$(2)$ 

 $(3)$ 

# Taylor's Expansion in a Distribution Algebra

L. BERG

Für eine gewisse Distributionenalgebra wird gezeigt, daß jedes Element der Algebra einen Wert besitzt sowie durch seine Taylorentwicklung dargestellt wird. Außerdem wird die Struktur der Teilalgebra der Werte untersucht.

Для некоторой дистрибуционной алгебры показывается, что каждый элемент алгебры имеет значение и представим разложением Тейлора. Кроме того исследуется структура подалгебры значений.

For a certain distribution algebra it is shown that every element of the algebra possesses a value and is representable by its Taylor's expansion. Moreover there is investigated the structure of the subalgebra of values.

By definition, a distribution algebra is an associative, but noncommutative differential algebra with at least one element  $h$  satisfying

and  $h' \neq 0$ . A good survey on some classes of distribution algebras with additional properties was given by MA KYIN MYINT [3]. Here we go back to the more general distribution algebras of [1].

The aim of this paper is to show that every element of the latter distribution algebras is representable by its Taylor's expansion, using the notion of values of such elements introduced in [2]. As a consequence we are able to determine the structure of the set of these values.

## **Preliminaries**

 $h^2 \doteq h$ 

In [1] there was considered the distribution algebra  $D_1$  with unit element 1 generated by two elements  $t, h$  with the properties.

$$
t'=1,\qquad th'=0
$$

and, of course, (1). The ring of scalars of this algebra is assumed to contain the rational numbers. According to Dirac the derivative h' is denoted by  $\delta$ , the more as this element shall be interpreted as a Schwartz distribution. As a consequence of  $(2)$ , in D. there are valid the well known relations

$$
t^n\delta^{(m)} = (-1)^n n! \binom{m}{n} \delta^{(m-n)}
$$

for all integers  $m, n \ge 0$  with  $\binom{m}{n} = 0$  for  $m < n$ . As a consequence of (1), in  $D_1$ 

there are valid the relations

L. BERO  
\nare valid the relations  
\n
$$
h^{(m)} = \sum_{n=0}^{m} {m \choose n} h^{(n)} h^{(m-n)},
$$
\nwhere the products on the right-hand

relations  $\binom{m}{n} h^{(n)} h^{(m-n)}$ , (4)  $\ldots$  of the right-hand side are no Schwartz distributions for too, where the products on the right-hand side are no Schwartz distributions for  $m > 1.$ 266 L. BERG<br>
there are valid the relations<br>  $h^{(m)} = \sum_{n=0}^{m} {m \choose n} h^{(n)} h^{(m-n)}$ ,<br>
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In [1] there was also constructed the extension  $D_2$  of  $D_1$  as the distribution algebra

$$
f = \sum_{n=0}^{\infty} (a_n t^n + b_n t^n h), \tag{5}
$$

BERG<br>
alid the relations<br>  $\binom{m}{n} = \sum_{n=0}^{n} \binom{m}{n} h^{(n)} h^{(m-n)}$ ,<br>  $\therefore$  the products on the right-hand side are no Schwartz distributions for<br>
ere was also constructed the extension  $D_2$  of  $D_1$  as the distribution a where  $a_n$ ,  $b_n$  are polynomials in  $\delta^{(k)}$  ( $k = 0, 1, 2, ...$ ), which are determined uniquely by *f* and can be prescribed arbitrarily. With the elements (5) all operations in  $D_2$  are to be carried out termwise. In  $D_2$  two elements *f*, *g* are defined to be equal if and only if  $\begin{array}{c}\n 01 \\
 \hline\n 01\n \end{array}$  $\begin{array}{r} \text{too,} \ \text{m} > \ \text{In} \ \text{of the} \ \text{of the} \ \text{by } f \text{ is} \ \text{to be} \ \text{if} \ \text{for } m \ \text{to the} \end{array}$ • *th*,  $h_n = 0$ <br> *time time ti* For the determined unit<br>
(5) all operations in<br>
ined to be equal if an<br>
ined to be equal if an<br>
ce  $f_n$  is defined to contained<br>
ions

