Zeitschrift für Analysis und ihre Anwendungen Bd. 2 (2) (1983), S. $\tilde{97} - 109$

 (1.1)

General Random Sum Limit Theorems for Martingales with large $\mathcal{O}-$ Rates

PAUL L. BUTZER and D. SCHULZ¹)

Die vorliegende Arbeit beschäftigt sich mit groß 0 Fehlerabschätzungen für die Konvergenz in Verteilung von zufälligen Summen nicht notwendig unabhängiger Zufallsvariablen. Als Anwendungen eines allgemeinen Satzes werden sowohl Versionen des zentralen Grenzwertsatzes als auch des schwachen Gesetzes der großen Zahlen für Martingaldifferenzenfolgen im Falle der zufälligen Summation durch spezielle Wahl der Grenzzufallsvariablen hergeleitet. Beide Sätze' werden mit O-Konvergenzraten versehen.

Работа посвящена О-оценкам погрешности для сходимости в распределении случайных сумм из случайных величин, которые не обязательно независимы. Применением некоторой общей теоремы выводятся варианты центральной предельной теоремы и слабого закона больших чисел для разностного ряда мартингалов в случае случайного суммирования посредством частного выбора предельной случайной переменной. В обеих тео ремах даются О-оценки для скорости сходимости.

This paper is concerned with large- θ error estimates for convergence in distribution of random sums of not necessarily independent random variables. As applications of a general theorem one obtains the random-sum versions of the central limit theorem and of the weak law of large numbers for martingale difference sequences by specializing the limiting random variable. Both theorems are equipped with $\mathcal{O}\text{-rates.}$

Dedicated to the memory of WOLFGANG RICHTER (1932-1972), a scholar of the theory of randomly indexed random variables.

1. Introduction and History

The central limit theorem (CLT), perhaps the most important limit theorem of probability theory, may be formulated as follows: Let $(X_i)_{i\in\mathbb{N}}$ $(N = \{1, 2, ..., \})$ be a sequence of real, independent, square-integrable random variables (r.vs.) defined on a

probability space $(\Omega, \mathfrak{A}, P)$, let $S_n := \sum X_i$ denote its nth partial sum, $E[S_n]$ the expectation, and Var [S_n] the variance of S_n . Then $(X_i)_{i\in\mathbb{N}}$ is said to satisfy the CLT provided the sequence $P_{\bar{T}_n}$ of distributions of the normalized sums,

$$
\overline{T}_n := (S_n - E[S_n]) / (\text{Var}[S_n])^{1/2}
$$

converges weakly to the standard normal distribution, i.e.,

$$
\lim_{n \to \infty} E[f(\overline{T}_n)] = E[f(X^*)] \qquad (f \in C_B),
$$

¹) The research of the second named author was supported by DFG grant Bu 166/37.

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where X^* is the standard normally distributed r.v., and $C_B \equiv C_B(\mathbf{R})$ the class of all bounded, uniformly continuous, real-valued functions f defined on the axis \mathbf{R} , 98 Paul L. Burzer and D. Schulz
where X^* is the standard normally distributed r.v., and C_B
all bounded, uniformly continuous, real-valued functions f de
endowed with norm $||f||_{C_B} := \sup_{x \in \mathbf{R}} |f(x)|$.
In 1948 H. ROBBIN endowed with norm $||f||_{C_B} := \sup_{x \in \mathbb{R}} |f(x)|$.
In 1948 H. ROBBINS [37] gave sufficient conditions for the validity of the random-

sum CLT. He generalized the classical CLT in the sense that he replaced the index ² \ldots of S_n by a positive, N-valued r.v. N_{λ} depending on a parameter $\lambda \in \mathbb{R}^+$, the family $(N_{i})_{i\in\mathbb{R}^+}$ being defined on the same probability space (Q, \mathfrak{A}, P) as the sequence $(X_{i})_{i\in\mathbb{N}^+}$. Relation (1.1) in the case of the CLT for randomly indexed sequences of r.vs. reads B. H. ROBBINS [37] gave suttict.
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ed with norm $||f||_{C_B} := \sup_{x \in \mathbf{R}} |f(x)|$.

948 H. ROBBINS [37] gave sufficient conditions in LT. He

$$
\lim_{\lambda \to \infty} E[f(\overline{T}_{N_{\lambda}})] = E[f(X^*)] \qquad (f \in C_{\beta}), \qquad (1.2)
$$

Where $\overline{T}_{N_{\lambda}} := (S_{N_{\lambda}} - E[S_{N_{\lambda}}]) / (Var[S_{N_{\lambda}}])^{1/2}, S_{N_{\lambda}} := \sum_{i=1}^{N_{\lambda}} X_i.$

Where $\overline{T}_{N_{\lambda}} := (S_{N_{\lambda}} - E[S_{N_{\lambda}}]) / (Var[S_{N_{\lambda}}])^{1/2}, S_{N_{\lambda}} := \sum_{i=1}^{N_{\lambda}} X_i.$

Whereas H. ROBBINS [37] assumed the r.vs. N_{λ} , $\lambda \in \mathbb{R}^+$, to be independent of the X_i , $i \in N$, F. J. ANSCOMBE [2] was the first to consider conditions for the validity of the random CLT without this restriction. These two classical papers were followed up by a series of papers on limit theorems for randomly indexed sequences of r.vs., such as A. *RENYI [34],* J. R. BLUM, D. L. HANSON, J. ROSENBLATT [7], W. RICHTER $[36]$, S. H. Straždinov – G. Orazov [43], Z. Rychlik [39] and D. J. Aldous [1]. An excellent survey on limit theorems in this connection is to be-found in the Habilitation $-$ thesis of RICHTER [35].

Another possibility to generalize (1.1) consists in dropping the independency assumption upon the r.vs. X_i , $i \in N$. Since it is generally difficult to find sufficient conditions for the convergence of arbitrary dependent r.vs., one usually restricts oneself to particular types of dependency. in this respect *martingale difference sequences* (MDS) and martingale difference arrays have been examined to an especially great extent. The pioneering papers and books here are those of P. Lévy [29, 30, 31: p. 242] and J. L. DOOB [18: p 383], which were followed up by [6, 25, 19,'9, 42, 28, 21], for example. By a MDS is meant the following: Let $(X_i)_{i\in \mathbf{N}}$ be a sequence of real r.vs. defined on $(\Omega, \mathfrak{A}, P)$, and let $(\mathfrak{F}_i)_{i \in P}$ ($\mathbf{P} := \mathbf{N} \cup \{0\}$) be an increasing sequence of sub- σ algebras of $\mathfrak A$ such that X_i is $\mathfrak F_i$ -measurable for each $i \in \mathbb N$. Then $(X_i, \mathfrak F_i)_{i \in \mathbf P}$, $X_0 := 0$ is called a MDS if **Example 18**: p. 38

By a MDS is
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E[X_i \mid \mathfrak{F}_{i-1}] = 0 \quad \text{a.s.} \qquad (i \in \mathbb{N}). \tag{1.3}
$$

For the martingàle random OLT the reader is referred to [16, 32, 17, 40].

The paper by *M.* CsöRoö [16] cited is, according to the best of our knowledge, the first ever concerned with. the CLT for martingales in the case of randomly indexed r:vs. The first result dealing with rates of convergence for the CLT for martingales is apparently due to I.A. IBRAGIMOV (see $[25]$). The latter paper is the forerunner of a series of results in this field (see e.g. [24, 22,3,44, 15: p. 314, 26, 8, 4,38,23: Sect. 3.6, 14], the rates in [4] being established for \mathbb{R}^n -valued r.vs., and in [38, 14] for Banachspace valued r.vs.).

The purpose of this paper is to study rates of convergence for martingales in the instance of randomly. indexed sequences of r.vs., a topic that has so far been considered only by B. L. S. PRAXASA RAO [33] and' J. STROBEL in his dissertation [44]. More concretely, the aim, is to deduce thd random CLT as well as the random *weak law of large numbers* (WLLN), both taken with rates, as-'applications of one general theorem. To carry out this unified approach, conditions are given which lead to large- θ estimates for the difference $\begin{array}{l} \text{so} \text{RGO} \ [16] \ (\text{with the CL}) \end{array}$
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 $|E[f(T_{N_1})]-E[f(Z)]|$

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$$
T_{N_{\lambda}} := \varphi(N_{\lambda}) S_{N_{\lambda}},
$$

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for all $f \in C_B^r(\mathbf{R})$. Here
 $T_{N_A} := \varphi(N_A) S_{N_A}$,
 $(X_i)_{i \in \mathbf{P}}$ are the first components of the MDS $(X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}$, *C_B*(**R**). Here
 T<sub>N₄ := $\varphi(N_i) S_{N_A}$, (1.5)

re the first components of the MDS $(X_i, \mathfrak{F}_i)_{i \in P}, \varphi : N \to \mathbb{R}^+$ is a positive,

ing function, and Z is a limiting r, v, that is assumed to be φ -decomposable.</sub> $(X_i)_{i\in P}$ are the first components of the MDS $(X_i, \mathfrak{F}_i)_{i\in P}, \varphi : \mathbb{N} \to \mathbb{R}^+$ is a positive, normalizing function, and Z is a limiting r.v. that is assumed to be φ -decomposable. $T_{N_1} := \varphi(N_1) S_{N_2},$
 $(X_i)_{i \in \mathbf{P}}$ are the first components of the MDS $(X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}, \varphi : N \to \mathbf{R}^+$ is a positive,

normalizing function, and *Z* is a limiting r.v. that is assumed to be φ -decomposable such that the distribution P_z of Z can be represented as *Cr_B*(**R**). Here
 P_{N_A: $= \varphi(N_1) S_{N_2}$, (1.5)
 P_{NA}: $= \varphi(N_1) S_{N_2}$, (1.6)
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 $(T_{N_1}) := \varphi(N_1) S_{N_2},$

re the first components of the MDS $(X_i, \mathfrak{F}_i)_{i \in P}, \varphi : N \to \mathbb{R}^+$ is a positive,

ing function, and Z is a limiting r.v. that is assumed to be φ -decomposable.

ns t $\{X_i\}_{i\in P}$ are the first components of the MDS $(X_i, \mathfrak{F}_i)_{i\in P}, \varphi : N \to \mathbb{R}^+$ is a position formalizing function, and Z is a limiting r.v. that is assumed to be φ -decomposition.
This means that for each $n \in \math$

$$
P_Z = P_{\varphi(n)\sum_{i=1}^n Z_i}
$$

Furthermore, for $r \in P$ we have set $C_B^0(\mathbf{R}) = C_B(\mathbf{R}),$

$$
C_B^r(\mathbf{R}) := \{f \in C_B(\mathbf{R}) : f^{(j)} \in C_B(\mathbf{R}), \quad 1 \leq j \leq r\}.
$$

For details concerning the relationship between the concepts of φ -decomposability •.

and infinite divisibility see [14].
In the sequel it will always be assumed that the r.vs. N_i , $\lambda \in \mathbb{R}^+$, and X_i , $i \in \mathbb{N}$, are independent, and that $N_{\lambda} \rightarrow \infty$ in probability for $\lambda \rightarrow \infty$.

