Zeitschrift für Analysis
und ihre Anwendungen
Bd. 2 (2) 1983, S. 135 – 144

On partial differential inequalities of the first order with a retarded argument

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Gegenstand der Arbeit sind partielle Differential-Ungleichungen erster Ordnung der Form

(i) $$z_t(t, x) \leqq f\big(t, x, z(t, z), z(\alpha(t, x), \beta(t, x)), z_x(t, x)\big),$$

wobei $x = (x_1, \ldots, x_n)$, $z_x(t, x) = \big(z_{x_1}(t, x), \ldots, z_{x_n}(t, x)\big)$ und

$$z(\alpha(t, x), \beta(t, x)) = \big(z(\alpha_1(t, x), \beta_1(t, x)), \ldots, z(\alpha_m(t, x), \beta_m(t, x))\big)$$

ist. Es wird vorausgesetzt, daß (i) eine Volterra-Ungleichung ist. Sei

$$E = \{(t, x)\colon \quad 0 \leqq t \leqq a, \quad |x_i - \hat{x}_i| \leqq b_i - M_i t \quad (i = 1, \ldots, n)\},$$
$$E_0 = \{(t, x)\colon \quad -\tau \leqq t \leqq 0, \quad |x_i - \hat{x}_i| \leqq b_i \quad (i = 1, \ldots, n)\},$$

und mögen $u, v \in C(E_0 \cup E, \mathbf{R})$ bezüglich (t, x) in E die Lipschitz-Bedingung und fast überall in E die Differential-Ungleichungen

$$u_t(t, x) \leqq f\big(t, x, u(t, x), u\,(\alpha(t, x), \beta(t, x)), u_x(t, x)\big),$$
$$v_t(t, x) \geqq f\big(t, x, v(t, x), v\,(\alpha(t, x), \beta(t, x)), v_x(t, x)\big)$$

erfüllen. Unter gewissen Voraussetzungen an die Funktionen f, α und β wird gezeigt, daß aus der Gültigkeit der Ungleichung $u(t, x) \leqq v(t, x)$ in E_0 die Gültigkeit dieser Ungleichung in E folgt.

В работе рассматриваются дифференциальные неравенства в частных производных вида

(i) $$z_t(t, x) \leqq f\big(t, x, z(t, x), z(\alpha(t, x), \beta(t, x)), z_x(t, x)\big),$$

где

$$x = (x_1, \ldots, x_n), \quad z_x(t, x) = \big(z_{x_1}(t, x), \ldots, z_{x_n}(t, x)\big),$$
$$z(\alpha(t, x), \beta(t, x)) = z(\alpha_1(t, x), \beta_1(t, x)), \ldots, z(\alpha_m(t, x), \beta_m(t, x)).$$

Предполагается, что (i) является неравенством типа Вольтерра. Пусть

$$E = \{(t, x)\colon \quad 0 \leqq t \leqq a, \quad |x_i - \hat{x}_i| \leqq b_i - M_i t \quad (i = 1, \ldots, n)\},$$
$$E_0 = \{(t, x)\colon \quad -\tau \leqq t \leqq 0, \quad |x_i - \hat{x}_i| \leqq b_i \quad (i = 1, \ldots, n)\},$$

и пусть $u, v \in C(E_0 \cup E, \mathbf{R})$ удовлетворяют условию Липшица по переменным (t, x) в E и удовлетворяют почти всюду в E дифференциальным неравенствам

$$u_t(t, x) \leqq f\big(t, x, u(t, x), u(\alpha(t, x), \beta(t, x)), u_x(t, x)\big),$$
$$v_t(t, x) \geqq f\big(t, x, v(t, x), v(\alpha(t, x), \beta(t, x)), v_x(t, x)\big).$$

При некоторых предположениях относительно функций f, α и β доказывается, что если неравенство $u(t, x) \leqq v(t, x)$ справедливо в E_0, то оно справедливо также в E.

This paper deals with first order partial differential inequalities of the form

(i) $$z_t(z, x) \leqq f\big(t, x, z(t, x), z(\alpha(t, x), \beta(t, x)), z_t(t, x)\big)$$

where $x = (x_1, \ldots, x_n)$, $z_x(t, x) = (z_{x_1}(t, x), \ldots, z_{x_n}(t, x))$ and

$$z(\alpha(t, x), \beta(t, x)) = \big(z(\alpha_1(t, x), \beta_1(t, x)), \ldots, z(\alpha_m(t, x), \beta_m(t, x))\big).$$

We assume that (i) is of the Volterra type. Let

$$E = \{(t, x): \quad 0 \leqq t \leqq a, \quad |x_i - \mathring{x}_i| \leqq b_i = M_i t \qquad (i = 1, \ldots, n)\}$$

and

$$E_0 = \{(t, x): \quad -\tau \leqq t \leqq 0, \quad |x_i - \mathring{x}_i| \leqq b_i \qquad (i = 1, \ldots, n)\}.$$

Assume that $u, v \in C(E_0 \cup E, \mathbf{R})$ satisfy on E the Lipschitz condition with respect to (t, x). Suppose that u and v satisfy almost everywhere on E the differential inequalities

$$u_t(t, x) \leqq f\big(t, x, u(t, x), u(\alpha(t, x), \beta(t, x)), u_x(t, x)\big),$$

$$v_t(t, x) \geqq f\big(t, x, v(t, x), v(\alpha(t, x), \beta(t, x)), v_x(t, x)\big)$$

and the initial inequality $u(t, x) \leqq v(t, x)$ on E_0. In the paper we prove that under certain assumptions concerning the functions f, α, β, the inequality $u(t, x) \leqq v(t, x)$ is satisfied on E.

