$\frac{1}{4}$

Modulation Spaces on the Euclidean *n***-Space**

H. **TRIEBEL**

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Die Arbeit bcschäftigt sich mit dem Problem der Spuren von Funktionen aus Modulationsräumen auf der Grundlage der Fourier-Analysis und der Maximalungleichungen. The Arbeit beschäftigt sich mit dem Problem der Spuren von Fur
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В работе рассматривается задача о следах функций из пространств модуляций на основании анализа Фурье и максимальных неравенств.

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1. Introduction

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$$
\mathbf{Z}_n = \{k \mid k \in \mathbf{R}_n, k = (k_1, ..., k_n), k_j \text{ integers} \} \tag{1}
$$

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<i>Z*iamen auf der Grundlage der Fourier-Analysis und der Maximalungleichungen.
 B paбore paccs are paracters agas a constrained and lution of unity in \mathbf{R}_n). Let $\mathfrak{a} = \{a_k\}_{k\in\mathbb{Z}_n}$ be a sequence of positive numbers. The paper deals with the problem of the traces of functions belonging to modulation spaces on

f Fourier analysis and maximal inequalities.

uction
 ≥ 0 be a compactly supported infinitely differentiable function on the
 n -sp $Z_n = \{k \mid k \in \mathbf{R}_n, k = (k_1, ..., k_n), k_j \text{ integ} \}$
 $= \varphi(x - k) \text{ with } k \in \mathbf{Z}_n. \text{ We assume that}$

mity in \mathbf{R}_n . Let $a = \{a_k\}_{k \in \mathbf{Z}_n}$ be a sequence

the spaces $B_{p,q}^a(\mathbf{R}_n)$ and $F_{p,q}^a(\mathbf{R}_n)$, which
 $\sum_{i \in \mathbf{Z}_n} a_k^q ||F^{-1}$

deals with the spaces $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ and $F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$, which are characterized by the quasinorms

$$
\left(\sum_{k\in\mathbb{Z}_n} a_k^q \left\|F^{-1}[\varphi_k F] \right\| L_p(\mathbf{R}_n) \right)^{1/q} \tag{2}
$$

and

$$
\left\| \left(\sum_{k \in \mathbb{Z}_n} a_k^q \left| (F^{-1}[\varphi_k F f] (\cdot) \right|^q \right)^{1/q} \middle| L_p(\mathbf{R}_n) \right\| \tag{3}
$$

respectively. Here $0 < p, q \leq \infty$ (with $p < \infty$ in the case of the spaces $F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$). If and F^{-1} stand for the Fourier transform and its inverse, respectively, on the Schwartz space $S(R_n)$. For our purpose it is sufficient to assume that f belongs to $S(R_n)$ with supp Ff compact. By a suggestion of H. G. Feichtinger we denote $B_{p,q}^{\alpha}(\mathbf{R}_n)$ and $F_{p,q}^{\alpha}(\mathbf{R}_n)$ as modulation spaces. The main aim of this paper is to study the trace problem: What can be said about the trace as modulation spaces. The main aim of this paper is to study the trace problem: What can be said about the trace operator R ,

$$
R: f(x) \rightarrow f(x', 0), \quad \text{where} \quad x = (x', x_n),
$$

as a mapping from $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ or $F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ onto corresponding spaces on \mathbf{R}_{n-1} ? Our main results are formulated in the Theorems 1 and 2, and in the Corollary. Furthermore, Theorem 3 contains a continuous version of (2) and (3). We give detailed proofs as far as the spaces $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ are concerned and outline the proofs for the (technically more complicated but also more interesting) spaces $F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$.

The interest on the above spaces comes from two quite different sources. $H. G.$ FEICHTINGER introduced in [2] spaces of Wiener type on locally compact abelian groups, cf. also [3, 4]. It comes out that $FB^{\mathfrak{a}}_{p,q}(\mathbf{R}_n)$ (the Fourier image of $B^{\mathfrak{a}}_{p,q}(\mathbf{R}_n)$) with $1 \leq p, q \leq \infty$ is a space of Wiener type on \mathbf{R}_n in the sense of Feichtinger. So one can try to look at spaces of type $B_{p,q}^a$ with $1 \leq p, q \leq \infty$ on locally compact abelian groups in the framework of the technique used there, cf. [5]. Our approach is restricted to \mathbf{R}_n , but includes the spaces $F_{p,q}^a(\mathbf{R}_n)$ and extends the range of p, q to $0 < p$, ded to \mathbf{R}_n , but includes the spaces $F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ and extends the range of p, q to $0 < p$
 $q \leq \infty$ $(p < \infty$ for $F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n))$. The other source is the spaces $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^s(\mathbf{R}_n)$ where again $0 < p, q \leq \infty$ and $-\infty < s < \infty$. These two scales include many wellknown spaces of functions and distributions, e.g. Holder-Zygmund spaces, Sobolev spaces, Besov spaces, Besse]-potential spaces, Hardy spaces and BMO. Let

