

## On stable space dependent stationary solutions of a competition system with diffusion

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Es wird ein aus zwei Gleichungen bestehendes, quasilineares parabolisches Reaktions-Diffusionsgleichungssystem behandelt, wobei nur in einer der beiden Gleichungen ein Diffusionsterm auftritt und Dirichletsche Randbedingungen vorgeschrieben werden. Untersucht wird die Bifurkation nichtnegativer stationärer Lösungen, ihr asymptotischer Stabilitätscharakter und ihre Abhängigkeit von der Raumvariablen.

Рассматривается реактивно-диффузионная система двух квазилинейных параболических дифференциальных уравнений, только в одном из которых имеется диффузионный член и поставлено краевое условие Дирихле. Исследуются бифуркация положительных стационарных решений, характер их асимптотической устойчивости и их зависимость от пространственной переменной.

A quasilinear parabolic system of two equations is considered, only one of which includes the diffusion term. The bifurcation of nonnegative stationary solutions is studied together with their stability character and their dependence on the "space" variables.

### 1. Introduction

Quasilinear parabolic systems arise in several biological and chemical models and are used to understand propagation phenomena, oscillations or stabilization towards a stationary state. While concerning the existence of stationary solutions — subject to rather general boundary conditions — quite general results are available [1, 6, 7, 18], convergence towards such distinguished solutions and their stability character present more difficult mathematical problems, and the results so far obtained do not yet claim to a comparable level of generality (for an extensive review, see [4]). In fact, in a number of *prima facie* naive and very simple models, the problem of asymptotic stability of (non-trivial) stationary solutions is still open — this is especially true if the system are supplemented with homogeneous boundary conditions which are not purely of Neumann type: in such case the search for non-trivial stationary solutions leads to quantities which are space-dependent [5], which makes their stability analysis considerably more difficult<sup>1)</sup>. These are, to our knowledge, not very many results in this sense [5, 9, 17]: our aim here is to present a contribution in this direction, concerning a competition model for two species, only one of which is subject to diffusion (and to Dirichlet homogeneous boundary condition). In spite of the apparent simplicity of the model, the resulting equations are not deprived of interest — they involve, by the way, a free boundary problem of non-variational type — and in fact, we shall display in a quite complete way nontrivial bifurcation and stability

<sup>1)</sup> Pure Neumann homogeneous boundary conditions allow, in case of constant coefficients, to look for constant stationary solutions, whose stability can be investigated in a relatively easy way: see [4] for a survey of results.

properties of (non-negative) stationary solutions. In addition, informations on the space structure of these solutions will be obtained.

Specifically, we shall be dealing with the following initial boundary value problem:

$$\begin{aligned} \partial_t u &= a_1 \Delta u + u(b_1 - c_1 u - d_1 v) && \text{in } (0, +\infty) \times \Omega, \\ \partial_t v &= v(b_2 - c_2 v - d_2 u) && \\ u &= 0 && \text{in } (0, +\infty) \times \partial\Omega, \\ u &= u_0, \quad v = v_0 && \text{in } \{0\} \times \Omega. \end{aligned} \quad (1.1)$$

Here  $\Omega \subseteq \mathbb{R}^n$  is an open bounded set with smooth boundary  $\partial\Omega$ ,  $a_i, b_i, c_i, d_i$  ( $i = 1, 2$ ) are positive constants and  $u_0, v_0$  are given nonnegative functions.

We are going to investigate existence, uniqueness and asymptotic stability properties of stationary solutions to (1.1), namely of solutions of the following elliptic problem:

$$\begin{aligned} a_1 \Delta u + u(b_1 - c_1 u - d_1 v) &= 0 \\ v(b_2 - c_2 v - d_2 u) &= 0 && \text{in } \Omega, \\ u &= 0 && \text{in } \partial\Omega. \end{aligned} \quad (1.2)$$

Of special interest will be the comparison with the properties of the ordinary differential system (the so-called space-clamp system) we formally obtain by dropping the diffusion term in the first equation in (1.1).

Throughout this paper we shall assume the following inequality to hold:

$$(C_0) \quad c_1 c_2 \geq d_1 d_2.$$

In terms of the same space-clamp system, assumption  $(C_0)$  implies the slope of the  $v$ -cline to be not steeper than that of the  $u$ -cline, thus ensuring the asymptotic stability of the solution (if any) having both components positive. If, in addition, we assume

$$(C_1) \quad \frac{b_1}{c_1} > \frac{b_2}{d_2},$$

the only nontrivial stationary solutions of the space-clamp system are known to be  $(b_1/c_1, 0)$  (which is stable and attracts the first open orthant) and  $(0, b_2/c_2)$  (which is unstable). Coexistence of both species at equilibrium is therefore impossible.

The main purpose of the present paper is to prove that the above picture can be destroyed if arbitrarily small diffusion is introduced. As we shall see, if the diffusion coefficient  $a_1$  is (nonzero but) small in a suitable sense, stationary solutions of (1.1) describing coexistence of both species arise, which have no counterpart in the ordinary differential case; moreover, one of these solutions enjoys attractivity and stability properties (with respect to solutions of (1.1)) in a sense to be made precise in the following. Such stable stationary solution exhibits space segregation (see [10, 19]) as a consequence of the assumed homogeneous Dirichlet boundary conditions: as a matter of fact, the non-diffusing species is allowed to survive near the boundary  $\partial\Omega$ , namely where the size of the competing population is controlled because of the condition  $u = 0$  on  $\partial\Omega$ .

It will also be proved that diffusion de-stabilizes the (unique) solution of (1.1) such that  $u > 0, v \equiv 0$  in  $\Omega$ , whenever it exists; on the other hand, if  $a_1$  is increased beyond a critical value (and the boundary  $\partial\Omega$  is connected), coexistence of both species is no longer possible and the stationary solution  $(0, b_2/c_2)$  of (1.1) becomes asymptotically stable in the uniform norm. In this respect, the situation just described can be viewed as a typical bifurcation phenomenon (for the de-stabilizing effect of diffusion see, for instance, [11, 14]).

2. Statement of Results

Let  $\Omega \subset \mathbb{R}^n$  be an open bounded set with smooth boundary  $\partial\Omega$ ; define  $\bar{\Omega} := \Omega \cup \partial\Omega$ . We shall work basically with the Banach spaces of continuous functions  $C(\bar{\Omega})$  and  $C_0(\bar{\Omega}) := \{u \in C(\bar{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$ , endowed with the supremum norm; the natural ordering in  $C(\bar{\Omega})$ , i.e.  $u \leq v$  ( $u, v \in C(\bar{\Omega})$ ) if  $u(x) \leq v(x)$  for any  $x \in \bar{\Omega}$ , will be used. We shall also be dealing with the Banach space  $C^1(\bar{\Omega})$  of continuously differentiable functions on  $\bar{\Omega}$ , with the Hölder spaces  $C^{k,\alpha}(\bar{\Omega})$  ( $k$  integer,  $\alpha \in (0, 1)$ ) and with the Sobolev spaces  $W^{2,p}(\Omega)$  ( $p > 1$ ).