This paper deals with first order partial differential inequalities of the form

$$z_t(t, x) \leqq f\big(t, x, z(t, x), z(\alpha(t, x), \beta(t, x)), z_x(t, x)\big) \tag{1}$$

where $x = (x_1, \ldots, x_n)$, $z_x(t, x) = \big(z_{x_1}(t, x), \ldots, z_{x_n}(t, x)\big)$ and $z\big(\alpha(t, x); \beta(t, x)\big) = \big(z(\alpha_1(t, x), \beta_1(t, x)), \ldots, z(\alpha_m(t, x), \beta_m(t, x))\big)$. We consider the above inequality almost everywhere in a pyramid

$$E = \{(t, x): \quad 0 \leqq t \leqq a, |x_i - \mathring{x}_i| \leqq b_i - M_i t \quad (i = 1, \ldots, n)\}$$

where $a, b_i, M_i > 0$ and $aM_i \leqq b_i$ $(i = 1, \ldots, n)$. For $\tau > 0$ let

$$E_0 = \{(t, x): \quad -\tau \leqq t \leqq 0, |x_i - \mathring{x}_i| \leqq b_i \quad (i = 1, \ldots, n)\}$$

be an initial set and $\tilde{E} = E - I_0$ where

$$I_0 = \{(t, x): \quad t = 0, |x_i - \mathring{x}_i| \leqq b_i \quad (i = 1, \ldots, n)\}.$$

We assume that (1) is of the Volterra type i.e. if $(t, x) \in E$ then $\big(\alpha_i(t, x), \beta_i(t, x)\big) \in E_t$, where $E_t = \{(s, y) \in E_0 \cup E : s \leqq t\}$. For $\delta \in (0, a)$ denote by H_δ a pyramid

$$H_\delta\{(t, x): \quad \delta \leqq t \leqq a, |x_i - \mathring{x}_i| \leqq b_i - M_i t \quad (i = 1, \ldots, n)\}.$$

Denote by $C(X, Y)$ the set of continuous functions defined in X taking values in Y. We denote by $\|\cdot\|$ the norm in $\mathbf{R}^n$.

Differential inequalities find numerous applications in the classical and generalized theory of first order partial differential equations. Such problems as: the estimation of solutions of partial differential equations, the estimation of the domain of the solution, the estimation of the difference between two solutions, criteria of uniqueness, Chaplygin's methods of approximation of a solution, and criteria of stability, are anongst the classical examples of the application of differential inequalities (cf. [4—8]).

First order partial differential inequalities were treated in the monograph [8] (see also [1, 5]). Some results concerning partial differential-functional equations and inequalities of the Volterra type can be found in [2, 3]. Our result is a generalization of the differential inequalities theorem of [4].

We introduce the following *Assumption H*:

1° The function f of the variables (t, x, u, q), $u = (u_0, u_1, \ldots, u_m)$, $q = (q_1, \ldots, q_n)$, is of class C^2 for $t \in [0, a]$, $x \in \mathbf{R}^n$, $u \in \mathbf{R}^{m+1}$, $q \in \mathbf{R}^n$.

2° The quadratic form $\sum_{i,j=1}^{n} f_{q_i q_j}(t, x, u, q)\, \xi_i \xi_j$ is positive for $(t, x) \in E$, $u \in \mathbf{R}^{m+1}$, $q \in \mathbf{R}^n$, i.e. for arbitrary $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$ we have

$$\sum_{i,j=1}^{n} f_{q_i q_j}(t, x, u, q)\, \xi_i \xi_j \geqq 0. \tag{2}$$

3° For $(t, x, u, q) \in E \times \mathbf{R}^{m+1+n}$ we have

$$f_{u_i}(t, x, u, q) \geqq 0 \qquad (i = 1, \ldots, m), \tag{3}$$

$$|f_{q_i}(t, x, u, q)| \leqq M_i \qquad (i = 1, \ldots, n). \tag{4}$$

4° $\alpha_i, \beta_i = (\beta_{i1}, \ldots, \beta_{in})$ $(i = 1, \ldots, m)$ are of class C^1 on E and $\big(\alpha_i(t, x), \beta_i(t, x)\big) \in E_t$ if $(t, x) \in E$ $\quad (i = 1, \ldots, n)$.

We prove the following theorem.

