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Taylor's Expansion in a Distribution Algebra

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Für eine gewisse Distributionenalgebra wird gezeigt, daß jedes Element der Algebra einen Wert besitzt sowie durch seine Taylorentwicklung dargestellt wird. Außerdem wird die Struktur der Teilalgebra der Werte untersucht.

Для некоторой дистрибуционной алгебры показывается, что каждый элемент алгебры имеет значение и представим разложением Тейлора. Кроме того исследуется структура подалгебры значений.

For a certain distribution algebra it is shown that every element of the algebra possesses a value and is representable by its Taylor's expansion. Moreover there is investigated the structure of the subalgebra of values.

By definition, a distribution algebra is an associative, but noncommutative differential algebra with at least one element h satisfying

$$h^2 = h \tag{1}$$

and $h' \neq 0$. A good survey on some classes of distribution algebras with additional properties was given by Ma Kyin Myint [3]. Here we go back to the more general distribution algebras of [1].

The aim of this paper is to show that every element of the latter distribution algebras is representable by its Taylor's expansion, using the notion of values of such elements introduced in [2]. As a consequence we are able to determine the structure of the set of these values.

Preliminaries

In [1] there was considered the distribution algebra D_1 with unit element 1 generated by two elements t, h with the properties

$$t' = 1, \quad th' = 0 \tag{2}$$

and, of course, (1). The ring of scalars of this algebra is assumed to contain the rational numbers. According to Dirac the derivative h' is denoted by δ, the more as this element shall be interpreted as a Schwartz distribution. As a consequence of (2), in D there are valid the well known relations

$$t^n\delta^{(m)} = (-1)^n\, n! \binom{m}{n} \delta^{(m-n)} \tag{3}$$

for all integers $m, n \geqq 0$ with $\binom{m}{n} = 0$ for $m < n$. As a consequence of (1), in D_1

there are valid the relations

$$h^{(m)} = \sum_{n=0}^{m} \binom{m}{n} h^{(n)} h^{(m-n)}, \tag{4}$$

too, where the products on the right-hand side are no Schwartz distributions for $m > 1$.

In [1] there was also constructed the extension D_2 of D_1 as the distribution algebra of the elements

$$f = \sum_{n=0}^{\infty} (a_n t^n + b_n t^n h), \tag{5}$$

where a_n, b_n are polynomials in $\delta^{(k)}$ $(k = 0, 1, 2, \ldots)$, which are determined uniquely by f and can be prescribed arbitrarily. With the elements (5) all operations in D_2 are to be carried out termwise. In D_2 two elements f, g are defined to be equal if and only if

$$(f - g)\,\delta^{(m)} = 0$$

for $m = 0, 1, 2, \ldots$ According to this definition a sequence f_n is defined to converge to the limit $f \in D_2$, if for every fixed $m \geqq 0$ there is

$$(f - f_n)\,\delta^{(m)} = 0$$

for suitable large n. With these definitions e.g. the equations

$$t^i h t^k = t^{i+k} h + k \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(i+j)(i+k+j)(j-1)!} \delta^{(j-1)} t^{i+k+j} \tag{6}$$

are valid in D_2.

Series

In what follows we need series with more general coefficients than in (5). For this case we consider the

Theorem 1: *For an arbitrary sequence f_k of elements from D_2 the series*

$$f = \sum_{k=0}^{\infty} f_k t^k \tag{7}$$

is always converging to an element from D_2.

Proof: According to (5) all elements f_k possess representations of the form

$$f_k = \sum_{i=0}^{\infty} (a_{ki} t^i + b_{ki} t^i h).$$

Hence in view of (6) we have

$$f_k t^k = \sum_{i=0}^{\infty} \left(a_{ki} t^{i+k} + b_{ki} \left(t^{i+k} h + k \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(i+j)(i+k+j)(j-1)!} \delta^{(j-1)} t^{i+k+j} \right)\right).$$