We shall denote by  $-\nu_0 < 0$  the principal eigenvalue of the Laplacian subject to Dirichlet homogeneous boundary conditions and by  $\phi_0$  the corresponding (normalized) eigenfunction:  $\Delta\phi_0 + \nu_0\phi_0 = 0$ ,  $\phi_0 > 0$  in  $\Omega$ ,  $\phi_0 = 0$  on  $\partial\Omega$ ,  $\int_{\Omega} \phi_0^2(x) dx = 1$ .

Concerning solutions of the initial boundary value problem (1.1), the following result can be easily proved.

**Theorem 0:** *For any nonnegative  $u_0 \in C_0(\bar{\Omega})$ ,  $v_0 \in C(\bar{\Omega})$  there exists a unique non-negative, global classical solution of (1.1) :  $u(t, \cdot) \in C^{2,\alpha}(\bar{\Omega}) \cap C_0(\bar{\Omega})$ ,  $v(t, \cdot) \in C(\bar{\Omega})$  for any  $t > 0$ . Moreover,*

$$\begin{aligned} \max_{x \in \bar{\Omega}} u(t, x) &\leq \max \left\{ b_1/c_1, \max_{x \in \bar{\Omega}} u_0(x) \right\} & (t \geq 0), \\ \max_{x \in \bar{\Omega}} v(t, x) &\leq \max \left\{ b_2/c_2, \max_{x \in \bar{\Omega}} v_0(x) \right\} & (t \geq 0). \end{aligned}$$

2A. Stationary Solutions: Existence and Uniqueness Results

By a *regular solution*  $(u, v)$  of system (1.2) we mean any solution such that  $u \geq 0$ ,  $v \geq 0$  in  $\Omega$  and  $u \in C^{2,\alpha}(\bar{\Omega}) \cap C_0(\bar{\Omega})$ ,  $v \in C(\bar{\Omega})$ . Solutions of (1.2) will be also referred to as *stationary solutions associated with problem (1.1)*. Stationary solutions having both components (resp. one component) non identically vanishing in  $\bar{\Omega}$  will be termed *coexistence (non-coexistence, respectively) stationary solutions*.

Regular non-coexistence solutions of problem (1.2) are immediately seen to exist: beside  $(0, b_2/c_2)$ , the solution  $(\bar{u}, 0)$  ( $\bar{u}$  denoting the unique strictly positive solution of the problem:  $a_1 \Delta \bar{u} + \bar{u}(b_1 - c_1 \bar{u}) = 0$  in  $\Omega$ ,  $\bar{u} = 0$  on  $\partial\Omega$ ) exists if  $a_1\nu_0 < b_1$ . As for coexistence stationary solution, the following theorem will be proved.

**Theorem A1:** *Assume  $(C_1)$  and*

$$(S_1) \quad a_1\nu_0 < b_1 - d_1 \frac{b_2}{c_2}.$$

*Then:*

a) *there exists a regular coexistence solution  $(u^*, v^*)$  of (1.2). Moreover,  $(u^*, v^*)$  is unique among regular stationary solutions  $(u, v)$  of (1.1) with  $u \not\equiv 0$ , which satisfy the following condition:*

$$(P) \quad u(x) \geq b_2/d_2 \quad \text{for any } x \in \Omega_1 := \{x \in \Omega \mid v(x) = 0\};$$

b) *if in addition the boundary  $\partial\Omega$  is connected,  $(u^*, v^*)$  is the unique regular coexistence solution of (1.2). Moreover,  $v^*$  vanishes in a closed non-empty subset of  $\Omega$  and is equal to  $b_2/c_2$  on  $\partial\Omega$ , provided  $a_1\nu_0$  is small enough.*

It is worthwhile pointing out that, under the assumptions  $(S_1)$ ,  $(C_1)$ , regular coexistence solutions of (1.2) exist when  $\partial\Omega$  is not connected, which do not satisfy assumption (P). This can be proved in the one-dimensional case by a direct calculation using phase-energy methods, as the following example shows.

Example: Let  $\Omega = (0, 1)$ ,  $c_i = d_i$  ( $i = 1, 2$ ) and  $b_1 - c_1 b_2 / c_2 > a_1 \pi^2$ . Then there exist  $\alpha \in (0, 1)$  and a regular coexistence solution  $(u, v)$  of (1.2) such that:  $u(x) > 0$  for any  $x \in (0, 1)$ ;  $v(0) = b_2 / c_2$ ,  $v(x) = 0$  for any  $x \in (\alpha, 1]$ .

Besides, weak solutions of (1.2) whose  $v$ -component is not continuous, which do not satisfy assumption (P), can be easily exhibited in the one-dimensional case.

Theorem A2: Assume  $(S_1)$  and

$$(C_2) \quad b_1 / c_1 < b_2 / d_2.$$

Then there exists a unique regular coexistence solution  $(\bar{u}, \bar{v})$  of (1.2); moreover  $\bar{u} > 0$  in  $\Omega$ ,  $\bar{v} > 0$  in  $\bar{\Omega}$ .

If the diffusion coefficient  $a_1$  is increased, the following situation prevails.

Theorem A3: Assume

$$(S_2) \quad a_1 v_0 > b_1 - d_1 b_2 / c_2.$$

Then the only regular solution of (1.2) such that  $v > 0$  in  $\Omega$  is  $(0, b_2 / c_2)$ .

A more refined, though less general result of the same kind is given in the following proposition.

Theorem A4: Assume  $(S_2)$ ,  $(C_1)$  and the connectedness of  $\partial\Omega$ . Then  $(0, b_2 / c_2)$  is the unique regular solution of (1.2) such that  $v \geq 0$ ,  $v \not\equiv 0$  in  $\Omega$ .

## 2 B. Stationary Solutions: Attractivity and Stability Results

Concerning asymptotic proportion of the above referred stationary solutions, the following theorem will be proved.