Theorem: *Suppose that*

1° *Assumption* H *is satisfied,*
2° $u, v \in C(E_0 \cup E, \mathbf{R})$, *u and v satisfy on E the Lipschitz condition with respect to* (t, x),
3° *there exists a function* $\lambda \in C((0, a], \mathbf{R}_+)$, $\mathbf{R}_+ = [0, +\infty)$, *such that the estimations*

$$\frac{u(t, x + l) - 2u(t, x) + u(t, x - l)}{\|l\|^2} \leqq \lambda(t),$$

$$\frac{v(t, x + l) - 2v(t, x) + v(t, x - l)}{\|l\|^2} \leqq \lambda(t)$$

are satisfied for $(t, x), (t, x + l), (t, x - l) \in \tilde{E}$, $l \neq 0$,
4° *there exists a constant* $K > 0$ *such that*

$$\|u_x(t, x)\|, \qquad \|v_x(t, x)\| \leqq K \tag{5}$$

almost everywhere on E,
5° *for* $(t, x) \in E_0$ *we have*

$$u(t, x) \leqq v(t, x), \tag{6}$$

6° *the differential-functional inequalities*

$$\begin{aligned} u_t(t, x) &\leqq f\big(t, x, u(t, x), u(\alpha(t, x), \beta(t, x)), u_x(t, x)\big) \\ v_t(t, x) &\geqq f\big(t, x, v(t, x), v(\alpha, (t, x), \beta(t, x)), v_x(t, x)\big) \end{aligned} \tag{7}$$

are satisfied almost everywhere on E.
Under these assumptions

$$u(t, x) \leqq v(t, x) \quad \text{for} \quad (t, x) \in E. \tag{8}$$

Proof: The proof of the theorem is divided into parts **I—V**. In **I—IV** we prove that for each $\varepsilon > 0$ there exists a $\delta \in (0, a)$ such that for an arbitrary function $\Phi \in C([0, a] \times \mathbf{R}^n, \mathbf{R}_-)$, $\mathbf{R}_- = (-\infty, 0]$, which satisfies $\Phi(t, x) = 0$ for $(t, x) \in ([0, a] \times \mathbf{R}^n) \setminus E$, there exists a function Ψ defined on $(0, a] \times \mathbf{R}^n$, with $\Psi(t, x) \geq 0$ on $\tilde{E}$ and $\Psi(t, x) = 0$ for $(t, x) \in ((0, a] \times \mathbf{R}^n) \setminus \tilde{E}$ such that

$$\int_{H_\delta} [u(t, x) - v(t, x)]\, \Phi(t, x)\, dt\, dx$$
$$+ \int_{H_\delta} \sum_{i=1}^{m} \int_0^1 f_{u_i}\big(P(t, x, s)\big)\, ds\, \big[u\big(\alpha_i(t, x), \beta_i(t, x)\big)$$
$$- v\big(\alpha_i(t, x), \beta_i(t, x)\big)\big]\, \Psi(t, x)\, dt\, dx \geq -\varepsilon \tag{9}$$

where

$$P(t, x, s) = \big(t, x, v(t, x) + s(u(t, x) - v(t, x)), v(\alpha(t, x), \beta(t, x))$$
$$+ s\big(u(\alpha(t, x), \beta(t, x)) - v(\alpha(t, x), \beta(t, x))\big), v_x(t, x) + s\big(u_x(t, x) - v_x(t, x)\big)\big). \tag{10}$$

In V we shall prove that (9) implies (8).

I. *An approximation of the functions u and v by means of the functions of class* C^∞. It follows from assumption 2° of the theorem that for some positive constant M we have $|u(t, x)| \leq M$ and $|v(t, x)| \leq M$ for $(t, x) \in E_0 \cup E$. Let $\tilde{u}, \tilde{v}$ be continuous functions on $[-\tau, a] \times \mathbf{R}^n$, such that $|\tilde{u}(t, x)| \leq M$, $|\tilde{v}(t, x)| \leq M$ and $\tilde{u}|_{E_0 \cup E} = u$, $\tilde{v}|_{E_0 \cup E} = v$ ($z|_{E_0 \cup E}$ is a restriction of the function z to the set $E_0 \cup E$). We choose sequences of functions $\{u^{(k)}\}$ and $\{v^{(k)}\}$ such that

a) for $k = 1, 2, \ldots$ we have

$$u^{(k)} \text{ and } v^{(k)} \text{ are of class } C^\infty \text{ on } [-\tau, a] \times \mathbf{R}^n,$$
$$|u^{(k)}(t, x)|,\ |v^{(k)}(t, x)| \leq M \quad \text{on} \quad [-\tau, a] \times \mathbf{R}^n \quad \text{and}$$
$$\|u_x^{(k)}(t, x)\|,\quad \|v_x^{(k)}(t, x)\| \leq K \quad \text{on} \quad [0, a] \times \mathbf{R}^n,$$

