_{|*x*} tors with respect to . the space X and *Y.* The *strong-type interpolation problem* consists in determining those spaces Z for which $ad(X, Y) \subset (Z)$, if X and Y are given. In the particular case that X, Y, Z are Lebesgue, spaces, this problem was solved by the convexity theorem of M. RrEsz/G. ThORIN [18,24]. In the frame work ofr.i. spaces If $X, Y \text{ }\subset \mathcal{M}(\Omega)$ are any two Banach function spaces, and $T \subset (X + Y)$, we say tha

T is admissable (compare [5]), if the restriction $T|_X$ of T to the space X belongs to (X

and, simultaneously, $T|_Y \in (Y)$. By $ad(X, Y$ **•** $W(X, Y) \equiv W(X(\Omega), Y(\Omega)) := \{T: A(X) + A(Y)$

• of weak types (X, X) and (Y, Y) .

• of weak types (X, X) and (Y, Y) .
 2. Necessity of the Rearrangement-Invariant Property
 If $X, Y \subset A(\Omega)$ are any two Banach function spaces, and solved by
r.i. spaces
rong-type If $X, Y \subset \mathcal{M}(\Omega)$ are any two Banach function spaces, and $T \subset (X + Y)$, we say that *T* is admissable (compare [5]), if the restriction $T|_X$ of *T* to the space *X* belongs to (X) and, simultaneously, $T|_Y \in (Y)$. By In the particular case that X, Y, Z are Lebesgue, spaces, this problem was solved b
the convexity theorem of M. RIESZ/G. THORIN [18, 24]. In the frame work of r.i. space
concrete methods of how to construct function space sists in determining those spaces Z for which $ad(X, Y) \subset (Z)$, if X a
In the particular case that X, Y, Z are Lebesgue_i spaces, this probler
the convexity theorem of M. RIESZ/G. THORIN [18, 24]. In the frame w
concrete met

Since $ad(X, Y) \subset W(X, Y)$, a harder problem is the *weak-type interpolation problem* which consists in finding those spaces Z for which $W(X, Y) \subset (Z)$. This problem

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was solved by J. **MAzcINKIEvIcz** [17] for Lebesgue spaces X, *Y, Z,* and by D. W. **BoYD;** $[2]$ in case that X, Y are Lebesgues spaces and Z is an arbitrary r.i. Banach function space. The purpose of this paper is to solve the weak-type interpolation problem for the case that X, Y are any abstract r.i. Banach function spaces and *Z* is any Banach function space. As a first step we now show that the space *Z* must necessarily also, be 114 F. FEHER

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[2] in case that X, Y are Lebesgues spaces and Z is an arbitrary r.i. Banach

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[2] in case that X, Y are Lebesgues spaces and Z is an arbitrary r.i. Banach function

space. The purpose of this paper is to solve the w

Theorem 2.1: Let $X, Y \subset \mathcal{M}(\Omega)$ be r.i. Banach function spaces such that $0 < \min$ **rearrangement invariant.**
Theorem 2.1: Let X, $Y \subset \mathcal{M}(\Omega)$ be r.i. Banach function spaces such that $0 < \min \{\beta_X, \beta_Y\} \leq \max \{\alpha_X, \alpha_Y\} < 1$, and $Z \subset \mathcal{M}(\Omega)$ be any Banach, function space. If $W(X, Y) \subset Z$, then Z is rearran

Proof: The idea of the proof consists in reducing the assertion to a particular result of A. P. CALDERÓN [5] for Lebesgue spaces by combining the interpolation theorems of M. Riesz/G. Thorin and of D. W. Boyd: By assumption, there exists a

$$
0<1/q<\min\{\beta_X,\beta_Y\}\leq \max\{\alpha_X,\alpha_Y\}<1.
$$

If we apply the interpolation theorem of $[2]$ twice, namely to the spaces X and Y , respectively, we can conclude that $W(L_1, L_q) \subset (X) \cap (Y)$. Since $(X) \cap (Y) \subset W(X)$, respectively, we can conclude that $W(L_1, L_q) \subset (X) \cap (Y)$. Since $(X) \cap (Y) \subset W(X, X)$, if follows that $W(L_1, L_q) \subset W(X, Y)$. On the other hand, the interpolation If we apply the interpolation theorem of [2] twice, namely to the spaces X and Y,

If we apply the interpolation theorem of [2] twice, namely to the spaces X and Y,

respectively, we can conclude that $W(L_1, L_q) \subset (X) \cap (Y)$ *Leorem of M. Kiesz/G. Thorn yields that* $aa(L_1, L_\infty) \subset [L_q] \subset (L_q)$ *, since* $aa(L_1, L_\infty) \subset (L_1)$ *and, obviously,* $(L_1) \cap (L_q) \subset W(L_1, L_q)$ *, we have* $ad(L_1, L_\infty) \subset W(L_1, L_q)$ *,* L_{∞}) \subset (L_1) and, obviously, (L_1) \cap (L_q) \subset $W(L_1, L_q)$, we have $ad(L_1, L_{\infty}) \subset W(L_1, L_q)$, and therefore finally, $ad(L_1, L_{\infty}) \subset W(X, Y)$. If, by assumption, $W(X, Y) \subset (Z)$, then necessarily $ad(L_1, L_\infty) \subset Z$). So by a theorem of [5] this implies that the space Z theorems of M. Riesz/G. Thorin and of
number $q \in (1, \infty)$ such that
 $0 < 1/q < \min \{\beta_X, \beta_Y\} \leq \max$
If we apply the interpolation theorem
respectively, we can conclude that W
 (Y) , if follows that $W(L_1, L_q) \subset W(X)$,
theorem of

With the above theorem in mind, our **next** aim is to show that the property of rearrangement-invariance is also sufficient for a weak-type interpolation theorem to hold.

3. The Generalized Average Operators

The basic idea of the interpolation theorem to be established is to try to characterize those r.i. spaces which solve the weak-interpolation problem by conditions upon their Boyd indices. As a link between Boyd indices and operators of weak type we now 3. The Generalized Average Operators
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rearrangem The basic idea of the i
those r.i. spaces which
their Boyd indices. As
briefly present two in
studied in detail in [8]
Definition 3.1: L
a) $(P_x f)(t) := \begin{align*}\n\text{subfin} \quad \text{and} \quad \text$ $\begin{aligned} &\text{ators}\ &\text{a} &\text{to} &\text{re} &\text{if}\ &\text{a} &\text{weak-int} &\text{if}\ &\text{e} &\text{in } B\text{odd in} \ &\text{if}\ &\$

 $(f \in \mathscr{M}(\Omega^*), t \in \Omega^*);$

Definition 3.1: Let $X = X_i(\Omega^*) \subset \mathcal{M}(\Omega^*)$ be a r.i. space. Then

$$
\mathbf{a})^{\top}
$$

b)
$$
(Q_Xf)(t) := \frac{1}{\tau_X(t)} \int\limits_t^\infty f(s) d\tau_X(s) \qquad (f \in \mathcal{M}(\Omega^*), t \in \Omega^*).
$$

 $\overline{r_X(t)} \int f^{(s)}$

Note that $\tau_{X_o(\Omega)}(Q) = \tau_{X_\lambda(Q^*)}$ if $X_\lambda(Q^*)$ is the Luxemburg representation of $X_o(Q)$. The operator P_X , used by L. MALIGRANDA [15] in connection with Hardy's inequality, is a generalization of the average operator P_{θ} of [2, 3]; the operator Q_{X} however is quite different to MALIGRANDA's [16] operator Q_{φ} . In the following lemmata we collect those properties of the operators P_X , Q_X which will be used in the sequel. (PxI) (b) $:= \frac{1}{\tau_X(t)} \int f(s) d\tau_X(s)$ $(f \in \mathcal{M}(S^{2+}), t \in S^{2+})$;

b) (Q_Xf) (b) $:= \frac{1}{\tau_X(t)} \int f(s) d\tau_X(s)$ $(f \in \mathcal{M}(\Omega^*), t \in \Omega^*)$.

Note that $\tau_{X_0(\Omega)}(\Omega) = \tau_{X_1(\Omega^*)}$ if $X_1(\Omega^*)$ is the Luxemburg representation of $X_0(\Omega)$.

A Marcinkievicz Interpolation Theorem' 115
 $X = X_{\lambda_1}(Q^*), Y = Y_{\lambda_2}(Q^*), Z = Z_{\lambda_3}(Q^*)$ are r.i. spaces. Lemma 3.1: *Assume that* $X = X_{\lambda_1}(Q^*), Y = Y_{\lambda_2}(Q^*), Z = Z_{\lambda_3}(Q^*)$ are r.i. spaces *a*) *II I I III* **III** *III III A ssume that* $X = X_{1}(Q^*), Y = Y_{1}(Q^*),$
 a) If $0 < \beta_{A(X)} \leq \alpha_{A(X)} < 1$, then for every $f \in Z, g \in Z'$

A Marcinkiewicz Interpolation Theorem' 11.
\nLemma 3.1: Assume that
$$
X = X_{1_1}(\Omega^*), Y = Y_{1_2}(\Omega^*), Z = Z_{1_2}(\Omega^*)
$$
 are r.i. spaces
\na) If $0 < \beta_{A(X)} \le \alpha_{A(X)} < 1$, then for every $f \in Z$, $g \in Z'$
\n
$$
\frac{\beta_{A(X)}}{1 - \beta_A(X)} \int_0^t f^*(t) (Q_{X'} g^*) (t) dt \le \int_0^t (P_X f^*) (t) g^*(t) dt
$$
\n
$$
\le \frac{\alpha_{A(X)}}{1 - \alpha_{A(X)}} \int_t^t f^*(t) (Q_{X'} g^*) (t) dt;
$$
\nb) $(P_X f)^*(t) \le (P_X f^*) (t)$ $(f \in \mathcal{M}(\Omega^*), t \in \Omega^*),$
\n $(Q_X f)^*(t) \le (Q_X f^*) (t)$ $(f \in \mathcal{M}(\Omega^*), t \in \Omega^*);$
\nc) the operator P_X is of weak type (X, X) ; if $\beta_{A(Y)} > 0$ and τ_Y/τ_X is a decreasing function, then P_X is also of weak type (Y, Y) ;
\nd) the operator Q_Y is of weak type (Y, Y) ; if $\beta_{A(X)} > 0$ and τ_Y/τ_X is decreasing, then Q_Y is also of weak type (X, X) ;
\ne) $P_X + Q_Y = S$ where S denotes the Calderón operator, defined by

 $(f\in \mathscr{M}(\varOmega^*), t\in \varOmega^*),$ *(P_xf)*(t)*
(Q_x*f*)*(t) \leq *(I \e M(* Ω^* *), t \e* Ω^* *);*

tion, then P_X *is also of weak type* (Y, Y) ;

d) the operator Q_Y *is of weak type* (Y, Y) ; *if* $\beta_{A(X)} > 0$ *and* τ_Y/τ_X *is decreasing, then* Q_Y *is also of weak type* (X, X) ;

e) $P_X + Q_Y = S$ where S denotes the Calderón operator, defined by

$$
\leq \frac{\alpha_{A(X)}}{1 - \alpha_{A(X)}} \int f^*(t) (Q_{X'} g^*)
$$

\n
$$
\leq (P_X f^*) (t) \qquad (f \in \mathcal{M}(\Omega^*), t \in \Omega^*),
$$

\n
$$
\leq (Q_X f^*) (t) \qquad (f \in \mathcal{M}(\Omega^*), t \in \Omega^*);
$$

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$$
\leq (Q_X f^*) (t) \qquad (f \in \mathcal{M}(\Omega^*), t \in \Omega^*);
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$$
\leq (Q_X f^*) (t) \qquad (f \in \mathcal{M}(\Omega^*), t \in \Omega^*);
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$$
\leq \alpha P_X \text{ is of weak type } (X, X); \text{ if } \beta_{A(Y)} > 0 \text{ and } \tau_Y/\tau_X \text{ is a de}
$$

\n
$$
P_X \text{ is also of weak type } (Y, Y); \text{ if } \beta_{A(X)} > 0 \text{ and } \tau_Y/\tau_X \text{ is a le}
$$

\n
$$
\text{weak type } (X, X);
$$

\n
$$
Q_Y = S \text{ where } S \text{ denotes the Calderon operator, defined by}
$$

\n
$$
(Sf) (t) := \int_0^t f(s) d \left(\min \left\{ \frac{\tau_X(s)}{\tau_X(t)}, \frac{\tau_Y(s)}{\tau_Y(t)} \right\} \right) \quad (f \in \mathcal{M}(\Omega^*), t \in \Omega^*).
$$

\n
$$
\text{proofs of these properties see [8]. In particular, the constant in}
$$

\n
$$
f \text{ and } \beta_{A(X)} \frac{\tau_X(t)}{t} \leq \frac{d\tau_X(t)}{dt} \leq \alpha_{A(X)} \frac{\tau_X(t)}{t},
$$

\n
$$
[10].
$$

\n
$$
\text{Holevo operator } S \text{ obtains its importance for interpolation that}
$$

\n
$$
[16].
$$

For the proofs of these properties see [8]. In particular, the constant in the "duality" relation of a) can be evaluated by recalling that

² proofs of these properties see [8]. In particular, the constant in the "duality"
of a) can be evaluated by recalling that

$$
\beta_{A(X)} \frac{\tau_X(t)}{t} \le \frac{d\tau_X(t)}{dt} \le \alpha_{A(X)} \frac{\tau_X(t)}{t},
$$
(3.1)
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compare [10].

