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Pseudo differential operators in Hardy-Triebel spaces

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Es wird bewiesen, daß Pseudodifferentialoperatoren der Klasse $L^0_{\varrho,\delta}$, $\varrho = 1$ und $0 \leqq \delta < 1$, in den Triebelschen Räumen $F^s_{p,q}$ stetig sind.

В статье доказано, что при $\varrho = 1$ и $0 \leq \delta < 1$ псевдодифференциальные операторы класса $L^0_{\varrho,\delta}$ непрерывны в пространствах Трибеля $F^s_{p,q}$.

Pseudo differential operators of class $L^0_{\delta,\varrho}$, $\varrho = 1$ and $0 \leqq \delta < 1$, are proved to be continuous in Triebel spaces $F^s_{p,q}$.

0. Introduction

Several results concerning the boundedness of pseudo differential operators in function spaces are known: From the results of HÖRMANDER [5, 6], CALDERON and VAILLANCOURT [2], and CHING [3] it follows that operators of class $L^0_{\varrho,\delta}$ (cf. Chapter 1) are bounded in L_2 if and only if $0 \leqq \delta \leqq \varrho \leqq 1$ and $(0, 0) \neq (\varrho, \delta) \neq (1, 1)$. ILLNER [7] proved the boundedness of operators of class $L^0_{1,\delta}$, $0 \leqq \delta < 1$, in L_p, $1 < p < \infty$.

In this paper we consider the Triebel spaces $F^s_{p,q}$ in $\mathbf{R}^n$. For the definition see Chapter 2. These spaces contain many classical spaces as special cases: For $1 < p < \infty$ we have $F^s_{p,2} = H_p^s$, the Bessel-potential spaces. If $s \in \mathbf{N} = \{1, 2, \ldots\}$ these are the usual Sobolev spaces. For $0 < p < 1$ we obtain the local Hardy spaces $h_p = F^0_{p,2}$ of GOLDBERG [4]. This was proved by BUI HUY QUI in [1].

Pseudo differential operators in Triebel spaces have previously been considered in [1] and [8]. The first result in this direction was due to GOLDBERG [4] who proved that the operators in $L^0_{1,0}$ are bounded in h_p (cf. also [9]). Bui Huy Qui extended this to $F^s_{p,q}$. Recently NILSSON [8] proved that also operators of class $L^0_{1,\delta}$, $0 < \delta < 1$, are bounded in h_p. Via interpolation he also succeeded to generalize this to $F^s_{p,q}$. However, his result contains some unnatural restrictions on the parameters p and q. The aim of this paper is to remove these restrictions and thus prove the following: Let $T \in L^m_{1,\delta}$, $0 \leqq \delta < 1$, $-\infty < m < \infty$. Then for all $0 < p, q < \infty$, $-\infty < s < \infty$

$$T : F^s_{p,q} \to F^{s-m}_{p,q}.$$

From this we get the above mentioned results of Illner, Goldberg, Bui Huy Qui and Nilsson as special cases. For further generalizations see Remark 3.7 in Chapter 3.

1. Definition of a pseudo differential operator

Let r be a polynomially bounded measurable complex valued function in $\mathbf{R}^n \times \mathbf{R}^n$. The pseudo differential operator $r(x, D)$ with symbol r is defined by the formula

$$r(x, D) f(x) = \int e^{ix\xi} r(x, \xi) \hat{f}(\xi) \, d\xi, \qquad x \in \mathbf{R}^n, \qquad f \in S, \tag{1.1}$$

where S denotes the Schwartz space in $\mathbf{R}^n$ and $\hat{f}$ is the Fourier transform of f (integrals without any integration limits are taken over all $\mathbf{R}^n$). We say that r belongs to the class $S^m_{\varrho,\delta}$, $m \in \mathbf{R}$, $0 \leqq \varrho, \delta \leqq 1$ if for each multi-index α and β there is a constant $c_{\alpha,\beta}$ such that

$$|D_\xi^\alpha D_x^\beta r(x, \xi)| \leqq c_{\alpha,\beta}(1 + |\xi|)^{m+\delta|\beta|-\varrho|\alpha|}$$