b) $\lim_{k\to\infty} u^{(k)} = \tilde{u}$, $\lim_{k\to\infty} v^{(k)} = \tilde{v}$ uniformly on E and $\lim_{k\to\infty} \operatorname{grad} u^{(k)} = \operatorname{grad} \tilde{u}$, $\lim_{k\to\infty} \operatorname{grad} v^{(k)} = \operatorname{grad} \tilde{v}$ in $L_2(E)$ norm, where $\operatorname{grad} z = (z_t, z_{x_1}, \ldots, z_{x_n})$,

c) for an arbitrary $l = (l_1, \ldots, l_n) \neq 0$, $l \in \mathbf{R}^n$, we have

$$\frac{\partial^2 u^{(k)}(t, x)}{\partial l^2} \leq \lambda(t), \qquad \frac{\partial^2 v^{(k)}(t, x)}{\partial l^2} \leq \lambda(t) \tag{11}$$

for $(t, x) \in \tilde{E}$ $(k = 1, 2, \ldots)$ where $\dfrac{\partial^2 z(t, x)}{\partial l^2}$ is the second directional derivative of the function z at the point (t, x). The sequences $\{u^{(k)}\}$ and $\{v^{(k)}\}$ can be defined in the following way (see [4]). Let $\omega \in C^\infty(\mathbf{R}, \mathbf{R}_+)$ be such a function that

(i) $\omega(t) = 0$ for $|t| \geq 1$ and $\omega(-t) = \omega(t)$ for $t \in \mathbf{R}$,

(ii) $\int_{\mathbf{R}^{n+1}} \omega(\|\eta\|_0)\, d\eta = 1$ where $\eta = (\eta_1, \ldots, \eta_{n+1})$ and $\|\cdot\|_0$

is the norm in $\mathbf{R}^{n+1}$.

If $z \in C([-\tau, a] \times \mathbf{R}^n, \mathbf{R})$, then we define for $\varepsilon > 0$

$$z_\varepsilon(t, x) = \frac{1}{\varepsilon^{n+1}} \int_{[-\tau, a] \times \mathbf{R}^n} z(s, y)\, \omega\left(\frac{\|y' - x'\|_0}{\varepsilon}\right) ds\, dy,$$

where $y' = (s, y)$, $x' = (t, x)$. It is easy to prove that $u^{(k)}(t, x) = \tilde{u}_{1/k}(t, x)$, $v^{(k)}(t, x) = \tilde{v}_{1/k}(t, x)$ $(k = 1, 2, \ldots)$ satisfy a)–c).

II. *We shall now prove a basic integral inequality for the functions* $u^{(k)}$ *and* $v^{(k)}$.

If z is a continuous function on $E_0 \cup E$ and satisfies the Lipschitz condition with respect to (t, x) on E then we define

$$F(z)(t, x) = f\big(t, x, z(t, z), z(\alpha(t, x), \beta(t, x)), z_x(t, x)\big).$$

We now define functions $\alpha^{(k)}$, $u_i^{(k)}$, and $v_i^{(k)}$ as follows:

$$\alpha^{(k)}(t, x) = F(v^{(k)})(t, x) - F(u^{(k)})(t, x) + F(u)(t, x) - F(v)(t, x) + u_t^{(k)}(t, x) - v_t^{(k)}(t, x) - u_t(t, x) + v_t(t, x),$$

$$u_i^{(k)}(t, x) = u^{(k)}\big(\alpha_i(t, x), \beta_i(t, x)\big), \quad v_i^{(k)}(t, x) = v^{(k)}\big(\alpha_i(t, x), \beta_i(t, x)\big),$$

$$i = 1, \ldots, m.$$

Let Φ be a continuous function on $[0, a] \times \mathbf{R}^n$ such that $\Phi(t, x) \leq 0$ for $(t, x) \in [0, a] \times \mathbf{R}^n$ and $\Phi(t, x) = 0$ on $([0, a] \times \mathbf{R}^n) \setminus E$. Now, we prove the fundamental inequality

$$\int_{H_\delta} [u^{(k)}(t, x) - v^{(k)}(t, x)]\, \Phi(t, x)\, dt\, dx + \int_{H_\delta} \sum_{i=1}^{m} \int_0^1 f_{u_i}\big(P_k(t, x, s)\big)\, ds\, [u_i^{(k)}(t, x) - v_i^{(k)}(t, x)]\, G^{(k)}(\Phi)(t, x)\, dt\, dx$$
$$\geq \int_{H_\delta} \alpha^{(k)}(t, x)\, G^{(k)}(\Phi)(t, x)\, dt\, dx - \int_{I_\delta} [u^{(k)}(\delta, x) - v^{(k)}(\delta, x)]\, G^{(k)}(\Phi)(\delta, x)\, dx \tag{12}$$

where

(i) $I_\delta = \{x: \ |x_i - \mathring{x}_i| \leq b_i - M_i\delta \ \ (i = 1, \ldots, n)\}, \qquad 0 < \delta < a,$

