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A Note on the Invertibility of Generalized Wiener-Hopf Operators

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Wir betrachten im Hilbertraum verallgemeinerte Wiener-Hopf-Operatoren der Form $T(A) := PA \upharpoonright R(P)$, wobei A und P lineare beschränkte Operatoren sind und P ferner eine Moore-Penrose-Inverse besitzt. Das Ziel der Untersuchungen besteht darin, ein Kriterium für die Invertierbarkeit von $T(A)$ in $R(p)$ anzugeben.

Исследуются обобщённые операторы Винера-Хопфа вида $T(A) := PA \upharpoonright R(P)$ в гильбертовом пространстве, причём A и P — ограниченные операторы и P имеет обратный Moore-Penrose оператор. Окончательая цель состоит в нахождении прямого и обратного значений инверсии $T(A)$.

We consider generalized Wiener-Hopf-operators of the form $T(A) := PA \upharpoonright R(P)$ in a Hilbert space, where A and P are linear, bounded operators, and P permits a Moore-Penrose inverse. The purpose of our investigations is to derive a criterion for the invertibility of $T(A)$ in $R(P)$.

1. Introduction

Let H be a Hilbert space, and suppose that A is a bounded linear operator in H and that, for the time being, P is an orthogonal projection onto a proper subspace M of H. Then the *part of A in M* is defined by

$$T(A) := PA_0, \tag{1}$$

where A_0 denotes the restriction of A onto $R(P)$ (the range of P). The operator $T(A)$, which we also write as $T_P(A)$ in order to indicate P, is well-known to be the prototype of a Wiener-Hopf operator. For example, if $\alpha \in \mathbf{R}$, $k \in L^1(\mathbf{R})$, and if A is the operator in $L^2(\mathbf{R})$ defined by

$$Af(x) := \alpha f(x) + \int_{\mathbf{R}} k(x-y)\, f(y)\, dy \qquad \left(f \in L^2(\mathbf{R}), x \in \mathbf{R}\right),$$

if further P is defined in $L^2(\mathbf{R})$ as the projection onto $L_+^2(\mathbf{R}) := \{f \mid f \in L^2(\mathbf{R}), f(x) = 0$ for $x < 0\}$, we obtain

$$T(A)\, g(x) := \alpha g(x) + \int_{\mathbf{R}_+} k(x-y)\, g(y)\, dy \qquad (g \in L_+^2(\mathbf{R}), x \in \mathbf{R}_+). \tag{2}$$

By the integral expression (2) a *special* Wiener-Hopf-operator $T(A)$ in $L_+^2(\mathbf{R})$ is defined [7].

Let us now in place of (2) consider an expression

$$T(A)\, g(x) := \alpha P^2 g(x) + P \int_{\mathbf{R}} k(x-y)\, Pg(y)\, dy \qquad \left(g \in L^2(\mathbf{R}), x \in \mathbf{R}\right), \tag{3}$$

where P is any bounded operator in $L^2(\mathbf{R})$ which has a (bounded) Moore-Penrose inverse P^+, e.g.

$$P = PP^+P. \tag{4}$$

Equation (4) is readily seen to be equivalent to $R(P)$ being closed (cf. Atkinson [1]). The operator $T(A)$ in $R(P)$ as defined by (3) is still of the form (1) except that P is in general not a projection. In this case, $T(A)$ is called a *generalized* Wiener-Hopf-operator (abbreviated, *GWH-operator*).

In a fundamental article Devinatz and Shinbrot [3] studied arbitrary Wiener-Hopf-operators (1) and derived necessary and sufficient conditions for these operators to be invertible and onto. The purpose of the present note is to establish such a criterion for GWH-operators. Before stating our main result, let us recall that a linear operator A in H is termed *invertible*, if there exists an operator $A^{-1}: R(A) \to H$ satisfying

$$A^{-1}Af = f \qquad (f \in H)$$

$$AA^{-1}f = f \qquad \big(f \in R(A)\big).$$

This definition differs from the one given in [3].

Our principal result is the following theorem.