holds for all x and ξ in $\mathbf{R}^n$. If $r \in S^m_{\varrho,\delta}$ then the corresponding pseudo differential operator $r(x, D)$ is said to be in class $L^m_{\varrho,\delta}$. If $r(x, D) \in L^m_{\varrho,\delta}$ then, clearly, it maps S continuously into itself. Hence we may extend it to a continuous operator from S' into S' by the formula

$$\langle r(x, D) f, \varphi\rangle = \langle \hat{f}, \tilde{\varphi}\rangle$$

where $\varphi \in S$ and $\tilde{\varphi}(\xi) = \int e^{ix\xi} r(x, \xi)\, \varphi(x)\, dx$. By S' we mean, of course, the space of tempered distributions in $\mathbf{R}^n$, the dual of S.

2. Function spaces

To define the Triebel spaces $F^s_{p,q}$ and the Besov spaces $B^s_{p,q}$ we choose a sequence of test functions $(\varphi_k)_{k=0}^\infty$ with the properties:

$$\operatorname{supp} \varphi_0 \subset \{\xi \mid |\xi| \leqq 2\},$$

$$\operatorname{supp} \varphi_k \subset \{\xi \mid 2^{k-1} \leqq \xi \leqq 2^{k+1}\}, \qquad k \in \mathbf{N},$$

$$\sum_{k=0}^{\infty} \varphi_k(\xi) = 1, \quad \text{for every} \quad \xi \in \mathbf{R}^n,$$

and for any multi-index α there is a constant c_α such that

$$|D^\alpha \varphi_k(\xi)| \leqq c_\alpha 2^{-|\alpha|k}.$$

For $0 < p, q < \infty$ and $-\infty < s < \infty$ we define $F^s_{p\,q}$ to be the space of all $f \in S'$ such that

$$\|f\|_{F^s_{p,q}} := \|(2^{sk}\varphi_k(D) f)_{k=0}^\infty\|_{L_p(l_q)} < \infty. \tag{2.1}$$

Notice that according to our notation (1.1) $\varphi_k(D) f = F^{-1}(\varphi_k \hat{f})$, where F stands for Fourier transform in S' and $\hat{f} = Ff$. By the norm $\|\cdot\|_{L_p(l_q)}$ we mean

$$\|(f_k)\|_{L_p(l_q)} = \left(\int \left(\sum_k |f_k(x)|^q\right)^{p/q} dx\right)^{1/p}.$$

If we change the roles of $\|\cdot\|_{l_q}$ and $\|\cdot\|_{L_p}$ in the right hand side of (2.1) we get the Besov spaces $B^s_{p,q}$ consisting of those $f \in S'$ for which

$$\|f\|_{B^s_{p,q}} := \|(2^{sk}\varphi_k(D) f)_{k=0}^\infty\|_{l_q(L_p)} < \infty.$$

Remark 2.1: For the properties of $F^s_{p,q}$ and $B^s_{p,q}$ see [10, 14, 15]. We only mention that different choices of the sequence $(\varphi_k)_{k=0}^\infty$ lead to equivalent (quasi) norms. For simplicity we also assume that $\varphi_k(\xi) = \varphi(2^{-k+1}\xi)$, $k \in \mathbf{N}$, where $\varphi = \varphi_1$ is an appropriate function and that $\sum_{k=0}^{\infty} \varphi_k(\xi) \equiv 1$.

Remark 2.2: Below we shall need another sequence of test functions $(\psi_k)_{k=0}^\infty$ with $\psi_k(\xi) = \psi(2^{-k}\xi)$, $k \in \mathbf{N}$, where ψ is chosen so that

$$\psi_k(\xi) = 1, \quad \text{for} \quad \xi \in \operatorname{supp} \varphi_k \quad \text{and} \quad \operatorname{supp} \psi_k \subset \{\xi \mid 2^{k-2} \leqq |\xi| \leqq 2^{k+2}\}$$

(with natural modification for $k = 0$). It is not hard to see that the use of this sequence instead of $(\varphi_k)_{k=s}^{\infty}$ in the definition of $F^s_{p,q}$ and $B^s_{p,q}$ leads to the same spaces and equivalent (quasi) norms (cf. [15: Chapter 2.1]).

