A Note on the Invertibility of Generalized Wiener-Hopf Operators

J. DONIG

Wir betrachten im Hilbertraum verallgemeinerte Wiener-Hopf-Operatoren der Form $T(A)$ $:= PA \upharpoonright R(P)$, wobei A und P lineare beschränkte Operatoren sind und P férner eine Moore-Penrose-Inverse besitzt. Das Ziel der Untersuchungen besteht darin, ein Kriterium für die Invertierbarkeit von $T(A)$ in $R(p)$ anzugeben.

Исследуются обобщённые операторы Винера-Хопфа вида $T(A) := PA \upharpoonright R(P)$ в гильбертовом пространстве, причём А и P - ограниченные операторы и P имеет обратный Moore-Penrose оператор. Окончательая цель состоит в нахождении прямого и обратного значений инверсии $T(A)$.

We consider generalized Wiener-Hopf-operators of the form $T(A) := PA \upharpoonright R(P)$ in a Hilbert space, where A and P are linear, bounded operators, and P permits a Moore-Penrose inverse. The purpose of our investigations is to derive a criterion for the invertibility of $T(A)$ in $R(P)$.

1. Introduction

Let H be a Hilbert space, and suppose that A is a bounded linear operator in H and that, for the time being, P is an orthogonal projection onto a proper subspace M of H. Then the part of A in M is defined by

$$
T(A) := PA_{0},
$$

 (1)

where A_0 denotes the restriction of A onto $R(P)$ (the range of P). The operator $T(A)$, which we also write as $T_P(A)$ in order to indicate P, is well-known to be the prototype of a Wiener-Hopf operator. For example, if $\alpha \in \mathbb{R}$, $k \in L^1(\mathbb{R})$, and if A is the operator in $L^2(\mathbf{R})$ defined by

$$
A f(x) := \alpha f(x) + \int_{\mathbf{R}} k(x-y) f(y) dy \qquad (f \in L^{2}(\mathbf{R}), x \in \mathbf{R}),
$$

if further P is defined in $L^2(\mathbf{R})$ as the projection onto $L^2(\mathbf{R}) := \{f \mid f \in L^2(\mathbf{R}), f(x) = 0\}$ for $x < 0$, we obtain

$$
T(A) g(x) := \alpha g(x) + \int_{\mathbf{R}_+} k(x - y) g(y)_i dy \qquad (g \in L_+^2(\mathbf{R}), x \in \mathbf{R}_+).
$$
 (2)

By the integral expression (2) a special Wiener-Hopf-operator $T(A)$ in $L_+^2(\mathbf{R})$ is defined [7].

Let us now in place of (2) consider an expression

$$
T(A) g(x) := \alpha P^2 g(x) + P \int_{\mathbf{R}} k(x-y) P g(y) dy \qquad (g \in L^2(\mathbf{R}), x \in \mathbf{R}), \quad (3)
$$

where P is any bounded operator in $L^2(\mathbf{R})$ which has a (bounded) Moore-Penrose inverse P^+ , e.g.

$$
P = \dot{P}P^+P.
$$

'4)

Equation (4) is readily seen to be equivalent to $R(P)$ being closed (cf. ATKINSON [1]). The operator $T(A)$ in $R(P)$ as defined by (3) is still of the form (1) except that P is in \mathcal{F}_{q} = \mathcal{F}_{q} . Downto \mathcal{F}_{q} . Downto \mathcal{F}_{q} is cally seen to be equivalent to $R(P)$ being closed (cf. ATKINSON [1]). The operator $T(A)$ in $R(P)$ as defined by (3) is still of the form (1) except that P general not a projection. In this case, $T(A)$ is called a *generalized* Wiener-Hopf-operator (abbreviated, GWH -operator). In a fundamental article DEVINATZ and SHINBROT [3] studied arbitrary Wiener-90 (**J.** Donic
 Equation (4) is readily seen to be equivalent to $R(P)$ being closed (cf. ATKINSON [1]).

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Hopf-operators (1) and derived necessary and sufficient conditions for these operators to be invertible and onto. The purpose of the present note is to establish such a criterion for GWH-operators. Before stating our main result, let us recall that a linear operator *A* in *H* is termed *invertible*, if there exists an operator $A^{-1}: R(A) \to H$ satisfying
 $A^{-1}Af = f \qquad (f \in H)$
 $A A^{-1}f = f \qquad (f \in R(A)).$ 90 J. Donie

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$$
\nThis definition differs from the one given in [3]
\nOur principal result is the following theorem.

This definition differs from the one given in [3].