Theorem 1: *Suppose that A is invertible and onto. Then $T(A)$ is invertible and onto if and only if there exists a bounded operator B in H which is invertible and maps $R(P^*)$ onto $R(P)$ and has the further property that AB is strongly accretive, e.g., for some $\gamma > 0$, we have*

$$\|f\|^2 \leqq \gamma \operatorname{Re}(ABf, f) \qquad (f \in H). \tag{5}$$

2. Preparatory Lemmata

We initially introduce some notation. Similar to the above definition of A_0, we put

$$(A^*)_0 := A^* \upharpoonright R(P^*).$$

We further denote by $T(A)^*$ the adjoint of $T(A)$ acting in $R(P)$, i.e.

$$T(A)^* := PP^+A^*P^* \upharpoonright R(P),$$

which we compare with

$$T_{P^*}(A^*) := P^*A^* \upharpoonright R(P^*).$$

Finally, we require a notion concerning the angle of two subspaces M and N of a Hilbert space H. For this purpose, we define the number

$$\varrho(M, N) := \sup |(f, g)|,$$

where f and g range over the unit balls of M and N, respectively. Following Helson and Szegö [5], we term M and N to be *at a positive angle*, if $\varrho(M, N) < 1$.

In what follows, P stands for an operator having a Moore-Penrose inverse, and $T(A)$ denotes a GWH-operator.

We first prove the following lemma.

Lemma 1: *The operator $T_{P^*}(A^*)$ permits a bounded inverse if and only if $T(A)^*$ shares this property.*

Proof: Let $T_{P^*}(A^*)$ have a bounded inverse. Then we can estimate, for an appropriate $\gamma > 0$,

$$\|f\| \leqq \|P^+\| \, \|P^*f\| \leqq \gamma \|P^+\| \, \|T_{P^*}(A^*) \, P^*f\| \leqq \gamma \|P^+\| \, \|P\| \, \|T(A)^*f\| \quad \big(f \in R(P)\big).$$

This shows that $T(A)^*$ has the desired properties.

Conversely, if $T(A)^*$ has a bounded inverse, then there is a $\gamma > 0$ such that

$$\|P^*f\| \leqq \gamma \|P\| \|T(A)^* f\| \leqq \gamma \|P\| \|P^+\| \|T_{P^*}(A^*) P^*f\| \quad (f \in R(P)).$$

Hence, using a result of GOHBERG [4: Lemma 1] which states that all elements g in $R(P^*)$ are representable in the form $g = P^*f$ with $f \in R(P)$, we arrive at

$$\|g\| \leqq \gamma' \|T_{P^*}(A^*) g\| \quad (g \in R(P^*)),$$

where $\gamma' > 0$. The lemma is therefore established ∎

In the same manner one proves the next lemma.

Lemma 2: *The operator $(T_{P^*}(A^*))^*$ permits a bounded inverse if and only if $T(A)$ has this property.*

The next stage in our development is to derive conditions in terms of A_0, $(A^*)_0$, P, and P^* which guarantee that $T(A)$ is invertible and onto.

Lemma 3: *The operator $T(A)$ permits a bounded inverse if and only of A_0 permits a bounded inverse, and if $R(A_0)$ and $N(P^+P)$ are at a positive angle.*

Proof: Assume first that $T(A)$ has a bounded inverse. Obviously, then the same holds true for A_0. In order to verify that $R(A_0)$ and $N(P^+P)$ are at a positive angle, let us suppose the contrary, i.e.

$$\varrho(R(A_0), N(P^+P)) = 1. \tag{6}$$

We show that for every $\varepsilon > 0$ one can find a $f \in R(P)$ satisfying

$$\|f\| < \varepsilon, \qquad \|A_0f\| = 1, \tag{7}$$

which is certainly inconsistent with the boundedness of A. To this end, we rewrite (6) by noting that PP^+ and $I - P^+P$ are projections onto $R(P)$ and $N(P)$, respectively [2]. We thus obtain

$$\sup \{|(A_0f, (I - P^+P) g)| : f \in R(P), f \in N(P), \|A_0f\| = 1, \|g\| = 1\} = 1. \tag{8}$$

Letting $\varepsilon > 0$, we conclude from (8) that there exists an $f \in R(P)$ with $\|A_0f\| = 1$, and

$$\|(I - P^+P) A_0f\| \geqq 1 - \varepsilon. \tag{9}$$

Hence, by employing the boundedness of $(T(A))^{-1}$ and inequality (9), we can estimate, for $\gamma > 0$,

$$\|f\|^2 \leqq \gamma^2 \|T(A) f\|^2 \leqq \gamma^2 \|P\|^2 (1 - \|(I - P^+P) A_0f\|^2) \leqq 2\gamma^2 \|P\|^2 \varepsilon.$$

This shows that the desired element f which satisfies both conditions (9) exists, and we have a contradiction.