Remark 2.3: We recall the following interpolation theorem:

$$B^s_{p,q} = (F^{s_0}_{p,2}, F^{s_1}_{p,2})_{\theta,q}, \qquad s = (1-\theta)\, s_0 + \theta s_1, \qquad s_0 \neq s_1.$$

For this result see [14: p. 72].

3. $F^s_{p,q}$ — estimates for pseudo differential operators

We start with the following result.

Theorem 3.1: *Let $r \in S^0_{1,\delta}$, $0 \leqq \delta < 1$ and $T = r(x, D)$ be the corresponding pseudo differential operator. Suppose additionally that $r(x, \xi)$ has compact support in x. Then*

$$T: F^s_{p,q} \to F^s_{p,q} \quad \textit{for} \quad 0 < p, \quad q < \infty. \tag{3.1}$$

More precisely, for $f \in S$

$$\|Tf\|_{F^s_{p,q}} \leqq c\, \|f\|_{F^s_{p,q}} \tag{3.2}$$

where c only depends on p, q, n, δ, s and on the Lebesgue measure of $\operatorname{supp}_x r(x, \xi)$.

Proof: For simplicity we suppose that $s = 0$. The general case follows similarly. We recall the Leibniz rule

$$\varphi_j(D)\, r(x, D) \sim \sum_\beta \frac{i^{-|\beta|}}{\beta!}\, r_{(\beta)}(x, D)\, \varphi_j^{(\beta)}(D). \tag{3.3}$$

Here $r_{(\beta)}(x, \xi) = (iD_x)^\beta\, r(x, \xi)$, $\varphi_j^{(\beta)}(\xi) = (iD_\xi)^\beta\, \varphi_j(\xi)$ and $\sim$ means that the operators coincide modulo a smoothing operator (cf. [13]).

Let $f \in S$. In the spirit of (3.3) we start with the expression

$$r_{(\beta)}(x, D)\, \varphi_j^{(\beta)}(D)\, f(x). \tag{3.4}$$

Since $\psi_j(\xi) = 1$ in $\operatorname{supp} \varphi_j$ this is equal to $r_{(\beta)}(x, D)\, \psi_j(D)\, \varphi_j^{(\beta)}(D)\, f(x)$. By denoting $\psi_j(D)\, f$ by f_j we obtain

$$r_{(\beta)}(x, D)\, \varphi_j^{(\beta)}(D)\, f(x) = \int K^j_{(\beta)}(x, y)\, f_j(y)\, dy$$

where

$$K^j_{(\beta)}(x, y) = \int e^{i(x-y)\xi}\, r_{(\beta)}(x, \xi)\, \varphi_j^{(\beta)}(\xi)\, d\xi. \tag{3.5}$$

In the following lemma we estimate the kernel $K^j_{(\beta)}(x, y)$.

Lemma 3.2: *For all $\lambda > 0$ there exists a constant $c = c_{\lambda,\beta}$ such that*

$$|K^j_{(\beta)}(x, y)| \leqq c\, \frac{2^{jn}}{(1 - 2^j\, |x-y|)^\lambda}. \tag{3.6}$$

Proof: Let first $j > 0$. Integrating (3.5) by parts one obtains for every multi-index α

$$|(x-y)^\alpha\, K^j_{(\beta)}(x, y)| = \left|\int e^{i(x-y)\xi} D_\xi^\alpha [r_{(\beta)}(x, \xi)\, \varphi_j^{(\beta)}(\xi)]\, d\xi\right|.$$

Hence by using the Leibniz rule we obtain

$$|(x-y)^\alpha K^j_{(\beta)}(x,y)| \leqq c_\alpha \int \sum_{\gamma\leqq\alpha} |r^{(\gamma)}_{(\beta)}(x,\xi)\, D_\xi^{\alpha-\gamma}\varphi_j^{(\beta)}(\xi)|\, d\xi$$

$$\leqq c_{\alpha,\beta} \sum_{\gamma\leqq\alpha} \int (1+|\xi|)^{-|\gamma|+\delta|\beta|}\, 2^{-j|\alpha+\beta-\gamma|} (D_\xi^{\alpha+\beta-\gamma}\varphi)\,(2^{-j}\xi)\, d\xi$$

$$\leqq c_{\alpha,\beta} 2^{jn} (1+2^j)^{\delta|\beta|}\, 2^{-j|\alpha+\beta|} \leqq c_{\alpha,\beta} 2^{jn} 2^{-j|\alpha|}.$$