Since the limiting r.v. *Z* can be chosen rather generally, in particular as the Gaussian $C_B^r(\mathbf{R}) := \{f \in C_B(\mathbf{R}) : f^{(i)} \in C_B(\mathbf{R}), \quad 1 \leq j \leq r\}.$ (1.7)

For details concerning the relationship between the concepts of φ -decomposability

and infinite divisibility see [14].

In the sequel it will always be a rates will be deduced as particular cases. The latter does not seem to have been con- $\Gamma_Z = \Gamma_{\varphi(n)\sum Z_i}$.

Furthermore, for $r \in P$ we have set C_B^0
 $C_B^r(R) := \{f \in C_B(R) : f^{(i)} \in C_B$

For details concerning the relationship

and infinite divisibility see [14].

In the sequel it will always be assum

independe General random sum limit theorems

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for all $f \in C'_\delta(\mathbf{R})$. Here
 $T_{N_1} := \varphi(N_1) S_{N_1}$,
 $(X_i)_{i\in \mathbf{R}}$ are the first components of the MDS; $(X_i, \mathfrak{F}_i)_{i\in \mathbf{P}}, \sigma : \mathbf{N} \to \mathbf{R}^+$ is a positive,

increase that

Inspite of our general approach for MDS, our convergence rates are even better than those for sums of independent r.vs. due to Z . RYCHLIK and D. SZYNAL [41] on account of our use of K-functional methods. They are indeed just as sharp as those of P. L. BUTZER and L. HAHN $[11, 12]$ in the case of non-random summation of independent r.vs. Returning to, the proofs again; our main theorem is based, upon a modification of the Trotter operator-theoretic method to the situation of not necessarily independent r.vs. as already applied in [3, 141. This time it is tailored to the situation of randomly indexed r.vs. X_i which are independent of the index variable N_i , $\lambda \in \mathbb{R}^+$. **P. L. BUTZER and L. HAHN [11, 12]** in th
pendent r.vs. Returning to the proofs aga
ification of the Trotter operator-theoretic
independent r.vs. as already applied in [3]
of randomly indexed r.vs. X_i which are in
As an count of our use of *K*-functional methods. They are indeed just as sharp as those of *L.* BUTZER and *L.* HAHN [11, 12] in the case of non-random summation of indent r.vs. Returning to the proofs again, our main theorem begins of the Trotter operator-theodology
dependent r.vs. as already applied in andomly indexed r.vs. X_i which are
As an illustration of our results let
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Let $(X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}$ be a MDS, and le THER and **r**, w.s. Re
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 σ $E[X_i^i]$
 $E[X_i^i]$ then, $E[X_i] \leq E[X_i] \leq E[X_i]$
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As an illustration
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As an illustration of our results let us formulate a particular case of our random T with θ -rates: V V

 $V \sim \alpha \geq 0.$ if $f \in \text{Lip}(\alpha, 0, 0, p)$ (see (2.0)) and

(1.8) V V $\begin{array}{l} \text{ion of} \ \text{s:} \ \text{be a} \ < \infty \ \text{s:} \ \text{for } \mathbb{R} \ \text$ $(i \in \mathbb{N})$, \bullet \bullet \bullet \bullet \bullet \bullet ^VV, ^V ' ' ' ' ' ^V . *E*[$|X_i|^3$] $< \infty$ (*i* \in N),
 $E[X_i|^3]$ $< \infty$ (*i* \in N),
 $E[X_i^j | \mathfrak{F}_{i-1}] = E[X^*j]$ a.s. $(1 \leq j \leq 2, i \in \mathbb{N}),$

as well as

$$
\mathbb{E}[X_i^j \mid \mathfrak{F}_{i-1}] = E[X^{*j}] \quad \text{a.s.} \qquad (1 \leq j \leq 2, i \in \mathbb{N}), \tag{1.9}
$$

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CLI with
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$$
-rates:

\nLet $(X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}$ be a MDS, and let $0 < \alpha \leq 3$. If $f \in \text{Lip}(\alpha; 3, C_B)$ (see (2.3)) and $E[|X_i|^3] < \infty$ (i $\in \mathbf{N}$),

\nas well as

\n
$$
E[X_i^* | \mathfrak{F}_{i-1}] = E[X^{*j}] \quad \text{a.s.}
$$
\n(1 \leq j \leq 2, i $\in \mathbf{N}$),

\nthen

\n
$$
|E[f(S_{N_A}/|\overline{N_A})] - E[f(X^*)]]
$$
\n
$$
\leq C_f \left\{ E\left[N_i^{-3/2} \sum_{i=1}^{N_A} (E[|X_i|^3] + E[|X^*|^3])\right] \right\}^{a/3}
$$
\n(\lambda \to \infty).

\n(1.10)

\nIn particular, if the r.vs, X_i are identically distributed, then the order of approximation in (1.10) is $\mathcal{O}([E[N_i^{-1/2}])^{a/3})$.

\nIn the case of independent, identically distributed r.vs. X_i , $i \in \mathbf{N}$, Z. RVCALI and D. SZYNAL [41] deduced the rate $\mathcal{O}(E[N_i^{-\beta/2}])$ with $0 < \beta \leq 1$ under corresponding assumptions upon the moments of X_i .

\nBy applying a result of V. M. ZOLOTAREV [47] on the Kolmogorov distance between the distribution functions F_X and F_Y of two r.vs. X and Y (see (4.2) below), the follow-

In particular, if the r.vs. X_i are identically distributed, then the order of approx-*V V*

By applying a result of V. M. ZOLOTAREV [47] on the Kolmogorov distance between the distribution functions F_X and F_Y of two r.vs. X and Y (see (4.2) below), the follow-

ing estimate concerning the strong convergence of distributions can be derived from

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ing estimate concerning the strong co

(1.10):

If $(X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}$ is a MDS and $f \in \text{Lip}$

(1.8), (1.9)
 \cdot $\sup_{t \in \mathbf{R}} |F_{S_{N_i}/\sqrt{N_i}}(t) - F_{X^*}(t)| \le$

10):
\nIf
$$
(X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}
$$
 is a MDS and $f \in$ Lip (3; 3; C_B), then one has under the assumptions
\n8), (1.9)
\n
$$
\sup_{t \in \mathbf{R}} |F_{S_{N_\lambda}/\sqrt{N_\lambda}}(t) - F_{X^*}(t)| \leq M \left\{ E \left[N_\lambda^{-3/2} \sum_{i=1}^{N_\lambda} (E[|X_i|^3] + E[|X^*|^3]) \right] \right\}^{1/4},
$$

and in the case of identically distributed r.vs. $(X_i)_{i\in \mathbf{P}}$ this estimate is of order $\mathcal{O}(\lambda^{-1/8})$ $\sup_{t \in \mathbf{R}} |F_{S_{N_{\lambda}}/(\sqrt{N_{\lambda}})}(t) - F_{X^*}(t)| \leq \lambda$
and in the case of identically distribut
 $\lambda \to \infty$, provided $E[N_{\lambda}^{-1/2}] = \mathcal{O}(\lambda^{-1/2})$.
The best possible convergence rate

The best possible convergence rate reached by D. **LANDERS** and L. ROGGE [27] in the case of independent, identically distributed r.vs., namely $\mathcal{O}(\lambda^{-1/2})$, cannot be achieved by our methods. This is due to the **ZOLOTABEV** estimate (see [47, 48]) used (see also Sec. 4).

Section 2 is concerned with questions of notation, the K-functional, moduli of continuity and Lipsehitz classes. Setion 3 is devoted to our general theorem on the convergence in distribution of the r.vs. T_{N_λ} towards a φ -decomposable r.v. Z. In Section 4 this theorem is applied to the strong convergence in distribution, and in • Sections 5 and 6 to the random CLT and WLLN, respectively. *Sec.* 4).

2 is concerned with questions of notation, the *K*-fu
 N and Lipschitz classes. Section 3 is devoted to our gen

ince in distribution of the r.vs. T_{N_λ} towards a φ -decon

this theorem is applied to t

2. Notations and Preliminaries

The K-funktional and modulus of continuity, defined in terms of the spaces C_B and C_B ^r, $r \in N$ (cf. (1.1) and (1.7)), need to be recalled. For any $f \in C_B$ and $t \geq 0$ the former' is defined by

$$
K(t; f; C_B, C_B^{\mathsf{T}}) := \inf_{g \in C_B^{\mathsf{T}}} \{ ||f - g||_{C_B} + t |g|_{C_B^{\mathsf{T}}}\},\tag{2.1}
$$

If C_B ^{*r*}, given by $|g|_{C_B}$ ^{*r*} := $||g^{(r)}||_{C_B}$, and the rth modulus of continuity by

i (cf. (1.1) and (1.7)), need to be recalled. For any
$$
f
$$

d by
\n $K(t; f; C_B, C_B^r) := \inf_{g \in C_B^r} {\left\{ ||f - g||_{C_B} + t |g|_{C_B^r} \right\}},$
\nthe semi-norm on C_B^r , given by $|g|_{C_B^r} := ||g^{(r)}||_C$
\ny by
\n $\omega_r(t; f; C_B) := \sup_{\{h\| \le t} \left\| \sum_{k=0}^r (-1)^{r-k} {r \choose k} f(u + kh) \right\}.$
\n $f \in C_B$ and each $t \ge 0$ the K-functional is equiv

For each $f \in C_B$ and each $t \geq 0$ the K-functional is equivalent to this modulus (see $[10:$ pp. $.192, 258]$), i.e., there are positive constants $c_{1,\tau}$ and $c_{2,\tau}$, independent of f and $t\geqq 0$, such that *k*(*t*; *f*; *C_B*, *C*_B^{*r*}) := inf_{{a}f} (||*f* - *g*||*c*_a^{*t*} + *l g*|_{*C*_a^{*t*}}, and the *r*th modulus of *y* by
 *w*_{*t*}(*t*; *f*; *C_B*) := sup $\left\| \sum_{i=0}^{r} (-1)^{r-k} {r \choose k} f(u + kh) \right\|$.
 f $\in C_B$ an For each $f \in C_B$ and each $t \ge 0$ the K-functional is equivalent to this modulus (see
 $[10: pp. 492, 258]$), i.e., there are positive constants $c_{1,r}$ and $c_{2,r}$, independent of f and
 $t \ge 0$, such that
 $c_{1,r}\omega_r(t^{1/r}; f;$ the semi-norm on C_B ^r, given by $|g|_{C_B}$: $= ||g^{(r)}||_{C_B}$, and the *r*th modulus of
y by
 $\omega_r(t; f; C_B) := \sup_{\|h\| \le t} \left\| \sum_{i=0}^r (-1)^{r-k} {r \choose k} f(u + kh) \right\|.$
 $f \in C_B$ and each $t \ge 0$ the *K*-functional is equivalent to this mo

$$
c_{1,r}\omega_r(t^{1/r};f;C_B) \le K(t;f,C_B,C_B^r) \le c_{2,r}\omega_r(t^{1/r};f;C_B). \tag{2.2}
$$

$$
\text{Lip } (\alpha; r; C_B) := \{ f \in C_B : \omega_r(t; f; C_B) \leq L_f t^{\alpha}, t > 0 \},\tag{2.3}
$$

 L_f being the Lipschitz constant.