Theorem B1: Assume  $(S_1)$ . Then:

a) if  $(C_1)$  holds, for any  $\sigma_1 \in (0, \bar{\sigma}_1)$  ( $\bar{\sigma}_1 > 0$ ) and any  $\sigma_2 > 1$  the set

$$\mathcal{A}(\sigma_1, \sigma_2) := \left\{ (u, v) \in C_0(\bar{\Omega}) \oplus C(\bar{\Omega}) \mid \sigma_1 \varphi_0 \leq u \leq \sigma_2 u^*, \right. \\ \left. \frac{1}{c_2} [b_2 - d_2 \sigma_2 u^*]_+ \leq v \leq \frac{1}{c_2} [b_2 - d_2 \sigma_1 \varphi_0]_+ \right\}$$

is invariant with respect to the evolution defined by system (1.1). Moreover, any solution of (1.1) with initial data  $(u_0, v_0) \in \mathcal{A}(\sigma_1, \sigma_2)$  approaches  $(u^*, v^*)$  in the  $C_0(\bar{\Omega}) \oplus C(\bar{\Omega})$ -norm as  $t$  diverges;

b) if  $(C_2)$  holds, the same result holds for  $(\bar{u}, \bar{v})$  with respect to the invariant set  $\{\sigma_1 \in (0, \bar{\sigma}_1), \bar{\sigma}_1 > 0; \sigma_2 > 1\}$ :

$$\mathcal{B}(\sigma_1, \sigma_2) := \left\{ (u, v) \in C_0(\bar{\Omega}) \oplus C(\bar{\Omega}) \mid \sigma_1 \varphi_0 \leq u \leq \sigma_2 \bar{u}, \right. \\ \left. \frac{1}{c_2} [b_2 - d_2 \sigma_2 \bar{u}]_+ \leq v \leq \frac{1}{c_2} [b_2 - d_2 \sigma_1 \varphi_0]_+ \right\}.$$

Let us observe that the lower bound  $\frac{1}{c_2} [b_2 - d_2 \sigma_2 u^*]_+ \leq v$  is satisfied for any nonnegative  $v \geq v^*$ ; on the other hand, the upper constraint  $v \leq \frac{1}{c_2} [b_2 - d_2 \sigma_1 \varphi_0]_+$  is in a way not very severe, as the  $v$ -component of any nonnegative solution of (1.1) satisfies the inequality  $v(t) \leq b_2/c_2 + \varepsilon$  for any  $\varepsilon > 0$ , provided  $t$  is large enough.

If conditions (S<sub>1</sub>) and (C<sub>2</sub>) hold (which imply the unique regular coexistence solution to have a strictly positive  $v$ -component in  $\bar{\Omega}$ ), a different attractivity result can be proved.

**Theorem B2:** *Assume (S<sub>1</sub>) and (C<sub>2</sub>). Then  $(\bar{u}, \bar{v})$  attracts in the  $C_0(\bar{\Omega}) \oplus C(\bar{\Omega})$ -norm any solution of (1.1) with initial data  $(u_0, v_0)$  such that:  $u_0 \in C_0(\bar{\Omega})$ ,  $u_0 \geq 0$ ,  $u \not\equiv 0$  in  $\Omega$ ;  $v_0 \in C^1(\bar{\Omega})$ ,  $v_0 > 0$  in  $\bar{\Omega}$ .*

Remark: A related result for predator-prey systems is contained in [15].

**Theorem B3:** *Assume (S<sub>2</sub>). Then  $(0, b_2/c_2)$  is asymptotically stable and attracts (in the  $C_0(\bar{\Omega}) \oplus C(\bar{\Omega})$ -norm) any solution of (1.1) with initial data  $(u_0, v_0)$  such that:  $u_0 \in C_0(\bar{\Omega})$ ,  $u_0 \geq 0$  in  $\Omega$ ;  $v_0 \in C(\bar{\Omega})$ ,  $v_0 > 0$  in  $\bar{\Omega}$ .*

As already remarked, regular coexistence solutions not satisfying assumption (P) may exist; however, the following theorem shows that they are unstable. In fact, instability will be proved for solutions of (1.2) violating (P) and having possibly discontinuous  $v$ -components — in which case the first equation in (1.2) must be interpreted in a weak sense, an appropriate function space being  $(W^{2,p}(\Omega) \cap C_0(\Omega)) \oplus L^\infty(\Omega)$  ( $p > 1$ ).

**Theorem B4:** *Let  $(\bar{u}, \bar{v}) \in (W^{2,p}(\Omega) \cap C_0(\bar{\Omega})) \oplus L^\infty(\Omega)$  ( $p \geq 1$ ) be a solution of (1.2) such that  $\bar{u} \geq 0$ ,  $\bar{u} \not\equiv 0$  and  $\bar{v} \geq 0$  almost everywhere in  $\Omega$ . Assume that*

(P) *The set  $\bar{\Omega}_1 := \{x \in \Omega \mid \bar{v}(x) = 0, \text{ and } \bar{u}(x) < b_2/d_2\}$  has positive measure.*

*Then  $(\bar{u}, \bar{v})$  is unstable in the  $C_0(\bar{\Omega}) \oplus L^\infty(\Omega)$ -norm.*

*As a consequence of Theorem B<sub>4</sub>, the non-coexistence solution  $(\bar{u}, 0)$ -whose counterpart in the space-clamp case is the solution  $(b_1/c_1, 0)$  — is always unstable.*

The situation outlined in the above theorems can be summarized as follows:

(i) (C<sub>1</sub>) holds. Then in the absence of diffusion (i.e., if  $a_1 = 0$ ) there exists a unique asymptotically stable stationary solution, namely  $(b_1/c_1, 0)$ . If  $a_1 v_0 \in \left(0, b_1 - d_1 \frac{b_2}{c_2}\right)$  (i.e., if “small” diffusion is introduced), the coexistence stationary solution  $(u^*, v^*)$  arises (which has no space-clamp analogous); such solution enjoys the uniqueness and attractivity properties stated in Theorems A1, B1 a). In particular, if  $\partial\Omega$  is connected, it is uniquely determined among the regular coexistence stationary solutions, and in that case it exhibits a marked space structure (see Theorem A1 b)). Yet even without assuming the connectedness of  $\partial\Omega$ ,  $(u^*, v^*)$  plays a unique role, since, according to Theorem B4, any other coexistence stationary solution of (1.1) is unstable. If diffusion is increased (i.e., if  $a_1 v_0 > b_1 - d_1 \frac{b_2}{c_2}$ ),  $(u^*, v^*)$  ceases to exist and  $\left(0, \frac{b_2}{c_2}\right)$  is asymptotically stable and attractive as asserted in Theorem B3 (see Fig. 1, where stable stationary solutions are depicted in the case  $\Omega = [0, 1]$ ,  $v(0) = v(1)$ ).