Consequently we have for all $\lambda > 0$

$$|(x-y)|^\lambda\, |K^j_{(\beta)}(x,y)| \leqq c_{\lambda,\beta} 2^{jn} 2^{-j\lambda}. \tag{3.7}$$

On the other hand it is clear that

$$|K^j_{(\beta)}(x,y)| \leqq c 2^{jn}. \tag{3.8}$$

Thus we have proved the lemma for $j > 0$. Evidently the claim holds also for $j = 0$ ∎

We turn back to the proof of the theorem. From Lemma 3.2 we get the following estimate for (3.4)

$$|r_{(\beta)}(x,D)\,\varphi_j^{(\beta)}(D)\, f(x)| \leqq c \int \frac{2^{jn}}{(1+2^j|x-y|)^\lambda}\, f_j(y)\, dy.$$

By introducing the *Fefferman-Stein maximal function* f_j^*,

$$f_j^*(x) = \sup_{y\in\mathbf{R}^n} |f_j(y)|\,(1+2^j|x-y|)^{-\mu},\quad \mu > \frac{n}{\min(p,q)}$$

we obtain

$$|r_{(\beta)}(x,D)\,\varphi_j^{(\beta)}(D)\, f(x)| \leqq c f_j^*(x).$$

Here we have taken $\lambda > \mu + n$.

Next we search for $\varphi_j(D)\, r(x,D)$ an expression similar to (3.3) and write

$$\varphi_j(D)\, r(x,D)\, f(x) := \sum_{|\beta|<N} \frac{i^{-|\beta|}}{\beta!}\, r_{(\beta)}(x,D)\,\varphi_j^{(\beta)}(D)\, f(x) + R_j^N(x) := g_j^0(x) + g_j^1(x).$$

For the sequence $(g_j^0)_{j=0}^\infty$ we get

$$\|(g_j^0(x))_{j=0}^\infty\|_{L_p(l_q)} \leqq c\|(f_j^*(x))_{j=0}^\infty\|_{L_p(l_q)} \leqq c\|(f_j(x))_{j=0}^\infty\|_{L_p(l_q)}.$$

The last inequality follows from a maximal inequality of Fefferman and Stein (cf. [11] or [15: p. 47]. Consequently

$$\|(g_j^0(x))_{j=0}^\infty\|_{L_p(l_q)} \leqq c\,\|f\|_{F^0_{p,q}}.$$

It remains to show the corresponding estimate for the remainder $R_j^N f$. Clearly, we may write

$$R_j^N f(x) = \int e^{ix(\eta+\xi)}\, \hat{f}(\xi)\, p_j^N(\eta,\xi)\, d\xi\, d\eta$$

where

$$p_j^N(\eta,\xi) = \hat{r}(\eta,\xi)\left(\varphi_j(\eta+\xi) - \sum_{|\beta|<N} \frac{i^{-|\beta|}}{\beta!}\,\varphi_j^{(\beta)}(\xi)\,\eta^\beta\right) \tag{3.9}$$

and $\hat{r}(\eta,\xi)$ is the Fourier transform of $r(x,\xi)$ with respect to x. In order to write (3.9) in a more convenient form we recall that $\sum_{\nu=0}^\infty \varphi_\nu(\xi) \equiv 1$ and that $\psi_\nu(\xi) = 1$ if ξ is in the support of φ_ν. This provides us with the formula

$$R_j^N f(x) = \sum_{\nu=0}^{\infty} \int e^{i(x-y)\xi} \varphi_\nu(\xi)\, q_j^N(x, \xi)\, f_\nu(y)\, dy\, d\xi \tag{3.10}$$

where $f_\nu = \psi_\nu(D) f$ and $q_j^N(x, \xi) = \int e^{ix\eta} p_j^N(\eta, \xi)\, d\eta$. In the following lemma we estimate the symbol $q_j^N(x, \xi)$.