$$
Q_k = \{x \mid x = (x_1, ..., x_n) \in \mathbb{R}_n, \ |x_j| \le 2^k\},
$$

$$
P_k = Q_{k+1} - Q_k \text{ if } k = 1, 2, 3, ... \text{ and } P_0 = Q_1.
$$

The corridors P_k with $k = 1, 2, \ldots$ can be divided in an obvious way in cubes with side-length 2^k . One obtains a covering of \mathbb{R}_n by "dyadic" cubes instead of a covering of **R**_n by congruent cubes, which underlies (2) and (3). Roughly speaking, the quasi-
norms of the spaces $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^s(\mathbf{R}_n)$ are defined in the same way as in (2) and ted to \mathbf{R}_n , but includes the spaces $F_{p,q}^a(\mathbf{R}_n)$ and extends the range of p, q to $\mathbf{V} = q$.
 $q \leq \infty$ ($p < \infty$ for $F_{p,q}^a(\mathbf{R}_n)$). The other source is the spaces $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^s(\mathbf{R}_n)$ of \mathbf{R}_n by congruent cubes, which underlies (2) and (3). Roughly speaking, the quasi-
norms of the spaces $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^s(\mathbf{R}_n)$ are defined in the same way as in (2) and
(3), respectively, where the from the just-described standpoint has been developed in the three books **[13, 15, 161.** It seems to be natural to ask whether the covering of \mathbf{R}_n by dyadic cubes can be replaced by other coverings. The first question is for what coverings of \mathbf{R}_n (or locally compact abelian groups etc.) definitions of type (2), **(3)** make sense. Considerations what coverings are admissible can be found in **[14:** Chapter 2] and [1, *7],* the latter one in locally compact spaces. The second question is whether one can characterize elements of corresponding B-spaces and/or F-spaces and their quasi-norms in other terms, e.g. via differences and derivatives of functions, approximation procedures or as traces of harmonic functions or temperatures (just as in the case of the spaces $B_{p,q}^{s}(\mathbf{R}_{n})$ and $F_{p,q}^{s}(\mathbf{R}_{n})$. Some work in this direction has been done. Beside [14: Chapter 21 we refer to the papers by M. L. **OOL'DMAN** [8, 91, G. A. **KAIJABnc** [11, 121 and *S. JANSON* [10]. The feeling is that some regularity assumptions for the admissible coverings of R , are necessary in order to get substantial results (cf. the papers by Kaljabin, Gol'dman and Janson). Probably the congruent covering is a limiting case for that purpose and of peculiar interest may be coverings which are "between" the congruent and the dyadic covering or coverings where the cubes (or more general, rectangles) grow even more rapid than in the dyadic case. In this sense this paper is also a contribution to the study of the limiting case "congruent covering". In particu-Interpendent and the dyadic covering or coverings where the cubes (or more general tangles) grow even more rapid than in the dyadic case. In this sense this paper is to a contribution to the study of the limiting case "co

lar we wish to show what is different in comparison with the "dyadic covering".
As far as the above spaces $B_{p,q}^{\alpha}(\mathbf{R}_n)$ with $1 \leq p \leq \infty$ (and $0 < q \leq \infty$) are concerned we refer to M. L. Gol'dman's paper [9]. His approach covers these spaces as a special case. Compare in particular Theorem I and the Corollary below with Theorem *7,* remark **1** on p. 57, and Subsection **4.1** in [9].

Adnowlcdgement: The first draft of this paper has been written during a visit of the author in November 1981 in Vienna. I take the opportunity to thank my colleagues in Austria for their hospitality and Dr. H. G. Feichtinger and P. Grobner from the University of Viennafor stimulating discussions about the subject of this paper. As an outgrowth we planned this paper and [5] (cf. also [6], which is a survey, including a description of further research).