Summing over k and substituting $i + k = n$ and $i + k + j = n$, respectively, we find that the element f from (7) possesses also the form (5) with

$$a_n = \sum_{i=0}^{n} \left(a_{n-i,i} + \sum_{j=1}^{n-i} b_{n-i-j,i} \frac{(n-i-j)(-1)^{j-1}}{(i+j)\,n(j-1)!} \delta^{(j-1)} \right) \quad \text{and} \quad b_n = \sum_{i=0}^{n} b_{n-i,i},$$

i.e. (7) is an element from D_2. All foregoing calculations with infinite series are possible in view of the definition of convergence, because all series terminate after multiplication by $\delta^{(m)}$, so that in fact no convergence problems arise ∎

Values

In [2] there were considered the series

$$f_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)} t^k \tag{8}$$

for some elements f of D_2'. Since by Taylor's theorem for holomorphic functions $f(t)$ and small numbers t the series (8) is equal to the value $f(0)$, the element f_0 from (8) is named the (*generalized*) *value of* f (at the point $t = 0$). In [2] there was already proved that $t_0 = 0$ and $(t^n h)_0 = 0$ for all natural numbers n,

$$h_0^2 = h_0, \qquad h_0 t = t h_0, \qquad \delta_0 t = t\delta_0, \tag{9}$$

$$\delta_0^{(m)} t - t\delta_0^{(m)} = m\delta_0^{(m-1)} \tag{10}$$

for $m \geqq 1$ as well as

$$\delta^{(n)}\delta^{(m)} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \delta_0^{(n+i)} \delta^{(m-i)}, \tag{11}$$

$$(\delta^{(n)}\delta^{(m)})_0 = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \delta_0^{(n+i)} \delta_0^{(m-i)} \tag{12}$$

or $m, n \geqq 0$ and

$$(f_0') = 0 \tag{13}$$

for $f = h^{(n)}$, $n \geqq 0$.

Theorem 2: *The value* (8) *exists for every element* $f \in D_2'$ *and has always the property* (13). *The value-operator is linear, i.e. it satisfies*

$$(\alpha f + \beta g)_0 = \alpha f_0 + \beta g_0 \tag{14}$$

in the case $\alpha' = \beta' = 0$. *Further relations for arbitrary* $f, g \in D_2$ *are*

$$(fg)_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)} g_0 t^k, \tag{15}$$

$$(ft)_0 = 0, \qquad (fh)_0 = f_0 h_0, \qquad (f\delta)_0 = f_0 \delta_0. \tag{16}$$

Proof: Since the series in (8) possesses the form (7), Theorem 1 implies the first assertion. The equations (13) and (14) are easily verified from (8). The equation

$$(fg)_0 = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (fg)^{(i)} t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{k=0}^{i} \binom{i}{k} f^{(k)} g^{(i-k)} t^i$$

$$= \sum_{k=0}^{\infty} \sum_{i=k}^{\infty} \frac{(-1)^i}{k!(i-k)!} f^{(k)} g^{(i-k)} t^i = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} g^{(j)} t^{j+k}$$

with $j = i - k$ shows in view of (8) with g instead of f that (15) is valid, too. From (15) and $t_0 = 0$ as well as (9) it follows (16). Hence Theorem 2 is proved ∎

Of course, both scalar multiples of the unit element 1 and the values themselves are their own values. These statements and the equations (13) and (14) show that the values (8) behave like ordinary constants in calculus. Equation (12) is a special application of (14) to equation (11). Note that formula (14) is, generally speaking, not valid in the case that the constants are standing at the right of the elements. A general consequence of (15) is the equation $(fg)_0 = (fg_0)_0$.

Taylor's expansion

Now we come to another property which underlines the right of terming the elements (8) *values*.

Theorem 3: *All elements $f \in D_2$ are respresentable by Taylor's expansion*

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} f_0^{(n)} t^n, \tag{17}$$

and they satisfy for arbitrary integers $m \geqq 0$ the equation

$$f\delta^{(m)} = \sum_{n=0}^{m} (-1)^n \binom{m}{n} f_0^{(n)} \delta^{(m-n)}. \tag{18}$$