The Calderón operator S obtains its importance for interpolation theory from the facts that (see [20]) $\begin{aligned} &\text{H}_{\text{H}}(\text{see [20]})\ &\text{S}\in W\big(X_{1,}(Q^*),\,Y_{1,}(Q^*)\big),\ &\text{Each}\,\,t\in W\big(X_{e_1}(Q),\,Y_{e_2}(Q^*)\big),\ &\text{and}\,\,Y_{1,}(Q^*)\,\,\text{are}\,\,\text{the}\,\,t\end{aligned}$

$$
S \in W(X_{\lambda_1}(\Omega^*), Y_{\lambda_2}(\Omega^*)), \tag{3.2}
$$

 $\text{and, for each } t \in W(X_{e_1}(\Omega), Y_{e_2}(\Omega)),$

$$
(Tf)^* \leq \text{const.} \, Sf^* \qquad \big(f \in A(X_{\mathfrak{e}_1}(Q)) + A(X_{\mathfrak{e}_1}(Q))\big),\tag{3.3}
$$

(find the first) is see [8]. In particular, the constant in the "duality"
 (find times)
 $\frac{\tau_X(t)}{t}$, (3.1)
 (find times)

(*fi* $\in A(X_{e_1}(\Omega)) + A(X_{e_2}(\Omega))$),
 (find times)
 (find times)
 (find times)
 (fight) If $X_{\lambda_1}(Q^*)$ and $Y_{\lambda_2}(Q^*)$ are the Luxemburg representations of $X_{\rho_1}(Q)$ and $Y_{\rho_2}(Q)$, respectively. On the other hand, the operators P_X and Q_Y are connected with the Boyd indices. In fact, for the case that $X, Z \subset \mathcal{M}(\Omega^*)$ are r.i. spaces of Lebesgue measurable functions on $\Omega^* = (0, l)$ such that $\beta_{A(X)} > 0$, the following holds; see [15], also [1, 3]. $\beta_{A(X)} \xrightarrow{\iota} \leq \dfrac{\gamma_{A(X)}}{t} \leq \alpha_{A(X)} \xrightarrow{\iota}$,

compare [10].

The Calderón operator *S* obtains its importance for interpretation (see [20])
 $S \in W(X_{\iota_1}(Q^*), Y_{\iota_2}(Q^*))$,

and, for each $t \in W(X_{\iota_1}(Q), Y_{\iota_2}(Q))$,
 $(T$ If α *LA* ℓ *LA zt. z*^{*} $\{f \in A(X_{e_1}(\Omega)) + A(X_{e_2}(\Omega))\}$,
 z^{*}) are the Luxemburg representations of $X_{e_1}(\Omega)$ and
 z, then hand, the operators P_X and Q_Y are connected v
 z, for the case that $X, Z \subset A(\Omega^*)$ are r.i. spaces

If
$$
\alpha_Z < \beta_{A(X)}
$$
, then $P_X \in [Z]$; if $P_X \in [Z]$, then $\alpha_Z \leq \alpha_{A(X)}$. (3.4)

An analogous assertion for the operator Q_r , which is not contained in [15], can be deduced from (3.4) by duality arguments, using Lemma 31a). Here we assume that $Y \subset \mathcal{M}(\Omega^*)$ is a r.i. space with $0 < \beta_{A(Y)} \leq \alpha_{A(Y)} < 1$. Then:

$$
If \alpha_{A(Y)} < \beta_Z, then Q_Y \in [Z]; \text{ if } Q_Y \in [Z], then \beta_{A(X)} < \beta_Z. \tag{3.5}
$$

For the proof recall that $\alpha_{X'} = 1 - \beta_X$ and $\beta_{X'} = 1 - \alpha_X$ for any r.i. space X, note that $A(X)' = M(Y')$, as well as $\tau_{M(Y')} = \tau_{A(Y')}$; see [21, 10]. Hence $\beta_{M(Y')} = \beta_{A(Y')}$, and

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116 F. FEHER
we can argue as follows: If $\alpha_{A(Y)} < \beta_Z$, then $1 - \beta_{A(Y)} = 1 - \beta_{M(Y')} = 1 - \beta_{A(Y')}$
= $\alpha_{A(Y)} < \beta_Z = 1 - \alpha_Z$, i.e. $\alpha_{Z'} < \beta_{A(Y')}$. So by (3.4) we have $P_{Y'} \in [Z']$, this being equivalent to $Q_Y \in [Z]$ on account of Lemma 3.1a). The second part of (3.5) is proved similarly. 6 F. FERER

9. can argue as follows: 1
 $\alpha_{A(Y)} < \beta_Z = 1 - \alpha_Z$, i.d.

uivalent to $Q_Y \in [Z]$ on a

milarly.

Concerning the operator

Lemma 3.2:

If $\alpha_Z < \beta_{A(X)}$, then $||P_X||$ 116 F. FERER
 we can argue as follows: If $\alpha_{A(Y)} < \beta_Z$, then $1 - \beta_{A(Y')} = \alpha_{A(Y)} < \beta_Z = 1 - \alpha_Z$, i.e. $\alpha_{Z'} < \beta_{A(Y')}$. So by (3.4) we have

equivalent to $Q_Y \in [Z]$ on account of Lemma 3.1a). The second is

similarly.

Concerni

Concerning the operator norms of P_Y and Q_X we have the following facts. I

equivalent to
$$
Q_Y \in [Z]
$$
 on account of Lemma 3.1a). The second part
similarly.
Concerning the operator norms of P_Y and Q_X we have the follow
Lemma 3.2:
a) If $\alpha_Z < \beta_{A(X)}$, then $||P_X||_{[Z]} \le \alpha_{A(X)} \int_0^1 ||E_s||_{[Z]} M(s, X) \frac{ds}{s} < \infty$;

b) *if* $\alpha_{A(Y)} < \beta_Z$, *then* $||P_X||_{\{Z\}} \leq \alpha_{A(X)} \int_0^1 ||E_s||_{\{Z\}} M(s, X) \frac{ds}{s}$
 b) if $\alpha_{A(Y)} < \beta_Z$, *then* $||Q_Y||_{\{Z\}} \leq \alpha_{A(X)} \int_0^1 ||E_s||_{\{Z\}} M(s, X) \frac{ds}{s}$
 b) if $\alpha_{A(Y)} < \beta_Z$, *then* $||Q_Y||_{\{Z\}} \leq \alpha_{A(Y)} \int_0^{\$ $< \infty$.

Here M(*s*, *X*) : = sup₈ $s_i \in \mathcal{Q} \cdot [\tau_X(st)/\tau_X(t)]$ and *M*(*s*, *Y*) analogously.

Proof: A weaker version of part a) — without the factor α_{A}

[16]: We confine ourselves to b), the proof of a) being similar.
 f Proof: A weaker version of part a) – without the factor $x_{A(X)}$ – was proved in [16]; We confine ourselves to b), the proof of a) being similar. For this purpose, let

Lemma 3.2:
\na) If
$$
\alpha_{Z} < \beta_{A(X)}
$$
, then $||P_X||_{|Z|} \leq \alpha_{A(X)} \int_0^1 ||E_s||_{|Z|} M(s, X) \frac{ds}{s} < \infty$;
\nb) if $\alpha_{A(Y)} < \beta_{Z}$, then $||Q_Y||_{|Z|} \leq \alpha_{A(Y)} \int_0^1 ||E_s||_{|Z|} M(s, Y) \frac{ds}{s} < \infty$.
\nHere $M(s, X) := \sup_{B,s,t \in \Omega^c} [\tau_X(st)/\tau_X(t)]$ and $M(s, Y)$ analogously.
\nProof: A weaker version of part a) — without the factor $\alpha_{A(X)}$ — was proved in [16]: We confine ourselves to b), the proof of a) being similar. For this purpose, let $t \in Z$, $g \in Z'$ such that $g \ge 0$, $||g||_{Z'} = 1$. Because of (3.1)
\n
$$
\int_0^t (Q_Y|f|) (t) g(t) dt \leq \alpha_{A(Y)} \int_0^1 \int_1^t |f(s)| \frac{\tau_Y(st)}{\tau_Y(t)} g(t) \frac{ds}{s} dt
$$
\n
$$
= \alpha_{A(Y)} \int_0^1 \int_1^t |f(st)| \frac{\tau_Y(st)}{\tau_Y(t)} g(t) \frac{ds}{s} dt
$$
\n
$$
\leq \alpha_{A(Y)} \int_0^1 \int_1^t |f(st)| \frac{\tau_Y(st)}{\tau_Y(t)} g(t) \frac{ds}{s} dt
$$
\n
$$
\leq \alpha_{A(Y)} \int_0^1 \int_1^t |E_s| f|) (t) M(s, Y) \frac{ds}{s} g(t) dt,
$$
\nyielding b), since $f \in Z$, $g \in Z'$ are arbitrary \mathbb{I}
\nBy means of the method of applying the Luxemburg representation, the assertions (3.4), (3.5), and Lemma 3.2 can be transferred to the more general situation of r. spaces in $\mathcal{M}(2)$.
\n π , spaces with $\beta_{A(X)} > 0$ and $0 < \beta_{A(Y)} \leq \alpha_{A(Y)} < 1$. Let $X_{1,1}(Q^*)$, $Y_{1,1}(Q^*)$, and $Z_{1,1}(Q^*)$ <

By means of the method of applying the Luxemburg representation, the assertions (3.4) , (3.5) , and Lemma 3.2 can be transferred to the more general situation of r.i. spaces in $\mathcal{M}(\Omega)$.

Theorem 3.3: Assume that $X = X_{q_1}(Q)$, $Y = Y_{q_2}(Q)$, $Z = Z_{q_3}(Q) \subset \mathcal{M}(Q)$ are $r.i. spaces with \ \beta_{A(X)} > 0 \ and \ 0 < \beta_{A(Y)} \leq \alpha_{A(Y)} < 1.$ Let $X_{A}(Q^*), Y_{A}(Q^*),$ and $Z_{A}(Q^*)$

a) If $\alpha_Z < \beta_{A(X)}$, then for every $f \in Z_{\varrho_s}(\Omega)$

yielding b), since
$$
f \in Z
$$
, $g \in Z'$ are arbitrary
\nBy means of the method of applying the Luxemburg representation, the assertions
\n(3.4), (3.5), and Lemma 3.2 can be transferred to the more general situation of r.i.
\nspaces in $\mathcal{M}(\Omega)$.
\nTheorem 3.3: Assume that $X = X_{c_1}(\Omega)$, $Y = Y_{c_2}(\Omega)$, $Z = Z_{c_1}(\Omega) \subset \mathcal{M}(\Omega)$ are
\nr.i. spaces with $\beta_{A(X)} > 0$ and $0 < \beta_{A(Y)} \leq \alpha_{A(Y)} < 1$. Let $X_{A_1}(\Omega^*)$, $Y_{A_1}(\Omega^*)$, and $Z_{A_2}(\Omega^*)$
\ndenote the Luxemburg representations of X, Y, and Z respectively.
\na) If $\alpha_Z < \beta_{A(X)}$, then for every $f \in Z_{c_2}(\Omega)$
\n $||P_Xf^*||_{Z_{A_2}(\Omega^*)}$ $\left(\alpha_{A(X)} \int_0^1 ||E_s||_{Z_{A_2}(\Omega^*)} \right) M(s, X) \frac{ds}{s}\right) ||f^*||_{Z_{A_2}(\Omega^*)}$; (3.6)
\nconversely, if (3.6) holds for every $f \in Z_{c_2}(\Omega)$, then $\alpha_Z < \alpha_{A(X)}$.
\nb) If $\alpha_{A(Y)} < \beta_Z$, then for every $f \in Z_{c_2}(\Omega)$
\n $||Q_Yf^*||_{Z_{A_2}(\Omega^*)} \leq \left(\alpha_{A(Y)} \int_1^{\infty} ||E_s||_{Z_{A_2}(\Omega^*)} \right) M(s, Y) \frac{ds}{s}\right) ||f^*||_{Z_{A_2}(\Omega^*)}$; (3.7)
\nconversely, if (3.7) holds for every $f \in Z_{c_2}(\Omega)$, then $\beta_{A(Y)} < \beta_Z$.
\n(3.7)

$$
\|\hat{Q}_Y f^*\|_{Z_{\lambda_{\mathbf{s}}}(\Omega^{\bullet})} \leqq \left(\alpha_{A(Y)} \int\limits_{1}^{\infty} \|E_s\|_{[Z_{\lambda_{\mathbf{s}}}(\Omega^{\bullet})]} M(s, Y) \frac{ds}{s}\right) \|f^*\|_{Z_{\lambda_{\mathbf{s}}}(\Omega^{\bullet})};\tag{3.7}
$$

conversely, if (3.7) holds for every $f \in Z_{\rho_a}(\Omega)$, then $\beta_{A(Y)} < \beta_Z$.