(ii) (C<sub>2</sub>) holds: in particular, let us discuss the significant case where, in addition,  $c_1 c_2 > d_1 d_2$  and  $b_1 - d_1 \frac{b_2}{c_2} > 0$  (observe that the last inequality is implied by (S<sub>1</sub>),

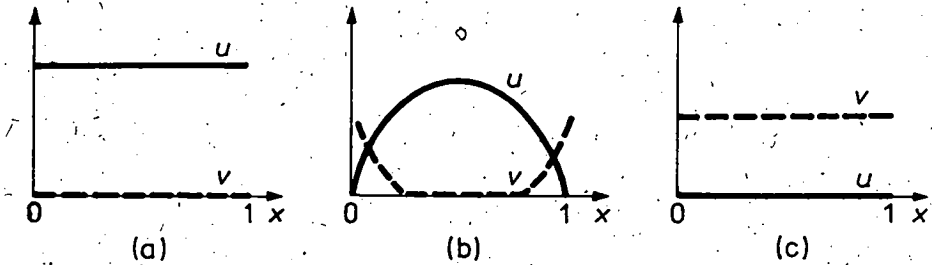


Fig. 1. Stable-stationary regular solutions for  $\Omega = (0, 1)$ ,  $v(0) = v(1)$  under hypothesis  $(C_1)$ :

(a) no diffusion:  $a_1 v_0 = 0$ ;

(b) small diffusion  $a_1 v_0 \in \left(0, b_1 - d_1 \frac{b_2}{c_2}\right)$ :  $\begin{cases} (b') a_1 v_0 \simeq 0 \\ (b'') a_1 v_0 \simeq b_1 - \frac{b_2}{c_2} d_1 \end{cases}$

(c) large diffusion:  $a_1 v_0 > b_1 - d_1 b_2/c_2$ .

while it is compatible with  $(S_2)$ ; the case  $b_1 - d_1 \frac{b_2}{c_2} < 0$  is not very interesting, since it makes  $(S_2)$  trivially satisfied: with respect to the space-clamp situation, no new feature is introduced by diffusion). Then, if diffusion is absent ( $a_1 = 0$ ), a unique asymptotically stable coexistence stationary solution exists. Such situation is preserved if "small" diffusion is introduced ( $a_1 v_0 \in \left(0, b_1 - d_1 \frac{b_2}{c_2}\right)$ ): in fact, the coexistence stationary solution  $(\tilde{u}, \tilde{v})$  arises, which enjoys the uniqueness and attractivity properties stated in Theorems A2, B1 b), B2. No other coexistence stationary solution exists. If strong diffusion is present ( $a_1 v_0 > b_1 - d_1 \frac{b_2}{c_2}$ ),  $(\tilde{u}, \tilde{v})$  disappears and the situation is much the same as that described under (i) above (see Fig. 2).

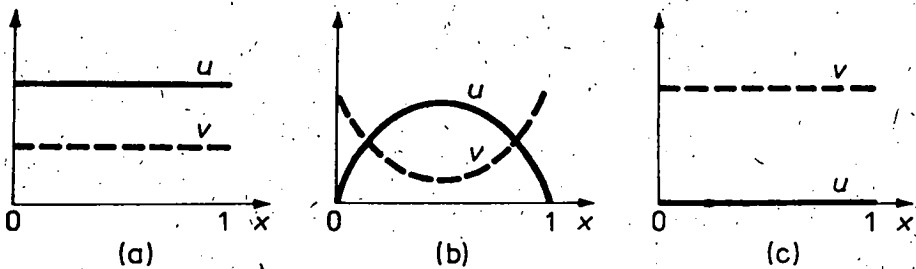


Fig. 2. The same under hypothesis  $(C_2)$ .

It can be said that introducing diffusion allows branches of attractive coexistence stationary solutions to exist, which connect asymptotically stable stationary solutions of the problem without diffusion: as a matter of fact,  $(u^*, v^*)$  converges to  $(b_1/c_1, 0)$  (uniformly on the compact subsets of  $\Omega$ ) as  $a_1 \rightarrow 0$  [12] and converges to  $(0, b_2/c_2)$  as  $a_1$  approaches from the left  $a^* := \frac{1}{v_0} \left(b_1 - d_1 \frac{b_2}{c_2}\right)$  (this can be proved by general

bifurcation results and checked by a direct calculation [13]); similar results hold true for  $(\bar{u}, 0)$  and  $(\bar{u}, \bar{v})$ . Consequently, the overall situation can be viewed as a typical bifurcation phenomenon with respect to the parameter  $a_1$  (see Figs. 3, 4).

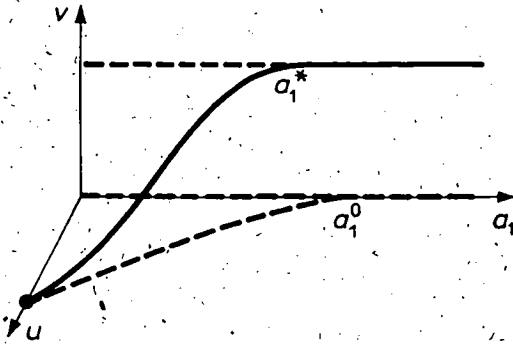


Fig. 3. Bifurcation diagram of regular stationary solutions in case of connected boundary, under hypothesis  $(C_1)$ ;  $a_1^* = \frac{1}{r_0} \left( b_1 - d_1 \frac{b_2}{c_2} \right)$ ;  $a_1^0 = \frac{b_1}{r_0}$ .

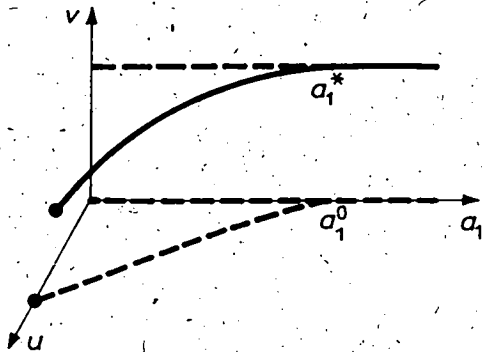


Fig. 4. The same under hypothesis  $(C_2)$ .

### 2C. Stationary Solutions: a Singular Perturbation Result

It is an open problem whether a situation similar to the above prevails when also the  $v$ -component undergoes small diffusion; to discuss this point amounts to investigating the elliptic problem

$$\begin{aligned}
 a_1 \Delta u + u(b_1 - c_1 u - d_1 v) &= 0 \\
 \epsilon_1 \Delta v + v(b_2 - c_2 v - d_2 u) &= 0 \quad \text{in } \Omega, \\
 u = v = 0 \quad \text{on } \partial\Omega,
 \end{aligned}
 \tag{2.1}$$

where  $0 < \epsilon \ll a_1$  (the case where  $\epsilon/a_1$  is not small and  $c_i = d_i$  ( $i = 1, 2$ ) was studied in [17]). However, the following theorem can be proved.