The concept of φ -decomposability, defined in (1.6), can be extended to randomly indexed r.vs. since the range of the index variable N_i is a subset of N. In fact, if Z is a [10: pp. 492, 258]), i.e., there are positive
 $t \geq 0$, such that
 $c_1, \omega_r(t^{1/r}; f; C_B) \leq K(t; f, C_B,$

This enables one to define a Lipschitz c

Lip $(\alpha; r; C_B) := \{f \in C_B : \omega_r(t)\}$
 L_f being the Lipschitz constant.

The concept $\text{Lip } (\alpha; r; C_B): \ L_f \text{ being the Lipschitz of \mathcal{P}-deconcept of \mathcal{P}-decomposable r-} \nu, \ \text{the \mathcal{P}-decomposable r-} \nu, \ \text{the \mathcal{P}-} \ P_Z = P_{\varphi(N_A)} \sum_{i=1}^{N_A} \nu_i.$ **Pa,** 2.388]), i.e., there are positive constants c_1 , and c_2 ,, independent of f and
 c_1 , $\omega_r(t^{1/r}; f; C_B) \leq K(t; f, C_B, C_B^r) \leq c_2$, $\omega_r(t^{1/r}; f; C_B)$. (2.2)

bles one to define a Lipschitz class of index $r \in \mathbb{N}$

$$
P_Z=P_{\varphi(N_A)}\sum_{i=1}^{N_A}Z_i\qquad (\lambda\in\mathbf{R}^+).
$$

If $p_n = p_n(\lambda)$ denotes the probability with which the index variable N_λ takes on the General random sum limit theorems 10

If $p_n = p_n(\lambda)$ denotes the probability with which the index variable N_λ takes on th

value $n \in \mathbb{N}$; then $\sum_{n=1}^{\infty} p_n = 1$, and so (2.4) yields that
 $P_z = \sum_{n=1}^{\infty} p_n P_{\varphi(n)}$ *i* the inde that $\frac{1}{2}$

$$
P_Z = \sum_{n=1}^{\infty} p_n P_{\varphi(n)} \sum_{i=1}^n Z_i
$$

provided the r.vs. Z_i , $i \in N$, are independent of N_i for each $\lambda \in \mathbb{R}^+$. Likewise one has

General random sum limit theorems
\n101
\nIf
$$
p_n = p_n(\lambda)
$$
 denotes the probability with which the index variable N_1 takes on the
\nvalue $n \in N$; then $\sum_{n=1}^{\infty} p_n = 1$, and so (2.4) yields that
\n
$$
P_Z = \sum_{n=1}^{\infty} p_n P_{\varphi(n)} \sum_{i=1}^n Z_i
$$
\nprovided the r.vs. Z_i , $i \in N$, are independent of N_1 for each $\lambda \in \mathbb{R}^+$. Likewise one has
\nfor the expectations $E[Z]$ and $E[T_{N_1}]$ of Z and T_{N_2} (recall (1.5)), respectively,
\n
$$
E[Z] = \sum_{n=1}^{\infty} p_n E\left[\varphi(n) \sum_{i=1}^n Z_i\right],
$$
\n(2.6)
\n
$$
E[T_{N_2}] = \sum_{n=1}^{\infty} p_n E[T_n].
$$

3. General Convergence Theorem for MDS with Rates

This section is concerned with the general limit theorem with rates described in the introduction. The proof is based upon the Trotter operator approach, first applied to the CLT in [45]. Hcivever, it has to be generalized so as to be applicable to MDS in. the case of random summation instead of just sequences of independent r.vs. Although the proof may appear rather long and technical, it is nevertheless elementary. It uses Taylor series expansions, the operational rules for conditional expectations, and standard K-functional arguments of approximation theory. $E[T_{N_1}] = \sum_{n=1} p_n E[T_n].$
 3. General Convergence Theorem for MDS with Rate

This section is concerned with the general limit the

introduction. The proof is based upon the Trotter of

the CLT in [45]. However, it has to phonomerned with the genes e proof is based upon the However, it has to be m summation instead of ppear rather long and is expansions, the operational arguments of applement of a photon of $Let (X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}$ be a *i* just sequences of independent r.vs. Although
 i technical, it is nevertheless elementary. It

attional rules for conditional expectations, and

proximation theory.
 MDS and Z be a q-decomposable r.v. with
 $\therefore E[Z_i$ *E*₁, *E*₂, *E*₂ with the general limit theorem with rates described in the
 *E*₁ is based upon the Trotter operator approach, first applied to
 *F*₁, it has to be generalized so as to be applicable to MDS in
 *E*₁, it has to be gen

Theorem 1: Let $(X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}$ be a MDS and Z be, a φ -decomposable r.v. with

$$
E[Z] = 0, \text{ such that}
$$

\n
$$
\zeta_{r,i} := E[|X_i|^r] < \infty, \qquad \xi_{r,i} := E[Z_i|^r] < \infty \qquad (i \in \mathbb{N}) \qquad (3.1; 3.2)
$$

\nfor some $r \in \mathbb{N}, r \ge 2$, as well as

$$
E[X_i^j \mid \mathfrak{F}_{i-1}] = E[Z_i^j] \quad \text{a.s.} \quad (i \in \mathbb{N}, 1 \leq j \leq r-1). \tag{3.3}
$$

a) *Under these hypotheses one has for* $f \in C_B$

standard K-functional arguments of approximation theory.
\nTheorem 1: Let
$$
(X_i, \tilde{y}_i)_{i \in \mathbf{P}}
$$
 be a MDS and Z be a φ -decomposable r.v. with
\n $E[Z] = 0$, such that
\n $\zeta_{r,i} := E[|X_i|^r] < \infty$, $\zeta_{r,i} := E[Z_i|^r] < \infty$ $(i \in \mathbf{N})$ $(3.1; 3.2)$
\nfor some $r \in \mathbf{N}, r \geq 2$, as well as
\n $E[X_i^j | \tilde{y}_{i-1}] = E[Z_i^j]$ a.s. $(i \in \mathbf{N}, 1 \leq j \leq r - 1)$ (3.3)
\na) Under these hypotheses one has for $f \in C_B$
\n $|E[(T_{N_A})] - E[f(Z)]| \leq c_{2,r}\omega_r \left(\left\{ E\left[(\varphi(N_i))^r \sum_{i=1}^{N_A} (\zeta_{r,i} + \zeta_{r,i}) \right] \right\}^{i/r}; f; C_B \right),$
\n $c_{2,r}$, being the constant of (2.2). In particular, if $f \in Lip(\alpha; r; C_B), \alpha \in (0, r]$, then
\n $|E[(T_{N_A})] - E[f(Z)]| \leq 2c_{2,r}L_f \left\{ E\left[(\varphi(N_i))^r \sum_{i=1}^{N_A} (\zeta_{r,i} + \zeta_{r,i}) \right] \right\}^{a/r}$ (3.5)
\nb) If the r.v.s. X_i , $i \in \mathbf{N}$, as well as the decomposition components Z_i , $i \in \mathbf{N}$, are identically distributed, then
\n $|E[f(T_{N_A})] - E[f(Z)]| \leq 2c_{2,r}\omega_r \left\{ \left\{ (\zeta_{r,1} + \zeta_{r,1}) \right\} [(\varphi(N_i))^r N_i] \right\}^{1/r}; f; C_B \right\}$ (3.6)

 c_2 , being the constant of (2.2). In particular, if $f \in \text{Lip } (x; r; C_B)$, $\alpha \in (0, r]$, then

/r E[((p(NA))r NA]}Ir . . ^S , . (3.7)

b) If the r.vs. X_i , $i \in \mathbb{N}$, as well as the decomposition components Z_i , $i \in \mathbb{N}$, are identi-

$$
|E[f(T_{N_{\lambda}})] - E[f(Z)]| \leq 2c_{2,r}\omega_{r}(\{(\zeta_{r,1} + \xi_{r,1}) E[(\varphi(N_{\lambda}))^{r} N_{\lambda}]\}^{1/r}; f; C_{B}).
$$
\n(3.6)

$$
|E[f(T_{N_{\lambda}})] - E[f(Z)]| \leq 2c_{2,r}L_{f} \left\{ E\left[(\varphi(N_{\lambda}))^{r} \sum_{i=1}^{N_{\lambda}} (\zeta_{r,i} + \xi_{r,i}) \right] \right\}^{a/r}.
$$
 (3.5
\n
$$
|E[f(T_{N_{\lambda}})] - E[f(Z)]| \leq 2c_{2,r}\omega_{r} \left\{ \left[(\zeta_{r,1} + \xi_{r,1}) E[(\varphi(N_{\lambda}))^{r} N_{\lambda}] \right\}^{1/r}; f; C_{B} \right\}.
$$
 (3.5
\n
$$
|E[f(T_{N_{\lambda}})] - E[f(Z)]| \leq 2c_{2,r}\omega_{r} \left\{ \left((\zeta_{r,1} + \xi_{r,1}) E[(\varphi(N_{\lambda}))^{r} N_{\lambda}] \right\}^{1/r}; f; C_{B} \right\}.
$$
 (3.6
\n
$$
|E(f) \left(\alpha; r; C_{B} \right), 0 < \alpha \leq r, \text{ then the left side of (3.6) has the bound}
$$

\n
$$
2c_{2,r}L_{f} \left\{ (\zeta_{r,1} + \xi_{r,1}) E[(\varphi(N_{\lambda}))^{r} N_{\lambda}] \right\}^{a/r}.
$$
 (3.7)

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Proof: a) Let $f \in C_B$, $g \in C_B$ be arbitrary. Making use of the fact that **PAUL L. BUTZER and D. SCHULZ**
 Proof: a) Let $f \in C_B$, $g \in C_B$ ^t be arbitra $|E[f(Z)]| \leq ||f||_{C_B}$, for all $f \in C_B$ and any *r.v. Z,*