$$
(f - g) \, \delta^{(m)} = 0
$$

for  $m = 0, 1, 2, \ldots$  According to this definition a sequence  $f_n$  is defined to converge to the limit  $f \in D_2$ , if for every fixed  $m \geq 0$  there is

$$
(f - f_n) \, \delta^{(m)} = 0
$$

for suitable large *n*. With these definitions e.g. the equations

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$$
f \in D_2
$$
, if for every fixed  $m \ge 0$  there is  
\n $(f - f_n) \delta^{(m)} = 0$   
\nfor suitable large *n*. With these definitions e.g. the equations  
\n $t^t h t^k = t^{i+k} h + k \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(i+j) (i+k+j) (j-1)!} \delta^{(j-1)} t^{i+k+j}$  (6)  
\nare valid in  $D_2$ .  
\nSeries  
\nIn what follows we need series with more general coefficients than in (5). For this  
\ncase we consider the  
\nTheorem 1: For an arbitrary sequence  $f_k$  of elements from  $D_2$  the series  
\n $f = \sum_{k=0}^{\infty} f_k t^k$  (7)  
\nis always converging to an element from  $D_2$ .  
\nProof: According to (5) all elements  $f_k$  possess representations of the form  
\n $t = \sum_{k=0}^{\infty} (a_k t^k + b_k t^k h)$ .

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\n
$$
f_k t^k = \sum_{i=0}^{\infty} \left( a_{ki}t^{i+k} + b_{ki} \left( t^{i+k}h + k \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(i+j) (i+k+j) (j-1)!} \delta^{(j-1)} t^{i+k+j} \right) \right).
$$
\nSumming over  $k$  and substituting  $i + k = n$  and  $i + k + j = n$ , respective find that the element  $f$  from (7) possesses also the form (5) with  
\n
$$
a_n = \sum_{i=0}^n \left( a_{n-i,1} + \sum_{j=1}^{n-i} b_{n-i-j, i} \frac{(n-i-j) (-1)^{j-1}}{(i+j) n (j-1)!} \delta^{(j-1)} \right)
$$
 and  $b_n = \sum_{i=0}^n$ 

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$$
\n
$$
i
$$
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$$
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$$
\nand\n
$$
b_n = \sum_{i=0}^n b_{n-i,i},
$$

i.e.  $(7)$  is an element from  $D_2$ . All foregoing calculations with infinite series are possible in view of the definition of convergence, because all series terminate after multiplication by  $\delta^{(m)}$ , so that in fact no convergence problems arise  $\blacksquare$ i.e. (7)<br>
sible in<br>
plicati<br> **Values** *li t .. . ' ' (8)* 

Tn [2] there were considered the series

$$
f_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)} t^k
$$
 (8)

for some elements *f* of  $D'_2$ . Since by Taylor's theorem for holomorphic functions  $f(t)$ and small numbers *t* the series (8) is equal to the value  $f(0)$ , the element  $f_0$  from (8) is named the *(generalized) value of f* (at the point  $t = 0$ ). In [2] there was already i.e. (7) is an element from  $D_2$ . All foregoing calculations with infinite series<br>sible in view of the definition of convergence, because all series terminate aft<br>plication by  $\delta^{(m)}$ , so that in fact no convergence pro ere were considered the series<br>  $f_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)} t^k$  (8)<br>
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nat  $t_{0} = 0$  and  $(t^{n$ 

$$
h_0^2 = h_0, \t h_0 t = th_0, \t \delta_0 t = t \delta_0,
$$
\t(9)

$$
\delta_0^{(m)}t - t\delta_0^{(m)} = m\delta_0^{(m-1)} \tag{10}
$$

for  $m \ge 1$  as well as

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proved that  $t_0 = 0$  and  $(t^nh)_0 = 0$  for all natural numbers *n*,  

$$
h_0^2 = h_0, \quad h_0 t = t h_0, \quad \delta_0 t = t \delta_0,
$$
 (9)  

$$
\delta_0^{(m)} t - t \delta_0^{(m)} = m \delta_0^{(m-1)}
$$
 (10)  
for  $m \ge 1$  as well as  