2. Definitions and Preliminaries

As in the Introduction, \mathbf{R}_n stands for the euclidean *n*-space and the lattice \mathbf{Z}_n is given by (1). Let $S(R_n)$ be the Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions on \mathbf{R}_n . The Fourier transform and its inverse on $S(\mathbf{R}_n)$ are denoted by F and F^{-1} , respectively. Let $S^c(\mathbf{R}_n)$ be the collection of all $f \in S(\mathbf{R}_n)$ such that Ff has a compact support. If $0 < p < \infty$ then

$$
||f| L_p(\mathbf{R}_n)|| = \left(\int_{\mathbf{R}_n} |f(x)|^p dx\right)^{1/p}
$$

and $||f||L_\infty(\mathbf{R}_n)|| = \sup |f(x)|.$ Finally, let $\{\varphi_k(x)\}_{k\in \mathbf{Z}_n}$ be essentially a smooth resolution $x \in \mathbb{R}$ $||f| L_p(\mathbf{R}_n)|| = \left(\int_{\mathbf{R}_n} |f(x)|^p dx\right)^{1/p}$

and $||f| L_\infty(\mathbf{R}_n)|| = \sup_{x \in \mathbf{R}_n} |f(x)|$. Finally, let $\{\varphi_k(x)\}_{k \in \mathbf{Z}_n}$ be essentially a smooth reso

of unity related to the lattice \mathbf{Z}_n , i.e.:

(i) $\varphi_k(x) \in S(\mathbf{R$ such that Ff has a compact support. If $0 < p < \infty$ then
 $||f| L_p(\mathbf{R}_n)|| = \left(\int_{\mathbf{R}_n} |f(x)|^p dx\right)^{1/p}$
 $\infty(\mathbf{R}_n)|| = \sup_{x \in \mathbf{R}_n} |f(x)|$. Finally, let $\{\varphi_k(x)\}_{k \in \mathbb{Z}_n}$ be essentially a smooth resolution

related to the

(i)
$$
\varphi_k(x) \in S(\mathbf{R}_n)
$$
, $\text{supp } \varphi_k \subset \{y \mid y = (y_1, \ldots, y_n) \in \mathbf{R}_n, |y_j - k_j| \leq 1\}$

of unity related to the lattice \mathbf{Z}_n , i.e.:

(i) $\varphi_k(x) \in S(\mathbf{R}_n)$, supp $\varphi_k \subset$

where $k = (k_1, ..., k_n) \in \mathbf{Z}_n$,

(ii) for every multi-index ν there exis

(ii) for every multi-index
$$
\gamma
$$
 there exists a number c_{γ} such that

$$
|D^r \varphi_k(x)| \leq c, \quad \text{for all} \quad k \in \mathbb{Z}_n,
$$

(iii) there exists a positive number *c* such that

$$
c \leq \sum_{k \in \mathbb{Z}_n} \varphi_k(x) \quad \text{for all} \quad x \in \mathbf{R}_n.1 \tag{4}
$$

(i) $\varphi_k(x) \in S(\mathbf{R}_n)$, $\text{supp } \varphi_k \subset \{y \mid y = (y_1, \ldots, y_n) \in \mathbf{R}_n, \ |y_j - k_j| \leq 1\}$

where $k = (k_1, \ldots, k_n) \in \mathbf{Z}_n$,

(ii) for every multi-index γ there exists a number c_r such that
 $|D^r \varphi_k(x)| \leq c_r$ for all $k \in \math$ bit more general version (4) is convenient for us. $\{\varphi_k(x)\}_{k\in \mathbb{Z}_n}$ with (i)-(iii) we call *admissible systems.* In we have a smooth resolution of unity.

rersion (4) is convenient for us. $\{\varphi_k(x)\}_k$

Let $a = \{a_k\}_{k \in \mathbb{Z}_n}$ be a sequence of positive
 $\frac{a_k}{a_k} \leq c_2 < \infty$ for all $k \in \mathbb{Z}_n$ and \tilde{k} $c \leq \sum_{k \in \mathbb{Z}_n} \varphi_k(x)$ for all $x \in \mathbb{R}_n$.¹)
 $c) \equiv 1$ then we have a smooth resolution of unity. For t

general version (4) is convenient for us. $\{\varphi_k(x)\}_{k \in \mathbb{Z}_n}$ w
 ke systems.

ition 1: Let $a = \{a_k\}_{k \in \$ e exists a positive number c s
 $c \leq \sum_{k \in \mathbb{Z}_n} \varphi_k(x)$ for all $x \in \mathbb{R}_n$
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 $0 < c_1 \leq \frac{a_k}{a_k} \leq c_$ *Ill in the net as mooth resolution of unit*
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 Ill ion 1: Let $a = \{a_k\}_{k \in \mathbb{Z}_n}$ be a sequence of posit
 $0 < c_1 \leq \frac{a_k}{a_k} \leq c_2 < \infty$ for all $k \in \mathbb{Z}_n$ and

with $|k - \tilde{k}| =$

Definition 1: Let $a = \{a_k\}_{k \in \mathbb{Z}_n}$ be a sequence of positive numbers with