• Proof: Note that the indices of a r.i. space coincide with the indices of its Luxemburg representation and, moreover, that inequality (3.6) for all $f \in Z_{e_0}(\Omega)$ is equivalent to $\tilde{P}_X \in [Z_{\lambda_1}(\Omega^*))$, in view of Lemma 3.1 b); similarly inequality (3.7) for all $f \in Z_{\lambda_1}(\Omega)$ is equivalent to $Q_Y \in [Z_{\lambda_1}(\Omega^*))$ **is equivalent to Ay Equivalent to Qy E [Z₄(Q*)]) a**
 is equivalent to C_X E (Z₄(Q*)], in view of Lemma 3.1b); similarly (3.6) for all $f \in Z_{e_1}(\Omega)$ is equivalent to $P_X \in [Z_{e_1}(\Omega^*)]$, in view of Lemma 3.1b); si **• Proof:** Note that the indices of a r.i. space coil

burg representation and, moreover, that inequality

to $P_X \in [Z_{l_2}(Q^*)]$, in view of Lemma 3.1b); simila

is equivalent to $Q_Y \in [Z_{l_2}(Q^*)]$
 4. The Interpolation

4. The Interpolation Theorem for Rearrangement-Invariant Spaces

Now we are ready to prove the following weak-type interpolation theorem.

Theorem 4.1: Assume that $X = X_{q_1}(Q)$, $Y = Y_{q_2}(Q)$, $Z = Z_{q_3}(Q) \subset \mathcal{M}(Q)$ are *4. The Interpolation Theorem for Hearrangement-Invariant spaces such that* $X \equiv X_{\rho_1}(\Omega)$ *,* $Y \equiv Y_{\rho_2}(\Omega)$ *,* $Z \equiv r.i.$ *spaces such that* $Z \subset A(X) + A(Y)$ *,* $\beta_{A(X)} > 0$ *,* $0 < \beta_{A(Y)} \leq c$ *is decreasing.* $x_{A(Y)} \leq 1$, and τ_Y/τ_X **4.** The Interpolation Theorem for Rearrangement-Invariant S

Now we are ready to prove the following weak-type interpol

Theorem 4.1: *Assume that* $X = X_{c_1}(\Omega)$, $Y = Y_{c_1}(\Omega)$, 2
 r.i. spaces such that $Z \subset A(X) + A(Y)$,

is decreasing.

a) *If* $\alpha_{A(Y)} < \beta_Z \leq \alpha_Z < \beta_{A(X)}$, then $W(X, Y) \subset (Z)$;

b) *If* $W(X, Y) \subset (Z)$, then $\beta_{A(Y)} < \beta_Z \leq \alpha_Z < \alpha_{A(X)}$.

Proof: a) Let $T \in W(X, Y)$ and $f \in Z$ be given. As above denote by $X_{\lambda}(\Omega^*),$ Proof: a) Let $T \in W(X, Y)$ and $f \in Z$ be given. As above denote by $X_{\lambda_1}(\Omega^*),$
 $Y_{\lambda_2}(\Omega^*)$, $Z_{\lambda_3}(\Omega^*)$ the Luxemburg representations of X, Y, Z, respectively. Since $f \in Z$, we have $f^* \in Z_{\lambda_2}(\Omega^*) \subset \mathcal{M}(\Omega^*)$ an we have $f^* \in Z_{\lambda_1}(Q^*) \subset \mathcal{M}(Q^*)$ and, by Lemma 3.1 e), $Sf^* = P_Xf^* + Q_Yf^*$. If the index condition of a) holds, we can apply Lemma 3.2 to conclude that

$$
||Sf^*||_{Z_{\lambda_1}(\Omega^*)} \leq (||P_X||_{[Z_{\lambda_1}(\Omega^*)]} + ||Q_Y||_{[Z_{\lambda_1}(\Omega^*)]}) ||f^*||_{Z_{\lambda_1}(\Omega^*)} < \infty.
$$

we have $f^* \in Z_{\lambda_1}(Q^*) \subset \mathcal{M}(Q^*)$ and, by Lemma 3.1 e), $Sf^* = P_X f^* + Q_Y f^*$. If the index condition of a) holds, we can apply Lemma 3.2 to conclude that $|Sf^*||_{Z_{\lambda_1}(Q^*)} \leq (||P_X||_{[Z_{\lambda_1}(Q^*)]} + ||Q_Y||_{[Z_{\lambda_1}(Q^*)]}) ||f^*||_{Z$ order to show that $j \in A(X_{e_1}(\Omega)) + A(Y_{e_2}(\Omega))$, we consider the norm of f^* in $A(X_{\lambda}(Q^*)) + A(Y_{\lambda}(Q^*))$. By definition of the norm of a sum of Banach spaces, This means,

where, since

order to sh
 $4(X_{\lambda_1}(Q^*))$
 $||f^*||$
 \cdots
 \cdots $||Sf^*||_{Z_{\lambda_1}(Q^*)} \leq (||P_X||_{[Z_{\lambda_1}(Q^*)]} + ||Q_Y||_{[Z_{\lambda_1}(Q^*)}] \ ||f^*||_{Z_{\lambda_1}(Q^*)} < \infty.$

This means, in particular, that $Sf^* \in Z_{\lambda_1}(Q^*)$, and hence (Sf^*) $(t) < \infty$ almowhere, since λ_3 is a r.i. norm. This implies that

Z₁,
$$
(Q^*)
$$
], in view of Lemma 3.1b); similarly inequality (3.7) for all
ent to $Q_Y \in [Z_{\lambda_1}(Q^*)]$ ■
terpolation Theorem for Rearrangement-Invariant Spaces
are ready to prove the following weak-type interpolation theorem
em 4.1: Assume that $X \equiv X_o(Q)$, $Y \equiv Y_o(Q)$, $Z \equiv Z_o(Q) \subset$.
s such that $Z \subset A(X) + A(Y)$, $\beta_{A(X)} > 0$, $0 < \beta_{A(Y)} \le \alpha_{A(Y)} < 1$,
ing.
 $\rho_Y < \beta_Z \le \alpha_Z < \beta_{A(X)}$, then $W(X, Y) \subset (Z)$;
 $X, Y) \subset (Z)$, then $\beta_{A(Y)} < \beta_Z \le \alpha_Z < \alpha_{A(X)}$.
: a) Let $T \in W(X, Y)$ and $f \in Z$ be given. As above denote by
Z₁, (Q^*) the Luxemburg representations of X, Y, Z, respectively. Si
 $f^* \in Z_{\lambda_1}(Q^*)$ the Luxemburg representations of X, Y, Z, respectively. Si
 $f^* \in Z_{\lambda_2}(Q^*) \subset \mathcal{M}(Q^*)$ and, by Lemma 3.1 e), $Sf^* = P_Xf^* + Q_Y$
addition of a) holds, we can apply Lemma 3.2 to conclude that
 $||Sf^*||_{Z_{\lambda_4}(Q^*)} \le (||P_X||_{[Z_{\lambda_4}(Q^*)]} + ||Q_Y||_{[Z_{\lambda_4}(Q^*)]}) ||f^*||_{Z_{\lambda_4}(Q^*)} < \infty$.
ans, in particular, that $Sf^* \in Z_1, (Q^*)$, and hence $(Sf^*) (t) < \infty$ all
nce λ_3 is a r.i. norm. This implies that $f^* \in \mathcal{A}(X_{\lambda_1}(Q^*)) + \mathcal{A}(Y$
show that $f \in \mathcal{A}(X_o(Q)) + \mathcal{A}(Y_{e_1}(Q))$, we consider the norm
 $||f^*||_{\mathcal{A}(X_{\lambda_4}(Q^*))+\mathcal{A}(Y_{\lambda_4}(Q^*))} = inf \{||g_1||_{\mathcal{A}(X_{\lambda_4}(Q^$

According to [5] there exists a measure preserving transformation $f^* \to f$ from $\mathcal{M}(\Omega^*)$ to $\mathcal{M}(\Omega)$ such that for each decomposition $f^* = g_1 + g_2$ with g_1, g_2 as above, there exist functions $f_1 \in A(X_{e_1}(\Omega)), f_2 \in \tilde{A}(Y_{e_1}(\Omega))$ with $f_1^* = g_1, f_2^* = g_2$ and $f = f_1 + f_2$.
Hence III* IIA(X(Q*))4(Y()) • . - , $|A(X_{\lambda_1}(\Omega^*))+A(Y_{\lambda_2}(\Omega^*))|$ = inf {||g₁|| $A(X_{\lambda_1}(\Omega^*))$ + ||g₂|| $A(Y_{\lambda_2}(\Omega^*))$:
 $f^* = g_1 + g_2, g_1 \in A((X_{\lambda_1}\Omega^*))$, $g_2 \in A(Y_{\lambda_1}(\Omega^*))$,
 $g_1 \geq 0, g_2 \geq 0$ }.

[5] there exists a measure preserving transformation f

$$
||f^*||_{A(\hat{X}_{\lambda_1}(\Omega^*))+A(Y_{\lambda_1}(\Omega^*))} = \inf \{ ||g_1||_{A(X_{\lambda_1}(\Omega^*))} + ||g_2||_{A(Y_{\lambda_1}(\Omega^*))} \}
$$
\n
$$
f^* = g_1 + g_2, g_1 \in A((X_{\lambda_1}Q^*)), g_2 \in A(Y_{\lambda_1}(\Omega^*))
$$
\n
$$
g_1 \geq 0, g_2 \geq 0 \}.
$$
\nAccording to [5] there exists a measure preserving transformation $f^* \to f$ from $A(\Omega)$ to $A(\Omega)$ such that for each decomposition $f^* = g_1 + g_2$ with g_1, g_2 as above, then exist functions $f_1 \in A(X_{e_1}(\Omega)), f_2 \in A(Y_{e_1}(\Omega))$ with $f_1^* = g_1, f_2^* = g_2$ and $f = f_1 + f$.
\nHence\n
$$
||f^*||_{A(X_{\lambda_1}(\Omega^*))} + A(Y_{\lambda_1}(\Omega^*))
$$
\n
$$
= \inf \{ ||f_1^*||_{A(Y_{\lambda_1}(\Omega^*))} + ||f_2^*||_{A(Y_{\lambda_1}(\Omega^*))} \}
$$
\n
$$
= |f_1| + f_2, f_1 \in A(X_{e_1}(\Omega)), f_2 \in A(Y_{e_1}(\Omega)) \}
$$
\n
$$
= ||f||_{A(X_{e_1}(\Omega)) + A(X_{e_1}(\Omega)),
$$
\nand therefore (3:3) can be applied to yielding that\n
$$
||Tf||_{Z_{e_1}(\Omega)} = ||(Tf)_e^*||_{Z_{\lambda_1}(\Omega^*)} \leq \text{const. } ||Sf^*||_{Z_{\lambda_1}(\Omega^*)}
$$
\n
$$
\leq \text{const. } ||f||_{Z_{e_1}(\Omega^*)} + ||Q_V||_{[Z_{\lambda_1}(\Omega^*)}||f^*||_{Z_{\lambda_2}(\Omega^*)}
$$
\n
$$
= \text{const. } ||f||_{Z_{e_1}(\Omega)} < \infty.
$$
\nTherefore we finally have $Tf \in Z_{e_1}(\Omega)$ and $||T||_{[Z_{e_1}(\Omega)]} < \infty$

This shows that $f \in A(X_{e_1}(\Omega)) + A(X_{e_2}(\Omega))$, and therefore (3.3) can be applied to f, $\frac{1}{z_{\varrho_{\mathfrak{s}}}(\varrho)}$

$$
||Tf||_{Z_{\rho_s}}(a) = ||(Tf)_{\circ}^*||_{Z_{\lambda_s}(\Omega^*)} \leq \text{const. } ||Sf^*||_{Z_{\lambda_s}(\Omega^*)}
$$

$$
\leq
$$
 const. $(||P_X||_{[Z_{\lambda_1}(Q^*)]} + ||Q_Y||_{[Z_{\lambda_1}(Q^*)]}) ||f^*||_{Z_{\lambda_3}(Q^*)}$

Therefore we finally have $Tf \in Z_{\varrho_{\mathbf{s}}}(\Omega)$ and $||T||_{[Z_{\varrho_{\mathbf{s}}}(\Omega)]} < \infty$.