Proof: Multiplying (17) by $\delta^{(m)}$ and considering (3) it follows (18). Hence, in view of the definition of equality in D_2 it suffices to prove (18). Applying (8) to $f^{(n)}$ instead of f we obtain

$$f_0^{(n)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(n+k)} t^k,$$

hence, in view of (3)

$$f_0^{(n)} \delta^{(m-n)} = \sum_{k=0}^{m-n} \binom{m-n}{k} f^{(n+k)} \delta^{(m-n-k)} = \sum_{j=n}^{m} \binom{m-n}{j-n} f^{(j)} \delta^{(m-j)}$$

with $j = n + k$ and therefore

$$\sum_{n=0}^{m} (-1)^n \binom{m}{n} f_0^{(n)} \delta^{(m-n)} = \sum_{j=0}^{m} \sum_{n=0}^{j} (-1)^n \binom{m}{n} \binom{m-n}{j-n} f^{(j)} \delta^{(m-j)}.$$

According to $\binom{m}{n}\binom{m-n}{j-n} = \binom{m}{j}\binom{j}{n}$ and

$$\sum_{n=0}^{j} (-1)^n \binom{j}{n} = \begin{cases} 1 & \text{for} \quad j = 0, \\ 0 & \text{for} \quad j > 0, \end{cases}$$

the foregoing sum is equal to $f\delta^{(m)}$ and Theorem 3 is proved ∎

For holomorphic functions f equation (18) is well known in the theory of Schwartz distributions. Equation (11) is a special case of (18) with $f = \delta^{(n)}$. Equation (18) reads for $m = 0$

$$f\delta = f_0\delta. \tag{19}$$

Conversely, if

$$f\delta = c\delta \tag{20}$$

with $c' = 0$, then it follows $c = f_0$, because (19) and (20) imply $(f_0 - c)\,\delta = 0$ and therefore $(f_0 - c)\,\delta^{(n)} = 0$ for every $n \geqq 0$, so that we obtain the assertion $f_0 = c$. Differentiating equation (19) n-times and considering the value of the result we obtain

$$f_0\delta_0^{(n)} = (f\delta)_0^{(n)}. \tag{21}$$

This equation implies inductively

$$\left.\begin{aligned} &f_0\delta_0^{(m)}\delta_0^{(n)} = \big((f\delta)^{(m)}\delta\big)_0^{(n)}, \\ &f_0\delta_0^{(k)}\delta_0^{(m)}\delta_0^{(n)} = \big(((f\delta)^{(k)}\,\delta)^{(m)}\,\delta\big)_0^{(n)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{22}$$

It should be possible to use Taylor's expansion also to define values of elements of D_2 in a point different from $t = 0$, however, up to now this is not worked out.

An isomorphism

The following theorem clarifies the structure of the set of values.

Theorem 4: *The values* (8) *form a subalgebra A of D_2, which is generated by the special values* 1 *and $h_0^{(n)}$ $(n \geqq 0)$, and which is isomorphic to the subalgebra D_0 generated by the elements* 1 *and $h^{(n)}$ $(n \geqq 0)$. The isomorphism $D_0 \to A$ is generated by the mapping*

$$1 \to 1, \qquad h^{(n)} \to n!\,h_0^{(n)}. \tag{23}$$

Proof: From (5), (16) and $t_0 = 0$ we find

$$f_0 = a_{00} + b_{00}h_0 \tag{24}$$

(the first zero is the index $n = 0$ from (5), the second zero indicates the value operation). Since (14) and (18) imply

$$(f\delta^{(m)})_0 = \sum_{n=0}^{m} (-1)^n \binom{m}{n} f_0^{(n)}\delta_0^{(m-n)}$$

and a_0, b_0 are polynomials in $\delta^{(k)}$, the values a_{00} and b_{00} are polynomials in $\delta_0^{(k)}$. Hence, (24) implies that all values f_0 are contained in the algebra A generated by $h_0^{(n)}$.

Next we derive from (4)

$$h_0^{(m)} = (hh^{(m)})_0 + \sum_{n=1}^{m-1} \binom{m}{n} (h^{(n)}h^{(m-n)})_0 + (h^{(m)}h)_0. \tag{25}$$

Considering $\delta = h'$, equation (12) can be written in the form

$$(h^{(n)}h^{(m)})_0 = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} h_0^{(n+i)}h_0^{(m-i)}$$

with $m, n \geqq 1$, so that with $j = n + i$ we have

$$\begin{aligned} \sum_{n=1}^{m-1} \binom{m}{n} (h^{(n)}h^{(m-n)})_0 &= \sum_{n=1}^{m-1} \sum_{i=0}^{m-n-1} \binom{m}{n} (-1)^i \binom{m-n-1}{i} h_0^{(n+i)}h_0^{(m-n-i)} \\ &= \sum_{n=1}^{m-1} \sum_{j=n}^{m-1} (-1)^{n+j} \binom{m}{n} \binom{m-n-1}{j-n} h_0^{(j)}h_0^{(m-j)}. \end{aligned}$$