$$
0 < c_1 \leq \frac{a_k}{a_{\tilde{k}}} \leq c_2 < \infty \quad \text{for all} \quad k \in \mathbb{Z}_n \quad \text{and} \quad \tilde{k} \in \mathbb{Z}_n
$$
\nwith $|k - \tilde{k}| = 1$,\nand c_2 are two suitable positive numbers. Let $\varphi = \{\varphi_k(x)\}_{k \in \mathbb{Z}_n}$ be an admissible in the above sense.\n\n $0 < p \leq \infty$ and $0 < q \leq \infty$. Then $S^c(\mathbb{R}_n)$ equipped with the quasi-norm $||f| B_{p,q}^a(\mathbb{R}_n)||^p = \left(\sum_{k \in \mathbb{Z}_n} a_k^q ||F^{-1}[\varphi_k F f]|| L_p(\mathbb{R}_n||^q)^{1/q} \right)$ \n\nand as $B_{p,q}^a(\mathbb{R}_n)$ (usual modification if $q = \infty$).\n\nIt $0 < p < \infty$ and $0 < q \leq \infty$. Then $S^c(\mathbb{R}_n)$ equipped with the quasi-norm $||f| F_{p,q}^a(\mathbb{R}_n)||^p = ||\left(\sum_{k \in \mathbb{Z}_n} a_k^q |F^{-1}[\varphi_k F f] (\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}_n)|| \qquad (7)$ \n\nand as $F_{p,q}^a(\mathbb{R}_n)$ (usual modification if $q = \infty$).

where c_1 and c_2 are two suitable positive numbers. Let $\varphi = {\varphi_k(x)}_{k \in \mathbb{Z}_n}$ be an admissible system in the above sense.

Given in the above sense.

\n(i) Let
$$
0 < p \leq \infty
$$
 and $0 < q \leq \infty$. Then $S^c(\mathbf{R}_n)$ equipped with the quasi-norm

\n
$$
||f| B_{p,q}^a(\mathbf{R}_n)||^p = \left(\sum_{k \in \mathbb{Z}_n} a_k^q \left[|F^{-1}[\varphi_k F f] \right] | L_p(\mathbf{R}_n||^q)^{1/q} \right)
$$

\n(6)

(ii) Let $0 < p < \infty$ and $0 < q \leq \infty$. Then $S(\mathbf{R}_n)$ equipped with the quasi-norm

is denoted as
$$
B_{p,q}^a(\mathbf{R}_n)
$$
 (usual modification if $q = \infty$). \n(ii) Let $0 < p < \infty$ and $0 < q \leq \infty$. Then $S^c(\mathbf{R}_n)$ equipped with the quasi-norm $||f| F_{p,q}^a(\mathbf{R}_n)||^q = \left\| \left(\sum_{k \in \mathbb{Z}_n} a_k^q |F^{-1}[\varphi_k F f] (\cdot)|^q \right)^{1/q} \left| L_p(\mathbf{R}_n) \right| \right\}$ (7)

is denoted as $F_{p,q}^{\alpha}(\mathbf{R}_n)$ (usual modification if $q = \infty$).

Remark 1: For sake of brevity we write $F^{-1}\varphi_kFf$ instead of $F^{-1}[\varphi_kFf]$ in the sequel. Of course, (6) and (7) make sense, in particular, by the Paley-Wiener-theorem, $F^{-1}\varphi_k F f$ is an analytic function (belonging to $S(\mathbf{R}_n)$). Of course, (6) and (7) are quasi-
norms (norms if $p \ge 1$ and $q \ge 1$). We recall that a quasi-norm $\|\cdot\|$ has all the propis denoted as $F_{p,q}^{\dagger}(\mathbf{h}_n)$ (usual modification if $q = \infty$).

Remark 1: For sake of brevity we write $F^{-1}\varphi_k Ff$ instead of $F^{-1}[\varphi_k Ff]$ in the sequel. Of course, (6) and (7) make sense. In particular, by the Pale

¹) All unimportant positive numbers are denoted by $c, c_1, ..., c'$, ... where the numerical values of these numbers may differ from formula to formula.

erties of a norm, only the triangle inequality is replaced by

a norm, only the triangle in
$$
||h_1 + h_2|| \leq c(||h_1|| + ||h_2||)
$$
, ≥ 1 is independent of h and h .

where $c \geq 1$ is independent of h_1 and h_2 .

Remark 2: For sake of simplicity we restrict our considerations to $S^{\rm c}({\bf R_n}).$ The idea is to extend the definitions of $B_{p,q}^{\alpha}(\mathbf{R}_n)$ and $F_{p,q}^{\alpha}(\mathbf{R}_n)$ to suitable distributions from $S'(\mathbf{R}_n)$ or from other appropriate spaces of distributions. (From that point of view it would, be better to denote $S^{c}(\mathbf{R}_{n})$ equipped with (6) by $\hat{B}_{p,q}^{a}(\mathbf{R}_{n})$, etc.) To find suitable distribution spaces for an extended definition of $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ and $F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ is a somewhat delicate question. A detailed discussion may be found in [9], cf. also [14:
2.2.3]. However it is quite clear that the assertions of this paper can be extended via $|h_1 + h_2| \le c(||h_1|| + ||h_2||)$,

where $c \ge 1$ is independent of h_1 and h_2 .