Paul L. ButzER and D. SCHULZ
\n
$$
f: a
$$
) Let $f \in C_B$, $g \in C_B^*$ be arbitrary. Making use of the fact that
\n≤ $||f||c_n$, for all $f \in C_B$ and any r.v. Z,
\n $|E[f(T_{N_A})] - E[f(Z)]|$
\n≤ $|E[f(T_{N_A})] - E[g(T_{N_A})]| + |E[g(T_{N_A})] - E[g(Z)]| + |E[g(Z)] - E[f(Z)]|$
\n≤ 2 $||f - g|| + |E[g(T_{N_A})] - E[g(Z)]|$. (3.8)
\nto estimate the second term, if $(Z_i)_{i \in N}$ is the sequence of independent r.r.s.
\nnot only can the $(Z_i)_{i \in N}$ be chosen to be independent of the r.s. N_i , $i \in R^+$
\nper choice of the underlying graphsblitt space (a f. 50 · p. 79(80), but also

In order to estimate the scond term, if $(Z_i)_{i\in\mathbb{N}}$ is the sequence of independent r.vs. of (1.6), not only can the, $(Z_i)_{i\in\mathbb{N}}$ be chosen to be independent of the r.vs. N_i , $\lambda \in \mathbb{R}^+$ by a proper choice of the underlying probability space (c.f. [20: p. 79/80]), but also $\leq P(f(X_N)) - E[g(X_N)] + |E[g(Y_{N_A})] - E[g(Z)]|$. (3.8)
 $\leq 2 ||f - g|| + |E[g(T_{N_A})] - E[g(Z)]|$. (3.8)

In order to estimate the scond term, if $(Z_i)_{i \in N}$ is the sequence of independent r.v

of (1.6), not only can the $(Z_i)_{i \in N}$ be chosen to be $\leq 2 ||f - g|| + |E[g(T_{N_2})] - E$
In order to estimate the scond term, if
of (1.6), not only can the $(Z_i)_{i \in \mathbb{N}}$ be chore
by a proper choice of the underlying p
the *σ*-algebras generated by the Z_i can
algebras \mathfrak{F}_i , **i**
 i order to estimate the scond term, if $(Z_i)_{i\in\mathbb{N}}$ is the sequence of independent r.vs.

(1.6), not only can the $(Z_i)_{i\in\mathbb{N}}$ be chosen to be independent of the r.vs. N_i , $\lambda \in \mathbb{R}^+$
 i a proper choic or (1.0), not only can the $(Z_i)_{i \in \mathbb{N}}$ be cho
by a proper choice of the underlying problem of the dependence of the underlying problem
the σ -algebras generated by the Z_i can
algebras \mathfrak{F}_i , $i \in \mathbb{N}$.
Sett $|E[f(T_{N_A})] - E[f(Z)]|$
 $\leq |E[f(T_{N_A})] - E[g(T_A)]$
 $\leq |E[f(T_{N_A})] - E[g(T_A)]$
 $\leq 2 ||f - g|| + |E[g(T_A)]$

In order to estimate the scond of (1.6), not only can the $(Z_i)_{i \in I}$

by a proper choice of the under

the σ -algebras generated by th

Taylor's formula for $g \in C_B$ ^r yields

Table 2 The graph of the graph. Since *g* ∈ *C_B* is the *g*(*T_n*) − *g*(*φ*(*n*) *Q_{n,i}* + *φ*(*n*) *X_i*) − *g*(*φ*(*n*) *R_{n,i}* + *φ*(*n*) *Z_i*)\n
$$
= \sum_{i=1}^{n} \sum_{j=1}^{r-1} \left\{ g(v)(p(n) R_{n,i}) \frac{(p(n) X_i)^j}{j!} - g^{(i)}(p(n) R_{n,i}) \frac{(p(n) Z_i)^j}{j!} \right\} + \sum_{i=1}^{n} \frac{1}{(r-2)!} \int_{0}^{1} (1-t)^{r-2} \left\{ g^{(r-1)}(p(n) R_{n,i} + t\varphi(n) X_i) - g^{(r-1)}(p(n) R_{n,i}) \right\} (\varphi(n) X_i)^{r-1} dt - \sum_{i=1}^{n} \frac{1}{(r-2)!} \int_{0}^{1} (1-t)^{r-2} \left\{ g^{(r-1)}(p(n) R_{n,i} + t\varphi(n) Z_i) - g^{(r-1)}(p(n) R_{n,i}) \right\} (\varphi(n) Z_i)^{r-1} dt.
$$
\nSince *g* ∈ *C_B* for one has *g*(*r* − 1) ∈ Lip (1; 1; *C_B*) with *L_g* := ||*g*(*r*)||, and so for 0 < *t* ≤ 1,
$$
\left\{ \left\{ g^{(r-1)}(p(n) R_{n,i} + t\varphi(n) X_i) - g^{(r-1)}(p(n) R_{n,i}) \right\} (\varphi(n) X_i)^{r-1} \right\} \leq ||g^{(r)}|| (\varphi(n)) |Y| X_i |^r
$$
 a.s., and analogously for the r.v.s. *Z_i*. In view of (2.5) and (2.6), this leads to

$$
\left| \{ g^{(r-1)}(\varphi(n) \, R_{n,i} + t \varphi(n) \, X_i) - g^{(r-1)}(\varphi(n) \, R_{n,i}) \} \, (\varphi(n) \, X_i)^{r-1} \right| \\ \leq ||g^{(r)}|| \, (\varphi(n))^r \, |X_i|^r \quad \text{a.s.,}
$$

and analogously for the r.vs. Z_i . In view of (2.5) and (2.6) this leads to

$$
-\sum_{i=1}^{n} \frac{1}{(r-2)!} \int_{0}^{1} (1-t)^{r-2} \{g^{(r-1)}(\varphi(n) R_{n,i} + t\varphi(n) Z_{i})
$$

\n
$$
-g^{(r-1)}(\varphi(n) R_{n,i})\} (\varphi(n) Z_{i})^{r-1} dt.
$$

\nSince $g \in C_{B}$ one has $g^{(r-1)} \in$ Lip (1; 1; C_{B}) with $L_{g} := ||g^{(r)}||$, and so for $0 < t \le 1$,
\n
$$
|\{g^{(r-1)}(\varphi(n) R_{n,i} + t\varphi(n) X_{i}) - g^{(r-1)}(\varphi(n) R_{n,i})\} (\varphi(n) X_{i})^{r-1}|
$$

\n
$$
\leq ||g^{(r)}|| (\varphi(n))^{r} |X_{i}|^{r} \text{ a.s.},
$$

\nand analogously for the r.s. Z_{i} . In view of (2.5) and (2.6) this leads to
\n
$$
|E[g(T_{N_{\lambda}})] - E[g(Z)]| \leq \sum_{n=1}^{\infty} p_{n} \left| E[g(T_{n})] - E\left[g\left(\varphi(n) \sum_{i=1}^{n} Z_{i}\right)\right] \right|
$$

\n
$$
\leq \sum_{n=1}^{\infty} p_{n} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{r-1} \left| E\left[g^{(j)}(\varphi(n) R_{n,i}) \frac{\left(X_{i}\varphi(n)\right)^{j}}{j!}\right] \right|
$$

\n
$$
- g^{(j)}(\varphi(n) R_{n,i}) \frac{\left(Z_{i}\varphi(n)\right)^{j}}{j!} \right|
$$

\n
$$
+ \frac{||g^{(r)}||}{(r-1)!} (\varphi(n))^{r} \sum_{i=1}^{n} \left(\zeta_{r,i} + \xi_{r,i}\right).
$$
 (3.9)

Let us now show that

$$
E[X_i ig^{(i)}(\varphi(n) R_{n,i})]
$$

= $E[Z_i ig^{(i)}(\varphi(n) R_{n,i})]$ $(1 \le i \le n, n \in \mathbb{N}; 1 \le j \le r).$ (3.10)

Setting $\mathfrak{A}_{i,n} := \mathfrak{A}(\mathfrak{F}_{i-1} \cup \mathfrak{A}(Z_{i+1},..., Z_n)), \mathfrak{A}(\mathfrak{E})$ and $\mathfrak{A}(X)$ being the σ -algebras generated by $\mathfrak E$ and X , respectively, where $\mathfrak E \subset \mathfrak P(\Omega)$, one has by standard arguments for the conditional expectation of real r.vs.,

$$
E[X_i^j g^{(j)}(\varphi(n) R_{n,i})] - E[Z_i^j g^{(j)}(\varphi(n) R_{n,i})]
$$

=
$$
E[g^{(j)}(\varphi(n) R_{n,i}) \{E[X_i^j | \mathfrak{A}_{i,n}] - E[Z_i^j | \mathfrak{A}_{i,n}]\}
$$

since $g^{(i)}(\varphi(n) R_{n,i})$ is measurable with respect to $\mathfrak{A}_{i,n}$. Since moreover $\mathfrak{A}(Z_{i+1},..., Z_n)$ is independent of $\mathfrak{A}(\mathfrak{F}_{i-1}\cup \mathfrak{A}(X_i)),$ one has (cf. [5 : p. 295]) $E[X_i^j\,|\,\mathfrak{A}_{i,n}]=E[X_i^j\,|\,\mathfrak{F}_{i-1}]$ a.s. As $\mathfrak{A}(Z_i)$ is independent of $\mathfrak{A}_{i,n}$, one finally has $E[Z_i] \mathfrak{A}_{i,n} = E[Z_i]$ a.s., and so $E[X_i]$ $\mathfrak{A}_{i,n} = E[Z_i] \mathfrak{A}_{i,n}$ a.s. on account of assumption (3.3). This establishes the validity of (3.10). Since the double sum in (3.9) vanishes, (3.8) yields

$$
|E[f(T_{N_{\lambda}})] - E[f(Z)]|
$$

\n
$$
\leq 2 ||f - g|| + \frac{||g^{(r)}||}{(r - 1)!} E\left[(\varphi(N_{\lambda}))^{r} \sum_{i=1}^{N_{\lambda}} (\zeta_{r,i} + \zeta_{r,i}) \right]
$$

on account of (2.5) and (2.6). But the left side of this inequality is independent of g, so that taking the infimum over all $g \in C_B$ ^r yields by (2.1)

$$
|E[f(T_{N_{\lambda}})] - E[f(Z)]|
$$

\n
$$
\leq 2K \left\langle \left\{ E\left[(\varphi(N_{\lambda}))^{\mathbf{r}} \sum_{i=1}^{N_{\lambda}} (\mathbf{r}_{\mathbf{r},i} + \xi_{\mathbf{r},i}) \right]^{1/\mathbf{r}} \right\}, f; C_{B}, C_{B}^{\mathbf{r}} \right\rangle.
$$

The first assertion, namely (3.4) , of part a) now follows immediately by (2.2) , and the second, namely (3.5), by (2.3). Part b) is a particular case of a) \blacksquare

Remark 1: Note that Theorem 1 covers the situation that the r.vs. X_i are independent (since such r.vs. with $E[X_i] = 0$ form a MDS). In this instance condition (3.3) reduces to $E[X_i] = E[Z_i]$, $i \in N$, $1 \leq j \leq r-1$, an assumption already used in [12], for example.