Lemma 3.3: *For each multi-index α and $L > 0$ there exist $N \in \mathbb{N}$ and a constant $c = C_{\alpha,L}$ such that*

$$|D_\xi^\alpha q_j^N(x, \xi)| \leqq c 2^{-j}(1 + |\xi|)^{-L}. \tag{3.11}$$

Proof: According to Leibniz's rule we have

$$|D_\xi^\alpha p_j^N(\eta, \xi)| \leqq c \sum_{\gamma \leqq \alpha} |D_\xi^\gamma \hat{r}(\eta, \xi)| \left| \varphi_j^{(\alpha-\gamma)}(\eta + \xi) - \sum_{|\beta|<N} \frac{i^{-|\beta|}}{\beta!} \varphi_j^{(\alpha-\gamma+\beta)}(\xi)\, \eta^\beta \right|.$$

By using the Lagrange remainder term in Taylor's formula we obtain

$$|D_\xi^\alpha p_j^N(\eta, \xi)| \leqq c \sum_{\gamma \leqq \alpha} |D_\xi^\gamma \hat{r}(\eta, \xi)| \left| \sum_{|\beta|=N} \varphi_j^{(\alpha-\gamma+\beta)}(\xi + \theta_\gamma \eta)\, \eta^\beta \right|,$$

where $0 > \theta_\gamma < 1$, $\gamma \leqq \alpha$. But because $r(x, \xi)$ has compact support in x it can easily be seen (cf. [5: Lemma 2.3]) that for each $M > 0$

$$|D_\xi^\gamma \hat{r}(\eta, \xi)| \leqq C_M (1 + |\xi|)^{-|\gamma|+\delta M} (1 + |\eta|)^{-M}. \tag{3.12}$$

Thus we can estimate as follows

$$\begin{aligned} |D_\xi^\alpha p_j^N(\eta, \xi)| &\leqq c \sum_{\gamma \leqq \alpha} |D_\xi^\gamma \hat{r}(\eta, \xi)|\, 2^{-j(N+|\alpha-\gamma|)}\, |\eta|^N \\ &\leqq C_M \sum_{\gamma \leqq \alpha} (1 + |\xi|)^{-|\gamma|+\delta M} (1 + |\eta|)^{-M+N}\, 2^{-j(N+|\alpha-\gamma|)}. \end{aligned} \tag{3.13}$$

We assume from now on that $j > 0$. The case $j = 0$ follows in the same manner. We also consider the two cases $|\xi| > 2\,|\eta|$ and $|\xi| \leqq 2\,|\eta|$ separately. Let us first assume that $|\xi| > 2\,|\eta|$. In this case we have

$$\frac{1}{2}\, |\xi| < |\xi + \theta\eta| < 2\,|\xi| \tag{3.14}$$

for every $0 \leqq \theta \leqq 1$. By taking into account this and the fact that $2^{j-1} \leqq |\xi + \theta_\gamma \eta| < 2^{j+1}$ we see that $|\xi| \sim 2^j$. Thus we get from (3.13)

$$|D_\xi^\alpha p_j^N(\eta, \xi)| \leqq c(1 + |\xi|)^{-L+n}\, 2^{-j}(1 + |\xi|)^{n-|\alpha|+L+\delta M-N+1} (1 + |\eta|)^{N-M}. \tag{3.15}$$

By taking first M large (e.g. $(1 - \delta) M > L + n + 1$) and afterwards N we see that

$$(1 + |\xi|)^{n-|\alpha|+L+\delta M-N+1} (1 + |\eta|)^{N-M} \leqq c(1 + |\eta|)^{-|\alpha|+L+1+(\delta-1)M} \leqq c$$

and hence we obtain

$$|D_\xi^\alpha p_j^N(\eta, \xi)| \leqq c(1 + |\xi|)^{-L-n}\, 2^{-j}. \tag{3.16}$$

On the other hand if $|\xi| \leqq 2\,|\eta|$ we get from (3.13)

$$\begin{aligned} |D_\xi^\alpha p_j^N(\eta, \xi)| &\leqq c(1 + |\xi|)^{-L} (1 + |\eta|)^{(\delta-1)M+L+N}\, 2^{-j} \\ &\leqq c(1 + |\xi|)^{-L} (1 + |\eta|)^{-(n+1)}\, 2^{-j} \end{aligned} \tag{3.17}$$

for M large enough.