$$
\delta^{(n)} \delta^{(m)} = \sum_{i=0}^m (-1)^i \binom{m}{i} \delta_0^{(n+i)} \delta^{(m-i)}
$$
 (11)  
or  $m, n \ge 0$  and  

$$
f_0' = 0
$$
 (13)  
or  $f = h^{(n)}, n \ge 0$ .  
Therefore  $2 : The value (8) exists for every element  $f \in D_2$  and has always the property  
(13). The value operator is linear, i.e. it satisfies  

$$
(\alpha f + \beta g)_0 = \alpha f_0 + \beta g_0
$$
 (14)  
in the case  $\alpha' = \beta' = 0$ . Further relations for arbitrary  $f, g \in D_2$  are  

$$
(fg)_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)} g_d^k,
$$
 (15)$ 

or 
$$
m, n \ge 0
$$
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\nTheorem 2: The value (8) write for every element  $f \in D$  and the solution the matrix

Theorem 2: *The value* (8) exists for every element  $f \in D_2$  and has always the property (13). The value-operator is linear, i.e. it satisfies

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(\alpha f + \beta g)_0 = \alpha f_0 + \beta g_0 \tag{14}
$$

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$$
\geq 0 \text{ and}
$$
\n
$$
(f_0') = 0
$$
\n
$$
h^{(n)}, n \geq 0.
$$
\n
$$
P(n^2), n \geq 0.
$$
\n
$$
(f + \beta g)_0 = \alpha f_0 + \beta g_0
$$
\n
$$
P(n^2), q \geq 0.
$$
\n
$$
P(n
$$

$$
(ft)_0 = 0, \qquad (fh)_0 = f_0h_0, \qquad (f\delta)_0 = f_0\delta_0. \tag{16}
$$

Proof: Since the series in (8) possesses the form (7), Theorem 1 implies the first

in the case 
$$
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$$
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\n
$$
(fg)_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)} g_d t^k,
$$
\n
$$
(f)_0 = 0, \quad (fh)_0 = f_0 h_0, \quad (f_0)_0 = f_0 \delta_0.
$$
\nProof: Since the series in (8) possesses the form (7), Theorem 1 implies the first assertion. The equations (13) and (14) are easily verified from (8). The equation  
\n
$$
(fg)_0 = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (fg)^{(i)} t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{k=0}^i \binom{i}{k} f^{(k)} g^{(i-k)} t^k.
$$
\n
$$
= \sum_{k=0}^{\infty} \sum_{i=k}^{\infty} \frac{(-1)^i}{k! (i-k)!} f^{(k)} g^{(i-k)} t^i = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} g^{(j)} t^{j+k}
$$
\nwith  $j = i - k$  shows in view of (8) with  $g$  instead of  $f$  that (15) is valid, too. From (15).  
\nand  $t_0 = 0$  as well as (9) it follows (16). Hence Theorem 2 is proved  $\blacksquare$ 

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Of course, both scalar multiples of the unit element 1 and the values themselves are their own values. These statements and the equations (13) and (14) show that the values (8) behave like ordinary constants in calculus. Equation (12) is a special application of  $(14)$  to equation  $(11)$ . Note that formula  $(14)$  is, generally speaking, not valid in the case that the constants are standing at the right of the elements. A general consequence of (15) is the equation  $(fg)_{0} = (fg)_{0}$ .

Taylor's expansion

Now we come to another property which underlines the right of terming the elements  $(8) *values*$ .

Theorem 3: All elements  $f \in D_2$  are respresentable by Taylor's expansion

$$
t = \sum_{n=0}^{\infty} \frac{1}{n!} f_0^{(n)} t^n,
$$
 (17)

and they satisfy for arbitrary integers  $m \geq 0$  the equation

$$
f\delta^{(m)} = \sum_{n=0}^{m} (-1)^n {m \choose n} f_0^{(n)} \delta^{(m-n)}.
$$
 (18)