**Theorem C:** *Let  $(u_\epsilon, v_\epsilon)$  denote any classical solution of (2.1) such that  $u_\epsilon \geq 0$ ,  $v_\epsilon \geq 0$  in  $\Omega$ . Then the set  $\{(u_\epsilon, v_\epsilon)\}$  is relatively compact in the topology defined by the  $C^1(\bar{\Omega})$*

(respectively,  $L^2(\Omega)$  weak) convergence of the first (respectively, second) component; any limiting point of  $\{(u_\varepsilon, v_\varepsilon)\}$  in such topology is a regular solution of (1.2) satisfying condition (P). In particular, if  $(S_1)$  and  $(C_1)$  (resp.  $(C_2)$ ) hold, the only limiting points are  $\left(0, \frac{b_2}{c_2}\right)$  and  $(u^*, v^*)$  ( $\left(0, \frac{b_2}{c_2}\right)$  and  $(\bar{u}, \bar{v})$ , respectively); if  $(S_2)$  holds,  $(u_\varepsilon, v_\varepsilon)$  converges to  $\left(0, \frac{b_2}{c_2}\right)$  in the above topology as  $\varepsilon$  goes to zero.

Let us remark that the above theorem makes no statements about the actual existence of nonnegative solutions of (2.1); singular perturbation methods such as those developed in [3] seem not applicable in the present context. In the case  $(S_1)$  and  $(C_1)$  hold, however, it highlights further the distinguished role of  $(u^*, v^*)$ : from Theorem B1 a) we already know about the attractivity property of such stationary solution, any other nonnegative (in particular, coexistence) solution of (1.2) being unstable; from Theorem C we learn that  $(u^*, v^*)$  is the only coexistence stationary solution to which solutions of (2.1) (if any) can converge as  $\varepsilon$  goes to zero.

### 3. Proof of Existence and Uniqueness Results

Let us first observe that the stationary system (1.2) involves a free boundary problem, the free boundary being the interface between the two regions where either factor of the second equation vanishes, namely  $\Omega_1 := \{x \in \Omega \mid v(x) = 0\}$  and  $\Omega_2 := \{x \in \Omega \mid b_2 - c_2v(x) - d_2u(x) = 0\}$ . However, because of the non-variational nature of system (1.2), the usual mathematical tools for dealing with free boundary problems cannot be used; therefore we shall proceed in a direct way.

To start with, let us observe that a class of regular coexistence solutions of (1.1) is given by couples  $(u, v)$  such that  $u$  is a nonnegative classical solution of the following problem:

$$\begin{aligned} a_1 \Delta u + u \left( b_1 - c_1 u - \frac{d_1}{c_2} [b_2 - d_2 u]_+ \right) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (3.1)$$

and  $v := \frac{1}{c_2} [b_2 - d_2 u]_+$ . As is immediately seen, an equivalent way of characterizing such solutions among all (regular) solutions of (1.2) is to say that they satisfy assumption (P).

**Lemma 3.1:** a) *If  $(S_1)$  holds, there is a unique nonnegative nontrivial solution  $u^*$  of (3.1).* b) *If  $(S_2)$  holds, no nonnegative nontrivial solution of (3.1) exists.*

**Proof:** Observe that problem (3.1) is of the form:  $a_1 \Delta u + u\psi(u) = 0$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , where  $\psi: [0, +\infty) \rightarrow \mathbf{R}$  is (i) Lipschitz continuous on bounded subsets, (ii) non-increasing, (iii) negative for  $u > b_1/c_1$ , (iv) differentiable for  $u > b_2/d_2$ , with negative derivative. As a consequence, we can apply to (3.1) the results of [12] relating the existence of a (unique) nontrivial nonnegative solution of (3.1) to the sign of  $(\psi(0) - a_1 v_0)$ . As  $\psi(0) = b_1 - d_1 \frac{b_2}{c_2}$ , the result follows ■

In the following lemmas, the space structure of regular coexistence stationary solutions is investigated.

**Lemma 3.2:** *Assume  $(C_1)$ . Then, for any regular coexistence solution  $(u, v)$  of (1.2),  $\Delta u \leq 0$  in  $\Omega$ . In particular,  $u > 0$  in  $\Omega$ .*



Proof: It suffices to observe that, due to assumption  $(C_1)$ ,  $b_1 - c_1u - d_1v \geq 0$  in  $\Omega$  ■

Lemma 3.3: Let  $(u, v)$  be a regular solution of (1.2). Then:

- a)  $\Omega = \Omega_1 \cup \Omega_2$ ,  $\Omega_1 \cup \Omega_2 \cap \partial\Omega = \phi$ ,
- b)  $v$  is constant on any connected component of  $\partial\Omega$ .

Proof: a) is obvious. b) follows from the assumed continuity of  $v$ ,  $v$  being zero or equal to  $b_2/c_2$  on  $\partial\Omega$  ■

Lemma 3.4: Assume  $(C_1)$ ; let moreover  $(u, v)$  be a regular coexistence solution of (1.2) and  $\partial\Omega$  consist of a single connected component. Then  $\partial\Omega \subset \Omega_2$  (namely,  $v = b_2/c_2$  on the whole of  $\partial\Omega$ ).

Proof: Assume the contrary: then, according to Lemma 3.3 b), there exists  $P \in \partial\Omega$  such that  $v(P) = 0$ , hence  $d_2u(P) + c_2v(P) \neq b_2$ . Let  $\mathcal{N}$  denote a neighbourhood of  $P$  in  $\Omega$  such that  $d_2u + c_2v \neq b_2$  in  $\mathcal{N}$  (such a neighbourhood exists, due to the assumed regularity of  $(u, v)$ ; thus  $v = 0$  in  $\mathcal{N}$ , i.e.  $\mathcal{N} \subset \Omega_1$ ). Let  $\mathcal{S}$  denote the connected component of  $P$  in  $\{x \in \Omega \mid d_2u(x) + c_2v(x) \neq b_2\} \supset \mathcal{N}$ , and set  $\partial\mathcal{S} = \Gamma_1 \cup \Gamma_2$ , where  $\Gamma_1 := \partial\mathcal{S} \cap \partial\Omega$ ,  $\Gamma_2 := \partial\mathcal{S} \cap \Omega$ . It is easily seen that:  $\Gamma_1 \neq \phi$  (by construction),  $\Gamma_2 \neq \phi$  (otherwise  $\Omega_1 = \Omega$  and  $(u, v)$  is not a coexistence stationary solution). By the very definition of  $\mathcal{S}$ ;  $u = b_2/d_2$  on  $\Gamma_2$ , hence on  $\bar{\Gamma}_2$ , follows; then  $\bar{\Gamma}_2 \cup \Gamma_1 = \phi$  and the assumed connectedness of  $\partial\Omega$  implies  $\Gamma_2 = \partial(\Omega \setminus \mathcal{S})$ . Due to Lemma 3.2 and the maximum principle, it follows  $u \geq b_2/d_2$ , hence  $v = 0$  on  $\Omega \setminus \mathcal{S}$ ; as a consequence,  $v = 0$  on the whole of  $\Omega$  and  $(u, v)$  cannot be a coexistence stationary solution. The contradiction proves the result ■

Lemma 3.5: Assume  $(C_1)$  and let  $\partial\Omega$  consist of a single connected component. Then every regular coexistence solution of (1.2) satisfies condition (P).