Theorem 1: Suppose that A is invertible and onto. Then T(A) is invertible and onto if and only if there exists a bounded operator B in H which is invertible and maps $R(P^*)$ *onto R(P) and has the further property that AB is strongly accretive, e.g., for some* $\gamma > 0$ *, we have* ible and onto. The purpose of the presen
WH-operators. Before stating our main is
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P and has the further property that AB is st
 $||f||^2 \leq \gamma$ Re (ABf, f) $(f \in H)$.

Therefore is and if the property of

$$
||f||^2 \leq \gamma \operatorname{Re}\left(ABf, f\right) \qquad (f \in H). \tag{5}
$$

2. Preparatory Lemmata

We initially introduce some notation. Similar to the above definition of A_0 , we put
 $(A^*)_0 := A^* \upharpoonright R(P^*)$.

We further denote by $T(A)^*$ the adjoint of $T(A)$ acting in $R(P)$, i.e.

$$
T(A)^* := PP^+A^*P^* \upharpoonright R(P),
$$

which we compare with

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 $T_{P^*}(A^*) := P^*A^* \upharpoonright R(P^*)$.

Finally, we require a notion concerning the angle of two subspaces M and N of α Finally, we require a notion concerning the angle of two subsp
Hilbert space *H*. For this purpose, we define the number $T_{P^*}(A^*) := P^*A^* \upharpoonright R(P^*)$.

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where */* and *g* range over the unit balls of *M* and *N,* respectively. Following **HELSON** and SzEGÖ [5], we term *M* and *N* to be *at a positive angle,* if $\rho(M, N) < 1$.

In what follows, P stands for an operator having a Moore-Penrose inverse, and $T(A)$ denotes a GWH-operator

Lemma 1: *The operator* $T_{P^{\bullet}}(A^*)$ permits a bounded inverse if and only if $T(A)^*$ shares this property.

Proof: Let $T_{P\bullet}(A^*)$ have a bounded inverse. Then we can estimate, for an appro-
priate $\gamma > 0$, Finally, we require a notion concerning the angle of two subspaces M and N of a
Hilbert space H. For this purpose, we define the number
 $e(M, N) := \sup |(f, g)|$,
where f and g range over the unit balls of M and N, respectively. F where f and g range over the unit balls of M and N, respectively. Following HELSON
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In what follows, P stands for an operator having a Moore-Penros

This shows that $T(A)^*$ has the desired properties.

Invertibility of generalized Wiener-Hopf-operators 91
Dounded inverse, then there is a $\gamma > 0$ such that Conversely, if $T(A)^*$ has a bounded inverse, then there is a $\gamma > 0$ such that

$$
||P^*f|| \leq \gamma ||P|| \, ||T(A)^* f|| \leq \gamma ||P|| \, ||P^*|| \, ||T_{P^*}(A^*) \, P^*f|| \qquad (f \in R(P)).
$$

Invertibility of generalized Wiener-Hopf-operators 91

Conversely, if $T(A)^*$ has a bounded inverse, then there is a $\gamma > 0$ such that
 $||P^*f|| \le \gamma ||P|| ||T(A)^* f|| \le \gamma ||P|| ||P^*|| ||T_{P^*}(A^*) P^*f|| \qquad (f \in R(P)).$

Hence, using a result of $R(P^*)$ are representable in the form $g = P^*f$ with $f \in R(P)$, we arrive at. $||P*f|| \leq \gamma ||P|| ||T(A)*f|| \leq \gamma$
sing a result of GoHBERG [4:
expresentable in the form g
 $||g|| \leq \gamma' ||T_{P^{\bullet}}(A^*) g||$ $(g \in$
 > 0 . The lemma is therefore. Invertibility of generalized Wiener-Hopf-operators 91

Conversely, if $T(A)^*$ has a bounded inverse, then there is a $\gamma > 0$ such that
 $||P^*|| \le \gamma ||P|| ||T(A)^*|| \le \gamma ||P|| ||P^*|| ||T_{P^*}(A^*) P^*||$ $(f \in R(P))$.

Hence, using a result of Gon

$$
||g|| \leq \gamma' ||T_{P^*}(A^*) g|| \qquad (g \in R(P^*)),
$$

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ICONVERSELY, if $T(A)^*$ has a bounded inverse, then there is a
 $||P^*|| \le \gamma ||P|| ||T(A)^* ||| \le \gamma ||P|| ||P^*|| ||T_{P^*}(A^*) P^*||$

Hence, using a result of GOHBER Lemma 2: The operator $(T_P(A^*))^*$ permits a bounded inverse if and only if $T(A)$ has *this property.* Conversely, if $T(A)^*$ has a bounded inverse, then there is a $\gamma > 0$ such that
 $||P^*|| \le \gamma ||P|| ||T(A)^* || \le \gamma ||P|| ||P^*|| ||P_{P^*}(A^*) P^*||$ $(f \in R(P))$.