Proof: Multiplying (17) by  $\delta^{(m)}$  and considering (3) it follows (18). Hence, in view of the definition of equality in  $D_2$  it suffices to prove (18). Applying (8) to  $f^{(n)}$  instead of f we obtain

$$
f_0^{(n)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(n+k)} t^k,
$$

hence, in view of  $(3)$ 

$$
f_0^{(n)}\delta^{(m-n)}=\sum_{k=0}^{m-n}\binom{m-n}{k}f^{(n+k)}\delta^{(m-n-k)}=\sum_{j=n}^m\binom{m-n}{j-n}f^{(j)}\delta^{(m-j)}
$$

with  $j = n + k$  and therefore

$$
\sum_{n=0}^{m} (-1)^n \binom{m}{n} f_0^{(n)} \delta^{(m-n)} = \sum_{j=0}^{m} \sum_{n=0}^{j} (-1)^n \binom{m}{n} \binom{m-n}{j-n} f^{(j)} \delta^{(m-j)}.
$$
  
according to 
$$
\binom{m}{n} \binom{m-n}{j-n} = \binom{m}{j} \binom{j}{n} \text{ and}
$$

$$
\sum_{n=0}^{j} (-1)^n \binom{j}{n} = \begin{cases} 1 & \text{for } j = 0, \\ 0 & \text{for } j > 0, \end{cases}
$$

the foregoing sum is equal to  $f_0^{(m)}$  and Theorem 3 is proved  $\blacksquare$ 

For holomorphic functions f equation (18) is well known in the theory of Schwartz distributions. Equation (11) is a special case of (18) with  $f = \delta^{(n)}$ . Equation (18) reads for  $m = 0$ 

$$
f\delta = f_0\delta.
$$

Conversely, if

A

$$
i\delta = c\delta
$$

 $(20)$ 

 $(19)$ 

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with  $c' = 0$ , then it follows  $c = f_0$ , because (19) and (20) imply  $(f_0 - c) \delta = 0$  and therefore  $(f_0 - c) \delta^{(n)} = 0$  for every  $n \ge 0$ , so that we obtain the assertion  $f_0 = c$ . Differentiating equation (19) *n-times* and considering the value of the result we Taylor's expansion<br>
with  $c' = 0$ , then it follows  $c = f_0$ , because (19) and<br>
therefore  $(f_0 - c) \delta^{(n)} = 0$  for every  $n \ge 0$ , so that<br>
Differentiating equation (19) *n*-times and consider<br>
obtain<br>  $f_0 \delta_0^{(n)} = (f \delta)_0^{(n)}$ .<br> *f f*<sub>0</sub>*o f***<sub>0</sub>***o f<sub>0</sub><i>o f<sub>0</sub><i>o f<sub>0</sub><i>o f<sub>0</sub><i>o c f<sub>0</sub><i>o f*<sub>0</sub>*o f*<sub>0</sub>*o f*<sub>0</sub>*o <i>f*<sub>0</sub>*o f*<sub>0</sub>*o f*<sub>0</sub>*o f*<sub>0</sub>*o f*<sub>0</sub>*o f*<sub>0</sub>*o f*<sub>0</sub>*o f*<sub>0</sub>*o f*<sub>0</sub>*o f*<sub>0</sub>*o f*<sub></sub> Taylor's expansion in a distribution algebra<br>
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therefore  $(f_0 - c) \delta^{(n)} = 0$  for every  $n \ge 0$ , so that we obtain the assertion<br>
Differentiatin Taylor's expansion in a distribution algebra<br>  $\begin{aligned}\n &\text{Taylor's expansion in a distribution algebra} \end{aligned}$ <br>  $\begin{aligned}\n &\text{Taylor's expansion in a distribution algebra} \end{aligned}$ <br>  $\begin{aligned}\n &\text{(for } (f_0 - c) \delta^{(n)} = 0 \text{ for every } n \geq 0, \text{ so that we obtain the assertion } f_0 = \text{a} \text{ is a function of } (19) \text{ n-times and considering the value of the result}\n \end{aligned}$ <br>  $\begin{aligned}\n &\text{if } f_0 \delta_0^{($ *(b)*  $(19)$  are  $(6 - c)$   $\delta^{(n)} = 0$  for every  $n \ge 0$ , so that iating equation (19) *n*-times and conside  $f_0\delta_0^{(n)} = (f_0)\delta_0^{(n)}$ .<br>
Ation implies inductively  $f_0\delta_0^{(n)}\delta_0^{(n)} = ((f_0)^{(m)}\delta_0^{(n)},$ <br>  $f_0\delta_0^{(k)}\delta_0^{(m$ 

$$
f_0\delta_0^{(n)}=(f\delta)_0^{(n)}.
$$

$$
f_0 \delta_0^{(m)} \delta_0^{(n)} = \left( (f \delta)^{(m)} \delta \right)_0^{(n)},
$$
  