 ${\bf Remark~2}\!$: The estimate in (3.4) is only of practical interest provided the modulus ω_r tends to zero for $\lambda \to \infty$, that is, if

$$
E\left[(\varphi(N_i))^{r}\sum_{i=1}^{N_{\lambda}}\left(E[|X_i|^{r}]+E[|Z_i|^{r}]\right)\right]
$$

tends to 0 for $\lambda \to \infty$. If the r.vs. X_i , $i \in \mathbb{N}$, and Z_i , $i \in \mathbb{N}$, are in particular identically distributed, and one sets $\varphi(N_1) = N_1 - r/2$, $r \ge 3$ (see Theorem 3b)), then the special bounds in (3.6) and (3.7) tend to zero since $\overline{E}[N_{\lambda}{}^{(2-r)/2}] \to 0$ for $\lambda \to \infty$ in view of the hypothesis that $N_{\lambda} \rightarrow \infty$ in probability.

4. Approximation Theorem for Distribution Functions with Rates

In this section we shall examine the rate of approximation for the strong convergence in distribution of the r.vs. $T_{N_{\lambda}}$ towards Z. For this purpose, it suffices to apply the following result of V. M. ZOLOTAREV, contained implicitly in [47]; it permits one to pass from weak convergence to strong convergence in distribution.

Lemma: Let Y be a real r.v. with distribution function F_Y for which there exists a *constant* $M_1 = M_Y > 0$ *such that* a: Let Y be a real r.v. with distributed $M_1 = M_Y > 0$ such that
 $|F_Y(t) - F_Y(s)| \leq M_1 |t - s|$

$$
|F_Y(t) - F_Y(s)| \le M_1 |t - s| \qquad (s, t \in \mathbb{R}, s < t), \tag{4.1}
$$

(bution function F_Y for which there exists a

(s, $t \in \mathbb{R}$, $s < t$), (4.1)

constant $M_2 > 0$ there exists a constant M

colmogorov distance between the distribution and let $r \in N$. For each r.v. X and each constant $M_2 > 0$ there exists a constant M $\equiv M(M_1, M_2) > 0$ such that the so-called Kolmogorov distance between the distribution functions F_X and F_Y , namely **PAUL L. BUTZER and D. SCHULZ**
 IA 1 *Eet Y* be a real r.v. with distribution function F_Y for which there exists a
 $|F_Y(t) - F_Y(s)| \leq M_1 |t - s|$ (s, $t \in \mathbb{R}, s < t$), (4.1)
 \in N. For each r.v. X and each constant $M_2 > 0$ **table 1. BUTZER and D. SCHULZ**
 a: Let Y be a real r.v. with distribution function F₁,
 $W_1 = M_Y > 0$ such that
 $F_Y(t) - F_Y(s) \leq M_1 |t - s|$ (s, $t \in \mathbb{R}, s < t$),
 $\in \mathbb{N}$. For each r.v. X and each constant $M_2 > 0$ to
 Example 1: *D***: Let** Y *be a real r.v. with distribution function* F_Y *for which there exists a*
 $M_1 = M_Y > 0$ such that
 $|F_Y(t) - F_Y(s)| \leq M_1 |t - s|$ (s, $t \in \mathbb{R}, s < t$), (4.1)
 $\in \mathbb{N}$. For each r.v. X and each co **Lemma:** Let Y be a real r.v. with distribution function F_Y for which ther

constant $M_1 = M_Y > 0$ such that
 $|F_Y(t) - F_Y(s)| \leq M_1 |t - s|$ $(s, t \in \mathbb{R}, s < t)$,

and let $r \in \mathbb{N}$. For each r.v. X and each constant $M_2 > 0$ the

$$
\sup_{t\in\mathbf{R}}|F_X(t)-F_Y(t)|\leq M\left\{\sup_{f\in D}|E[f(X)]-E[f(Y)]|\right\}^{1/(r+1)},\tag{4.2}
$$

the function class $D \equiv D(M_2, r)$ *being defined by*

$$
D := \{f \in C_B^{r-1}; f^{(r-1)} \in \text{Lip}_{M_1} (1; 1; C_B) \},\tag{4.3}
$$

As an immediate consequence of Theorem 1 and (4.2) we have the following theorem.

Theorem 2: Let the assumptions of Theorem 1 *be satisfied, and let (4.1) hold 9or the limiting r.v. Z. Then*

$$
\sup_{t \in \mathbf{R}} |F_X(t) - F_Y(t)| \leq M \left\{ \sup_{f \in D} |E[f(X)] - E[f(Y)]| \right\}^{1/(r+1)}
$$

\nthe function class $D \equiv D(M_2, r)$ being defined by
\n
$$
D := \{ f \in C_B^{r-1} ; f^{(r-1)} \in \text{Lip}_{M_1} (1; 1; C_B) \},
$$

\nthe Lipschitz constant $L_{f^{(r-1)}}$ being uniformly bounded by M_2 .
\nAs an immediate consequence of Theorem 1 and (4.2) we have the
\nTheorem 2: Let the assumptions of Theorem 1 be satisfied, and
\nlimiting r.v. Z. Then
\na) $\sup_{t \in \mathbf{R}} |F_{T_{N_2}}(t) - F_Z(t)| \leq M \left\{ E \left[(\varphi(N_1))^r \sum_{i=1}^{N_2} (\zeta_{r,i} + \zeta_{r,i}) \right] \right\}^{1/(r+1)}$
\nb) If the r.v.s. X, and Z_i, $i \in \mathbf{N}$, are additionally identically
\n $\varphi(N_1) := N_1^{-1/2}$, then
\n $\sup_{t \in \mathbf{R}} |F_{T_{N_2}}(t) - F_Z(t)| = \mathcal{O} \left(\left(E \left[N_1^{\frac{2-r}{2}} \right] \right)^{1/(r+1)} \right) \qquad (\lambda \to \infty)$
\nThe order of approximation deduced for the result, such that

Theorem 2: Let the assumptions of Theorem 1 be satisfied, and let (4.1) hold for the limiting r.v. Z. Then

a) $\sup_{t \in \mathbf{R}} |F_{T_{N_\lambda}}(t) - F_Z(t)| \leq M \left\{ E \left[(\varphi(N_\lambda))^r \sum_{i=1}^{N_\lambda} (\zeta_{r,i} + \xi_{r,i}) \right] \right\}^{1/(r+1)}$

b) If the r.vs. X , are a
 $\left(\begin{matrix} \begin{matrix} E \end{matrix}\right| N_1 \end{matrix}\right)$

$$
\sup_{t\in\mathbf{R}}|F_{T_{N_{\lambda}}}(t)-F_{Z}(t)|=\tilde{\phi}\left(\left(E\left[N_{\lambda}^{\frac{2-r}{2}}\right]\right)^{1/(r+1)}\right) \qquad (\lambda\to\infty).
$$

The order of approximation deduced for the weak convergence can in general not be transferred to the associated strong convergence in distribution although it is known that both types of convergence are equivalent when considering convergence per se (without rates). (Further details to this and to other assertions equivalent to the convergence in distribution without as well as with rates may be found in BUTZER-
HAHN [13]). Hender of a proximation of the right side of (4.2) over all *f* $\in D$ is smaller than the same supermunitally $\lim_{t \to \infty} |F_{T_{N_1}}(t) - F_2(t)| \leq M \left\{ E\left[(\varphi(N_1))^r \sum_{i=1}^{N_1} (\zeta_{r,i} + \xi_{r,i}) \right] \right\}^{1/(r+1)}$.

b) If the r.vs. X $\varphi(N_A) := N_A^{-1/2}$, then
 $\sup_{t \in \mathbf{R}} |F_{T_{N_A}}(t) - F_Z(t)| = \sigma \left(\left(E \right) \right)$

The order of approximation deduced for

be transferred to the associated strong c

known that both types of convergence are

per se (without rates). vaer or approximation deduced for the weak conv
ferred to the associated strong convergence in that both types of convergence are equivalent whe
ithout rates). (Further details to this and to other a
nce in distribution w

Since the supremum of the right side of (4.2) over all $f \in D$ is smaller than the same supremum taken over all $f \in C_B^r(\mathbf{R})$, it is to be expected that the power $1/(r + 1)$ cannot be dropped in estimates of type (4.2) for general r.vs. *Y* that need not satisfy condition (4.1). This will now be shown by means of a simple example taken from $[46]$. **The metric solution** is the metric and the metric in the metric convergence in distribution without as well as with rates may be for **HARN** [13].

Since the supremum of the right side of (4.2) over all $f \in D$ is same sup same supremum of the right side of (4.2) over all $f \in D$ is small
same supremum taken over all $f \in C_B(\mathbf{R})$, it is to be expected that the
condition (4.1). This will now be shown by means of a simple example
(46).
If one

If one considers the distribution function F_{X_0} of the degenerate r.v. X_0 (cf. Section 1), and if $F_{\bar{x}_0}$ is a further distribution function of the form $F_{\bar{x}_0}(t) = (1 - \varepsilon)$ c dropped in estimates of type (4.2) for general r.vs. Y that need not satisfy

1 (4.1). This will now be shown by means of a simple example taken from

considers the distribution function F_{X_0} of the degenerate r.v.

$$
\times F_{X_s}(t) + \varepsilon F_{X_s}(t - \varepsilon), t \in \mathbb{R}, \text{ for some } \varepsilon \in (0, 1), \text{ then one has immediately}
$$
\n
$$
\sup_{t \in \mathbb{R}} |F_{X_s}(t) - F_{\overline{X}_s}(t)| = \varepsilon. \tag{4.4}
$$
\n
$$
\text{Now with the help of the metric } \varepsilon_s \text{ defined in [46] by}
$$
\n
$$
x_s(X, Y) := s \int |t|^{s-1} |F_X(t) - F_Y(t)| dt \quad (s \ge 1)
$$

$$
\varkappa_s(X, Y) := s \int\limits_{\mathbf{R}} |t|^{s-1} |F_X(t) - F_Y(t)| dt \qquad (s \ge 1)
$$

$$
+ \epsilon r_{X_0}(t - \epsilon), t \in \mathbf{R}, \text{ for some } \epsilon \in (0, 1), \text{ then one has immediately}
$$
\n
$$
\sup_{t \in \mathbf{R}} |F_{X_0}(t) - F_{\overline{X}_0}(t)| = \epsilon.
$$
\n
$$
\text{a the help of the metric } \mathbf{x}_s \text{ defined in [46] by}
$$
\n
$$
\mathbf{x}_s(X, Y) := s \int_{\mathbf{R}} |t|^{s-1} |F_X(t) - F_Y(t)| dt \quad (s \ge 1)
$$
\n
$$
\text{eal } r.\text{vs. } X, Y, Z \text{olotarev established the inequality}
$$
\n
$$
\sup_{t \in D} |E[f(X)] - E[f(Y)]| \le \frac{1}{\Gamma(r)} \mathbf{x}_r(X, Y), \tag{4.5}
$$
\n
$$
\text{N is taken as in the definition of the class } D \text{ of (4.3)}.
$$

where $r \in N$ is taken as in the definition of the class *D* of (4.3).