Remark 2: For sake of simplicity we restrict our considerations to $S^c(\mathbf{R}_n)$. The

idea is to extend the definitions of $B_{p,q}^a(\mathbf{R}_n)$ erties of a norm, only the triangle inequality is replaced by
 $||h_1 + h_2|| \leq c(||h_1|| + ||h_2||)$,

where $c \geq 1$ is independent of h_1 and h_2 .

Remark 2: For sake of simplicity we restrict our consideration

idea is to ex

Remark 3: As has been said in the introduction the dyadic counterpart of $B_{p,q}^a(\mathbf{R}_n)$ and $F_{n,q}^{\alpha}(\mathbf{R}_n)$ are the spaces $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^s(\mathbf{R}_n)$, respectively, $-\infty < s < \infty$. For these spaces we have an elaborated theory at hand, cf. 115, 16]. Some parts of that theory are independent of the underlying covering of R_n if it is regular enough (e.g. congruent or dyadic coverings). Tn particular, for given *p's* and *q's* the quasi-norms (6) for all admissible systems φ are pairwise equivalent (i.e. for two admissible systems φ the quotients of the corresponding quasi-norms (6) can be estimated from above and from below by positive constants, which are independent of the elements of $S^c(\mathbf{R}_n)$. Similarly for the quasi-norms (7). In this sense we write $||f| B_{p,q}^a(\mathbf{R}_n)||$ and $||f| \, F_{p,q}^{\alpha}(\mathbf{R}_n)||$ instead of $||f| \, B_{p,q}^{\alpha}(\mathbf{R}_n)||^{\varphi}$ and $||f| \, F_{p,q}^{\alpha}(\mathbf{R}_n)||^{\varphi}$, respectively, in the sequel. Furthermore, it is no problem to prove Fourier multiplier theorems and maximal inequalities for the spaces $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ and $F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ in the same way as this has been
done in [16] for the spaces $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^s(\mathbf{R}_n)$ (cf. the Proposition below). Other properties are different, e.g. the traces on hyperplanes, which is the subject of this paper. *(***)**. Similarly for the quasi-norms (7). In this sense we write $||f|B_{p,q}^{\alpha}(\mathbf{R}_n)||$
 $\int_{p,q}^{\infty}(\mathbf{R}_n)||$ instead of $||f|B_{p,q}^{\alpha}(\mathbf{R}_n)||^{\sigma}$ and $||f|F_{p,q}^{\alpha}(\mathbf{R}_n)||^{\sigma}$, respectively, in the
 *pr*thermore, it *(f)* for the spaces $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^s(\mathbf{R}_n)$ (cf. the Proposition below). Other

if if for the spaces $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^s(\mathbf{R}_n)$ (cf. the Proposition below). Other

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Definition 2 *(Maximal functions)*: Let $b > 0$, $d > 0$ and $f \in S(\mathbf{R}_n)$. Let $\{\varphi_k(x)\}_{k \in \mathbf{Z}_n}$. $\frac{1}{2}$ be a set of and

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\ntion 2 (*Maximal functions*): Let
$$
b > 0
$$
, $d > 0$ and $f \in S(\mathbf{R}_n)$. Let $\{\varphi_k(x)\}_{k \in \mathbb{Z}_n}$
\nisisible system in the above sense. Then
\n
$$
\varphi_k f)^*(x) = \sup_{y \in \mathbf{R}_n} \frac{|(F^{-1}\varphi_k Ff)(x - y)|}{1 + |y|^b}, \quad x \in \mathbf{R}_n, \quad k \in \mathbb{Z}_n
$$
\n(8)

Definition 2 (Maximal functions): Let
$$
b > 0
$$
, $d > 0$ and $f \in S(\mathbf{R}_n)$. Let $\{\varphi_k(x)\}_{k \in \mathbf{Z}_n}$ be an admissible system in the above sense. Then

\n
$$
(\varphi_k f)^* (x) = \sup_{y \in \mathbf{R}_n} \frac{|(F^{-1}\varphi_k Ff)(x - y)|}{1 + |y|^b}, \quad x \in \mathbf{R}_n, \quad k \in \mathbf{Z}_n \tag{8}
$$
\nand

\n
$$
(\varphi_k f)_n^* (x) = \sup_{t \in \mathbf{R}_n} \frac{|(F^{-1}\varphi_k Ff)(x', x_n - t)|}{1 + |t|^d}, \quad x = (x', x_n) \in \mathbf{R}_n, \quad k \in \mathbf{Z}_n. \tag{9}
$$