\n
$$
f_0 \delta_0^{(k)} \delta_0^{(m)} \delta_0^{(n)} = \left( ((f \delta)^{(k)} \delta)^{(m)} \delta \right)_0^{(n)},
$$
  
\n
$$
\tag{22}
$$

It should be possible to use Taylor's expansion alsp to, define values of elements of  $D_2$  in a point different from  $t = 0$ , however, up to now this is not worked out.

### An isomorphism

 

The following theorem clarifies the structure of the set of values.

Theorem 4: The values (8) form a subalgebra  $A$  of  $D_2$ , which is generated by the *special values* 1 *and*  $h_0^{(n)}$  ( $n \ge 0$ ), and which is isomorphic to the subalgebra  $D_0$  generated by the elements  $1$  and  $h^{(n)}$   $(n\geqq 0).$  The isomorphism  $D_0\rightarrow A$  is generated by the mapping *'1 i n!h0. Ming theorem clarines the structure of tr*<br> *em 4: The values* (8) *form a subalgebra is discomnents* 1 *and*  $h_0^{(n)}$  ( $n \ge 0$ ), *and which is isomnents* 1 *and*  $h^{(n)}$  ( $n \ge 0$ ). *The isomorphism* if  $1 \rightarrow 1$ ,  $h^{(n)} \$ An isomorphism<br>
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special values 1 and  $h_0^{(n)}$  ( $n \ge 0$ ), and which is isomorphic

$$
\rightarrow 1, \qquad h^{(n)} \rightarrow n! h_0^{(n)}.
$$

**Proof:** From (5), (16) and  $t_0 = 0$  we find

$$
f_0=a_{00}+b_{00}h_0
$$

the first zero is the index  $n = 0$  from (5), the second zero indicates the value opera-**Proof:** From (5), (16) and  $t_0 = 0$  we find<br>  $f_0 = a_{00} + b_{00}h_0$  (24)<br>
(the first zero is the index  $n = 0$  from (5), the second zero indicates the value operation). Since (14) and (18) imply<br>  $(f\delta^{(m)})_0 = \sum_{n=0}^m (-1)^n \binom$ 

$$
f_0 = a_{00} + b_{00}h_0
$$
  
zero is the index  $n = 0$  from (5), the  
ace (14) and (18) imply  

$$
(f\delta^{(m)})_0 = \sum_{n=0}^m (-1)^n \binom{m}{n} f_0^{(n)} \delta_0^{(m-n)}
$$