To end up the proof of the lemma we conclude from (3.16) and (3.17) that

$$\int\limits_{|\eta|\leq\frac{1}{2}|\xi|} |D^\alpha p_j^N(\eta,\xi)|\,d\eta + \int\limits_{|\eta|\geq\frac{1}{2}|\xi|} |D^\alpha p_j^N(\eta,\xi)|\,d\eta$$

$$\leq c|\xi|^n (1+|\xi|)^{-L-n} 2^{-j} + (1+|\xi|)^{-L} 2^{-j} \int (1+|\eta|)^{-n-1}\,d\eta$$

$$\leq c(1+|\xi|)^{-L} 2^{-j}$$

which is the desired estimate ∎

To complete the proof of the theorem we write (3.10) as follows

$$|R_j^N f(x)| \leq \left|\sum_{\nu=0}^{\infty} \int \varphi_\nu(D)\, f(y)\, K_\nu^j(x,y)\,dy\right|$$

where

$$K^j(x,y) = \int e^{i(x-y)\xi} q_j^N(x,\xi)\,\psi_\nu(\xi)\,d\xi.$$

By taking $L = \lambda + 1$ in Lemma 3.3 we obtain for all $\lambda > 0$ that

$$K_\nu^j(x,y) \leq c\,\frac{2^{-j}2^{\nu n}2^{-\nu}}{(1+2^\nu|x-y|)^\lambda}.$$

Thus we have the estimate

$$|R_j^N f(x)| \leq c2^{-j} \sum_{\nu=0}^{\infty} 2^{-\nu} f_\nu^*(x).$$

Obviously for $0 < q \leq 1$

$$\sum_{\nu=0}^{\infty} 2^{-\nu} f_\nu^*(x) \leq c\left(\sum_{\nu=0}^{\infty} |f_\nu^*(x)|^q\right)^{1/q}. \tag{3.18}$$

For $1 < q < \infty$, (3.18) follows from Hölder's inequality. Hence for any $0 < q < \infty$

$$\|(R_j^N f(x))_{j=0}^{\infty}\|_{l_q} \leq c\,\|(f_\nu^*(x))_{\nu=0}^{\infty}\|_{l_q}$$

and finally the Fefferman-Stein maximal inequality yields (take λ large enough)

$$\|(R_j^N f(x))_{j=0}^{\infty}\|_{L_p(l_q)} \leq c\,\|f\|_{F_{p,q}^0}.$$

This gives (3.2) and consequently the proof is complete ∎

In the following theorem we are going to abandon the restriction made on $\operatorname{supp}_x r$.

Theorem 3.4: *Let $T = r(x, D)$ be in $L_{1,\delta}^0$, $0 \leq \delta < 1$. Then for all $0 < p, q < \infty$ and s sufficiently large $T : F_{p,q}^s \to F_{p,q}^s$.*

Proof: Let φ be a C^∞-function supported in $|x| \leq 1$. Furthermore, let ψ be another C^∞-function with $\psi(x) = 1$ in $|x| \leq 2$ and $\operatorname{supp}\psi \subset \{x \mid |x| \leq 4\}$. We put $\varphi_k(x) = \varphi(x - g_k)$ and $\psi_k(x) = \psi(x - g_k)$ where g_k, $k = 1, 2, \ldots$, goes through all the lattice points in $\mathbf{R}^n$. We also assume that $\sum \varphi_k \equiv 1$. Because of the known local representation of $F_{p,q}^s$-spaces [16] we have

$$\|u\|_{F_{p,q}^s}^p \sim \sum_k \|\varphi_k u\|_{F_{p,q}^s}^p \sim \sum_k \|\psi_k u\|_{F_{p,q}^s}^p \tag{3.20}$$

for s large enough.