Remark 4: We have

$$
(\varphi_k f)^* (x) \geq (\varphi_k f)_n^* (x) \geq |(F^{-1} \varphi_k F f)(x)|, \qquad x \in \mathbb{R}_n.
$$
 (10)

Of course, $(\varphi_k f)^* (x)$ and $(\varphi_k f)_n^* (x)$ depend on *b* and *d*, respectively. However this is unimportant (under the restrictions formulated below). So we omit *b* and *d* as indices. Of course, (8) and (9) make also sense if (4) is not satisfied. course, $(\varphi_k f)^* (x)$ and $(\varphi_k f)_n^* (x)$ depend on *b* and *d*, respectively
important (under the restrictions formulated below). So we omit *l*
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Proposition: (i) $(\varphi_k f)^* (x) = \sup_{y \in \mathbf{R}}$

and
 $(\varphi_k f)_n^* (x) = \sup_{i \in \mathbf{R}}$

Remark 4: We have
 $(\varphi_k f)^* (x) \geq (\varphi_k f)^*$

Of course, $(\varphi_k f)^* (x)$ and

unimportant (under the

Of course, (8) and (9) m

Proposition: (i) Let

in (9). Then
 $\$ $(Ff) (x - y)$
 $\overline{f} + |y|^b$
 $\overline{f} + |t|^d$
 $\geq |(F^{-1}\varphi_k Ff)|$
 $\geq |(F^{-1}\varphi_k Ff)|$
 (x) depend on

sense if (4)
 $\leq \infty$ and 0
 $\left| \int_0^a f \right|^{1/a}$

in (8) and $d>\frac{1}{p}$

b) a minimum of
$$
(\varphi_k f)^* (x) = \sup_{y \in \mathbb{R}_n} \frac{|(F^{-1}\varphi_k F f)(x - y)|}{1 + |y|^b}, x \in \mathbb{R}_n, k \in \mathbb{Z}_n
$$
 (8)
\nand
\n $(\varphi_k f)^* (x) = \sup_{y \in \mathbb{R}_n} \frac{|(F^{-1}\varphi_k F f)(x - y)|}{1 + |y|^b}, x \in \mathbb{R}_n, k \in \mathbb{Z}_n$ (9)
\nRemark 4: We have
\n $(\varphi_k f)^* (x) \geq (\varphi_k f)_n^* (x) \geq |(F^{-1}\varphi_k F f)(x)|, x \in \mathbb{R}_n$. (10)
\nIf $(\varphi_k f)^* (x) \geq (\varphi_k f)_n^* (x)$ and $(\varphi_k f)_n^* (x)$ depend on b and d, respectively. However this is
\nunimportant (under the restrictions formulated below). So we omit b and d a is indices.
\nOf course, (8) and (9) make also sense if (4) is not satisfied.
\nProposition: (i) Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $b > \frac{n}{p}$ in (8) and $d > \frac{1}{p}$
\n $\therefore \left(\sum_{k \in \mathbb{Z}_n} a_k^q ||(\varphi_k f)^* |L_p(\mathbb{R}_n)||^q \right)^{1/q}$
\nand
\n $\left(\sum_{k \in \mathbb{Z}_n} a_k^q ||(\varphi_k f)_n^* |L_p(\mathbb{R}_n)||^q \right)^{1/q}$ (11)

are equivalent quasi-norms in $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ (modification if $q = \infty$). If (4) is not ensured *then the expressions from* (11) can be estimated from above by the quasi-norm in $B_{p,q}^a(\mathbf{R}_n)$.

Modul
 Let quasi-norms in $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ (modification if $q = \infty$),
 n the expressions from (11) can be estimated from above by the quality of $\lim_{n \to \infty} \log p \leq \infty$ and $0 < q \leq \infty$. Let $b > \frac{n}{\min(p, q)}$

(9). T *and (1>*

and

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 are equivalent quasi-norms in $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ (modification if $q = \infty$). If (4) is not ensured

then the expressions from (11) can be estimated from above by the quasi-norm in $B_{p,q}^{\mathfr$ $\sum_{k \in \mathbb{Z}_n} a_k^q \cdot |(\varphi_k f)^* (\cdot)|^q \Big)^{1/q} \left[L_p(\hat{\mathbf{R}}_n) \right]$ $\sum_{k \in \mathbb{Z}_n} a_k^q \cdot |(\varphi_k f)^* (\cdot)|^q)^{1/q} |L_p(\hat{\mathbf{R}}_n)|$
 $\sum_{k \in \mathbb{Z}_n} a_k^q |(\varphi_k f)_n^* (\cdot)|^q)^{1/q} |L_p(\mathbf{R}_n)|$ J (12)

are equivalent quasi-norms in $F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ (modification if $q = \infty$). If (4) is not ensured *then the expressions from* (12) *can be estimated from above by the quasi-norm in* $F_{p,q}^{\mathfrak{a}}(\mathbf{R_n})$.