(ii) $G^{(k)}(\Phi)$ is solution of the initial problem

$$g_t(t, x) + \int_0^1 f_{u_0}\big(P_k(t, x, s)\big)\, ds\, g(t, x) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\int_0^1 f_{q_i}\big(P_k(t, x, s)\big)\, ds\, g(t, x)\right) = \Phi(t, x), \quad (t, x) \in (0, a] \times \mathbf{R}^n,$$

$$g(a, x) = 0 \quad \text{for} \quad x \in \mathbf{R}^n,$$

(iii) $P_k(t, x, s) = \big(t, x, v^{(k)}(t, x) + s[u^{(k)}(t, x) - v^{(k)}(t, x)],$
$v^{(k)}(\alpha(t, x), \beta(t, x)) + s\big[u^{(k)}(\alpha(t, x), \beta(t, x)) - v^{(k)}(\alpha(t, x), \beta(t, x))\big], v_x^{(k)}(t, x) + s[u_x^{(k)}(t, x) + v_x^{(k)}(t, x)]\big).$

If Φ satisfies the above conditions then $G^{(k)}(\Phi)$ is defined on $(0, a] \times \mathbf{R}^n$, $G^{(k)}(\Phi)(t, x) \geq 0$ on $\tilde{E}$ and $G^{(k)}(\Phi)(t, x) = 0$ on $((0, a] \times \mathbf{R}^n) \setminus \tilde{E}$.

If follows from (7) that

$$u_t^{(k)}(t, x) - v_t^{(k)}(t, x) \leq F(u^{(k)})(t, x) - F(v^{(k)})(t, x) + \alpha^{(k)}(t, x) \tag{13}$$

almost everywhere on E. Ineqaulity $G^{(k)}(\Phi)(t,x) \geqq 0$ on $\tilde{E}$, together with (13) yields

$$\int_{H_\delta} [u_t^{(k)}(t,x) - v_t^{(k)}(t,x)]\, G^{(k)}(\Phi)(t,x)\,dt\,dx$$
$$\leqq \int_{H_\delta} \alpha^{(k)}(t,x), G^{(k)}(\Phi)(t,x)\,dt\,dx + \int_{H_\delta} [F(u^{(k)})(t,x) - F(v^{(k)})(t,x)]$$
$$\times G^{(k)}(\Phi)(t,x)\,dt\,dx. \tag{14}$$

We integrate (14) by parts and, taking into account Hadamard's meen value theorem, we get

$$-\int_{H_\delta} [u^{(k)}(t,x) - v^{(k)}(t,x)]\, G_t^{(k)}(\Phi)(t,x)\,dt\,dx$$
$$-\int_{I_\delta} [u^{(k)}(\delta,x) - v^{(k)}(\delta,x)]\, G^{(k)}(\Phi)(\delta,x)\,dx$$
$$\leqq \int_{H_\delta}\int_0^1 f_{u_0}\big(P_k(t,x,s)\big)\,ds\, G^{(k)}(\Phi)(t,x)\,[u^{(k)}(t,x) - v^{(k)}(t,x)]\,dt\,dx$$
$$+\int_{H_\delta} \sum_{i=1}^m \int_0^1 f_{u_i}\big(P_k(t,x,s)\big)\,ds G^{(k)}(\Phi)(t,x)\,[u_i^{(k)}(t,x) - v_i^{(k)}(t,x)]\,dt\,dx$$
$$-\int_{H_\delta} \sum_{i=1}^m \frac{\partial}{\partial x_i}\left(\int_0^1 f_{q_i}\big(P_k(t,x,s)\big)\,ds\, G^{(k)}(\Phi)(t,x)\right)$$
$$\times [u^{(k)}(t,x) - v^{(k)}(t,x)]\,dt\,dx + \int_{H_\delta} \alpha^{(k)}(t,x)\, G^{(k)}(\Phi)(t,x)\,dt\,dx,$$

which implies (12).

III. *Now we prove that to every $\varepsilon > 0$ there corresponds a $\delta \in (0,a)$ and an N such that for $k \geqq N$ we have*

$$\int_{H_\delta} [u^{(k)}(t,x) - v^{(k)}(t,x)]\,\Phi(t,x)\,dt\,dx$$
$$+\int_{H_\delta} \sum_{i=1}^m \int_0^1 f_{u_i}\big(P_k(t,x,s)\big)\,ds\,[u_i^{(k)}(t,x) - v_i^{(k)}(t,x)]\, G^{(k)}(\Phi)(t,x)\,dt\,dx$$
$$\geqq -\varepsilon - \int_{H_\delta} \alpha^{(k)}(t,x)\, G^{(k)}(\Phi)(t,x)\,dt\,dx. \tag{15}$$

Since

$$|f_{u_0}\big(P_k(t,x,s)\big)| \leqq \tilde{M}, \qquad (t,x) \in E, \quad s \in [0,1], \qquad k = 1,2,\ldots,$$
$$\left|\int_{\mathbf{R}^n} \Phi(t,x)\,dx\right| \leqq \tilde{N}, \qquad t \in [0,a]$$

for some constants $\tilde{M}, \tilde{N}$, the function $h(t) = \int_{\mathbf{R}^n} G^{(k)}(\Phi)\, t(,x)\,dx$ satisfies the differential inequality $h'(t) \geqq -\tilde{M}h(t) - \tilde{N}$, $t \in [0,a]$. This estimation and the condition $h(a) = 0$ imply that

$$h(t) \leqq \frac{\tilde{N}}{\tilde{M}}\,(e^{\tilde{M}(a-t)} - 1), \qquad t \in [0,a],$$

and, consequently, it follows that

$$\int_{\mathbf{R}^n} G^{(k)}(\Phi)(t, x)\,dx \leqq M_0, \qquad t \in [0, a], \tag{16}$$

for $M_0 = \tilde{N}\tilde{M}^{-1}(e^{\tilde{M}a} - 1)$.