Now we break up T into two parts, $T = T_0 + T_1$, where $T_0 = \sum_{k=1}^{\infty} \varphi_k T \psi_k$. By using 3.20) we obtain

$$\|T_0 u\|^p_{F^s_{p,q}} \leqq c \sum_j \|\varphi_j T_0 u\|^p_{F^s_{p,q}} \leqq c \sum_j \Big\| \sum_k \varphi_j \varphi_k T \psi_k u \Big\|^p_{F^s_{p,q}}$$

and hence by Theorem 3.1

$$\|T_0 u\|^p_{F^s_{p,q}} \leqq c \sum_j \|\psi_j u\|^p_{F^s_{p,q}} \sim c\, \|u\|^p_{F^s_{p,q}}.$$

To estimate T_1 write $\chi_k = 1 - \psi_k$ and $T_1 = \sum \varphi_k T \chi_k$. Let $K(x, \cdot)$ denote the Fourier transform of $r(x, \xi)$ with respect to ξ. We get

$$T(\chi_k u)(x) = \int K(x, x - z)\, \chi_k(z)\, u(z)\, dz. \tag{3.21}$$

If $\varphi_k(x)\, T(\chi_k u)(x) = 0$ we must have $|x - z| \geqq 1$ in (3.21). Thus we may assume that $K(x, z) = 0$ for $|z| \leqq 1/2$. Note that this also means a modification to $r(x, \xi)$. But, for $|\gamma|$ sufficiently large $|z^\gamma D_x^\beta D_z^\alpha K(x, z)| \leqq c_{\alpha\beta\gamma}$ and hence we obtain for all $N \in \mathbf{N}$

$$|D_x^\beta D_z^\alpha K(x, z)| \leqq c_{\alpha\beta N}(1 + |z|)^{-N}.$$

Consequently, we also have

$$|D_x^\beta D_\xi^\alpha r(x, \xi)| \leqq c_{\alpha\beta N}(1 + |\xi|)^{-N}$$

for each N and therefore $r(x, D) \in L^{-\infty}_{1,0}$ and $T_1 \in L^{-\infty}_{1,0}$. Thus $T_1 : F^s_{p,q} \to F^s_{p,q}$ which proves the theorem ∎

Having now done all the hard work we may prove the following assertion.

Theorem 3.5: *Let $T = r(x, D)$ be a pseudo differential operator of class $L^m_{1,\delta}$, $-\infty < m < \infty$, $0 \leqq \delta < 1$. Then for all $0 < p, q < \infty$ and $-\infty < s < \infty$*

$$T : F^s_{p,q} \to F^{s-m}_{p,q}$$

Proof: The claim follows reading from the following basic facts: If σ_s is the pseudo differential operator $(1 - \Delta)^{s/2}$ then $\sigma_s \in L^s_{1,0}$. Moreover, $\sigma_s : F^{s'}_{p,q} \to F^{s'+s}_{p,q}$. Finally, if $S \in L^m_{1,\delta}$ and $T \in L^{m'}_{1,\delta}$ then $ST \in L^{m+m'}_{1,\delta}$ (cf. [13: p. 225]) ∎

Corollary 3.6: *If T is as in Theorem 3.5 then $T : B^s_{p,q} \to B^{s-m}_{p,q}$ for all $-\infty < s$, $m < \infty$ and $0 < p, q < \infty$.*

Proof: Use Theorem 3.5 and Remark 2.3 ∎

Remark 3.7: The question arises whether the result in Theorem 3.5 can be extended for the values $0 < \varrho < 1$ as in L_2. The answer is negative because there are symbols in $S^0_{\varrho,0}$, $0 < \varrho < 1$, independent of x which are not Fourier multipliers in L_p, $p \neq 2$. This can be seen, as noted by *P.* Nilsson, in the following way. Let $\|\cdot\|$ denote the multiplier norm in L_p and assume that

$$\|m\| \leqq c \sup_\alpha \|m^\alpha(\xi)|/(1 + |\xi|)^{-\varrho|\alpha|}.$$

To obtain a contradiction replace m by $m(\cdot/\varepsilon)$ and observe that the left hand side does not depend on ε.

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