According to

$$\sum_{n=1}^{j} (-1)^{n+j} \binom{m}{n} \binom{m-n-1}{j-n} = 1 - \binom{m-1}{j} (-1)^j,$$

this implies

$$\sum_{n=1}^{m-1} \binom{m}{n} (h^{(n)} h^{(m-n)})_0 = \sum_{j=0}^{m-1} \left(1 - \binom{m-1}{j} (-1)^j\right) h_0^{(j)} h_0^{(m-j)},$$

and from (18) with $f = h$, $\delta = h'$, (16) and (25) we obtain equation

$$h_0^{(m)} = \sum_{n=0}^{m} h_0^{(n)} h_0^{(m-n)}. \tag{26}$$

Obviously, the mapping (23) transfers equation (4) into equation (26). Since D_0 is nothing else than the free algebra generated by $h_0^{(n)}$, which is endowed with the relations (4) (cf. [1]), and since the corresponding relations (26) under the mapping (23) are also valid, D_0 has at least a homomorphic image in A. The fact that all elements of D_0 possess the normal form $a + bh$ with polynomials a, b in $\delta^{(k)}$ implies that all elements of A possess by the mapping (23) the normal form $c + dh_0$ with polynomials c, d in $\delta_0^{(k)}$. The equations (22) show that these polynomials are always values of polynomials a_0, b_0 in $\delta^{(k)}$. Hence, every element of A is a value of an element of D_0.

It remains to show that the homomorphism generated by (23) is one-to-one. Assume that this is not the case, i.e. that for an element $a + bh \neq 0$ in D_0 the image $c + dh_0$ in A is vanishing. Let $c = a_{00}$, $d = b_{00}$ as before. Then according to $\delta = h\delta + \delta h$, (19) and (16) we have

$$0 = (a_{00} + b_{00} h_0)\, \delta = a_0 \delta + b_{00} h\delta = (a_0 + b_0)\, \delta - b_0 \delta h,$$

and this implies $b_0 = a_0 = 0$. The polynomial c has the form

$$c = \sum \gamma_m \delta_0^{(k_{m1})} \delta_0^{(k_{m2})} \ldots \delta_0^{(k_{mn_m})},$$

so that according to (22)

$$a_0 = \sum \gamma_m \left(\ldots (\delta^{(k_{m1})} \delta)^{(k_{m2})} \ldots \delta\right)^{(k_{mn_n})} = 0. \tag{27}$$

Considering in this sum the addends with maximal n_m, from these once more the addends with maximal k_{mn_m}, from these the addends with maximal k_{m,n_m-1} etc., we find an addend, from which the term

$$\gamma_m \delta^{(k_{m1})} \delta^{(k_{m2})} \ldots \delta^{(k_{mn_m})}$$

appearing after evaluating the brackets in (27) cannot be canceled against another term in the sum. Hence, this special γ_m must vanish, and therefore all γ_m must vanish. This means that c is the zero-polynomial and so must be a. Analogously, d is the zero-polynomial and b, too, but this contradicts the assumption $a + bh \neq 0$. So Theorem 4 is proved ∎

The algebra D_0 is a distribution algebra, but the algebra A not, though it is in view of (13) a trivial differential algebra. The isomorphism (23) allows to transfer all relations from D_0 into relations in A. For example from $h\delta^2 = \delta^2 h$, $h\delta h = 0$ we immediately obtain $h_0 \delta_0^2 = \delta_0^2 h_0$, $h_0 \delta_0 h_0 = 0$.

REFERENCES

[1] BERG, L.: Construction of distribution algebras. Math. Nachr. 82 (1978), 255–262.
[2] —: Values in a distribution algebra. Demonstratio Math. 12 (1979), 639–644.
[3] MA KYIN MYINT: Distribution Algebra (M. Sc.-Thesis). Rangoon: Arts and Science University 1979.

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