Since $u(0, x) \leqq v(0, x)$ for $(0, x) \in I_0$, it follows that to every $\varepsilon > 0$ there corresponds a δ, $0 < \delta < a$, and an N such that for $0 \leqq t \leqq \delta$ and $|x_i - \mathring{x}_i| \leqq b_i - M_i t$ $(i = 1, \ldots, n)$ we have

$$u^{(k)}(t, x) - v^{(k)}(t, x) \leqq \frac{\varepsilon}{\max(1, M_0)}, \qquad k \geqq N.$$

Let $\varepsilon > 0$ be fixed and δ be a sufficiently small constant that the above condition is satisfied. Then, because of the relations (12) and (16), we have (15).

IV. *We shall show that the functions $G^{(k)}(\Phi)$ are equibounded on H_δ.*

The function $G^{(k)}(\Phi)$ is a solution of the equation

$$g_t(t, x) + A^{(k)}(t, x)\,g(t, x) + \sum_{i=1}^{n} B_i^{(k)}(t, x)\,g_{x_i}(t, x) = \Phi(t, x)$$

where

$$\begin{aligned} A^{(k)}(t, x) &= \int_0^1 f_{u_0}\big(P_k(t, x, s)\big)\,ds - \sum_{i=1}^{n}\int_0^1 f_{q_i x_i}\big(P_k(t, x, s)\big)\,ds \\ &- \sum_{i=1}^{n}\int_0^1 f_{q_i u_0}\big(P_k(t, x, s)\big)\big[v^{(k)}_{x_i}(t, x) + s\big(u^{(k)}_{x_i}(t, x) - v^{(k)}_{x_i}(t, x)\big)\big]\,ds \\ &- \sum_{i=1}^{n}\sum_{j=1}^{m}\int_0^1 f_{q_i u_j}\big(P_k(t, x, s)\big)\frac{\partial}{\partial x_i}\big[v_j^{(k)}(t, x) + s\big(u_j^{(k)}(t, x) - v_j^{(k)}(t, x)\big)\big]\,ds \\ &+ \sum_{i,j=1}^{n}\int_0^1 f_{q_i q_j}\big(P_k(t, x, s)\big)\,ds\,\big[v^{(k)}_{x_i x_j}(t, x) + s\big(u^{(k)}_{x_i x_j}(t, x) - v^{(k)}_{x_i x_j}(t, s)\big)\big]\,ds, \end{aligned}$$

$$B_i^{(k)}(t, x) = -\int_0^1 f_{q_i}\big(P_k(t, x, s)\big)\,ds, \qquad i = 1, \ldots, n.$$

First we prove that there exists a function $\tilde{a}$ continuous on $(0, a]$ such that

$$|A^{(k)}(t, x)| \leqq \tilde{a}(t), \qquad (t, x) \in \tilde{E}, \qquad k = 1, 2, \ldots \tag{17}$$

Assume that $\lambda_1, \ldots, \lambda_n$ is a system of eigenvalues of the matrix

$$f_{qq}\big(P_k(t, x, s)\big) = \big[f_{q_i q_j}\big(P_k(t, x, s)\big)\big]_{i,j=1,\ldots,n}$$

and $l_1, \ldots, l_n$ is a corresponding system of eigenvectors of norm 1. It follows from conditions 1° and 2° of Assumption H that $0 < \lambda_i < \overline{M}$ $(i = 1, \ldots, n)$ for some constant $\overline{M}$ and

$$\begin{aligned} &\sum_{i,j=1}^{n} f_{q_i q_j}\big(P_k(t, x, s)\big)\big[v^{(k)}_{x_i x_j}(t, x) + s\big(u^{(k)}_{x_i x_j}(t, x) - v^{(k)}_{x_i x_j}(t, x)\big)\big] \\ &= \sum_{i=1}^{n} \lambda_i \left[\frac{\partial^2 v^{(k)}(t, x)}{\partial l_i^2} + s\left(\frac{\partial^2 u^{(k)}(t, x)}{\partial l_i^2} - \frac{\partial^2 v^{(k)}(t, x)}{\partial_i^2}\right)\right]. \end{aligned}$$

Now, we obtain estimation (17) by condition 1° of Assumption H and (11).