Proof: By Lemma 3.4,  $v = b_2/c_2$  on the whole of  $\partial\Omega$  under the present assumptions; let  $\mathcal{C}$  denote the connected component of  $\partial\Omega$  in  $\{x \in \Omega \mid v(x) > 0\}$  and set  $\Gamma := \partial\mathcal{C} \cap \Omega$ ,  $\partial\mathcal{C} = \partial\Omega \cup \Gamma$ ; thus, if  $\Gamma = \phi$ ,  $\Omega_1 = \phi$  and condition (P) is trivially satisfied. If  $\Gamma \neq \phi$ ,  $u = b_2/d_2$  on  $\Gamma$  by the very definition of  $\mathcal{C}$ , so that  $u \geq b_2/c_2$  on  $\Omega \setminus \mathcal{C}$  by Lemma 3.2 and the maximum principle; then  $v = 0$  on  $\Omega \setminus \mathcal{C}$  and the result follows ■

Proof of Theorem A.1: We gather claim a) from Lemma 3.1 a), claim b) from Lemma 3.2 a) and Lemma 3.5. The last claim follows because, as  $a_1v_0 \rightarrow 0$ ,  $\max u \rightarrow b_1/c_1 > b_2/d_2$  (see [12: Thm. 1.3]) ■

Proof of Theorem A.2: Due to assumption  $(C_2)$  and the maximum principle, by which  $u \leq b_1/c_1$ , the  $v$ -component of any regular coexistence solution of (1.2) satisfies the inequality  $v \geq b_2 - d_2 \frac{b_1}{c_1} > 0$  in  $\Omega_2$ , which implies  $\Omega_1 = \phi$  and  $v > 0$  in  $\bar{\Omega}$ . Then regular coexistence solutions of (1.2) are in one-to-one correspondence with nontrivial nonnegative solutions of the problem

$$\begin{aligned}
 a_1 \Delta u + u \left\{ \left( b_1 - d_1 \frac{b_2}{c_2} \right) - \frac{1}{c_2} (c_1c_2 - d_1d_2) u \right\} &= 0 \quad \text{in } \Omega, \\
 u &= 0 \quad \text{in } \partial\Omega
 \end{aligned}
 \tag{3.2}$$

whence the result follows by assumption  $(S_1)$  ■

**Proof of Theorem A.3:** Looking for regular solutions of (1.2) such that  $v > 0$  in  $\Omega$  amounts again to looking for nonnegative solutions of (3.2), the  $v$ -component being consequently determined; under assumption  $(S_2)$  no such solutions of (3.2) exist but the trivial one, which proves the claim ■

**Proof of Theorem A.4:** According to Lemma 3.5, assumption  $(C_1)$  and the connectedness of  $\partial\Omega$  ensure regular coexistence solutions of (1.2) to be in one-to-one correspondence with nonnegative (classical) solutions of (3.1); according to Lemma 3.1 b), the unique solution of this kind is the trivial one in the present case, whence the claim follows ■

#### 4. Proof of Attractivity and Stability Results

Let us first prove Theorem B1: Since system (1.2) is quasi-monotone [16], let us look for upper-lower and lower-upper solutions.

It is well known (and easy to verify) that, whenever  $\sigma > 1$ ,  $u^\sigma := \sigma u^*$  is an upper solution to

$$\begin{aligned} \Delta u + u \left( b_1 - c_1 u - \frac{d_1}{c_2} [b_2 - d_2 u]_+ \right) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega; \end{aligned} \quad (4.1)$$

defining  $v_\sigma := \frac{1}{c_2} [b_2 - d_2 u^\sigma]_+$ , it is immediately seen that

$$\begin{aligned} \Delta u^\sigma + u^\sigma (b_1 - c_1 u^\sigma - d_1 v_\sigma) &\leq 0 \quad \text{in } \Omega, \\ u^\sigma &= 0 \quad \text{in } \partial\Omega, \\ v_\sigma (b_2 - c_2 v_\sigma - d_2 u^\sigma) &= 0, \end{aligned}$$

so that  $(u^\sigma, v_\sigma)$  is an upper-lower solution to (1.2). Similarly, since  $\sigma\phi_0$  is a lower solution to (4.1) for  $\sigma > 0$ ,  $\sigma$  small enough, setting  $u_\sigma := \sigma\phi_0$ ,  $v^\sigma := \frac{1}{c_2} [b_2 - d_2 u_\sigma]_+$ ,  $(u_\sigma, v^\sigma)$  is a lower-upper solution to (1.2), provided  $\sigma > 0$  is small enough.

To complete the proof we have to adapt to the present case well-known monotone methods. Since the lower-upper solution  $(u_\sigma, v^\sigma)$  satisfies  $u_\sigma \geq 0$ ,  $u_\sigma \not\equiv 0$ ,  $v^\sigma \geq 0$ , we know [16] that the solution of (1.1) with Cauchy data  $u_0 = u_\sigma$ ,  $v_0 = v^\sigma$  has the following properties:

1.  $(u(t))(x)$  increases in  $t$ , for any  $x \in \Omega$ ; due to the *a priori* bound of Theorem 0,  $u(t) \rightarrow \bar{u}$  pointwise as  $t \rightarrow \infty$ ,  $\bar{u} > 0$  in  $\Omega$ ;
2. likewise,  $(v(t))(x)$  decreases in  $t$  for any  $x \in \Omega$  and  $v(t) \rightarrow \bar{v}$  pointwise as  $t \rightarrow \infty$ ,  $\bar{v} \geq 0$  in  $\bar{\Omega}$ .

Using the regularizing properties of the equation for  $u(\cdot)$ , it is easily seen that  $u(t) \rightarrow \bar{u}$  in the supremum norm, thus  $\bar{u} \in C_0(\bar{\Omega})$ .