Hence, (24) implies that all values  $f_0$  are contained in the algebra *A* generated by  $h_0^{(n)}$ .<br>Next we derive from (4) the elements 1 and  $h^{(n)}$  ( $n \ge 0$ ). The isomorphism  $D_0 \rightarrow A$  is generated by th<br>  $1 \rightarrow 1$ ,  $h^{(n)} \rightarrow n!h_0^{(n)}$ .<br>
Proof: From (5), (16) and  $t_0 = 0$  we find<br>  $f_0 = a_{00} + b_{00}h_0$ <br>
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tion). Since (14) and (18) imply<br>  $(f\delta^{(m)})_0 = \sum_{n=0}^m (-1)^n \binom{m}{n} f_0^{(n)} \delta_0^{(n)}$ <br>
and  $a_0$ ,  $b_0$  are polynomials in  $\delta^{(k)}$ , the<br>
Hence, (24) implies that all values  $f_0$ *(i.b.)*  $\int_0^{\lambda} f_0^{(n)} \delta_0^{(m-n)}$ <br> *(i.b.)*, the values  $a_{00}$  and  $b_{00}$  are polynomials in  $\delta_0^{(k)}$ .<br>  $\int_0^{\lambda} f_0^{(n)} \delta_0^{(n)} dx$ <br>  $\int_0^{\lambda} (h^{(n)}h^{(m-n)})_0 + (h^{(m)}h)_0.$  (25)<br>
(25)<br>
(27) can be written in the form  $f_0 = a_{00} + b_{00}h_0$ <br>
(the first zero is the index  $n = 0$  from (5), the second zero indicates the value<br>
tion). Since (14) and (18) imply<br>  $(f\delta^{(m)})_0 = \sum_{n=0}^m (-1)^n \binom{m}{n} f_0^{(n)} \delta_0^{(m-n)}$ <br>
and  $a_0$ ,  $b_0$  are polynomial  $\delta^{(k)}$ , the va<br>
es  $f_0$  are cor<br>  $\binom{m}{n} (h^{(n)}h^{(n)}$ <br>
2) can be w<br>  $\binom{m-1}{i} h_0$ 5), the second zero indicates the value<br>  $(m-n)$ <br>
• values  $a_{00}$  and  $b_{00}$  are polynomials in<br>
contained in the algebra A generated b<br> *•*  $h_b(m-n)$ <sub>0</sub> +  $(h_c(m)h)$ <sub>0</sub>.<br>
• written in the form<br>  $h_b(n+1)h_b(m-i)$ <br>
we have

we derive from (4)  
\n
$$
h_0^{(m)} = (hh^{(m)})_0 + \sum_{n=1}^{m-1} {m \choose n} (h^{(n)}h^{(m-n)})_0 + (h^{(m)}h)_0.
$$
\n
$$
\text{ing } \delta = h', \text{ equation (12) can be written in the form}
$$
\n
$$
(h^{(n)}h^{(m)})_0 = \sum_{i=0}^{m-1} (-1)^i {m \over i} h_0^{(n+i)}h_0^{(m-i)}
$$
\n
$$
n \ge 1. \text{ So that with } j = n + i \text{ we have}
$$

Considering  $\delta = h'$ , equation (12) can be written in the form

idering 
$$
\delta = h'
$$
, equation (12) can be written in t\n $(h^{(n)}h^{(m)})_0 = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} h_0^{(n+i)}h_0^{(m-i)}$ \n $m, n \geq 1$ , so that with  $j = n + i$  we have\n
$$
\sum_{n=1}^{m-1} \binom{m}{n} (h^{(n)}h^{(m-n)})_0 = \sum_{n=1}^{m-1} \sum_{i=0}^{m-n-1} \binom{m}{n} (-1)
$$

$$
(10^{10} \tcdot 7_0 - \frac{1}{n=0} (-1)^n \binom{n}{n} f_0^{n-1} \tcdot 0_0^{n-1}
$$
  
and  $a_0, b_0$  are polynomials in  $\delta^{(k)}$ , the values  $a_{00}$  and  $b_{00}$  are polynomials in  $\delta_0^{(k)}$ .  
Hence, (24) implies that all values  $f_0$  are contained in the algebra A generated by  $h_0^{(m)}$ .  
Next we derive from (4)  

$$
h_0^{(m)} = (hh^{(m)})_0 + \sum_{n=1}^{m-1} \binom{m}{n} (h^{(n)}h^{(m-n)})_0 + (h^{(m)}h)_0.
$$
  
Considering  $\delta = h'$ , equation (12) can be written in the form  

$$
(h^{(n)}h^{(m)})_0 = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} h_0^{(n+i)}h_0^{(m-i)}
$$
  
with  $m, n \ge 1$ , so that with  $j = n + i$  we have  

$$
\sum_{n=1}^{m-1} \binom{m}{n} (h^{(n)}h^{(m-n)})_0 = \sum_{n=1}^{m-1} \sum_{i=0}^{m-1} \binom{m}{n} (-1)^i \binom{m-n-1}{i} h_0^{(n+i)}h_0^{(m-n-i)}
$$

$$
= \sum_{n=1}^{m-1} \sum_{j=n}^{m-1} (-1)^{n+j} \binom{m}{n} \binom{m-n-1}{j-n} h_0^{(j)}h_0^{(m-j)}.
$$