General random sum limit.theorems 105

In the case of our example, the r.vs. X_0 and \overline{X}_0 are at a distance $\varkappa_r(X, \overline{X}_0)$ In the case of our example, the r.vs. X_0 and \overline{X}_0 are at a distance $\varkappa_r(X,\,\overline{X}_0)$ $= (r/(r+1)) \varepsilon^{r+1}$ apart. So one has by (4.4) Fig. 2. Case of our example, the r.vs. X_0 and \overline{X}_1
 $\sup_{t \in \mathbb{R}} |F_{X_0}(t) - F_{\overline{X}_0}(t)| \leq \{2 \varkappa_r(X_0, \overline{X}_0)\}^{1/(r+1)}$
 $\lim_{t \in \mathbb{R}} |F_{X_0}(t) - F_{\overline{X}_0}(t)| \leq \frac{2 \varkappa_r(X_0, \overline{X}_0)\}^{1/(r+1)}$ $\begin{aligned} \text{In} \quad & \text{t} \\ & = \frac{r}{r} \end{aligned}$

$$
\sup_{t\in\mathbf{R}}|F_{X_0}(t)-F_{\bar{X}_0}(t)|\leq \{2\varkappa_r(X_0,\bar{X}_0)\}^{1/(r+1)}\qquad (r\in\mathbf{N}).
$$
\n(4.6)

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 o are at a distance $\varkappa_r(X, \overline{X}_0)$
 $(r \in N)$. (4.6)

en passing from the distance

tric sup_{ter} $|F_{X_0}(t) - F_{\overline{X}_0}(t)|$, A comparison of (4.5) with (4.6) therefore shows that when passing from the distance $\sup_{t\in D} |E[f(X_0)] - E[f(\overline{X}_0)]|$ to the Kolmogorov-metric $\sup_{t\in R} |F_{X_0}(t) - F_{X_0}(t)|$, the rate of convergence becomes poorer, at least for those r.vs. Z which do not ne-. cessarily satisfy a Lipschitz condition of type (4.1) . Indeed, the r.v. X_0 does not satisfy (4.1) . However, whether the estimate (4.2) and so the convergence rate in Theorem 2 could possibly be improved (in the sense that the power $1/(r + 1)$) in the estimate of Theorem 2 could be dropped or increased) for those limiting r.vs. Z satisfying (4.1) is a fact that is unknown to the authors. A comparison of (4.5) with (4.6) therefore shows that when passing from the distance $\log_{10} E[f(X_0)] - E[f(X_0)]$ to the Kolonggrov-metric supergal $F_X(t) - F_X^*(t) - F_X(t)$
the rate of convergence becomes poorer, at least for those r. **Example 12.13** and the extendance of the sensative statistic condition of type (4.1). Indeed, the r.v. X_0 and so the convergence Theorem 2 could possibly be improved (in the sense that the power $1/(r +$ estimate of The

5. The Random - Sum CLT for MDS with Rates

Let us apply our two theorems to a concrete limiting r.v. Z, namely to the Gaussian distributed r.v. X^* with mean zero and variance 1. Theorem 1 yields the following *Efferant True Contention is* the confident mining i.v. *Z*, namely to the caussiant of r, \mathbf{x}^* with mean zero and variance 1. Theorem 1 yields the following LLT with large- \mathcal{O} rates for martingales.

em 3: Let

Theorem 3: Let $(X_i, \mathfrak{F}_i)_{i\in \mathbf{P}}$ be a MDS, $r \in \mathbf{N}$ and $(a_i)_{i\in \mathbf{N}}$ any sequence of positive random CLT with large- \emptyset rates for martingales.
Theorem 3: Let $(X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}$ be a MDS, $r \in \mathbf{N}$ real numbers. Assume that (3.1) holds, i.e.,

$$
\zeta_{r,i} := E[|X_i|^r] < \infty \quad , \quad (i \in \mathbb{N}),
$$

as well a-s

$$
E[X_i^j | \mathfrak{F}_{i-1}] := a_i^j E[X^{*j}] \quad \text{a.s.} \quad (i \in \mathbf{N}, 1 \leq j \leq r-1).
$$

a) Under these hypotheses one has for $f \in C_B$,

EEl (ASNA)] — E[/(X)]I* **/1 1 Ir N Th** *^ 2cC,* **rWr** (I E E **(r.i** *+ atTE[JX*V])IF ;* /; *C), ' (5.2)* **\ L 1=1** u **/** — E[/(X*)]I . • '' ^S

 \mathcal{L}_{2} , being the constant of (2.2), and $A_{N_\lambda}:=\Big(\sum^{N_\lambda}a_i{}^2\Big)^{1/2}$. In particular, if $f\in \mathrm{Lip}\left(\alpha\right)$ $\binom{N}{i} E[|X^*]$
 $\binom{N}{i=1}$
 $\binom{N}{i=1}$
the form $\alpha \in (0, r]$, then the bound in (5.2) takes on the form $\leq 2c_2, \omega_r \left\{ \left\{ E \left[A_{N_1}^{T_1} \sum_{i=1}^{n} (\zeta_{r,i} + a_i t E[|X^*|^r]) \right] \right\} \right\} ; f; C_5 \right\},$ (5.2)

being the constant of (2.2), and $A_{N_4} := \left(\sum_{i=1}^{N_1} a_i^2 \right)^{1/2}$. In particular, if $f \in \text{Lip}(\alpha; r; C_5)$.

(0, r], then

the constant of (2.2), and
$$
A_{N\lambda} := \sum_{i=1}^{\infty} a_i
$$
;
\nthen the bound in (5.2) takes on the form
\n $2c_2, L_f \left\{ E \left[A_{N\lambda}^{-r} \sum_{i=1}^{N\lambda} (\zeta_{r,i} + a_i^* E[|X^*|^r]) \right] \right\}.$

b) If, in addition, the r.vs. X_i , $i \in \mathbb{N}$, are identically distributed, and (5.1) holds for $a_i = 1, i \in \mathbb{N}$, then

1, then the bound in (5.2) takes on the form
\n
$$
2c_{2,r}L_{f}\left\{E\left[A_{N_{\lambda}}^{-r}\sum_{i=1}^{N_{\lambda}}(\zeta_{r,i}+a_{i}^{*}E[|X^{*}|^{r}])\right]\right\}.
$$
\n
$$
in addition, the r.v.s. X_{i}, i \in \mathbb{N}, are identically distributed, and (5.1) holds for\n
$$
|E[f(S_{N_{\lambda}}|N_{\lambda}^{1/2})]-E[f(X^{*})]|
$$
\n
$$
\leq 2c_{2,r}\omega_{r}(\langle\zeta_{r,1}+E(|X^{*}|^{r})E[N_{\lambda}^{(2-r)/2}])|^{1/r}; f; C_{B}).
$$
\n
$$
p(\alpha; r; C_{B}), \alpha \in (0, r], then the bound in (5.3) reads
$$
\n
$$
2c_{2,r}^{b}L_{f}(\langle\zeta_{r,1}+E[|X^{*}|^{r}])E[N_{\lambda}^{(2-r)/2}]^{\alpha/r}.
$$
\n
$$
f: The r.v. X^{*} is \varphi-decomposable for each $n \in \mathbb{N}$ into n independent, non-
\nstributed r.v.s. $Z_{i}, 1 \leq i \leq n$, namely $Z_{i} = a_{i}X^{*}$. Moreover, one can ensure
$$
$$

 $I f f \in \text{Lip } (\alpha; r; C_B)$, $\alpha \in (0, r]$, then the bound in (5.3) reads

$$
2\partial_{2,r}^{\nu}L_f(\zeta_{r,1} + E[|X^*|^{r}]) E[N_1^{(2-r)/2}]^{\alpha/r}.
$$
 (5.4)

Proof: The r.v. X^* is φ -decomposable for each $n \in \mathbb{N}$ into *n* independent, normally distributed r.vs. Z_i , $1 \leq i \leq n$, namely $Z_i = a_i X^*$. Moreover, one can ensure

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as in the proof of Theorem 1 that the Z_i , $i \in N$, N_λ , $\lambda \in \mathbb{R}^+$, as well as the sub- σ -al-

gebras \mathfrak{F}_i , $i \in N$, are all independent. So X^* can be decomposed according to gebras \mathfrak{F}_i , $i \in \mathbb{N}$, are all independent. So X^* can be decomposed according to (1.6). $\sqrt{(-1)^{n-1}}$ (i=1)
tion (3.2) is satisfied on account of (5.1). So Theorem 1 may be applied since the with $\varphi(n) :=$ of Theorem 1 that the Z_i , $i \in \mathbb{N}$, N_λ , $\lambda \in \mathbb{R}^+$, as well as the sub- σ -al-
 \sum_i , are all independent. So X^* can be decomposed according to (1.6)
 $\sum_{i=1}^n a_i^2$ Since $E[Z_i^*] = E[a_i^*X^{*i}] = a_i^j E[X^{*i}]$ L. BUTZER and D. SCHULZ

of Theorem 1 that the 2

N, are all independent. S
 $\sum_{i=1}^{n} a_i^2$
 $\left(\sum_{i=1}^{n} a_i^2\right)^{-1/2}$. Since $E[Z_i]$

itisfied on account of (5.

1) exist here, too 1

the previous literature in

the pr moments of (4.1) exist here, too \blacksquare 106 PAUL L. BUTZER and D. SCHULZ

as in the proof of Theorem 1 that the Z_i , $i \in N$, Δ

gebras \mathfrak{F}_i , $i \in N$, are all independent. So X^* can

with $\varphi(n) := \left(\sum_{i=1}^n a_i^2\right)^{-1/2}$. Since $E[Z_i^j] = E[a_i^j X^*$

tion $\begin{aligned} \mathcal{L}(\mathcal{L}_1) &= \left(\frac{Z}{i-1}\right)^2 \end{aligned}$. Since $E[Z_i^i] = E[a_i^i \mathcal{A}^{-i}] = a_i^i$
 I is satisfied on account of (5.1). So Theorem 1

of (4.1) exist here, too \blacksquare

ining the previous literature in the matter, the
 IE with $\varphi(n) := \left(\sum_{i=1}^{n} a_i^2\right)^{-1/2}$. Since $E[Z_i^j] = E[a_i^j X^*$
tion (3.2) is satisfied on account of (5.1). So The
moments of (4.1) exist here, too \blacksquare
Concerning the previous literature in the matt
authors is that by

Concerning the previous literature in the matter, the only paper known to the Concerning the previous literature in the matter, the only paper known to the authors is that by BasU [4], who in the classical case of non-random summation deduced the estimate

$$
|E[f(S_n/n^{1/2})] - E[f(X^*)]| = O(n^{-(\alpha-2)/2}) \qquad (n \to \infty)
$$

provided that $(X_i)_{i\in\mathbb{N}}$ defines a MDS of stationary r.vs. satisfying the assumptions of Theorem 3b) in the particular case $r = 3$ and $f \in Lip(\alpha; 3; C_B)$, $\alpha \in (2, 3]$. Comparing this estimate with ours, (5.4) for $r = 3$ and non-random summation gives the order $\mathcal{O}(n^{-\alpha/6})$, $\alpha \in (2, 3]$, even in the case of non-stationary sequences $(X_i)_{i\in \mathbb{N}}$.