Remark 5: The Proposition is an easy consequence of Theorem 1.4.1 and Theorem 1.6.2 in [16]. It is the counterpart of corresponding maximal inequalities for the spaces $B_{p,q}^s(\mathbf{R}_n)$ and $F_{p,q}^s(\mathbf{R}_n)$ in [15, 16].

3. Traces of $B_{p,q}^{\alpha}(\mathbf{R}_{n})$

3.1. Main Assertion

The trace operator *R* is given by

$$
Rf = f(x', 0)
$$
, where $f(x) \in S^{c}(\mathbf{R}_{n})$ and $x = (x', x_n)$, $x' \in \mathbf{R}_{n-1}$, (13)

3.1. Main Assertion

The trace operator R is given by
 $Rf = f(x', 0)$, where $f(x) \in S^c(\mathbb{R}_n)$ and $x = (x', x_n)$, $x' \in \mathbb{R}_{n-1}$. (13)

Our aim is to find a space $B_{p,q}^a(\mathbb{R}_{n-1})$ with a suitable sequence $a' = \{a_k'\}_{k' \in \math$ tive numbers such that *R* is a linear and bounded operator from the given space $B_{p,q}^{\alpha}(\mathbf{R}_n)$ onto $B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})$. Furthermore, *R* is called a *retraction* if there exists a linear and bounded operator *T* from $B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})$ into $B_{p,q}^{\alpha}(\mathbf{R}_n)$ such that *R* **B**^c_{*P*_{*R*}} (**R**_n)
 R F = *f*(*x'*, 0), where *f*(*x*) \in *S*^c(**R**_n) and *x s n s s s s t n a space B*²_{*P*_{*n*}</sup>_{*g*}(**R**_{n-1}) with a suitable *s* beers such that *R* is a l} *At* $f = f(x', 0)$, where $f(x) \in S^c(\mathbf{R}_n)$ and $x = (x', x_n)$, $x' \in \mathbf{R}_{n-1}$. (13)

is to find a space $B_{p,q}^c(\mathbf{R}_{n-1})$ with a suitable sequence $a' = \{a_k'\}_{k' \in \mathbf{Z}_{n-1}}$ of posi-

ibers such that *R* is a linear and bou

$$
RT = I \quad (\text{identity in } B_{p,q}^{\alpha'}(R_{n-1})) \tag{14}
$$

(In that case it is clear that *R* maps "onto".) We call *T* a coretraction (to the retraction *R)* or an extension operator.

Theorem 1: Let $0 < p \le \infty$, $\bar{p} = \min (1, p)$ and $0 < q \le \infty$. Let $a = \{a_k\}_{k \in \mathbb{Z}_n}$ be a sequence of positive numbers with (5). If $k \in \mathbb{Z}_n$ then we put $k = (k', k_n)$ with
 $k' \in \mathbb{Z}_{n-1}$ and $k_n \in \mathbb{Z}_1$. Let $k' \in \mathbb{Z}_{n-1}$,
 $a'_k = \inf_{l \in \mathbb{Z}_1} a_{(k',l)} > 0$ if $0 < q \leq \overline{p}$

and
 $a'_{k'} = \sum_{l=-\infty}^{\$ *a* $B_{p,q}^{\alpha}(\mathbf{R}_{n-1})$. Furthermore, *R* is called a *retraction* if there exists a linear ded operator *T* from $B_{p,q}^{\alpha}(\mathbf{R}_{n-1})$ into $B_{p,q}^{\alpha}(\mathbf{R}_n)$ such that $RT = I$ (identity in $B_{p,q}^{\alpha}(\mathbf{R}_{n-1})$). (14)

$$
a'_{k'} = \inf_{l \in \mathbb{Z}_1} a_{(k',l)} > 0 \quad \text{if} \quad 0 < q \leq \overline{p} \tag{15}
$$

and

$$
a'_{k'} = \inf_{l \in \mathbb{Z}_1} a_{(k',l)} > 0 \quad \text{if} \quad 0 < q \leq \overline{p} \tag{15}
$$
\n
$$
and
$$
\n
$$
a'_{k'} = \sum_{l = -\infty}^{\infty} a_{(k',l)}^{-\sigma} < \infty \quad \text{with} \quad \frac{1}{\sigma} = \frac{1}{\overline{p}} - \frac{1}{q} \quad \text{if} \quad \overline{p} < q \leq \infty. \tag{16}
$$
\n
$$
\text{Then } B_{p,q}^{\alpha'}(\mathbf{R}_{n-1}) \text{ is a space in the sense of Definition 1 (i). If there exists a positive num-
$$

ber A such that $a'_{k'} \geq A a_{k'}$ *on for all k'* $\in \mathbb{Z}_{n-1}$ *, then R is a retraction from* $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ *onto* $B_{p,q}^{a'}(\mathbf{R}_{n-1})$. Furthermore, there exists a corresponding coretraction T in the sense of (14), *which is independent of a, p and q.*