 $(23)$ 

 $(24)$ 

0 -

270 L. BERG<br>According to 270 L. B<br>  $\text{According to}$ <br>  $\begin{array}{c}\n\cdot & \cdot \\
\cdot & \cdot \\
\hline\n\end{array}$ 

270 L. BERC  
\nAccording to  
\n
$$
\sum_{n=1}^{j} (-1)^{n+j} \binom{m}{n} \binom{m-n-1}{j-n} = 1 - \binom{m-1}{j} (-1)^j,
$$
\nthis implies  
\n
$$
\sum_{n=1}^{m-1} \binom{m}{n} (h^{(n)}h^{(m-n)})_0 = \sum_{j=0}^{m-1} \left( 1 - \binom{m-1}{j} (-1)^j \right) h_0^{(j)} h_0^{(m-j)},
$$

$$
\sum_{n=1}^{m-1} {m \choose n} (h^{(n)}h^{(m-n)})_0 = \sum_{j=0}^{m-1} \left(1 - {m-1 \choose j} (-1)^j\right) h_0^{(j)}h_0^{(m-j)}
$$
  
1 (18) with  $f = h$ ,  $\delta = h'$ , (16) and (25) we obtain equation  

$$
h_0^{(m)} = \sum_{n=0}^{m} h_0^{(n)}h_0^{(m-n)}.
$$

and from (18) with  $f = h$ ,  $\delta = h'$ , (16) and (25) we obtain equation

$$
h_0^{(m)} = \sum_{n=0}^{m} h_0^{(n)} h_0^{(m-n)}.
$$
 (26)

Obviously, the mapping  $(23)$  transfers equation  $(4)$  into equation  $(26)$ . Since  $D_0$  is nothing else than the free algebra generated by  $h_0^{(n)}$ , which is endowed with the relations  $(4)$  (cf.  $[1]$ ), and since the corresponding relations  $(26)$  under the mapping  $(23)$ are also valid,  $D_0$  has at least a homomorphic image in  $A$ . The fact that all elements of  $D_0$  possess the normal form  $a + bh$  with polynomials a, *b* in  $\delta^{(k)}$  implies that all elements of *A* possess by the mapping (23) the normal form  $c + dh_0$  with polynomials *c, d* in  $\delta_0^{(k)}$ . The equations (22) show that these polynomials are always values of polynomials  $a_0$ ,  $b_0$  in  $\delta^{(k)}$ . Hence, every element of *A* is a value of an element of *D*<sub>0</sub>. nothing else than the free algebra generated by  $h_0^{(n)}$ , which is endowed with the lations (4) (cf. [1]), and since the corresponding relations (26) under the mapping ( $2$  are also valid,  $D_0$  has at least a homomorphi and from (18) with  $f = h$ ,  $\delta = h'$ , (16) and (25) we obtain equation<br>  $h_0^{(m)} = \sum_{n=0}^{m} h_0^{(n)} h_0^{(m-n)}$ . (26)<br>
Obviously, the mapping (23) transfers equation (4) into equation (26). Since  $D_0$  is<br>
nothing else than the  $h_0^{(m)} = \sum_{n=0}^{m} h_0^{(n)} h_0^{(m-n)}$ .<br>
Obviously, the mapping (23) transfers enothing else than the free algebra generate<br>
lations (4) (cf. [1]), and since the correspon<br>
are also valid,  $D_0$  has at least a homomorp<br>
of *C = /mO OO(km) ... 6 0( kmn.) ,*  S 270 L. Buse<br>
According to<br>  $\sum_{k=1}^{n} (-1)^{n+k} {m \choose k} {n - n - 1 \choose k} - 1 = {m - 1 \choose k} (-1)^k$ ,<br>
this implies<br>  $\sum_{k=1}^{n} {m \choose k} (h^{(k)}k^{(k-1)})_n = \sum_{k=1}^{n} \left(1 - {m - 1 \choose k} (-1)^k\right) h_0^{(k)}h_0^{(k-1)}$ ,<br>
and from (18) with  $j = \lambda$ ,  $\lambda$  is a  $\lambda$ 