Since $G^{(k)}(\Phi)(a, x) = 0$ for $x \in \mathbf{R}^n$, using comparison theorems for first order partial equations (see [8: Chapter VII] and [5: Chapter IX]) we obtain:

$$G^{(k)}(\Phi)(t, x) \leqq c(a-t)\exp\left(\int_t^a \tilde{a}(s)\,ds\right), \qquad k = 1, 2, \ldots \tag{18}$$

where $c = \max_{(t,x)\in E} |\Phi(t, x)|$. It is readily seen that there exists a constant C such that

$$\int_{H_\delta} [G^{(k)}(\Phi)(t, x)]^2\,dt\,dx \leqq C, \qquad k = 1, 2 \ldots \tag{19}$$

Now we prove (9). It follows from (19) that there exists a sequence $\{k_r\}$ such that $\{G^{(k_r)}(\Phi)\}$ is convergent in the $L_2(H_\delta)$ norm. Let $\Psi = \lim_{r\to\infty} G_{(k_r)}(\Phi)$ in $L_2(H_\delta)$. As a result of (15) and (18) we get (9).

V. *We prove that* (9) *implies* (8).

Suppose that the inequality (8) is false. Then there exists a point $(t^*, x^*) \in \tilde{E}$ such that $u(t^*, x^*) - v(t^*, x^*) > 0$. Let $\delta > 0$ be a sufficiently small constant that $(t^*, x^*) \in H_\delta$. By the continuity of u and v on $E_0 \cup E'$ it follows that there exists a neighbourhood Q of (t^*, x^*) such that

$$u(t, x) - v(t, x) > 0 \quad \text{for} \quad (t, x) \in Q \cap H_\delta. \tag{20}$$

Now, there are two cases to be distinguished.

1° Suppose that there exists a domain $\tilde{Q} \subset Q \cap H_\delta$ such that for $i = 1, \ldots, m$ we have

$$u\big(\alpha_i(t, x), \beta_i(t, x)\big) - v\big(\alpha_i(t, x), \beta_i(t, x)\big) \leqq 0, \qquad (t, x) \in \tilde{Q}. \tag{21}$$

Let Φ be a continuous function on $[0, a]\times \mathbf{R}^n$, $\Phi(t, x) < 0$ for $(t, x) \in \tilde{Q}$ and $\Phi(t, x) = 0$ for $(t, x) \in ([0, a] \times \mathbf{R}^n) \setminus \tilde{Q}$. Then $\Psi(t, x) \geqq 0$ for $(t, x) \in \tilde{Q}$ and $\Psi(t, x) = 0$ for $(t, x) \in H_\delta - \tilde{Q}$. From (9) it follows that

$$\begin{aligned} I_{\tilde{Q}}(u, v) = {} & \int_{\tilde{Q}} [u(t, x) - v(t, x)]\,\Phi(t, x)\,dt\,dx \\ & + \int_{\tilde{Q}} \sum_{i=1}^m \int_0^1 f_{u_i}\big(P(t, x, s)\big)\,ds\,\big[u\big(\alpha_i(t, x), \beta_i(t, x)\big) - v\big(\alpha_i(t, x), \beta_i(t, x)\big)\big] \\ & \times \Psi(t, x)\,dt\,dx \geqq -\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $I_{\tilde{Q}}(u, v) \geqq 0$, which contradicts (20) and (21).

2° Suppose that there exists a domain $\tilde{Q} \subset Q \cap H_\delta$ such that for $(t, x) \in \tilde{Q}$ we have

$$u\big(\alpha_i(t, x), \beta_i(t, x)\big) - v\big(\alpha_i(t, x), \beta_i(t, x)\big) \leqq 0, \qquad i = 1, \ldots, \tilde{k}, \tag{22}$$

and

$$u\big(\alpha_i(t, x), \beta_i(t, x)\big) - v\big(\alpha_i(t, x), \beta_i(t, x)\big) \geqq 0, \qquad i = \tilde{k} + 1, \ldots, m, \tag{23}$$

where $1 \leqq \tilde{k} < m$. It follows from (15) that

$$\begin{aligned} & \int_{H_\delta} [u^{(k_r)}(t, x) - v^{(k_r)}(t, x)]\,\Phi(t, x)\,dt\,dx \\ & + \int_{H_\delta} \sum_{i=1}^m \int_0^1 f_{u_i}\big(P_{k_r}(t, x, s)\big)\,ds[u_i^{(k_r)}(t, x) - v_i^{(k_r)}(t, x)]\,G^{(k_r)}(\Phi)(t, x)\,dt\,dx \\ & \geqq -\varepsilon - \int_{H_\delta} \alpha^{(k_r)}(t, x)\,G^{(k_r)}(\Phi)(t, x)\,dt\,dx, \qquad k_r \geqq N. \end{aligned} \tag{24}$$