Hence, using a result of Gonsenc [4: Lemma 1] which states that all elements
 $R(P^*)$

The next stage in our development is to derive conditions in terms of A_0 , $(A^*)_0$, P , and P^* which guarantee that $T(A)$ is invertible and onto.

Lemma 3: The operator $T(A)$ permits a bounded inverse if and only of A_0 permits a bounded inverse, and if $R(A_0)$ and $N(P^+P)$ are at a positive angle.

Hence, using a result of Goursne, $f_A(P^*)$ are representable in the form $g = P^*f$ with $f \in R(P)$, we arrive at $||g|| \leq \gamma' ||T_{P^*}(A^*) g||$ $(g \in R(P^*))$,
where $\gamma' > 0$. The lemma is therefore established \blacksquare
In the same manner Proof: Assume first that *T(A)* has a bounded inverse. Obviously, then the same holds true for A_0 . In order to verify that $R(A_0)$ and $N(P^*P)$ are at a positive angle, $||g|| \leq \gamma' ||T_{P^*}(A^*) g||$ $(g \in R(P^*))$,

where $\gamma' > 0$. The lemina is therefore established **I**

In the same manner one proves the next lemma.

Lemma 2: The operator $(T_{P^*}(A^*))^*$ permits a bounded inverse if and only if T . > 0. The lemma is therefore established ■

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and P^* which guarantee that $T(A)$ is invertible and onto.
Lemma 3: The operator $T(A)$ permits a bounded inverse if and only of A
a bounded Let us suppose the contrary, i.e.
 $\rho(R(A_0), N(P^*P)) = 1.$

We show that for every $\varepsilon > 0$ one can find a $f \in R(P)$ satisfying rator $(T_{P^*}(A^*))^*$ permits a bounded inverse if and only if $T(A)$ has

ir development is to derive conditions in terms of A_0 , $(A^*)_0$, P ,

tee that $T(A)$ is invertible and onto.
 rator $T(A)$ permits a bounded inver

$$
\rho(R(A_0), N(P^+P)) = 1. \tag{6}
$$

$$
||f|| < \varepsilon, \qquad ||A_0f|| = 1, \tag{7}
$$

which is certainly inconsistent with the boundedness of *A*. To this end, we rewrite (6)
by noting that PP^+ and $I - P^+P$ are projections onto $R(P)$ and $N(P)$, respectively
[2]. We thus obtain
sup $\{[(A_0 f, (I - P^+P) g)] : f \in$ by noting that PP^+ and $I - P^+P$ are projections onto $R(P)$ and $N(P)$, respectively. and P^* which guarantee that $T(A)$ is invertible and onto.

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a bounded inverse, and if $R(A_0)$ and $N(P^*P)$ are at a positive angle.

Proof: Assume fi [2]. We thus obtain verify that $R($

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by noting that PP^+ and $I - P^+P$ are projections onto $R(P)$ and I
[2]. We thus obtain
sup $\left\{ \left\| (A_0 f, (I - P^+P) g) \right\| : f \in R(P), f \in N(P), \|A_0 f\| = 1, \|t\| \right$

$$
\sup\left\{\left|\left(A_0f,(I-P^*P)\,g\right)\right|:f\in R(P),\,f\in N(P),\,\|A_0f\|=1,\,\|g\|=1\right\}=1.\tag{8}
$$

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\sup \{[(A_0f, (I - P^+P)g)] : f \in R(P), f \in N(P), ||A_0f|| = 1, ||g|| = 1\} = 1. \tag{8}
$$
\n
$$
\text{Letting } \varepsilon > 0 \text{, we conclude from (8) that there exists an } f \in R(P) \text{ with } ||A_0f|| = 1 \text{, and}
$$
\n
$$
||(I - P^+P)A_0f|| \ge 1 - \varepsilon. \tag{9}
$$
\n
$$
\text{Hence, by employing the boundedness of } (T(A))^{-1} \text{ and inequality (9), we can estimate.}
$$

Hence, by employing the boundedness of $(T(A))^{-1}$ and inequality (9), we can estimate, for $v > 0$ Letting $\varepsilon > 0$, we conclude from (8) that there exists an $f \in R(P)$ with $||A_0f|| = 1$, and $||(I - P^*P) A_0f|| \ge 1 - \varepsilon$. (9)