It remains to show that the homomorphism generated by (23) is one-to-one. Assume<br>
It remains to show that the homomorphism generated by (23) is one-to-one. Assume<br>
at this is not the case, i.e. that for an element  $a + bh + 0$ that this is not the case, i.e. that for an element  $a + bh + 0$  in  $D_0$  the image  $c + dh_0$  in  $\vec{A}$  is vanishing. Let  $c = a_{00}$ ,  $d = b_{00}$  as before. Then according to  $\delta = h\delta + \delta h$ , (19) *(A6)* the mapping (23) the normal form  $c + dh_0$  with polynomials<br>
ons (22) show that these polynomials are always values of<br> *Chence, every element of A* is a value of an element of  $D_0$ .<br> *Chence, every element of A* is

$$
0 = (a_{00} + b_{00}h_0)\delta = a_0\delta + b_{00}h\delta = (a_0 + b_0)\delta - b_0\delta h,
$$

and this implies  $b_0 = a_0 = 0$ . The polynomial *c* has the form

$$
c = \sum \gamma_m \delta_0^{(k_{m1})} \delta_0^{(k_{m2})} \dots \delta_0^{(k_{mn})}
$$

so that according to (22)

$$
a_0 = \sum \gamma_m(\dots (\delta^{(k_{m1})}\delta)^{(k_{m2})}\dots \delta)^{(k_{mn})} = 0.
$$

 $\begin{aligned} 0&=(a_{\bf 00}+b_{\bf 00}h)\ \text{implies}\ b_{\bf 0}^{\prime}&=a_{\bf 0}\ c&=\sum \gamma_m\delta_{\bf 0}^{(k_{\bf m1})}\ \text{according to (22)}\ a_{\bf 0}&=\sum \gamma_m(\ldots)\ \text{ng}\ \text{ in this sum} \end{aligned}$ Considering in this sum the addends with maximal  $n_m$ , from these once more the addends with maximal  $k_{mn_m}$ , from these the addends with maximal  $k_{m,n_m-1}$  etc., we find an addend, from which the term *ng* in this sum<br>with maximal<br>ddend, from wl<br> $\gamma_m \delta^{(k_m)} \delta^{(k_m)}$ ...

$$
\nu_m \delta^{(k_{m1})} \delta^{(k_{m2})} \ldots \delta^{(k_{mn})}
$$

appearing after evaluating the brackets in  $(27)$  cannot be canceled against another term in the sum. Hence, this special  $\gamma_m$  must vanish, and therefore all  $\gamma_m$  must vanish. This means that c is the zero-polynomial and so must be a. Analogously, *d* is the zeropolynomial and *b*, too, but this contradicts the assumption  $a + bh + 0$ . So Theorem 4 is proved  $\blacksquare$  $c = \sum \gamma_m \delta_0^{(k_{m1})} \delta_0^{(k_{m2})} \dots \delta_0^{(k_{mn})}$ ,<br>
so that according to (22)<br>  $a_0 = \sum \gamma_m (\dots (\delta^{(k_{m1})}\delta)^{(k_{m1})} \dots \delta)^{(k_{mn_n})}$ <br>
Considering in this sum the addends with n<br>
addends with maximal  $k_{mn_m}$ , from these the<br>
find an a Considering in this sum the addends with maximal  $n_m$ , from these once more<br>addends with maximal  $k_{m,n_m}$ , from these the addends with maximal  $k_{m,n_m-1}$  etc.,<br>find an addend, from which the term<br> $\gamma_m \delta^{(k_{m1})} \dots \delta^{(k_{mn})}$ 

The algebra  $D_0$  is a distribution algebra, but the algebra  $A$  not, though it is in view of (13) a trivial differential algebra. The isomorphism (23) allows to transfer all relations from  $D_0$  into relations in *A*. For example from  $h\delta^2 = \delta^2 h$ ,  $h\delta h = 0$  we immediately obtain  $h_0\delta_0^2 = \delta_0^2 h_0$ ,  $h_0\delta_0 h_0 = 0$ .

 

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 $\frac{1}{2}\sum_{\mathbf{k}}\left(\frac{\mathbf{k}}{2}\right)^{2}$ 

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Manuskripteingang: 06. 09. 1982<br>
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