As an application of Theorem 2 to X^* we have:

$$
\mathcal{O}(n^{-a/6}), \alpha \in (2, 3], \text{even in the case of non-stationary sequences } (X_i)_{i \in \mathbb{N}}.
$$

As an application of Theorem 2 to X^* we have:
 Theorem 4: a) Under the assumptions of Theorem 3 one has
\n
$$
\sup_{t \in \mathbb{R}} |F_{S_{N,l}}|A_{N,l}(t) - F_{X^*}(t)| \leq M \left\{ E \left[A_{N,l}^{-r} \sum_{i=1}^{N_l} (\zeta_{r,i} + a_i^* E[|X^*|]) \right] \right\}^{1/(r+1)}.
$$

\nb) If the r.v.s. X_i are identically distributed, and (5.1) holds for $a_i = 1, i \in \mathbb{N}$, then
\n
$$
\sup_{t \in \mathbb{R}} |F_{S_{N,l}}|A_{N,l}^{(1)}(t) - F_{X^*}(t)| = \mathcal{O}((E[N_2^{(2-r)/2}])^{1/(r+1)}) \quad (2 \to \infty).
$$

\nIf $r = 3$, and $E[N_2^{-1/2}] = \mathcal{O}(\lambda^{-1/2})$, $\lambda \to \infty$, then the latter estimate is of order $\mathcal{O}(\lambda^{-1/8})$.
\nRemark 3: In his dissertation [44] StrOBEL obtains under additional assumptions
\nas a corollary of results in a more general setting the estimate
\n
$$
\sup_{t \in \mathbb{R}} |F_{S_{\nu_n}/\sqrt{n}}(t) - F_{X^*}(t)| = \mathcal{O}(n^{-1/8} + P(\{\nu_n > n\})) \quad (n \to \infty)
$$

b) If the r.vs. X_i are identically distributed, and (5.1) holds for $a_i = 1$, $i \in \mathbb{N}$, then

$$
\sup_{t \in \mathbf{R}} |F_{S_{N_1}/N_1^{1/4}}(t) - F_{X^*}(t)| = \mathcal{O}((E[N_1^{(2-r)/2}])^{1/(r+1)}) \qquad (\lambda \to \infty).
$$

If $r = 3$, and $E[N_i^{-1/2}] = \mathcal{O}(\lambda^{-1/2})$, $\lambda \to \infty$, then the latter estimate is of order $\mathcal{O}(\lambda^{-1/8})$.

Remark 3:.In his dissertation [44] **STROBEL** obtaihs under additional assumptions

, and
$$
E[N_1-1/2] = \mathcal{O}(\lambda^{-1/2}), \lambda \to \infty
$$
, then the latter estimate is of or
\n $\lambda \cdot k$ 3: In his dissertation [44] STROBEL obtains under additional a
\nollary of results in a more general setting the estimate
\n
$$
\sup_{t \in \mathbf{R}} |F_{S_{\nu_n}/\sqrt{n}}(t) - F_{\chi \bullet}(t)| = \mathcal{O}(n^{-1/8} + P(\{\nu_n > n\})) \qquad (n \to \infty)
$$

for a sequence of independent, identically distributed, normalized r.vs. $(X_i)_{i\in\mathbb{N}}$, and a sequence of stopping rules $(\nu_n)_{n\in\mathbb{N}}$ for $\mathfrak{F}_n:=\mathfrak{A}(\{X_1,...,X_n\}).$

sup $|F_{S_{N,l}/N,l^{1,l}}(t) - F_{X^*}(t)| = \mathcal{O}((E[N_1^{\alpha-1/2}])^{1/(r+1)})$ $(\lambda \to \infty)$.

If $r = 3$, and $E[N_1^{-1/2}] = \mathcal{O}(\lambda^{-1/2})$, $\lambda \to \infty$, then the latter estimate is of order $\mathcal{O}(\lambda^{-1/2})$.

Remark 3: In his dissertation [44] STRO The final application of Theorem 1 will be the WLLN of the title with θ -error estima-6. The Random – Sum WLLN for MDS with Rates
The final application of Theorem 1 will be the WLLN of the title with θ -error estima-
tes, a result that does not seem to have been considered before. When examining the
WLLN WLLN in connection with r.vs. T_{N_1} defined in (1.5), one normally thinks of stochastic convergence of F_{N_1} (i T_{N_2} = ε) = 0 ($\varepsilon > 0$).

Experimentally distributed, normalized r.vs. $(X_i)_{i \in N}$, and
 $\sup_{t \in \mathbb{R}} |F_{S_{\tau_n}/|\sqrt{n}}(t) - F_{\mathcal{X}^*}(t)| = \mathcal{O}(n^{-1/8} + P(\{v_n > n\}))$ $(n \to \infty)$

for a sequence of tes, a result that does not seem to have been considered before. When examining the WLLN in connection with r.vs. T_{N_A} defined in (1.5), one normally thinks of stochastic convergence of T_{N_A} towards zero, namely **WLLN for MDS with Rates**
 Solution WELN for MDS with Rates
 Solution Theorem 1 will be the WLLN of the title with 0-error estimate

t seem to have been considered before. When examining the

h r.vs. T_{N_λ} defined *(I ECB -) - - " (6.2)*

$$
\lim_{\epsilon \to \infty} P(|T_{N_{\lambda}}| \geq \epsilon) = 0 \qquad (\epsilon > 0). \qquad (6.1)
$$

Instead, we consider a formulation of the random WLLN which can be shown to be equivalent to (6.1), just as in the classical situation (see e.g. [5: p. 220]), namely lint $P({|T_{N_\lambda}| \ge \varepsilon}) = 0$ ($\varepsilon >$
we consider a formulation of the to (6.1), just as in the classi
lint $|E[f(T_{N_\lambda})] - f(0)| = 0$

$$
\lim_{k\to\infty}|E[f(T_{N_k})]-f(0)|=0 \qquad (f\in C_B^{\bullet})
$$

•

for any $r \in \rm N$. Version (6.2) has the advantage that Theorem 1 can be applied directly; one just needs to choose the limiting r.v. Z as the degenerate r.v. X_0 defined in Seneral random sum limit theorems.

for any $r \in N$. Version (6.2) has the advantage that Theorem 1 can be applied

ly; one just needs to choose the limiting r.v. Z as the degenerate r.v. X_0 def

Section 1. In this fram *A*. Version (6.2) has

needs to choose the

n this frame the ra

n 5: Let $(X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}$

Lip $(\alpha; 2; C_B)$, then
 $f(T_{N_2})$] $- f(0)$ \leq 2

the assumption **b) Controllering** $\mathbf{v} \in \mathbb{R}^N$. Version (6.2) has the advantage that Theorem 1 can \mathbf{v} , one just needs to choose the limiting $\mathbf{r} \cdot \mathbf{v} \cdot Z$ as the degenerate \mathbf{r} choin 1. In this frame the random *Na Natural* \in N. Version (6.2) has the advantage that Theorem 1 can be applied direct-

and the coloose the limiting r.v. Z as the degenerate r.v. X_0 defined in
 *Na In this frame the random WLLN for MDS with * for any $r \in N$. Version (6.2) has the advantage that Theoren
 ly; one just needs to choose the limiting r.v. Z as the deger

Section 1. In this frame the random WLLN for MDS with
 Theorem 5: Let $(X_i, \mathfrak{F}_i)_{i \in \mathbf{P$

Section 1. In this frame the random WLLN for MDS with θ -rates reads
Theorem 5: Let $(X_i, \mathfrak{F}_i)_{i \in \mathbf{P}}$ be a MDS such that $E(|X_i|^2) < \infty$, $i \in \mathbf{N}$. *a*) *If* $f \in \text{Lip}$ (α ; 2; C_B), then

$$
|E[f(T_{N_1})] - f(0)| \leq 2c_2 \cdot L_f \left\{ E\left[(\varphi(N_1))^2 \sum_{i=1}^{N_2} E[|X_i|^2] \right] \right\}^{a/2}
$$

I

$$
\sum_{i=1}^{N_2} E[|X_i|^2] = o((\varphi(N_1))^{-2}) \quad \text{a.s.} \quad (\lambda \to \infty),
$$
\n
$$
\lim_{\lambda \to \infty} P(\{ |T_{N_\lambda}| \ge \varepsilon \}) = 0 \quad (\varepsilon > 0).
$$
\n(6)

 $(\epsilon > 0)$.

In particular, if $\varphi(N_1) = N_1^{-1}$ *in (6.3), then* $S_{N_1}/N_1 \rightarrow 0$ *in probability for* $\lambda \rightarrow \infty$ *, i.e.*, $(X_i)_{i \in \mathbb{N}}$ *satisfies the random WLLN.*

Proof: First note that $E[f(X_0)] = \int f(x) dP_{X_0}(x) = f(0)$. The distribution P_{X_0} can **co R** $\sum_{i=1}^{N_1} E[|X_i|^2] = o((\varphi(N_1))^{-2})$ a.s. $(\lambda \to \infty)$, (6.3)

one has
 $\lim_{\lambda \to \infty} P(|T_{N_1}| \ge \varepsilon) = 0$ $(\varepsilon > 0)$.
 In particular, if $\varphi(N_1) = N_1^{-1}$ in (6.3), then $S_{N_1}/N_1 \to 0$ in probability for $\lambda \to \infty$,
 i.e., *r'(n)EZ* $\int |x|^s dP_{x_0}(x) = 0$ for any $s > 0$, condition (3.2) of Theorem 1 is satisfied. Then assump- ${\bf B}$.
tion (3.3) follows immediately from (1.3) and the distribution of the r.vs. Z_i . Im $P(\{|I_{N_i}| \leq \varepsilon\}) = 0$ $(\varepsilon > 0)$.

In particular, if $\varphi(N_i) = N_i^{-1}$ in (6.3), then $S_{N_i}/N_i \to 0$ in probability for $\lambda \to \infty$.

i.e., $(X_i)_{i\in\mathbb{N}}$ satisfies the random WLLN.