118 F. F_{EHÉR}

b) Since the σ -finite measure space (Ω, Σ, μ) was assumed to be nonatomic, there exists a measure preserving transformation $\pi: \Omega \to \Omega^*$. By means of this transformation π the operators P_X and Q_Y on $\mathcal{M}(\Omega^*)$ can be transferred into operators, say \mathbf{P}_x and \mathbf{Q}_Y , on $\mathcal{M}(\Omega)$. For $v \in \Omega$ we introduce two kernels k_1 and k_2 by. inite measure space (Ω, Σ)

oreserving transformation

ators P_X and Q_Y on $\mathcal{M}(\Omega)$
 Ω). For $v \in \Omega$ we introducted
 $\begin{cases} \tau_X(s)/s & \text{if } v \in \pi^{-1}(\{s\}) \\ 0 & \text{elsewhere,} \end{cases}$ (Q, Σ, μ) was assuation $\pi : \Omega \to \Omega^*$
 $\mathscr{M}(\Omega^*)$ can be transformed that $\mathscr{M}(\Omega^*)$
 $k_2(v) := \Omega$ Finite measure space (Ω, Σ, μ) was assumed to be nonatomic, the preserving transformation $\pi : \Omega \to \Omega^*$. By means of this trans ators P_X and Q_Y on $\mathcal{M}(\Omega^*)$ can be transferred into operators (Ω) . For $v \in \Omega$ we HER
 σ -finite measure space (Ω, Σ, μ) was assumed to luminon $\pi : \Omega \to \Omega^*$. By measured for $\mathcal{M}(\Omega)$. For $v \in \Omega$ we introduce two kernels k_1 and
 $\mathcal{M}(\Omega)$. For $v \in \Omega$ we introduce two kernels k_1 and
 $\$ $\begin{align*} \text{1}\ \text{1}\ \text{1}\ \text{2}\ \text{1}\ \text{2}\ \text{2}\ \text{2}\ \text{2}\ \text{2}\ \text{3}\ \text{3}\ \text{4}\ \text{4}\ \text{5}\ \text{6}\ \text{6}\ \text{7}\ \text{7}\ \text{8}\ \text{8}\ \text{9}\ \text{1}\ \text{1}\ \text{1}\ \text{1}\ \text{2}\ \text{2}\ \text{2}\ \text{2}\ \text{3}\ \text{3}\ \text{4}\ \text{4}\ \text{5}\ \text{5}\ \text{6}\ \text{6}\ \text{6}\ \text{6}\ \text{7}\ \text{7}\ \text{8$ $\begin{array}{r} \mathbf{ex} \\ \mathbf{ex} \\ \mathbf{me} \\ \mathbf{P}_x \end{array}$

$$
k_1(v) := \begin{cases} \tau_X(s)/s & \text{if } v \in \pi^{-1}(\{s\}) \\ 0 & \text{elsewhere,} \end{cases} \qquad k_2(v) := \begin{cases} \tau_Y(s)/s & \text{if } v \in \pi^{-1}(\{s\}) \\ 0 & \text{elsewhere,} \end{cases}
$$

and then define the operators $\mathbf{P}_\mathcal{X}$ and \mathbf{Q}_Y for $f \in \mathscr{M}(\varOmega)$, $v \in \varOmega$ by

b) Since the
$$
\sigma
$$
-finite measure space (Ω, Σ, μ) was assumed to be no
exists a measure preserving transformation $\pi : \Omega \to \Omega^*$. By means of
mation π the operators P_X and Q_Y on $\mathcal{M}(\Omega^*)$ can be transferred into
 P_X and Q_Y , on $\mathcal{M}(\Omega)$. For $v \in \Omega$ we introduce two kernels k_1 and k_2 by

$$
k_1(v) := \begin{cases} \tau_X(s)/s & \text{if } v \in \pi^{-1}(\{s\}) \\ 0 & \text{elsewhere,} \end{cases}
$$
and Q_Y for $f \in \mathcal{M}(\Omega)$, $v \in \Omega$ by
and then define the operators P_X and Q_Y for $f \in \mathcal{M}(\Omega)$, $v \in \Omega$ by

$$
(P_Xf)(v) := \begin{cases} \frac{1}{\tau_X(t)} \int_{\pi^{-1}(0, t)} f k_1 d\mu & \text{if } v \in \pi^{-1}(\{t\}) \end{cases}
$$

$$
(Q_Yf)(v) := \begin{cases} \frac{1}{\tau_Y(t)} \int_{\pi^{-1}(t, t)} f k_2 d\mu & \text{if } v \in \pi^{-1}(\{t\}) \end{cases}
$$

$$
(Q_Yf)(v) := \begin{cases} \frac{1}{\tau_Y(t)} \int_{\pi^{-1}(t, t)} f k_2 d\mu & \text{if } v \in \pi^{-1}(\{t\}) \end{cases}
$$
elsewhere.
In case $\Omega = \Omega^*$, the operators P_X and Q_Y are equivalent to P_X and Q_X
In the general case

$$
(P_X\bar{f})^*(t) \ge \frac{1}{\alpha_{\mathcal{A}(X)}} (P_Xf^*)(t) \qquad (f \in \mathcal{M}(\Omega), t \in \Omega^*)
$$
,

$$
(Q_Y\bar{f})^*(t) \ge \frac{1}{\alpha_{\mathcal{A}(X)}} (Q_Yf^*)(t) \qquad (f \in \mathcal{M}(\Omega), t \in \Omega^*)
$$
,
where $\bar{f}(v) := f^*(s)$ if v

In case $\varOmega=\varOmega$ *, the operators \mathbf{P}_X and \mathbf{Q}_Y are equivalent to P_X and Q_X respectively.

$$
\begin{aligned}\n\begin{cases}\n0 & \text{elsewhere.} \\
0 & \text{elsewhere.}\n\end{cases} \\
\text{where.} \\
\Omega = \Omega^*, \text{ the operators } P_X \text{ and } Q_Y \text{ are equivalent to } P_X \text{ and } Q_X \text{ respectively.}\n\end{aligned}
$$
\n
$$
(\mathbf{P}_X \tilde{f})^* (t) \ge \frac{1}{\alpha_{A(X)}} (P_X f^*) (t) \qquad (f \in \mathcal{M}(\Omega), t \in \Omega^*), \tag{4.1}
$$

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$$
(\mathbf{Q}_Y \bar{f})^* (t) \geq \frac{1}{\alpha_{A(Y)}} (Q_Y f^*) (t) \qquad (f \in \mathcal{M}(\Omega), t \in \Omega^*), \qquad (4.2)
$$

S
S where $\bar{f}(v) := f^*(s)$ if $v \in \pi^{-1}(\{s\})$, and $\bar{f}(v) = 0$ elsewhere. Indeed since $(\bar{f})^* = f^*$ because of the measure preserving property of π , we can estimate $(\mathbf{P}_i)^*$ from below by

In the general case
\n
$$
(\mathbf{P}_A \tilde{f})^* (t) \geq \frac{1}{\alpha_{A(X)}} (P_X f^*) (t) \qquad (f \in \mathcal{M}(\Omega), t \in \Omega^*),
$$
\n
$$
(\mathbf{Q}_Y \tilde{f})^* (t) \geq \frac{1}{\alpha_{A(Y)}} (Q_Y f^*) (t) \qquad (f \in \mathcal{M}(\Omega), t \in \Omega^*),
$$
\n
$$
(\mathbf{Q}_Y \tilde{f})^* (t) \geq \frac{1}{\alpha_{A(Y)}} (Q_Y f^*) (t) \qquad (f \in \mathcal{M}(\Omega), t \in \Omega^*),
$$
\n
$$
\text{where } \tilde{f}(v) := f^*(s) \text{ if } v \in \pi^{-1}(\{s\}), \text{ and } \tilde{f}(v) = 0 \text{ elsewhere. Indeed since } (\tilde{f})^* = f^* \text{ because of the measure preserving property of } \pi, \text{ we can estimate } (\mathbf{P}_X f)^* \text{ from below by}
$$
\n
$$
\cdot \qquad (\mathbf{P}_X \tilde{f})^* (t) = \left\{ \frac{1}{\tau_X(\cdot)} \int_{0}^{t} f^*(s) \frac{\tau_X(s)}{s} ds \right\}^* (t) \geq \left\{ \frac{1}{\alpha_{A(X)}} (P_X f^*) \right\}^* (t)
$$
\n
$$
= \frac{1}{\alpha_{A(X)}} (P_X f^*) (t).
$$
\nHere we used (3.1) and the fact that $P_X f^*$ is decreasing (see [8]). Inequality (4.2) is proved analogously. Note that $\alpha_{A(X)} > 0$ and $\alpha_{A(Y)} > 0$ by assumption.
\nConversely, it can similarly be shown that
\n
$$
(\mathbf{P}_X f)^* (t) \leq \frac{1}{\beta_{A(X)}} (P_X g)^* (t) \qquad (f \in \mathcal{M}(\Omega), t \in \Omega^*),
$$
\n
$$
(\mathbf{Q}_Y f)^* (t) \leq \frac{1}{\beta_{A(Y)}} (Q_Y g)^* (t) \qquad (f \in \mathcal{M}(\Omega), t \in \Omega^*),
$$
\n
$$
\text{with } g(t) := f(v) \text{ if } v \in \pi^{-1}(\{t\}), \text{and } g
$$

Here we used (3.1) and the fact that P_Xf^* is decreasing (see [8]). Inequality (4.2) is proved analogously. Note that $\alpha_{A(X)} > 0$ and $\alpha_{A(Y)} > 0$ by assumption. used (3.1) and the fact that P_X
nalogously. Note that $\alpha_{A(X)} > 0$
sely, it can similarly be shown t
 $(P_X f)^*$ (t) $\leq \frac{1}{\beta_{A(X)}} (P_X g)^*$ (t)

$$
(P_Xf)^*(t) \leq \frac{1}{\beta_{A(X)}} (P_Xg)^*(t) \qquad \big(f \in \mathscr{M}(\Omega), t \in \Omega^*\big), \tag{4.3}
$$

Proved analogously. Note that
$$
\alpha_{A(X)} > 0
$$
 and $\alpha_{A(Y)} > 0$ by assumption.

\nConversely, it can similarly be shown that

\n
$$
(\mathbf{P}_X f)^* (t) \leqq \frac{1}{\beta_{A(X)}} (P_X g)^* (t) \quad \text{if } t \in \mathcal{M}(\Omega), t \in \Omega^* \text{,}
$$
\n
$$
(\mathbf{Q}_Y f)^* (t) \leqq \frac{1}{\beta_{A(Y)}} (Q_Y g)^* (t) \quad \text{if } t \in \mathcal{M}(\Omega), t \in \Omega^* \text{,}
$$
\n
$$
(4.4)
$$

with $g(t) := f(v)$ if $v \in \pi^{-1}(\{t\})$, and $g(t) = 0$ elsewhere.

Next we benefit from the fact that P_X , as stated in Lemma 3.1c) is of weak type $(X_{\lambda_1}(Q^*), X_{\lambda_1}(Q^*))$, in order to show that the new operator P_X on $\mathcal{M}(\Omega)$ is of weak type $(X_{e_i}(Q), X_{e_i}(Q))$. In fact, multiplying (4.3) by $\tau_X(t)$ and passing to the supremum, we

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A Marcinkiewicz Interpolation Theorem 119
\nhave by Lemma 3.1b) and c) that
\n
$$
\sup_{t \in \Omega^*} (\mathbf{P}_X f)^* (t) \tau_X(t) \leq \frac{1}{\beta_{A(X)}} \sup_{t \in \Omega^*} (P_X g)^* (t) \tau_X(t)
$$
\n
$$
\leq \frac{1}{\beta_{A(X)}} \sup_{t \in \Omega^*} (P_X g)^* (t) \tau_X(t)
$$
\n
$$
\leq \frac{1}{\beta_{A(X)}} \sup_{t \in \Omega^*} (P_X g)^* (t) \tau_X(t)
$$
\n
$$
\leq \frac{\text{const.}}{\beta_{A(X)}} ||g^*||_{A(X_{\lambda_1}(\Omega^*))} = \frac{\text{const.}}{\beta_{A(X)}} ||f||_{A(X_{\rho_1}(\Omega))},
$$
\nsince $g^* = f^*$, and $X_{\lambda_1}(\Omega^*)$ is the Luxemburg representation of $X_{\rho_1}(\Omega)$. If we multiply (4.3) by $\tau_Y(t)$ instead of $\tau_X(t)$, an analogous calculation leads to
\n
$$
\sup_{t \in \Omega^*} (P_X f)^* (t) \tau_Y(t) \leq \frac{\text{const.}}{\beta_{A(X)}} ||f||_{A(Y_{\rho_1}(\Omega))}.
$$
\nStarting with (4.4) instead of (4.3), one can similarly show that

since $g^* = f^*$, and $X_{\lambda_i}(Q^*)$ is the Luxemburg representation of $X_{\rho_i}(Q)$. If we multiply (4.3) by $\tau_Y(t)$ instead of $\tau_X(t)$, an analogous calculation leads to

$$
\sup_{t\in\Omega^*} \left(\mathbf{P}_x f\right)^{\neq}(t) \ \tau_Y(t) \leq \frac{\text{const.}}{\beta_{A(X)}} \ \Vert f \Vert_{A(Y_{\varrho_a}(Q))}.
$$

Starting with (4.4) instead of (4.3), one can similarly show that

$$
\mu_{A(X)} \in \mathcal{F}
$$
\n
$$
\leq \frac{\text{const.}}{\beta_{A(X)}} ||g^*||_{A(X_{\ell_1}(Q^*))} = \frac{\text{const.}}{\beta_{A(X)}} ||f||_{A(X_{\ell_1}(Q))},
$$
\nsince $g^* = f^*$, and $X_{\ell_1}(Q^*)$ is the Luxemburg representation of $X_{\ell_1}(Q)$. I
\ntiply (4.3) by $\tau_Y(t)$ instead of $\tau_X(t)$, an analogous calculation leads to
\n
$$
\sup_{t \in Q^*} (P_X f)^*(t) \tau_Y(t) \leq \frac{\text{const.}}{\beta_{A(X)}} ||f||_{A(Y_{\ell_1}(Q))}.
$$
\nStarting with (4.4) instead of (4.3), one can similarly show that
\n
$$
\sup_{t \in Q^*} (Q_Y f)^*(t) \tau_X(t) \leq \frac{\text{const.}}{\beta_{A(Y)}} ||f||_{A(Y_{\ell_1}(Q))},
$$
\n
$$
\sup_{t \in Q^*} (Q_Y f)^*(t) \tau_Y(t) \leq \frac{\text{const.}}{\beta_{A(Y)}} ||f||_{A(Y_{\ell_1}(Q))}.
$$
\nCollecting all these estimates we have that $P_X, Q_Y \in W(X_{\ell_1}(Q), Y_{\ell_1}(Q))$,
\n $P_X, Q_Y \in (Z)$, on account of the assumption upon Z .
\nNow, let $f \in Z_{\ell_1}(Q)$ be given and construct \tilde{f} as above. Since $P_X \in (Z)$ and
\nwe have
\n
$$
||P_X \tilde{f}||_{Z_{\ell_1}(Q)} \leq \text{const.} ||\tilde{f}||_{Z_{\ell_1}(Q)} = \text{const.} ||f^*||_{Z_{\ell_1}(Q^*)}.
$$
\nPassing to f and applying (4.1), we therefore have the estimate
\n
$$
||P_X f^*||_{Z_{\ell_1}(Q^*)} \leq \alpha_{A(X)} ||P_X \tilde{f}||_{Z_{\ell_2}(Q)} \leq \alpha_{A(X)} \text{const.} ||f^*||_{Z_{\ell_1}(Q)}.
$$
\nThis implies $\alpha_Z < \alpha_{A(X)}$ by Theorem 3.3 a), as maintained.
\nConcerning the first part of the index condition asserted

Collecting all these estimates we have that P_X , $Q_Y \in W(X_{e_1}(\Omega), Y_{e_2}(\Omega))$, and hence.