On the other hand, since

$$v_t(t) = v(t) (b_2 - c_2 v(t) - d_2 \bar{u}) + f(t) \quad (t > 0)$$

where  $f(t) := d_2 v(t) (\bar{u} - u(t)) \rightarrow 0$  for  $t \rightarrow \infty$ , an easy argument shows that  $v(\cdot)$  behaves, for large  $t$ 's, as the solution  $w(\cdot)$  of the equation

$$\partial_t w(t) = w(t) [b_2 - c_2 w(t) - d_2 \bar{u}] \quad (t > 0).$$

Hence we get

$$\bar{v} = \frac{1}{c_2} [b_2 - d_2 \bar{u}]_+,$$

which proves  $v \in C(\bar{\Omega})$  (and  $v(t) \rightarrow \bar{v}$  in the supremum norm as  $t \rightarrow \infty$  by Dini's theorem). As a consequence,  $(\bar{u}, \bar{v})$  is a regular stationary solution, satisfying (P). Since  $\bar{u} \geq u_\sigma > 0$  in  $\Omega$ , Theorem A.1 applies, yielding  $(\bar{u}, \bar{v}) = (u^*, v^*)$ .

A similar argument holds for the upper-lower solution  $(u^\sigma, v_\sigma)$ , which satisfies  $u^\sigma \geq u^*, v_\sigma \geq 0, v_\sigma \not\equiv 0$  in  $\bar{\Omega}$ ; this completes the proof. ■

Let us now turn to the proof of Theorem B2; to this end, we permit a technical lemma.

**Lemma 4.1:** Let  $(u, v)$  be an arbitrary classical solution of (1.1) such that  $u(t) \geq 0, v(t) \geq 0$  for any  $t \geq 0$ .

- a)  $\frac{\Delta u(t, \cdot)}{u(t, \cdot)}$  is well defined as a continuous function on  $\bar{\Omega}$ , for any  $t > 0$ .
- b) Let  $(C_2)$  hold; moreover, assume  $u_0 \in C_0(\bar{\Omega}), u_0 \geq 0, u_0 \not\equiv 0; v_0 \in C^1(\bar{\Omega}), v_0 > 0$  on  $\bar{\Omega}$ . Then, for any  $\delta > 0$  the trajectory

$$\Gamma_\delta = \{(u(t), v(t)) \mid t \geq \delta\}$$

is a relatively compact set in  $C_0(\bar{\Omega}) \oplus C(\bar{\Omega})$ .

**Proof:** a) We have to check that  $\frac{\Delta u(t, \cdot)}{u(t, \cdot)}$  is well behaved near  $\partial\Omega$ : By a local change of coordinates we can suppose  $\partial\Omega$  to be the hyperplane  $x_1 = 0$ ; then  $u(t, x)$  has a Taylor expansion near  $\partial\Omega$ :

$$u(t, x) = c_1(t, x_2, \dots, x_n) x_1 + R(t, x) \quad (t \geq 0; x = (x_1, x_2, \dots, x_n) \in \Omega).$$

Because of Hopf's maximum principle  $c_1(t, x_2, \dots, x_n) \neq 0$ , so that

$$\partial_t u(t, x) = \partial_t c_1(t, x_2, \dots, x_n) + \partial_t R(t, x)$$

whence  $\frac{\partial_t u(t, \cdot)}{u(t, \cdot)}$  is well defined (and in fact, a smooth function) on  $\partial\Omega$ . Using now the equation, it is immediately seen that the claim follows.

To prove b) observe that, because of the assumptions,  $v(t) \in C^1(\bar{\Omega}), v(t) > 0$  on  $\bar{\Omega}$ . It is a standard matter to find an *a priori* estimate for  $|\partial_i u(t, x)|, t \geq \delta > 0, x \in \bar{\Omega}$  [2].

To find a parallel estimate for  $\partial_i v$ , observe that, since  $0 \leq u \leq b_1/c_1, v_i \geq v(b_2 - c_2 v - d_2 \frac{b_1}{c_1})$ , and, because of  $(C_2)$ ,  $v$  is bounded below above zero:  $v(t, x) \geq \mu > 0$ . Consequently,  $w = \log v$  is well defined and satisfies  $\partial_i w = b_2 - c_2 e^w - d_2 u$ ; thus  $z_i = \partial_i w$  fulfils  $\partial_i(z_i) = (-c_2 v) z_i - d_2 \partial_i u$ . Now,  $-c_2 v \leq -c_2 \mu$  and, as noted above,  $|\partial_i u|$  is uniformly bounded for  $t \geq \delta > 0$  and  $x \in \bar{\Omega}$ . Then it is easy to see that  $|z_i|$  is uniformly bounded, too, whence the claim follows. ■

We shall also need the following result.

**Lemma 4.2:** Assume  $(S_1)$  and  $(C_2)$ . Then the quantity

$$V := \int_{\Omega} \bar{u} \left\{ u - \bar{u} - \bar{u} \log \frac{u}{\bar{u}} + \frac{d_1}{d_2} (v - \bar{v} \log v) \right\} dx$$

has the following properties:

a)  $V : C_0(\bar{\Omega}) \oplus C(\bar{\Omega}) \rightarrow \mathbf{R}$  is well defined and continuous for any  $(u, v) \in C_0(\bar{\Omega}) \oplus C(\bar{\Omega})$  such that  $u/\bar{u} > 0, v > 0$  in  $\bar{\Omega}$ ;

b) the map  $t \rightarrow V(u(t), v(t))$  (where  $(u(t), v(t))$  is any solution of (1.1) in  $C_0(\bar{\Omega}) \oplus C(\bar{\Omega})$  with Cauchy data  $u \geq 0, v > 0$  in  $\bar{\Omega}$ ) is nonincreasing; in fact, it is differentiable along such trajectories of (1.1), and

$$\begin{aligned} \frac{d}{dt} V(u(t), v(t)) = & - \int_{\bar{\Omega}} \left| \text{grad } \bar{u} - \text{grad } u(t) \frac{\bar{u}}{u(t)} \right|^2 dx - \int_{\bar{\Omega}} \bar{u} \left[ c_1(u(t) - \bar{u})^2 \right. \\ & \left. + 2d_1(u(t) - \bar{u})(v(t) - \bar{v}) + \frac{d_1}{d_2} c_2(v(t) - \bar{v})^2 \right] dx. \quad (4.2) \end{aligned}$$

**Proof:** Claim a) is straightforward. Proceeding as in Lemma 4.1 a), we see that  $u/\bar{u} > 0$  in  $\bar{\Omega}$  on the trajectories, so that  $V(u(t), v(t))$  is well defined for  $t \geq \delta > 0$ , as well as its time derivative (see again Lemma 4.1 a)). Moreover, due to  $(C_0)$ , the integrand of the second term in the right-hand side of (4.2) is a semidefinite positive quadratic form (it is definite positive if strict inequality holds in  $(C_0)$ ). Then claim b) follows by a direct calculation ■

**Remark:** The definition of  $V$  was suggested by [8], where a discrete model was considered; see as well [15].

**Proof of Theorem B2:** Due to the well known La Salle's invariance argument and Lemma 4.1 b), the result will follow from the investigation of the critical set of  $V$ . Let us distinguish two cases:

(i) Strong inequality holds in  $(C_0)$ . Then the critical set of  $V$  is easily seen to shrink to the unique point  $\{(\bar{u}, \bar{v})\}$ .