Hence, by employing the boundedness of $(T(A))^{-1}$ and inequality (9), we can estimate, for $\gamma > 0$,
 $\frac{1}{2}$ Hence $\frac{1}{2}$ for $\frac{1}{2}$ This shave a $\frac{1}{2}$ have $\frac{1}{2}$ Then $\frac{1}{2}$

$$
||f||^2 \leq \gamma^2 ||T(A) f||^2 \leq \gamma^2 ||P||^2 (1 - ||(I - P^+P) A_0 f||^2) \leq 2\gamma^2 ||P||^2
$$

To prove sufficiency, we start out from the fact that, for $\gamma > 0$ and $\varepsilon \in (0, 1]$, we have $||f||^2 \leq \gamma^2 ||T(A)$
This shows that the desir
have a contradiction.
To prove sufficiency,
have
 $||f|| \leq \gamma ||A_0 f||$
Then we estimate
 $||f|| < \frac{\gamma}{\gamma}$ ($||A_0 f||$

$$
||f||^2 \leq \gamma^2 ||T(A) f||^2 \leq \gamma^2 ||P||^2 (1 - ||(I - P^+P) A_0 f||^2) \leq 2\gamma^2 ||P||^2 \varepsilon.
$$

shows that the desired element f which satisfies both conditions (9) exists, and we
a contradiction.
prove sufficiency, we start out from the fact that, for $\gamma > 0$ and $\varepsilon \in (0, 1]$, we

$$
||f|| \leq \gamma ||A_0 f|| \qquad (f \in R(P)), \text{ and } \varrho (R(A_0), N(P^+P)) = 1 - \varepsilon.
$$

we estimate

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that the desired element
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 which satisfies both conditions (c) class, and we
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\nWe sufficiency, we start out from the fact that, for $\gamma > 0$ and $\varepsilon \in (0, 1]$, we
estimate

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$$
||f|| \leq \gamma ||A_0 f|| \qquad (f \in R(P)), \text{ and } \varrho (R(A_0), N(P^+P)) = 1 - \varepsilon.
$$
\n
$$
||f|| \leq \frac{\gamma}{\varepsilon} (||A_0 f|| - (1 - \varepsilon) ||A_0 f||) = \frac{\gamma}{\varepsilon} (||A_0 f|| - \varrho (R(A_0), N(P^+P)) ||A_0 f||)
$$
\n
$$
\leq \frac{\gamma}{\varepsilon} (||A_0 f|| - ||(I - P^+P) A_0 f||) \leq \frac{\gamma}{\varepsilon} ||P^+|| ||T(A) f|| \qquad (f \in R(P)).
$$
\n, $T(A)$ has a bounded inverse, and the lemma is verified \blacksquare

Therefore, $T(A)$ has a bounded inverse, and the lemma is verified \blacksquare $\ddot{}$

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(i)
$$
T(A)
$$
 is invertible and onto,

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Corollary 1: *The following are equivalent:*
(i) $T(A)$ is invertible and onto,
(ii) $T_{p^*}(A^*)$ is invertible and onto,
(iii) A_2 and (A^*) , have a hounded inverse, and $R(A_2)$ and $N(P^*)$ 92 J. Donto

Corollary 1: The following are eq.

(i) $T(A)$ is invertible and onto,

(ii) $T_p(A^*)$ is invertible and onto,

(iii) A_0 and $(A^*)_0$ have a bounded if

and $N(PP^+)$ are at a positive angle. (ii) $T_{p^e}(A^*)$ *is invertible and onto*,
(iii) A_0 *and* $(A^*)_0$ *have a bounded inverse, and* $R(A_0)$ *and* $N(P^+P)$ *as well as* $R((A^*)_0)$ and $N(PP^+)$ are at a positive angle. 3. Dostary 1: The following are equivalent:

(i) $T(A)$ is invertible and onto,

(ii) $T_{p^*}(A^*)$ is invertible and onto,

(iii) A_0 and $(A^*)_0$ have a bounded inverse, and $R(A_0)$ and $N(P^*P)$

and $N(P^+')$ are at a posit Corollary 1: The following are equivalent:

(i) $T(A)$ is invertible and onto,

(ii) $T_p(A^*)$ is invertible and onto,

(iii) A_q and A^* h have a bounded inverse, and $R(A_0)$ and $N(P^*P)$ as well as $R((A^*)_0)$

and $N(PP^+)$

• Let $T(A)$ be invertible and onto. Then, by Corollary 1, $T_{P^{\bullet}}(A^{\bullet})$ is invertible and onto of Theorem 1
be invertible an
is invertible, we
 $A^* = US$,

$$
A^* = US.
$$

where *U* is unitary, and *S* is a bounded invertible operator mapping $R(P^*) = R(P^+P)$ $A^* = US$,
where U is unitary, and S is a bounded invertible operator mapping $R(P^*) = R(P^*P)$
onto itself. We thus conclude that $T_{P^*}(U)$ again is invertible and onto. Consequently,
by Corollary 1, we obtain
 $||(I - PP^+) U P^*P|| <$ (i) $T(A)$ is invertible and onto,

(ii) $T_{P}(A^*)$ is invertible and onto,

(iii) A_0 and $(A^*)_0$ have a bounded inverse, and $R(A_0)$ and $N(P^*P)$ as u

and $N(P^F^*)$ are at a positive angle.