 $Q_Y \in (Z)$, on account of the assumption upon Z.

Now, let $f \in Z_{e_2}(\Omega)$ be given and construct \overline{f} as above. Since $P_X \in (Z)$ and $(\overline{f})^* = f^*$,
 $\|P_X \overline{f}\|_{Z_{e_2}(\Omega)} \leq \text{const.} \|\overline{f}\|_{Z_{e_2}(\Omega)} = \text{const.} \|\overline{f}\|_{Z_{\lambda$ we have $\sup_{t \in \Omega^*} (\mathbf{Q}_Y f)^* (t) \tau_Y(t) \leq \frac{\text{const.}}{\beta_{A(Y)}} ||f||_{A(Y_{\rho_4}(Q))}$.

Collecting all these estimates we have that \mathbf{P}_X , $\mathbf{Q}_Y \in W(X_{\rho_4}(P_X, \mathbf{Q}_Y, \mathbf{Q}_Y))$.
 \mathbf{P}_X , $\mathbf{Q}_Y \in (Z)$, on account of the assumption upo

$$
\|\mathbf{P}_X\overline{f}\|_{Z_{\varrho_\mathbf{s}}(\Omega)} \leqq \text{const. } \|\overline{f}\|_{Z_{\varrho_\mathbf{s}}(\Omega)} = \text{const. } \|f^*\|_{Z_{\lambda_\mathbf{s}}(\Omega^*)}.
$$

$$
\|\mathbf{P}_X f^*\|_{Z_{\lambda_{\mathbf{s}}}(\Omega^*)} \leq \alpha_{A(X)} \|\mathbf{P}_X \overline{f}\|_{Z_{\varrho_{\mathbf{s}}}(\Omega)} \leq \alpha_{A(X)} \text{ const. } \|f^*\|_{Z_{\lambda_{\mathbf{s}}}(\Omega)}.
$$

This implies $\alpha_Z < \alpha_{A(X)}$ by Theorem 3.3 a), as maintained.

Concerning the first part of the index condition asserted, it follows from (4.2) and $\mathbf{Q}_Y \in \mathbf{(Z)}$ that $z_{\lambda_1}(\rho^*)$ the estimate $||f^*||_Z$
 constant $||f^*||_Z$
 standard
 asserted , index
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 $\|\mathbf{Q}_Y f^*\|_{Z_{\lambda}(Q^*)} \leq \alpha_{A(Y)} \text{ const. } \|f^*\|_{Z_{\lambda}(Q^*)},$

and hence $\beta_{\Lambda(Y)} < \beta_Z$ by Theorem 3.3b). This concludes the proof of Theorem 4.1

If the spaces X and Y are of fundamental type (see [10] and § 5) then $\alpha_{A(X)} = \alpha_X$,

 $\beta_{A(X)} = \beta_X$, and $\alpha_{A(Y)} = \alpha_Y$, $\beta_{A(Y)} = \beta_Y$, and we have in addition Corollary 4.2: In addition to the assumptions of Theorem 4.1 fundamental type such that $\alpha_X = \beta_X$ and $\alpha_Y = \beta_Y$. Then $W(X, Y)$ $\beta_Y < \beta_Z \leq \alpha_Z < \alpha_X$. *Corollary 4.2: In addition to the assumptions of Theorem 4.1 let X and Y be of fundamental type such that* $\alpha_X = \beta_X$ and $\alpha_Y = \beta_Y$. Then $W(X, Y) \subset Z$ *if and only if* $\beta_Y < \beta_Z \leq \alpha_Z < \alpha_X$.

Note that most of the known \vec{r} . spaces such as Lebesgue spaces, Lorentz spaces, and Orlicz spaces are of fundamental type, see [7]. **In** particular, Corollary 42 contains fundamental type such that $\alpha_X = \beta_X$ and $\alpha_Y = \beta_Y$. Then $W(X, Y) \subset (Z)$ if and only if $\beta_Y < \beta_Z \leq \alpha_Z < \alpha_X$.

Note that most of the known r.i. spaces such as Lebesgue spaces, Lorentz spaces, and Orlicz spaces are of fundamen the interpolation theorems of D.W. BOYD [2] (for $X = L_p$, $Y = L_q$, Z arbitrary, $1 \le p < q < \infty$), of A. P. CALDERÓN [5] (for $X = L_p$, $Y = L_q$, $Z = L_r$, $1 \le p < r$ $<\overline{q}<\infty$) and its weaker version of J. E. MARCINKIEVICZ [17], as well as the theorem of M. RIESZ/G. THORIN; since each bounded operator on L_p is a fortiori of weak type

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 (L_p, L_p) . Moreover, part b) of Theorem 4.1 might be regarded as an answer to Con-
jecture 5.4 in [25].

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120 F. FEHER
 (L_p, L_p) . Moreover, part b) of

jecture 5.4 in [25].

Explicitely, Theorem 4.1 a)
 (X, X) , we denote by $||T||_{W(Y)}$.

similarly we define $||T||_{W(Y)}$. Explicitely, Theorem 4.1 a) can be reformulated as follows: If *T* is of weak type (X, X) , we denote by $||T||_{W(X)}$ the lowest positive constant such that (1.1) holds; similarly we define $||T||_{W(Y)}$. part b) of Theorem 4.1 might be reg
 *i*rem 4.1 a) can be reformulated as f
 by $||T||_{W(X)}$ the lowest positive con-
 $||T||_{W(Y)}$.
 Under the assumptions of Theorem 4.1
 $\leq ||E_{1/2}||_{Z_{A_s}(\mathcal{Q}^*)}$ max $\{||T||_{W(X)}, ||T_{W(Y)}||\$

Corollary 4.3: Under the asuinptions of Theorem 4.1 a) *we have*

part b) of Theorem 4.1 might be rega
\nem 4.1 a) can be reformulated as fol
\nby
$$
||T||_{W(X)}
$$
 the lowest positive const
\n $T||_{W(Y)}$.
\n $||T||_{W(X)}$ have a
\n $||T||_{W(X)}$, $||T||_{W(Y)}$

Here we used Lemma 3.2, Lemma 3.1 e), and (3.3) with the constant being equalto $||E_{1/2}||_{Z_{1,2}(\mathcal{Q}^*)}$ max $||T||_{W(X)}$, $||T||_{W(Y)}$ as a slight modification of Sharpley's [20] argument shows. Concerning the distribution function $D_{T_I}(\sigma) := \mu(x \in \Omega : |Tf| \ (x) > \sigma)$ of Tf we have the estimate *(d)*
(d)
(d) (d) (d) 3.1 e), and (3.3)
as a slight modi
ation function *1*
 $|g_1||_{A(X)}$ + τ_Y^{-1}
the $g \in A(Y)$ $\frac{ds}{s}$

(with the constant being e

fication of Sharpley's [20] a
 $2r_r(\sigma) := \mu\{x \in \Omega : |Tf|(x)\}$
 $\left(\frac{2||T||_{W(Y)}}{\sigma} ||g_2||_{A(Y)}\right)$
 $\in \mathcal{A}(Y)$. Indeed, on one has Here we used Lemma 3.2, Lemma 3.1 e), and (3.3) with the constant being equal

to $||E_{1/2}||_{Z_{14}(\mathcal{Q}^*)}$ max $(||T||_{W(X)}||_{T||_{W(Y)}}||_{T||_{W(Y)}}$ as a slight modification of Sharpley's [20] argument shows. Concerning the distribu

$$
(D_{Tf})\,(\sigma) \leq \tau_X^{-1}\left(\frac{2\,||T||_{W(X)}}{\sigma}\,||g_1||_{A(X)}\right) + \tau_Y^{-1}\left(\frac{2\,||T||_{W(Y)}}{\sigma}\,||g_2||_{A(Y)}\right) \qquad (4.6)
$$

for any representation $f = g_1 + g_2$ with $g_1 \in A(X)$, $g_2 \in A(Y)$. Indeed, on one hand it (D_{Tf}) $(\sigma) \leq \tau_X^{-1} \left(\frac{2||T||_{\mathcal{W}(X)}}{\sigma} ||g_1||_{\mathcal{A}(X)} \right) + \tau_Y^{-1} \left(\frac{2||T||_{\mathcal{W}(Y)}}{\sigma} ||g_2||_{\mathcal{A}(Y)} \right)$ (4.6)
for any representation $f = g_1 + g_2$ with $g_1 \in \mathcal{A}(X), g_2 \in \mathcal{A}(Y)$. Indeed, on one hand it
is known that weak type (X, X) , then $\begin{split} \text{presentation } f & = g_1 + g_2 \text{ with } g_1 \in A(X), \, g_2 \in \text{that } D_{Tf}(\sigma) \leqq D_{Tg_1}(\sigma/2) + D_{Tg_2}(\sigma/2) \text{ for } \sigma > 0 \colon (X, \, X), \text{ then} \ \text{sup } \sigma \tau_X(D_{Tg_1}(\sigma)) & = \text{sup } \left(Tg_1 \right)^* \left(t \right) \, \tau_X(t) \leqq \text{if } \sigma \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \math$ For any representation $f = g_1 +$

is known that $D_{Tf}(\sigma) \leq D_{Tg_1}(\sigma)^t$

weak type (X, X) , then
 $\sup_{0 \leq \sigma \leq \infty} \sigma x \left(D_{Tg_1}(\sigma) \right) =$

If T is of weak type (Y, Y) then
 $\times ||g_2||_{\mathcal{A}(Y)}$. Therefore
 (D_{Tg_1}) $(\sigma/2) \leq$ x₁, $||T||_{W(Y)}$ as a slight modification of Sharpley's

the distribution function $D_{T_f}(\sigma) := \mu\{x \in \Omega : |T\}$
 $\left(\frac{2||T||_{W(X)}}{\sigma} ||g_1||_{A(X)}\right) + \tau_Y^{-1}\left(\frac{2||T||_{W(Y)}}{\sigma} ||g_2||_{A(Y)}\right)$
 $= g_1 + g_2$ with $g_1 \in A(X), g_2 \in A(Y)$. Indeed, on ibution function $D_{T_f}(\sigma)$
 $\frac{d\mathbf{x}(\cdot)}{d\mathbf{x}(\cdot)}$ $\frac{d\mathbf{x}(\cdot)}{d\mathbf{x}(\cdot)}$
 $\frac{d\mathbf{x}(\cdot)}{d\mathbf{x}(\cdot)}$ $\frac{d\mathbf{x}(\cdot)}{d\mathbf{x}(\cdot)}$ $\frac{d\mathbf{x}(\cdot)}{d\mathbf{x}(\cdot)}$
 $\frac{d\mathbf{x}(\cdot)}{d\mathbf{x}(\cdot)}$
 $\frac{d\mathbf{x}(\cdot)}{d\mathbf{x}(\cdot)}$
 $\frac{d\mathbf{x}(\cdot)}$ $(D_{Tf}) (\sigma) \leq \tau_X^{-1} \left(\frac{2||T||_{W(X)}}{\sigma} ||g_1||_{A(X)} \right) + \tau_Y^{-1} \left(\frac{2||T||_{W(Y)}}{\sigma} ||g_2||_{A(Y)} \right)$ (4.6)

epresentation $f = g_1 + g_2$ with $g_1 \in A(X)$, $g_2 \in A(Y)$. Indeed, on one hand it

that $D_{Tf}(\sigma) \leq D_{Tg_1}(\sigma/2) + D_{Tg_2}(\sigma/2)$ for $\$

$$
\sup_{0\leq\sigma\leq\infty}\sigma_{X}\big(\overline{Q}_{Tg_1}(\sigma)\big)=\sup_{0\leq t\leq\infty}(Tg_1)^*(t)\,\tau_X(t)\leq\|T\|_{W(X)}\,\|\dot{g_1}\|_{A(X)}.
$$

weak type (X, X) , then
 $\sup_{0 < s < \infty} \sigma_X(D_{Tg_1}(\sigma)) = \sup_{0 < t < \infty} (Tg_1)^*$ $(t) \tau_X(t) \le ||T||_{W(X)} ||g_1||_{A(X)}$.