(ii) Equality holds in  $(C_0)$ . Then the second term on the right-hand side of (4.2) vanishes if and only if  $c_1 u(t) + d_1 v(t) = c_1 \bar{u} + d_1 \bar{v}$  ( $t \geq 0$ ). As for the first term, it vanishes if and only if  $u(t) = \gamma(t) \bar{u}$  for some smooth positive function  $\gamma(\cdot)$  ( $t \geq \delta > 0$ ). As a consequence, the largest subset of the critical set of  $V$  which is invariant with respect to (1.1) consists of  $\{(\bar{u}, \bar{v})\}$ . This completes the proof ■

The proof of Theorem B3 is similar and will be omitted. Finally, let us prove the instability result asserted in Theorem B4.

**Proof of Theorem B4:** Pick a stationary solution  $(\bar{u}, \bar{v})$  satisfying the hypothesis. If  $(u(\cdot), v(\cdot))$  is a solution of (1.1), the deviations  $h(\cdot) := u(\cdot) - \bar{u}, k(\cdot) := v(\cdot) - \bar{v}$  satisfy the system

$$\partial_t h = a_1 \Delta h + h(b_1 - 2c_1 \bar{u} - d_1 \bar{v}) + d_1 \bar{u} k - h(c_1 h + d_1 k)$$

$$\partial_t k = k(b_2 - 2c_2 \bar{v} - d_2 \bar{u}) - d_2 \bar{v} h - k(c_2 k + d_2 h),$$

(plus Dirichlet homogeneous boundary conditions for  $h$ ). Note first that, as a consequence of the maximum principle, if  $h(0) \leq 0, k(0) \geq 0$ , then  $h(t) \leq 0, k(t) \geq 0$  for all  $t \geq 0$ . Pick now  $h(0) = 0, k(0) = \sigma q$ , where  $q$  is a non negative non zero function with  $\text{supp } q \subset \bar{\Omega}_1$ , and  $\sigma$  is a positive number. Since for any  $x \in \text{supp } q, \bar{v}(x) = 0$  and  $b_2 - d_2 \bar{u}(x) > 0$ , the component  $k$  solves, for any such  $x$ , the problem

$$\partial_t k = k(t, x) \{b_2 - d_2 \bar{u}(x) - d_2 h(t, x)\} - c_2 k^2(t, x),$$

$$k(0, x) = \sigma q(x).$$

Since  $b_2 - d_2\bar{u}(x) - d_2h(t, x) \geq b_2 - d_2\bar{u}(x) > 0$ , it follows that  $k(t, \cdot)$  does not remain, for all  $t \geq 0$ , in an arbitrarily fixed  $C(\bar{\Omega})$ -neighbourhood of 0, however small  $\sigma$  is chosen: thus  $(\bar{u}, \bar{v})$  is unstable ■

### 5. Small Diffusion on the $v$ -Component

Our aim in this section is to prove Theorem C: For this purpose let us proceed stepwise, denoting by  $(u_\varepsilon, v_\varepsilon)$  a solution of (2.1) such that  $u_\varepsilon \geq 0, v_\varepsilon \geq 0$  in  $\Omega$ .

(i) Because of the maximum principle there exists  $k > 0$  such that

$$\max \left\{ \max_{x \in \bar{\Omega}} |u_\varepsilon(x)|, \max_{x \in \bar{\Omega}} |v_\varepsilon(x)| \right\} \leq k.$$

Then from system (2.1) we get, with a suitable  $k' > 0$ ,  $\max_{x \in \bar{\Omega}} |\Delta u_\varepsilon(x)| \leq k'$ . As a consequence,  $\{u_\varepsilon\}$  is relatively compact in  $C^1(\bar{\Omega})$ ; similarly for  $\{v_\varepsilon\}, \{v_\varepsilon^2\}$  in the  $L^2(\Omega)$  weak topology, so that we have, along suitable sequences (we shall label by the same index  $\varepsilon$  for notational simplicity),

$$u_\varepsilon \rightarrow u \text{ in } C^1(\bar{\Omega}); \quad v_\varepsilon \rightarrow v, \quad v_\varepsilon^2 \rightarrow w \text{ in } L^2(\Omega).$$

(ii) Observe that  $w \geq v^2$  almost everywhere in  $\Omega$ ; this follows easily from the estimate (which holds for almost every  $x \in \Omega$ )

$$v_\varepsilon^2(x) \geq v^2(x) + 2v(x)(v_\varepsilon(x) - v(x)), \quad \text{as } v_\varepsilon - v \text{ in } L^2(\Omega).$$

(iii) As  $v_\varepsilon$  is the principal eigenfunction of the elliptic operator  $\varepsilon \Delta + (b_2 - c_2v_\varepsilon - d_2u_\varepsilon)$ , the inequality

$$\int_{\Omega} \phi [\varepsilon \Delta \phi + (b_2 - c_2v_\varepsilon - d_2u_\varepsilon) \phi] dx \leq 0$$

holds for any  $\phi \in H_0^1(\Omega)$ . As  $\varepsilon \rightarrow 0$  we get

$$-\int_{\Omega} \phi^2 (b_2 - c_2v - d_2u) dx \leq 0,$$

which implies

$$c_2v + d_2u \geq b_2 \tag{5.1}$$

almost everywhere in  $\Omega$ , due to the arbitrariness of  $\phi$ .

(iv) Let us prove that in fact  $w = v^2$  (almost everywhere in  $\Omega$ ). Taking the limit in the sense of distributions as  $\varepsilon \rightarrow 0$  of the second equation in (2.1), we get  $b_2v - c_2w - d_2uv = 0$  almost everywhere in  $\Omega$ ; this in turn implies, due to (5.1),  $w \leq v^2$  in the same sense. Then the claim follows from step (ii).

(v) It follows from (i), (iv) above that the limiting point  $(u, v)$  satisfies the first equation of (1.2) in the weak sense, the second almost everywhere in  $\Omega$ . On the other hand, inequality (5.1) entails  $v = \frac{1}{c_2} [b_2 - d_2u]_+$ , thus  $v \in C(\bar{\Omega})$  and  $(u, v)$  is a regular solution of (1.2) which satisfies condition (P). Then the remaining claims follow by Theorem A 1–A 3, thus completing the proof ■

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