We choose a function $\tilde{\Psi}$ continuous on $[0, a] \times \mathbf{R}^n$ such that $\tilde{\Psi}(t, x) < 0$ for $(t, x) \in \tilde{Q}$ and $\tilde{\Psi}(t, x) = 0$ for $(t, x) \in ([0, a] \times \mathbf{R}^n) \setminus \tilde{Q}$. Let us consider for each $k_r \geqq N$ the equation

$$\Phi(t, x) + \sum_{i=\tilde{k}+1}^{m} \int_0^1 f_{u_i}\big(P_{k_r}(t, x, s)\big)\, ds[u_i^{(k_r)}(t, x) - v_i^{(k_r)}(t, x)] \times [u^{(k_r)}(t, x) - v^{(k_r)}(t, x)]^{-1} G^{(k_r)}(\Phi)(t, x) = \tilde{\Psi}(t, x). \tag{25}$$

(We choose the sequences $\{u^{(k_r)}\}$, $\{v^{(k_r)}\}$ such that $u^{(k_r)}(t, x) - v^{(k_r)}(t, x) > 0$ for $(t, x) \in \tilde{Q}$). Let $\Phi^{(k_r)}$ be a solution of (25). It is easy to see that $\Phi^{(k_r)}(t, x) < 0$ for $(t, x) \in Q$ and $\Phi^{(k_r)}(t, x) = 0$ for $(t, x) \in ([0, a] \times \mathbf{R}^n) \setminus \tilde{Q}$. Now, we have from (24)

$$\int_{H_\delta} [u^{(k_r)}(t, x) - v^{(k_r)}(t, x)]\, \Phi^{(k_r)}(t, x)\, dt\, dx + \int_{H_\delta} \sum_{i=\tilde{k}+1}^{m} \int_0^1 f_{u_i}\big(P_{k_r}(t, x, s)\big)\, ds[u_i^{(k_r)}(t, x) - v_i^{(k_r)}(t, x)]\, G^{(k_r)}(\Phi^{(k_r)})(t, x)\, dt\, dx$$
$$+ \int_{H_\delta} \sum_{i=1}^{\tilde{k}} \int_0^1 f_{u_i}\big(P_{k_r}(t, x, s)\big)\, ds[u^{(k_r)}(t, x) - v_i^{(k_r)}(t, x)]\, G^{(k_r)}(\Phi^{(k_r)})(t, x)\, dt\, dx \geqq -\varepsilon - \int_{H_\delta} \alpha^{(k_r)}(t, x)\, G^{(k_r)}(\Phi^{(k_r)}(t, x)\, dt\, dx, \quad k_r \geqq N. \tag{26}$$

It is readily seen that there exists a sequence $\{k_{r_s}\}$ such that $\{G^{(k_{r_s})}(\Phi)^{(k_{r_s})})\}$ is convergent in $L_2(H_\delta)$ norm to a function Ψ_0 such that $\Psi_0(t, x) \geqq 0$ in $\tilde{Q}$ and $\Psi_0(t, x) = 0$ in $H_\delta \setminus \tilde{Q}$. As a result of (24)—(26) we get (making k_{r_s} tend to ∞ in (24))

$$\tilde{I}_{\tilde{Q}}(u, v) = \int_{\tilde{Q}} [u(t, x) - v(t, x)]\, \tilde{\Psi}(t, x)\, dt\, dx + \int_{\tilde{Q}} \sum_{i=1}^{\tilde{k}} \int_0^1 f_{u_i}\big(P(t, x, s)\big)\, ds\big[u\big(\alpha_i(t, x), \beta_i(t, x)\big) - v\big(\alpha_i(t, x), \beta_i(t, x)\big)\big] \times \Psi_0(t, x)\, dt\, dx \geqq -\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\tilde{I}_{\tilde{Q}}(u, v) \geqq 0$ which contradicts (20) and (22). This completes the proof ∎

The above Theorem implies the following remarks.

Remark 1: (*Uniqueness of a solution of the Cauchy problem.*)
Suppose that

1° Assumption H is satisfied,

2° $u, v \in C(E_0 \cup E, \mathbf{R})$, u and v satisfy on E the Lipschitz condition with respect to (t, x),

3° u and v satisfy almost everywhere on E the differential equation with a retarded argument

$$z_t(t, x) = f\big(t, x, z(t, x), z(\alpha(t, x), \beta(t, x)), z_x(t, x)\big)$$

and $u(t, x) = v(t, x)$ for $(t, x) \in E_0$,

4° conditions 3° and 4° of Theorem are satisfied.

Under these assumptions $u(t, x) = v(t, x)$ on E.

Remark 2: The above results can be extended to the system of first order partial differential inequalities of the form

$$u_t^{(i)}(t, x) \leqq f^{(i)}\big(t, x, u(t, x), u(\alpha(t, x), \beta(t, x)), u_x^{(i)}(t, x)\big), \qquad i = 1, \ldots, k,$$

where $u(t, x) = \big(u^{(1)}(t, x), \ldots, u^{(k)}(t, x)\big)$.

One could formulate and prove analogous results for the overdetermined system of partial differential inequalities considered in [5, 8].

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Manuskripteingang: 24. 9. 1981; in revidierter Fassung: 2. 03. 1982

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