If *T* is of weak type (Y, Y) then analogously it follows that $\sup_{0 < s < \infty} \sigma_{Y}(D_{Tg_1}(\sigma)) \le$
 $\times ||g_2||_{A(Y)}$. Therefore $\|T\|_{W(Y)}$

\n for any representation
$$
f = g_1 + g_2
$$
 with $g_1 \in A$ is known that $D_{Tf}(\sigma) \leq D_{Tg_1}(\sigma/2) + D_{Tg_2}(\sigma/2)$:\n

\n\n weak type (X, X) , then\n

\n\n
$$
\sup_{0 < \sigma < \infty} \sigma \tau_X(D_{Tg_1}(\sigma)) = \sup_{0 < t < \infty} (Tg_1)^* (t)
$$
\n

\n\n If T is of weak type (Y, Y) then analogously it\n

\n\n If T is of weak type (Y, Y) then analogously it\n

\n\n
$$
\times ||g_2||_{A(Y)}
$$
. Therefore\n

\n\n
$$
(D_{Tg_1})(\sigma/2) \leq \tau_X^{-1} \frac{||T||_{W(X)}||g_2||_{A(X)}}{\sigma/2}
$$
\n

\n\n if $T \in W(X, Y)$, yielding (4.6) . Let us remark and L_r with $1/r = \theta/q + (1 - \theta)/p$, $0 < \theta < \theta$ start $||T||_{W(L_p)}^{\theta}$ in Macinkiewicz's theorems\n

\n\n 5. A publications\n

\n\n 5.1. Applications to particular spaces\n

V

 $\frac{2}{\pi} \left\|g_1\|_{A(X)}\right\} + \tau_Y^{-1}\left(\frac{2\|T\|_{W(Y)}}{\sigma}\|g_2\|_{A(Y)}\right)$ (4.6)

with $g_1 \in A(X), g_2 \in A(Y)$. Indeed, on one hand it
 $D_{Tg_1}(\sigma/2)$ for $\sigma > 0$. On the other hand, if T is of
 $(Tg_1)^*(t) \tau_X(t) \leq \|T\|_{W(X)} \|\dot{g}_1\|_{A(X)}$.
 if $T \in W(X, Y)$, yielding (4.6). Let us remark that in case of Lebesgue spaces L_p , L_q , and *L_r* with, $1/r = \theta/q + (1 - \theta)/p$, $0 < \theta < 1$, this leads to the well known constant $||T||_{W(L_2)}^{\theta} ||T||_{W(L_2)}^{\theta}$ in Marcinkievicz's theorem. Is known that $D_{Tf}(\sigma) \leq D_{Tg_1}(\sigma/2) + D_{Tg_2}(\sigma/2)$ for $\sigma > 0$. On th

weak type (X, X) , then
 $\sup_{0 < \sigma < \infty} \sigma_X(D_{Tg_1}(\sigma)) = \sup_{0 < t < \infty} (Tg_1)^*(t) \tau_X(t) \leq ||T||_{W(X)}$

If *T* is of weak type (Y, Y) then analogously it follows t $(D_{Tg_1}) (\sigma/2) \leq r_Y^{-1} \left(\frac{||T||_{W(X)} ||g_2||_{A(X)}}{\sigma/2} \right)$

if $T \in W(X, Y)$, yielding (4.6) . Let us remark that

and L_r with $1/r = \theta/q + (1 - \theta)/p, 0 < \theta < 1$, t

stant $||T||_{W(L_q)}^{\theta} ||T||_{W(L_p)}^{\theta}$ in Marcinkievicz's theorem.

5. Appli

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 $\epsilon_{\rm v}$

Of particular interest with respect to applications is the case when the space $Z = Z_{\rho}$. (Ω) is of fundamental type, i.e., when $\|\overline{E}_s\|_{\llbracket Z_{\perp}(\Omega^*)} = M(1/s, Z)$. In this case, it follows that $||E_s||_{[Z_\lambda(Q^\bullet)]} = M(1/s, A(Z))$, since $\tau_{A(Z)} = \tau_{Z}$ is valid even for any r.i. space Z. From this equality it then can easily be deduced that $\alpha_z = \alpha_{A(z)}$ and $\beta_z = \beta_{A(z)}$. For a more detailed discussion of spaces of fundamental type see [10].

Concerning the weak-interpolation problem we now show that for this important 'class of r.i. spaces (which contains the Lebesgue-, Lorentz-, and Orlicz spaces) there exist two further conditions which are equivalent to the index condition of Corollary $\frac{1}{2}$
Class of
exist ty
4.2:

Theorem 5.1: As in Theorem 4.1 let X, Y, Z $\subset \mathcal{M}(\Omega)$ be r.i. spaces such that $\beta_{\mathcal{A}(X)}$ **Example 1.1.** Spaces (which conducts the Ecologie, Ecology, and Chickelpheory,
 A.2:

Theorem 5.1: As in Theorem 4.1 let X, Y, Z $\subset \mathcal{M}(\Omega)$ be r.i. spaces such that
 $>0, 0 < \beta_{A(Y)} \leq \alpha_{A(Y)} < 1$ and τ_Y/τ_X decreasin $> 0, 0 < \beta_{A(Y)} \leq \alpha_{A(Y)} < 1$ and τ_Y/τ_X decreasing. Further assume that $\alpha_{A(X)} = \beta_{A(X)}$, $\alpha_{A(Y)} = \beta_{A(Y)}$, and Z is of fundamental type. Then the following statements are equiv*alent:* A Marcinkievicz Interpolation Theorem.

(Concerning the weak-interpolation problem we now show that for this important

class of r.i. spaces (which contains the Lebesgue-, Lorentz-, and Orlicz spaces) there

exist two fur A Marcinkie

• Concerning the weak-interpolation problem we

class of r.i. spaces (which contains the Lebesgue-

• exist two further conditions which are equivalent

4.2:

• Theorem 5.1: As in Theorem 4.1 let X, Y, Z
 >0 *f* i. spaces (which contains the Lebesgue-, Lorentz-, and Orlicz spaces) there
further conditions which are equivalent to the index condition of Corollary
em 5.1: As in Theorem 4.1 let X, Y, Z $\subset \mathcal{A}(\Omega)$ be r.i. space provide the polarity of the text of the set of the set of text of the set of the set of assumed to be of fundamental ty

provide the set of text of the set of as

:

-
- (b) $W(X, Y) \subset (A(Z));$

(c) $\beta_{A(Y)} < \beta_Z \le \alpha_Z < \alpha_{A(X)};$
- (d) there exists a finite number $A > 0$ such that

$$
\int_{0}^{1} F(s, t) \, dx_{Z}(t) \leq A \tag{5.1}
$$

uniformly in s $\in \Omega^*$ *, the function F* : $\Omega^* \times \Omega^* \to \mathbf{R}$ being defined by

$$
y \text{ in } s \in \Omega^*, \text{ the function } F: \Omega^* \times \Omega^* \to \mathbf{R} \text{ being defined by},
$$
\n
$$
F(s, t) := \min \left\{ \frac{\tau_X(s)}{\tau_X(t)}, \frac{\tau_Y(s)}{\tau_Y(t)} \right\} / \tau_Z(s) \qquad (s, t \in \Omega^*).
$$
\n
$$
(5.2) \cdot \mathbf{R} \cdot \mathbf{
$$

righta regional type. Then
<i>r $A > 0$ *such that*
<i>r $F: \Omega^* \times \Omega^* \to \mathbb{R}$
r_y(t)} $\left| \frac{\tau_z(s)}{\tau_y(t)} \right| \neq \tau_z(s)$ (*s*, ow follows readily.
Z is assumed to botally. The proof of this theorem now follows readily. The equivalence of (a) with (c) is essentially Corollary 4.2. Since Z is assumed to be of fundamental type, (c) can be uniformly in $s \in \mathbb{S}^2$, the function $F: \mathbb{S}^2 \times \mathbb{S}$
 $F(s, t) := \min \left\{ \frac{\tau_X(s)}{\tau_X(t)}, \frac{\tau_Y(s)}{\tau_Y(t)} \right\} / \tau_Z(s)$

The proof of this theorem now follows

essentially Corollary 4.2. Since Z is assum

rewritten as $\beta_{A(Y)} < \beta_{$ rewritten as $\beta_{A(Y)} < \beta_{A(Z)} \leq \alpha_{A(Z)} < \alpha_{A(X)}$. Therefore, the equivalence of (b) and (c) 'is again Corollary 4.2, but now applied to *A(Z)* instead of *Z.* Finally, the equivalence, of (b) and (d) was proved in [20]. Note that in addition to the theorem of [20] we now also have the equivalence of (d) with (a) \blacksquare $\int_{0}^{1} F(s, t) dx_{Z}(t) \leq A$

uniformly in $s \in \Omega^*$, the function $F : \Omega^* \times \Omega$
 $F(s, t) := \min \left\{ \frac{\tau_X(s)}{\tau_X(t)}, \frac{\tau_Y(s)}{\tau_Y(t)} \right\} / \tau_Z(s)$

The proof of this theorem now follows re

essentially Corollary 4.2. Since Z is assume

rewri The proof of this theorem now follows readily. The fundamental type, (e) with (c)

Sesentially Corollary 4.2. Since Z is assumed to be of fundamental type, (c) can

rewritten as $\beta_{A(X)} \leq \beta_{A(Z)} \leq \alpha_{A(X)}$. Therefore, the **rewritten as** $\beta_{A(Y)} \leq \beta_{A(Z)} \leq \alpha_{A(Z)} < \alpha_{A(X)}$ **. Therefore, the equivalence of (b) and (is again Corollary 4.2, but now applied to** $A(Z)$ **instead of Z. Finally, the equivalence of (b) and (d) was proved in [20]. Note that**

As a more concrete example we next consider the case when X and Y are Lebesgue

corollary 5.2: *If Z* is of fundamental type and $1 \leq p < q < \infty$, then the following tensors and *Z* is arbitrary.

Corollary 5.2: *If Z* is of fundamental type and $1 \leq p < q < \infty$, then the following tensors are equivale *statements are equivalent:* • spaces, and Z is arbitrary.

Corollary 5.2: If Z is of fundamental type and $1 \leq p < q <$

statements are equivalent:

(a) $W(L_p, L_q) \subset (Z)$;

(b) $W(L_p, L_q) \subset (A(Z))$;

(c) $1/q < \beta_Z \leq \alpha_Z < 1/p$;

(d) there exists a finite number $A >$

- (a) $W(L_p, L_q) \subset (Z)$;
(b) $W(L_p, L_q) \subset (A(Z));$
-
- (c) $1/q <$
 d) there $\frac{s}{\tau_2}$
 *r*₂
-

$$
We then this are equivalent:\nW(Lp, Lq) \subset (Z);\nW(Lp, Lq) \subset (A(Z));\n1/q < \beta_Z \leq \alpha_Z < 1/p;
$$

\nthere exists a finite number $A > 0$ such that uniform
\n
$$
\frac{s^{1/q}}{\tau_Z(s)} \int_0^s t^{-1/q} dx_Z(t) + \frac{s^{1/p}}{\tau_Z(s)} \int_s^1 t^{-1/p} dx_Z(t) \leq A.
$$
\nIndeed, since $\tau_{L_q}(t) = t^{1/q}$ and $\tau_{L_p}(t) = t^{1/p}$, the function
\nby if $p < q$. Moreover, $\beta_{A(L_p)} = \beta_{L_p} = 1/p > 0$ and

 $t^{1/q}$ and $\tau_{L_p}(t) = t^{1/p}$, the function τ_{L_p}/τ_{L_q} is decreasing if and $\frac{S^{1/4}}{\tau_Z(s)} \int_{0}^{s} t^{-1/q} dt_Z(t) + \frac{S^{1/2}}{\tau_Z(s)} \int_{s}^{s} t^{-1/p} dt_Z(t) \leq A.$

Indeed, since $\tau_{L_g}(t) = t^{1/q}$ and $\tau_{L_g}(t) = t^{1/p}$, the function τ_{L_g}/τ_{L_g} is decreasing if and only if $p < q$. Moreover, $\beta_{A(L_p)} = \beta_{L_p} = 1/p > 0$ Corollary 5.2: If Z is of fundamental type and $1 \le p < q < \infty$, then the
statements are equivalent:
(a) $W(L_p, L_q) \subset (Z)$;
(b) $W(L_p, L_q) \subset (A(Z))$;
(c) $1/q < \beta_z \le \alpha_z < 1/p$;
(d) there exists a finite number $A > 0$ such that uniformly , since $\tau_{L_q}(t) = t^{1/q}$ and $\tau_{L_p}(t) = t^{1/p}$, the function τ
 $\langle q, \text{ Moreover}, \beta_{A(L_p)} = \beta_{L_p} = 1/p > 0$ and $0 < \theta$

observe that in this case
 $F(s, t) = \min\left\{ (s/t)^{1/p}, \quad (s/t)^{1/q} / \tau_z(s) \right\}$
 $= \begin{cases} (s/t)^{1/q} / \tau_z(s) & \text{if } 0 < t < s \\ (s/t)^{$ (c) $1/q < \beta_z \leq \alpha_z < 1/p$;

(d) there exists a finite number $A > 0$ such that uniformly in $s \in \Omega^*$,
 $\frac{s^{1/e}}{\tau_z(s)} \int_0^s t^{-1/q} \, d\tau_z(t) + \frac{s^{1/p}}{\tau_z(s)} \int_0^t t^{-1/p} \, d\tau_z(t) \leq A$.