Proof: First note that $E[f(X_0)] = \int f(x) dP$

Part b) follows from a), noting the equivalence of (6.1) and (6.2) **^U**

The authors would like to thank $Dr. L. H_{A}$ for-kindly supervising the early stages of the work of the second named author, as well as Prof. M. Csöaoö, Ottawa, art b) follows from a), noting the equivalence of (6.1)
he authors would like to thank Dr. L. HAHN for ki
es of the work of the second named author, as well as
ada, for his generous help in connection with the liter
FER

REFERENCES

- [1] ALDOUS, D. J.: Weak convergence of randomly indexed sequences of random variables.
Proc. Cambridge Philos. Soc. 83 (1978), 117-126.
- **[2] ANSCOMBE,** F. J.: Large-sample theory of sequential estimation. Proc. Cambridge Philos. Soc. 48 (1952), 600 - 607.
- (3] **BA5U,** A. K.: On' the rate of convergence to normalityof sums of dependent random variables. Acta Math. Acad. Sci. Hungar. 28 (1976), 261-265.
- variables. It is On the rate of approximation in the central limit theorem for dependent random variables and random vectors. J. Multivariate Anal. 10 (1980), 565–578.

[5] BAUER, H.: Wahrscheinlichkeitstheorie und Grundzü random variables and random vectors. J. Multivariate Anal. 10 (1980), 565–578.
- **[5] BAUER,** H.: \\Tahrscheinlichkeitstheorie und Grundzuge der MaBtheori. Berlin 1978,
- [6] BILLINGSLEY, P.: The Lindeberg-Lévy theorem for martingales. Proc. Amer. Math. Soc. 12 (1961), 788-792.
- [7] BLUM, J. R., HANSON, D. L., and J. I. ROSENBLATT: On the central limit theorem for the sum of a random number of independent random variables. Z. Wahrscheiñlichkeitstheorie und verw. Gebiete 1 (1963), 389-393.
- **[8] BOLTHAUSEN,** E.: Exact convergence rates in some martingale central limit theorems. Preprint (1979).
- [9] BROWN, B. M.: Martingale central limit theorems. Ann. Math. Statist. 42 (1971), 59-66.
- [10] BtJrzER, P. L., and H. **BERENS,** Semi-groups of Operators and Approximation. Berlin 1967.
- [11] BUTZER, P. L., and L. HAHN: General theorems on the rate of convergence in distribution of random variables. 1: General limit theorems. J. Multivariate Anal. 8 (1978), 181-201.
- [12] BUTZER, P. L., and L. HAHN: General theorems on the rate of convergence in distribution of random. variables. II: Applications to the stable limit laws and weak law of large [12] BUTZER, P. L., and L. HAHN: General theorems on the rate of convergence in distribution
of random variables. II: Applications to the stable limit laws and weak law of large
'numbers. J. Multivariate Anal. 8 (1978),
	- 'numbers. J. Multivariate Anal. 8 (1978), $202-221$.
- [13] BUTZER,'P. L., find L. HAHN: On the connections between the rates of norm and weak convergence in the central limit theorem. Math. Nachr. 91 (1979), 245-251.
- \mathcal{O} -rates for martingales in Banach spaces. In print (1982).
- [15] CHOW, Y. S., and H. TEICHER: Probability Theory. New York 1978.
- $[16]$ Csör G_6 , M.: On'the strong law of large numbers and the central limit theorem for martingales. Trans. Amer. Math. Soc. 131 (1968), 259-275.
- [17] CSöRGO, M., and R. FIscmER: Some examplesand results in the theory of mixing and random $-$ sum central limit theorems. Period. Math. Hungar. 3 (1973), 41 -57 .
- [18] DooB, J. L.: Stochastic Processes. New York 1953.
- [19] DVORETZEY, A.: Asymptotic normality for sums of dependent random variables. In: Proc. Sixth Berkeley Symp. on Math. Stat, and Prob, Vol. II, Berkeley 1970, p. 513-535. DVORETZKY, A.: Asymptotic normality for sums

DVORETZKY, A.: Asymptotic normality for sums

Sixth Berkeley Symp. on Math. Stat. and Prob

GAENSSLER, P., and W. STUTE: Wahrscheinlich

GAENSSLER, P., STUTE, W., and J. STROBE
- [20] CAENSSLER, P., and W. STUTE : Wahrscheinlichkeitstheorie. Berlin 1977.
- [21] GAENSSLER, P., STUTE, W., and J. STROBEL: On central limit theorems for martingale triangular arrays. Acta Math. Acad. Sci. Hung. 31 (1978), 205-216.
- [22] GRAMS, W. F.: Rates of convergence in the central limit theorem for dependent random variables. Dissertation. Florida State University 1972.
- [23] HALL, P., and C. C. HEYDE: Martingale Limit Theory and its Applications. New York 1980. variables. Dissertation. Florida State University 1972.

HALL, P., and C. C. HEYDE: Martingale Limit Theory and its Applications.

1980.

HEYDE, C. C., and R. M. BROWN: On the departure from normality of a cert

HEYDE, C.
- [24] HEYDE, C. C., and R. M. BROWN: On the departire from normality of a certain class of martingales. Ann. Math. Stat. 41 (1970), 2161-2165.
- [25] TRRAGIOV, I. A.: *A* central limit theorem for a class of dependent random vaiiables. Theor. Probability App!. 8 (1963), 83-89.
- [26] KATO, Y.: Convergence rates in central limit theorem for martingale differences: Bull.
- [27] LANDERS, D., and L. ROGGE; The exact approximation order in the central-limit-theorem 1. for random summation. Z. \Vahrscheinlichkeitstheorie verw. Gebiete **36** (1976), 269-283.
- [28] Mc LEISH, D. L.: Dependent central limit theorems and invariance principles. Ann. Prob. $2(1974), 620 - 628.$
- [29] Lévx, P.: Propriétés asymptotiques des sommes de variables aléatoires enchainées.
Bull. Sci. Math. 59, sér. 2 (1935), 84–96, 109–128.
- [30] LEvy, P.: Propriétés ásymtotiques des sommes de variables aléatoires indépendantes ou enchainées. J. Math. Pures AppI. 14, sér. 9 (1935), 347-402.
- [31] LEvy,' P.: Théorie de l'addition des variables léatoires. Paris 1937.
- [32] PRAKASA RAO, B. L. S.: Random central limit theorems for martingales. Acta Math: Acad. Sci. Hung. $20'(1969)$, $217-222$.
- TREACIMOV, I. A.: A central limit theorem for a class of dependent rate of the and science in the central limit theorem for a class of dependent rate of Math. Stat. 18 (1963), 83–89.

KATO, Y.: Convergence rates in centra [33] PRAKASA RAO, B. L. S.: On the rate of convergence in the random central limit theorem for martingales. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 22 (1974), 1255 to Math. Stat. 18 (1979), 1-9.

Math. Stat. 18 (1979), 1-9.

LANDERS, D., and L. Rocce_{ia} The exact approximation order

for random summation. Z. Wahrscheinlichkeitstheorie verw.

Mc LEISH, D. L.: Dependent central limit the Bull. Sci. Math. 59, sér. 2 (1935), 84-96, 109-128.

Lévy, P.: Propriétés asymptotiques des sommes de variables aléatoires indépendante

enchanées. J. Math. Pures Appl. 14, sér. 9 (1935), 347-402.

Lévy, P.: Théorie de l'a PRAKASA RAO, B. L. S.: On the rate of convergence in the rand
for martingales. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom
1260.
RÉNYI, A.: On the central limit theorem for the sum of a randor
random variables. Acta M
- [34] REx yi, A.: On the central limit theorem for the sum of a random number of independent random variables. Acta Math. Acad. Sci. Hung. 11 (1960), 97-102.
- [35] RICHTER, W.: Grenzwertsätze für Folgen zufälliger Elemente mit zufälligen Indizes. Habilitatj onsschrift. Dresden 1964.
- [36] RICHTER, W.: Ubertragung von Grenzaussagen für Folgen zufälliger Elemente auf'Folgen
- [37] ROBBINS, H.: The asymptotic distribution of the sum of a random number of summands.. Bull. Amer. Math. Soc. 54 (1948), 1151-1161.
- [38] ROECKERATH, M. TH.: Der, Zentrale Grenzwertsatz und das Schwache Gesetz der GroBen Zahlen mit Konvergenzraten für Martingale in Banachräumen. Dissertation. Aachen 1980.
- [39] RYCHLIK, *Z.:* A central limit theorem for sums of a random number of independent random
- [40] RYCHLIK, *Z.:* Martingale random central limit theorems. Acta Math. Acad. Sci. Hung. 34 $(1979), 129-139.$

V V

 $S_{\rm 3}$

- [41] RYCHLIK, Z., and D. SZYNAL: On the rate of approximation in the random-sum central limit theorem. Theor. Probability Appl. 24 (1979), $614-620$.
- $[41]$
 $[42]$ [42] Scorr, D. J.: Central limit theorems for martingales and for processes with stationary increments using a Skorokhod representation approach. Advances in AppI. Probability 5 Gene

RYCHLIK, Z., and D. SzYNAL: On the rate of a

limit theorem. Theor. Probability Appl. 24 (1978

SCOTT, D. J.: Central limit theorems for marting

crements using a Skorokhod representation app

(1973), 119-137.

Сигаж
- $-$ [43] Сираждинов, С. Х., и Г. Оразов: Обобщение одной теоремы Г. Роббинса. В сб. генения авида
1973), 119—137.
Предельные теоремы и статистические выводы. Ташкент 1966, стр. 154—162.
Предельные теоремы и статистические выводы. Ташкент 1966, стр. 154—162.
Sтвоввы. J.: Konvergenzraten in zentralen Grenz
	- [44] STROBEL, J.: Konvergenzraten in zentralen Grenzwertsätzen für Martingale. Dissertation. Bochum '1978.
	- [45]TROTTER, H. F.: An elementary proof of the central limit theorem. Arch. Math. 10 *(1959),* $226 - 234.$
	- [46] ZOLOTAREV, V. M.: Some new inequalities in probability connected with Lévy's metric. Soviet Math. Dokl. 11 (1970), 231-234.
	- [47] ZOLOTAREV, V. M.: Approximation of distributions of sums of independent random variables with values in infinite-dimensiohal spaces. Theor; Probability AppI. **21 (1976),** $721 - 737.$ Im 1978.

	YER, H. F.: An elementary proof of t

	234.

	YAREV, V. M.: Some new inequalitie

	Math. Dokl. 11 (1970), 231–234.

	YAREV, V. M.: Approximation of dis

	with values in infinite-dimensions

	737.

	XAREV, V. M.: Propert
- [48] ZOLOTAREV, V. M.: Properties and relations of certain types of metrics. J. Soviet Math. 1 ables with values

721–737.

[48] ZOLOTAREV, V. M.:

(1981), 2218–2232. $\frac{481 \times 50107 \times 881 \times 1.218 - 2232}{(1981), 2218 - 2232}.$

Manuskripteingang: 2. 4. 1982

V • V

), 2218–2232.

Manuskripteingang: 2.4.1982

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