To prove sufficiency, we start out from the fact that, for $\gamma > 0$ and $\varepsilon \in (0, 1]$, we have

$$\|f\| \leqq \gamma \|A_0f\| \quad (f \in R(P)), \quad \text{and} \quad \varrho(R(A_0), N(P^+P)) = 1 - \varepsilon.$$

Then we estimate

$$\|f\| \leqq \frac{\gamma}{\varepsilon} (\|A_0f\| - (1 - \varepsilon) \|A_0f\|) = \frac{\gamma}{\varepsilon} (\|A_0f\| - \varrho(R(A_0), N(P^+P)) \|A_0f\|)$$

$$\leqq \frac{\gamma}{\varepsilon} (\|A_0f\| - \|(I - P^+P) A_0f\|) \leqq \frac{\gamma}{\varepsilon} \|P^+\| \|T(A) f\| \quad (f \in R(P)).$$

Therefore, $T(A)$ has a bounded inverse, and the lemma is verified ∎

Corollary 1: *The following are equivalent:*

(i) $T(A)$ *is invertible and onto,*
(ii) $T_{P^*}(A^*)$ *is invertible and onto,*
(iii) A_0 *and* $(A^*)_0$ *have a bounded inverse, and* $R(A_0)$ *and* $N(P^+P)$ *as well as* $R\big((A^*)_0\big)$ *and* $N(PP^+)$ *are at a positive angle.*

3. Proof of Theorem 1

Let $T(A)$ be invertible and onto. Then, by Corollary 1, $T_{P^*}(A^*)$ is invertible and onto. Since A^* is invertible, we can apply a lemma of SHINBROT [6: p. 400] asserting that

$$A^* = US,$$

where U is unitary, and S is a bounded invertible operator mapping $R(P^*) = R(P^+P)$ onto itself. We thus conclude that $T_{P^*}(U)$ again is invertible and onto. Consequently, by Corollary 1, we obtain

$$\|(I - PP^+)\, UP^+P\| < 1, \qquad \|PP^+U(I - P^+P)\| < 1. \tag{10}$$

Let us now put

$$C := PP^+UP^+P + (I - PP^+)\, U(I - P^+P). \tag{11}$$

We show that by (11) an operator C is defined in H, which is invertible and onto. It is, as can easily be seen, enough to show that the operators

$$C_1 := PP^+U \restriction R(P^*), \qquad C_2 := (I - PP^+)\, U \restriction R(P^*)^\perp$$

are invertible and, respectively, map $R(P^*)$ onto $R(P)$ and $R(P^*)^\perp$ onto $R(P)^\perp$. We limit ourselves to verifying this for C_1. Suppose that there exists an $f \in R(P^*)$ satisfying $\|f\| = 1$ and $C_1 f = 0$. Then we obtain

$$(I - PP^+)\, UP^+Pf = Uf,$$

whence we have

$$\|(I - PP^+)\, UP^+Pf\| = 1.$$

Since this contradicts to the first condition (10), C_1 is invertible. In order to verify that C_1 is onto, one shows in the same manner as before that C_1^* is invertible.

Now it is not difficult to demonstrate, by using both conditions (10), that a $\gamma > 0$ exists satisfying

$$\|f\|^2 \leqq \gamma\big((C^*U + U^*C)\, f, f\big) \qquad (f \in H).$$

Putting $B := CS$, then B meets the asserted properties, and we obtain (5).

The sufficiency of (5) follows from standard properties of strongly accretive operators in combination with the mapping properties of B. The Theorem is established. ∎

REFERENCES

[1] ATKINSON, F. V.: On relatively regular operators. Acta Sci. Math. (Szeged) **15** (1953), 38—56.
[2] DESOER, C. A., and B. H. WHALEN: A note on pseudoinverses. SIAM J. Appl. Math. **11** (1963), 442—447.
[3] DEVINATZ, A., and M. SHINBROT: General Wiener-Hopf-operators. Trans. Amer. Math. Soc. **145** (1969), 467—494.

[4] GOHBERG, I.: On linear equations in Hilbert spaces. Dokl. Akad. Nauk SSSR 76 (1951), 9–12.
[5] HELSON, H., and G. SZEGÖ: A problem in prediction theory. Ann. Math. Pura Appl. (4), 51 (1960), 107–138.
[6] SHINBROT, M.: In singular integral operators. J. Math. Mech. 13 (1964), 395–406.
[7] WIENER, N., und E. HOPF: Über eine Klasse singulärer Integralgleichungen. Sitz.-Ber. Preuss. Akad. Wiss. 1931, 696–706.

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