Indeed, since $\tau_{L_o}(t) = t^{1/q}$ and $\tau_{L_o}(t) = t^{1/p$

observe that in this case
\n
$$
F(s, t) = \min \{ (s/t)^{1/p}, (s/t)^{1/q} / \tau_z(s) \}
$$
\n
$$
= \begin{cases}\n (s/t)^{1/q} / \tau_z(s) & \text{if } 0 < t < s \\
 (s/t)^{1/p} / \tau_z(s) & \text{if } t \ge s,\n\end{cases}
$$

Remark 5.1: For the equivalence of (a) and (c) it is not necessary that Z be of fundamental. type. Then this equivalence is the theorem of Boyd.

Corollary 5.3: If $1 \leq p < q < \infty$, $1 < r < \infty$, and Z is one of the spaces L_r, L_{rs} , $L^r(\log^+ L)$ or the Marcinkievicz space $M_{1-1/r}$, then the following statements are equiva*lent:* (122 F. FEHER

(122 F. FEHER

(122 F. FOT the equivalence of (a) and (c) it is

fundamental type. Then this equivalence is the theorem

(Corollary 5.3: If $1 \le p < q < \infty$, $1 < r < \infty$, and Z
 $L'(\log^+ L)$ or the Marcinkievicz sp 122 F. FERRER

Remark 5.1: For the equivalence of (a) and (c) if

if if modamental type. Then this equivalence is the theore

Corollary 5.3: If $1 \leq p < q < \infty, 1 < r < \infty, a$

L'(log⁺ L) or the Marcinkievicz space $M_{1-1/r}$, then t *^fF(s, t) drz(t)* = *Ty(S) [Pz(11ry)] +* **y(8)** [Qz(1/tx)] (a), (5.3)

(a) $W(L_p, L_q) \subset (Z)$;

1

(b)
$$
W(L_p, L_q) \subset (L_{r_1});
$$

 $\frac{c^*}{c}$

Condition (d) is trivial in this case. This corollary follows from the preceeding corollary since in any case $\tau_Z(t) = t^{1/r}$, and hence $A(Z) = L_{r1}$.

Remark 5.2: The implication $(c) \Rightarrow (a)$ contains the theorems of Riesz/Thorin and of Marcinkievicz, whereas $(c) \Rightarrow (b)$ is the theorem of Calderón.

Let us conclude this paragraph with the relation

$$
\int_{0}^{t} F(s, t) d\tau_{Z}(t) = \tau_{Y}(s) [P_{Z}(1/\tau_{Y})] + \tau_{X}(s) [Q_{Z}(1/\tau_{X})](s), \qquad (5.3)
$$

between the integral of (5.1) and the average operators of Section 3.

5.2. Applications to Particular Operators

• As a first example we consider the Hardy-Littlewood maximal operator θ on \mathbf{R}_n , **5.2. Applications to Particular Operators**
As a first example we consider the Hardy-Lit
 $n \ge 1$, in its spherical form, given for $f \in \mathbb{R}_n$ by

Remark 5.2: The implication
$$
(c) \Rightarrow (a)
$$
-contains the theorems of Riesz/Thorin
and of Marcinkicvicz, whereas $(c) \Rightarrow (b)$ is the theorem of Calderón.
Let us conclude this paragraph with the relation

$$
\int_{0}^{1} F(s, t) d\tau_{Z}(t) = \tau_{Y}(s) [P_{Z}(1/\tau_{Y})] + \tau_{X}(s) [Q_{Z}(1/\tau_{X})] (s),
$$
(5.3)
between the integral of (5.1) and the average operators of Section 3.
5.2. Applications to Particular Operators

$$
\therefore \quad \text{As a first example we consider the Hardy-Littlewood maximal operator } \theta \text{ on } \mathbb{R}_{n},
$$

$$
n \geq 1, \text{ in its spherical form, given for } f \in \mathbb{R}_{n} \text{ by}
$$

$$
(\theta f) (v) := \sup_{B(v)} \frac{1}{m((B(v))} \int_{B(v)} |f(u)| du. \tag{5.4}
$$

Here the supremum has to be taken over all balls $B(v)$ with positive radius and center
 v . If $Q(v)$ is the circumscribed cube of $B(v)$ with its sides parallel to the coordinate

(vial in this case. This corollary
 (vial in this case. This corollary
 (c) \Rightarrow *(a)* \Rightarrow *(a)* \Rightarrow *(a)* \Rightarrow *(a)* \Rightarrow *(a)* \Rightarrow *(a)* \Rightarrow *(b) is the theore is paragraph with the relation* \cdot *<i>(t)* $= \tau_Y$ Here the supremum has to be taken over all balls $B(v)$ with positive radius and center v. If $Q(v)$ is the circumscribed cube of $B(v)$ with its sides parallel to the coordinate *a*. If $Q(v)$ is the circumscribed cube of $D(v)$ with its sides parallel to the coordinate axes, then there exists a constant $A_n > 0$, depending only on *n* (with $A_1 = 1$) such that $m(Q(v)) \leq A_n m(B(v))$, see [23], As an appli that $m(Q(v)) \leq A_n m(B(v))$, see [23], As an application of Corollaries 4.2 and 4.3 we obtain the following mapping theorem for the maximal operator. α *18z*, then
 α that $m(Q(v))$
 α obtain the
 $0 < \beta_Z \leq 0$ ns to Particular Operators

nple we consider the Hardy-Littlewood maxin

therical form, given for $f \in \mathbf{R}_n$ by
 $:= \sup_{B(v)} \frac{1}{m((B(v))} \int_{B(v)} |f(u)| du$.

num has to be taken over all balls $B(v)$ with posi-

circumscribed cube *2*^{**1**} $\lim_{B(e)} \frac{1}{m((B(v))} \int_{B(e)} |f(u)| du.$ (5.4)
 20 $\lim_{B(e)} \frac{1}{m((B(v))} \int_{B(e)} |f(u)| du.$ (5.4)
 A m has to be taken over all balls $B(v)$ with positive radius and center
 A x 3*i A x* 3*i A z* 9*i A a z*

 α_{Z}

$$
\begin{aligned}\n\text{rem 5.4:} \quad & \text{If } Z \equiv Z_e(\mathbf{R}_n), \ n \geq 1, \text{ has } \text{indiv} \\
&\leq \alpha_Z < 1, \text{ then } \theta \in (Z) \text{ and} \\
\|\theta\|_{(Z)} \leq 2^n A_n \, \|E_{1/2}\|_{(Z_i(\Omega^*))} \int_0^1 \|E_e\|_{(Z_i(\Omega^*))} \, ds,\n\end{aligned}
$$

where $Z_1(\Omega^*)$ is the Luxemburg representation of Z.

Proof: In order to derive this theorem from Corollary 4.2, choose $X := L_1(\mathbf{R}_n)$ and $Y := L_q(\mathbf{R})$ with $1 < q < \infty$. It is well known that the maximal operator θ is of weak type (X, X) and a bounded operator on *Y*. More precisely, if $D_{\theta}(\sigma)$ *••* $\mathcal{Z}_1(\Omega^*)$ *is the*
 Proof: In order
 $Y := L_q(\mathbf{R}) \text{ with}$
 $\text{weak type } (X, X)$
 $= \mu |x \in \mathbf{R}_n : |\theta(f)|$
 $D_{\theta f}(\sigma) \leq \frac{1}{2}$
 $||\theta||_{(Y)} \leq 2$

obtain the following mapping theorem for the maximal operator.
\nTheorem 5.4: If
$$
Z = Z_e(\mathbf{R}_n)
$$
, $n \ge 1$, has indices strictly between 0 and 1, i.e., $0 < \beta_Z \le \alpha_Z < 1$, then $\theta \in (Z)$ and
\n $||\theta||_{(Z)} \le 2^n A_n ||E_{1/2}||_{(Z_i(\Omega^*))} \int_{0}^{1} ||E_g||_{(Z_i(\Omega^*))} ds$,
\nwhere $Z_1(\Omega^*)$ is the Luxemburg representation of Z.
\nProof: In order to derive this theorem from Corollary 4.2, choose $X := L_1(\mathbf{R}_n)$ and
\n $Y := L_q(\mathbf{R})$ with $1 < q < \infty$. It is well known that the maximal operator θ is of
\nweak type (X, X) and a bounded operator on Y. More precisely, if $D_{\theta f}(\sigma)$
\n $:= \mu |x \in \mathbf{R}_n : |\theta(f) (x)| > \sigma$ is the distribution function of θf , then (see [23])
\n $D_{\theta f}(\sigma) \le \frac{2^n A_n}{\sigma} ||f||_X$ $(f \in X)$,
\n $||\theta||_{(Y)} \le 2 \left(\frac{q2^n A_n}{q-1}\right)^{1/q}$.
\n(5.5)

A Marcinkievicz Interpolation Theorem 123

that $(\theta f)^*(t) \leq (2^n A_n/t) ||f||_X = (2^n A_n/t) ||f||_{11}$ for $f \in X$. Recal-From (5.6) it follows that $(\theta f)^*(t) \leq (2^n A_n/t) ||f||_X = (2^n A_n/t) ||f||_{11}$ for $f \in X$. Recalmore general sense of (1.1) with the weak norm *A* Marcinkievicz Interpolation Theorem 123
 B) it follows that $(\theta f)^* (t) \leq (2^n A_n/t) ||f||_X = (2^n A_n/t) ||f||_1$ for $f \in X$. Recal-
 $\tau_X(t) = t$ and $A(X) = L_{11}$, we can conclude that θ is of weak type in the

real sense of (1.1)

$$
\|\theta\|_{W(X)} \le 2^n A_n. \tag{5.8}
$$

ling that $\tau_X(t) = t$ and $A(X) = L_{11}$, we can conclude that θ is of weak type in the

more general sense of (1.1) with the weak norm
 $||\theta||_{W(X)} \leq 2^n A_n$. (5.8)

On the other hand, $\sigma^q D_{\theta f}(\sigma) \leq ||\theta f||_g^q$ for $f \in Y$, on the other hand, $\sigma^q D_{\theta f}(\sigma) \leq \|\theta f\|_{p}^q$ for $f \in Y$, see [23]; hence $(\theta f)^* (t) \leq (\|\theta\|_{(Y)}/t^{1/q})$
 $\|\theta\|_{W(X)} \leq 2^n A_n$.