3. Proof of Theorem 1

Let T by Corollary 1, $T_{P^*}(A^*)$ is invertible and onto.

lemma of SHINBROT [6: p. 400] asserting that

invertible operator mapping $R(P^*) = R(P^+P)$

U) again is invertible and onto. Consequently,
 $||PP^*U(I - P^+P)|| < 1.$ (10) and $N(PP^+)$ are at a positive angle.

3. Proof of Theorem 1

Let $T(A)$ be invertible and onto. Then, by Corollary 1, $T_{P^*}(A)$

Since A^* is invertible, we can apply a lemma of SHINBROT
 $A^* = US$,

where U is unitary, a **Theorem 1**
 2: invertible and onto. Then, by Corollary 1, $T_{P^*}(A^*)$ is invertible and onto.

invertible, we can apply a lemma of SHINBROT [6: p. 400] asserting that
 $f = US$,

initary, and *S* is a bounded invertible o

$$
||(I - PP^+)UP^+P|| < 1, \qquad ||PP^+U(I - P^+P)|| < 1.
$$
 (10)

$$
C := PP^+UP^+P + (I - PP^+) U(I - P^+P). \tag{11}
$$

We show that by (11) an operator C is defined in $H,$ which is invertible and onto. It is, as can easily be seen, enough to show that the operators *C*: $= PP^+UP^+P + (I - PP^+) U(I - P^+P).$

that by (11) an operator *C* is defined in *H*, which is invert

sily be seen, enough to show that the operators
 $C_1 := PP^+U \upharpoonright R(P^*), \qquad C_2 := (I - PP^+) U \upharpoonright R(P^*)$

tible and, respectively, man

$$
C_1 := PP^{\perp}U \upharpoonright R(P^*), \qquad C_2 := (I - PP^+) U \upharpoonright R(P^*)^{\perp}
$$

are invertible and, respectively, map $R(P^*)$ onto $R(P)$ and $R(P^*)$ ¹ onto $R(P)^1$. We limit ourselves to verifying this for C_1 . Suppose that there exists an $f \in R(P^*)$ satisfying. $||f|| = 1$ and $C_1 f = 0$. Then we obtain as can easily be scen, enough to show that the operators
 $C_1 := PP^t U \upharpoonright R(P^*)$, $C_2 := (I - PP^+) U \upharpoonright R(P^*)^{\perp}$

are invertible and, respectively, map $R(P^*)$ onto $R(P)$ and $R(P^*)^{\perp}$ onto $R(P^*)$

limit ourselves to verifying

$$
(I - PP^+) \; UP^+ P f = U f,
$$

whence we have

$$
|| (I - PP^+) \; UP^+ P f || = 1.
$$

whence we have
 $||(I - PP^+) UP^+Pf|| = 1.$

Since this contradicts to the first condition (10), C_1 is invertible. In order to verify
that C_1 is onto, one shows in the same manner as before that C_1 ^{*} is invertible. that C_1 is onto, one shows in the same manner as before that C_1^* is invertible.

Now it is not difficult to demonstrate, by using both conditions (10), that a $\gamma > 0$ exists satisfying 113 s contradicts to the first condition (10)

12 s onto, one shows in the same manner

is not difficult to demonstrate, by usin

12 is $\gamma((C^*U + U^*C) f, f)$ $(f \in H).$

13 := CS, then B meets the asserted pro (1963), 442-447.

Cerams Convarieties to the Handmoni (10), C_1 is invertible for that C_1 is into the shows in the same manner as before that C_1 ^{*} is invertible Now it is not difficult to demonstrate, by using bo

$$
||f||^2 \leq \gamma((C^*U + U^*C)f, f) \qquad (f \in H).
$$

Putting $B := CS$, then *B* meets the asserted-properties, and we obtain (5).

The sufficiency of (5) follows froni standard properties of strongly accretive opera tors in combination with the'mapping properties of *B.* The Theorem is.established. **^I**

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