On the other hand, $\sigma^q D_{\theta f}(\sigma) \leq \|\theta f\|_{p}^q$ for $f \in Y$, see [23]; hence $(\theta f)^* (t) \leq (\|\theta$ $\|\theta\|_{W(X)} \leq 2^n A_n.$
On the other hand, $\sigma^q D_{\theta f}(x)$
 $\times \|f\|_Y \leq (\|\theta\|_{(Y)} / t^{1/q}) \|f\|_{q_1}.$
with (5.7) – means that i $X \Vert f \Vert_Y \leq (\Vert \theta \Vert_{(Y)} f^{1/q}) \Vert f \Vert_{q_1}$. On account of $\tau_Y(t) = t^{1/q}$ and $A(Y) = L_{q_1}$, this $-$ together with (5.7) – means that θ is of weak type (Y, Y) with the weak-norm *a* that $\tau_X(t) = t$ is
 $\tau_X(t) = t$ is
 $\|\theta\|_{W(X)} \leq t$

the other hand,
 $\|f\|_Y \leq (\|\theta\|_{(Y)}/t^{1/q})$
 $\|f\|_{W(Y)} \leq t$
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and $A(X) =$
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 $2^n A_n$.
 $\sigma^q D_{\theta f}(\sigma) \le$
 $\frac{2}{\theta} \left(\frac{q 2^n A_n}{q-1}\right)^{1/n}$
can say that *q* $\begin{aligned} \mathbf{1}_n/t) \, \|\boldsymbol{f}\|_1 \ \text{at} \ \theta \text{ is a} \end{aligned}$
at θ is connected $(\theta f) = L$
is weak-

$$
\|\theta\|_{W(Y)} \le 2\left(\frac{q2^n A_n}{q-1}\right)^{1/q}.\tag{5.9}
$$

From (5.6) it follows that $(\theta f)^*$ (t) $\leq (2^n A_n/t) ||f||_X = (2^n A_n/t) ||f||_1$ for $f \in X$. Recalling that $\tau_X(t) = t$ and $A(X) = L_{11}$, we can conclude that θ is of weak type in the more general sense of (1.1) with the weak norm are of fundamental type with indices $\alpha_X = \beta_X = 1$ and $\alpha_Y = \beta_Y = 1$ a is decreasing since $q > 1$, so that all the assumptions of Corollary 4.2 are satisfied. For any r.i. space *Z* with $1/q < \beta_Z \le \alpha_Z < 1$ we therefore have by Corollary 4.2 that $\theta \in (Z)$, and by Corollary 4.3 that Firm (3.5) in lows that $\left| \varphi \right|_{\mathcal{V}} = \left| \frac{2^{-n}A_n(t)}{2} \right| \left| \varphi \right|_{\mathcal{V}}$ for $f \in X$

ling that $\tau_X(t) = t$ and $A(X) = L_{11}$, we can conclude that θ is of weak typ

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Here we used that
$$
M(s, X) = s
$$
 and $M(s, Y) = \frac{1}{2}PY = 1/q$. Further, $U(Y)$ is decreasing since $q > 1$, so that all the assumptions of Corollary 4.2 are satisfifier any r.i. space Z with $1/q < \beta_Z \leq \alpha_Z < 1$ we therefore have by Corollary 4.2 if $\theta \in (Z)$, and by Corollary 4.3 that

\n
$$
\|\theta\|_{(Z)} \leq \|E_{1/2}\|_{[Z_2(\Omega^*)]} \max \left\{2^n A_n, 2\left(\frac{q2^n A_n}{q-1}\right)^{1/q}\right\}
$$

\nHere we used that $M(s, X) = s$ and $M(s, Y) = s^{1/q}$. Note that the latter two in g is an infinite on account of the index condition assumed, see [3]. Letting q tend infinity, we obtain (5.5), observing that $A_n \geq 1$, and the second integral is decreasing in q .

\nRemark 5.3: A result similar to Theorem 5.4 could be deduced for the cubic main operator by the same methods. The only change is that the factor A_n in (5.5), (5.9) is omitted.

\nFor concrete spaces Z the norm estimate of (5.5) can be evaluated explicitly. Indeed, we have

\nCorollary 5.5: a) If $Z = A(\phi, \phi)$, $n > 1$ is a uniformly conver to g for g and g for g and g for g and g are not a real. For A_n is a unit of A_n and A_n is a unit of A_n for A_n and A_n are not a real, and A_n and A_n are not a real, and A_n and A_n

Here we used that $M(s, X) = s$ and $M(s, Y) = s^{1/q}$. Note that the latter two integrals are finite on account of the index condition assumed, see [3]. Letting q tend to be used that $M(s, X) = s$ and $M(s, Y) = s^{n/2}$. Note that

e finite on account of the index condition assumed, see

we obtain (5.5), observing that $A_n \ge 1$, and the second

ark 5.3: A result similar to Theorem 5.4 could be de

mal operator by the same methods. The only change is that the factor A_n in (5.5), ..., (5.9) is omitted. **EXECUTE:** For concrete spaces *Z* the norm estimate of (5.5) can be evaluated explicitely. In-
For concrete spaces *Z* the norm estimate of (5.5) can be evaluated explicitely. In-

deed, we have

Corollary 5.5: a) *If* $Z = A(\phi, p)$, $p > 1$, *is a uniformly convex Lorentz space, then.* For concrete
deed, we have
 Corollary 5.
 $(\theta) \in (Z)$ and spaces Z the nor
 $\overline{5}$: a) If $Z = A(4)$
 $\leq 2^n A \cdot N(1/2)$

$$
\theta \|_{(Z)} \le 2^n A_n N(1/2) \int_0^1 N(s)^{1/p} ds \le 2^{n+1/p} A_n \frac{p}{p-1} \tag{5.10}
$$

 $i \text{ with } N(s) := \sup_{t \in \Omega^{\bullet}} \left[\int_{0}^{t} \phi(u) \ du \middle/ \int_{0}^{st} \phi(u) \ du \right].$ $\begin{align} \text{Area} \ \ (\theta) \in \ \ (\theta) \in \ \text{with} \ \text{b)} \ \text{that} \ \ \mathbf{b} \ \$

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b) *If* $Z = L_M \varphi$ *is an Orlicz space with strictly increasing Young function* φ *such*

Let so notice spaces
$$
B
$$
 the right-estimate of (0.5) can be evaluated explicitly. Indeed, we have

\n $Corollary 5.5: a) If Z = A(\phi, p), p > 1, is a uniformly convex Lorentz space, then$

\n $|\theta||_{(z)} \leq 2^n A_n N(1/2) \int_0^1 N(s)^{1/p} ds \leq 2^{n+1/p} A_n \frac{p}{p-1}$

\nwith $N(s) := \sup_{t \in \Omega^*} \left[\int_0^t \phi(u) du \int_0^s \phi(u) du \right]$.

\nb) If $Z = L_{M\Psi}$ is an Orlicz space with strictly increasing Young function Ψ such that $L_{M\Psi}$ is reflexive, then

\n $||\theta||_{(z)} \leq \frac{2^n A_n}{K_{\Psi}^{-1}(1/2)} \int_0^1 ds/K_{\Psi}^{-1}(s),$

\n $K_{\Psi}^{-1}(s)$ denoting the right-continuous inverse of $K_{\Psi}(s) := \sup_{t \in \Omega^*} \left[\Psi(st)/\Psi(t) \right]$. This corollary follows from Theorem 5.4 by inserting the norm $||E_x||_{(z, Q\Phi)}$ of the

 $K_{\Psi}^{-1}(s)$ *denoting the right-continuous inverse of* $K_{\Psi}(s) := \sup_{t \in \Omega^*} [\Psi(st)/\Psi(t)].$

This corollary follows from Theorem 5.4 by inserting the norm $||E_s||_{[Z_A(\Omega^*)]}$ of the $K_{\Psi}^{-1}(s)$ denoting the right-continuous inverse of $K_{\Psi}(s) := \sup_{l \in \Omega^*} [\Psi(st)/\Psi(t)]$.

This corollary follows from Theorem 5.4 by inserting the norm $||E_s||_{[z]_1(\Omega^*)}$ of the

respective space. If $Z = A(\phi, p)$, then $||E_s||_{[z]}$ respective space. If $Z = A(\phi, p)$, then $||E_{s}||_{Z} = N(s)^{1/p}$, and for $Z = L_{M\Psi}$ one has $||E_{s}||_{Z} = 1/K_{\Psi}^{-1}(s)$. Concerning the index conditions note that for Lorentz spaces

 $Z = A(\phi, p)$ one has $0 < \beta_Z \leq \alpha_Z < 1$ if and only if *Z* is uniformly convex, whereas in case of Orlicz spaces $Z = L_{M\Psi}$ this condition is equivalent to the reflexivity of $L_{M\Psi}$. 124 F. FEHER,
 $Z = A(\phi, p)$ one has 0

in case of Orlicz spaces .

The second estimate in
 $= p/(p - 1)$ for $p > 1$.

proves the estimate $||\theta||$

Remark 5.4: If $Z =$

this case to
 $||\theta||_{(L_r)} \le 2^n A_n$

which for $n = 1$ differen

The second estimate in (5.10) follows from $N(1/2) \leq 2^{1/p}$ and $\int N(s)^{1/p} ds \leq \int s^{-1/p} d s$

 $p/(p-1)$ for $p> 1$. For the one-dimensional case $A_1 = 1$, and (5.10) therefore im-The second estimate in (5.10) $\equiv p/(p-1)$ for $p > 1$. For the proves the estimate $\|\theta\|_{(z)} \le$ proves the estimate $\|\theta\|_{\mathbf{Z}} \leq 2 \cdot 2^{1/p} p/(p-1)$ as stated in [13].

 ${\rm Remark~5.4}\colon {\rm If}\ Z=L_r,\ r>1,$ one should expect (5.7), but Theorem 5.4 leads in

The estimate
$$
||\sigma||(z) \geq 2 \cdot 2^{1/p}
$$
.\n\nThe sum of the values $|S| \leq 2^n$, $r > 1$.\n\nTo $||\theta||_{(L_r)} \leq 2^n A_n 2^{1/r} \cdot \frac{r}{r-1}$.

which, for $n=1$, differs from the classical constant of Hardy-Littlewood by the methodic factor $2^{1/r}$. For arbitrary n, the constant given in [22], namely $2^{2n}r/(r-1)$, is larger than the above constant by the factor 2^{n-1} *''* $/A_n$.

For $n = 1$ and $\Omega = (0, l)$, $0 < l < \infty$, $\mu = m =$ Lebesgue measure it is well known [12] that (θf^*) (*t*) = $(1/t)$ $\int f^*(s) ds$. By means of this representation we now can

give an application to ergodic theory. For this purpose, let *&* be a measure space with finite measure \bm{l} , and let $\bar{\bm{G}}$ be an ergodic group of one to one measure preserving trans formations g of $\mathcal E$. Further assume that for each measurable function f on $\mathcal E$ and each. $g \in G$ the product fg is measurable on $G \times \mathcal{E}$. Then the expression pplication to ergodic theory. For this purpose
isure *l*, and let *G* be an ergodic group of one to
is *g* of *E*. Further assume that for each measur
product *fg* is measurable on $G \times \mathcal{E}$. Then the
 $(p_a f)(v) := \frac{1}{|N_a|}$

$$
(p_a f)(v) := \frac{1}{|N_a|} \int_{N_a} f(gv) \, dg \qquad (v \in \mathcal{E}, a > 0)
$$

'exists for any nonnegative function $f \in L_1(\mathscr{E})$. Here $\{N_a : a > 0\}$ denotes a family of compact, symmetric neighbourhoods of the identity of G such that $N_aN_b\subset N_{a,b}$, α and α *I* and *I* approach α and $|N_{2a}| \leq K |N_a|$ where $|N_a|$ is the left invariant measure of N_a . Concerning the operator any nonnegative function $f \in L$,
symmetric neighbourhoods of
 $\leq K |N_a|$ where $|N_a|$ is the left
(*Pf*) (*v*) := ess sup *Pa*(|*f*|) (*v*)
 \leq (1) (1) give an application to ergodic theory. For this purpose

finite measure *l*, and let *G* be an ergodic group of one to

formations *g* of *E*. Further assume that for each measure
 $y \in G$ the product *fg* is measurable on *(P_df)* (*v*) $:=$ $\frac{1}{|N_a|} \int_{N_a} f(gv) \, dg$ (*v* $\in \mathcal{E}, a$ >
 (P_df) (*v*) $:=$ $\frac{1}{|N_a|} \int_{N_a} f(gv) \, dg$ (*v* $\in \mathcal{E}, a$ >

any nonnegative function $f \in L_1(\mathcal{E})$. Here

symmetric neighbourhoods of the ident
 \le

$$
(Pf)(v) := \operatorname*{ess\,sup}_{a>0} p_a(|f|) (v) \qquad \bigl(f \in L_1(\mathscr{E}), v \in \mathscr{E}\bigr).
$$

$$
(Pf)^{*}(t) \leq \frac{K^2}{t} \int_{0}^{t} |f|^{*}(s) ds \qquad (t \in (0, t)).
$$

Hence, *P* is of weak type $(L_1(\mathscr{E}), L_1(\mathscr{E}))$ and $(Pf)^*(t) \leq K^2(\theta f^*)(t)$. Since θ is bounded on $L_q(\mathscr{E})$ for $1 < g < \infty$, the latter implies that P is also bounded and, a fortiori, of weak type $(L_q(\mathscr{E}), L_q(\mathscr{E}))$. By Theorem 5.4 we therefore have $(Pf)^*(t) \leq \frac{K^2}{t} \int_{0}^t |f|^*(s) ds \qquad (t \in (0, l)).$

There P is of weak type $\left(L_1(\mathscr{E}), L_1(\mathscr{E})\right)$ and $(Pf)^*(t) \leq K^2(0f^*)$ (*t*). Since θ is bounded $L_q(\mathscr{E})$ for $1 < g < \infty$, the latter implies that P is also bounde $(Pf)(v) := \operatorname{ess} \sup_{a>0} p_a(|f|) (v)$ $(f \in A)$
 on 6
 o o i $(Pf)* (t) \leq \frac{K^2}{t} \int_0^t |f|^{*}(s) ds$ $(t \in (t \in A)$
 CPf)* $(t) \leq \frac{K^2}{t} \int_0^t |f|^{*}(s) ds$ $(t \in A)$
 CPf)* $(t) \leq \frac{K^2}{t} \int_0^t |f|^{*}(s) ds$ $(t \in A)$
 Corollary C

Remark 5.5: Similar considerations could be applied e.g. to the Hilbert transform, to the conjugate operator, the Poisson operator or, more generally, to kernel operators with a kernel which is homogeneous of degree -1 . In these cases Theorem' 5.4 would furnish mapping properties of these operators on Lorentz $-$ or Orlicz spaces, such as Hardy- and Hardy-Schur inequalities. In particular by means of the conjugate operator a theorem [11] about norm convergence of Fourier series on r.i. spaces could he reestablished of.

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