Proof: *Step* **1.** It is easy to see that the sequence $a' = \{a'_k\}_{k' \in \mathbb{Z}_{n-1}}$ satisfies (5) with $n-1$ instead of *n*. Hence, $B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})$ is a space in the sense of Definition 1 (i).

Step 2. Notations with respect to $\mathbf{R_{n-1}} = \{y \mid y \in \mathbf{R_n}, y = (y', 0)\}$ are indicated by $'$. In particular, F' and F^{-1} are the Fourier transform and its inverse on $S(\mathbf{R}_{n-1})$. If $f(x) \in S^{c}(\mathbf{R}_{n})$ then $f(x', 0) \in S^{c}(\mathbf{R}_{n-1})$. Let $\varphi' = {\varphi'_k(x')}_{k' \in \mathbf{Z}_{n-1}}$ be an admissible system in the above sense (with respect to \mathbf{R}_{n-1}). We have to calculate

$$
\{F'^{-1}\varphi'_{k'}F'(f(x', 0))\}(y') = \sum_{l\in \mathbb{Z}_p} \{F'^{-1}\varphi'_{k'}F'[(F^{-1}\varphi_lFf)(x', 0)]\}(y')
$$

with $y' \in \mathbf{R_{n-1}}.$ By elementary arguments we have

$$
\{F'^{-1}\varphi'_k F'[(F^{-1}\varphi_l F f)(x', 0)]\} (y') = (F^{-1}\varphi'_k \varphi_l F f)(y', 0),
$$

where the left-hand side is considered as a function in \mathbf{R}_{n-1} and the right-hand side as $\{F'^{-1}\phi'_{k'}F'[(F^{-1}\phi_{l}F')(x', 0)]\} (y') = (F^{-1}\phi'_{k'}\phi_{l}F')(y',$
where the left-hand side is considered as a function in \mathbf{R}_{n-1} and
a function in \mathbf{R}_{n} . Obviously, $\phi'_{k'}(x')\phi_{l}(x', x_{n}) = 0$ if $k' = (k_{1},$
and $\max |k_{i}$ and max $|k_t - l_t| > 1$. Consequently **£=1...** n—i and side is considered as a function in \mathbf{R}_{n-1} and the right-hand
 i, Obviously, $\varphi'_{k'}(x') \varphi_{l}(x', x_{n}) = 0$ if $k' = (k_1, ..., k_{n-1}), l = (l - l_t) > 1$. Consequently
 $F'((r, 0)) (x') = \sum_{m=-\infty}^{\infty} (F^{-1} \varphi'_{k'} \varphi_{k',m} F f)(x', 0) + \$

$$
F'^{-1}\varphi'_{k'}F'\big(f(\cdot,0)\big)(x') = \sum_{m=-\infty}^{\infty} (F^{-1}\varphi'_{k'}\varphi_{(k',m)}F\big)(x',0) + \cdots = h_{k'}(x') + \cdots \qquad (17)
$$

where $+ \cdots$ indicates sums of the above type where $\varphi_{(k',m)}$ must be replaced by $\varphi_{(l',m)}$ $F'^{-1}\varphi'_{k'}F'((f(\cdot, 0))(x') = \sum_{m=-\infty} (F^{-1}\varphi'_{k'}\varphi_{(k',m)}F') (x', 0) + \cdots = h_{k'}(x') + \cdots$

where $+\cdots$ indicates sums of the above type where $\varphi_{(k',m)}$ must be replaced by $\varphi_{(l,m)}$

with max $|k_l - l_l| = 1$. Of course, $l' = (l_1, \ld$ t—i....**.** n—i with $h_k(x')$ in the sequel. The above Proposition can be applied to $\{\psi_k(x)\}_{k\in \mathbb{Z}_n}$ with where $+ \cdots$ indicates sums of the above type where $\varphi_{(k',m)}$ must be replaced by $\varphi_{(l,m)}$
with max $|k_i - l_i| = 1$. Of course, $l' = (l_1, \ldots, l_{n-1})$. It will be sufficient to deal only
with $h_{k'}(x')$ in the sequel. The a *(FigreFfilingFig)* in the above type where $\varphi_{(k',m)}$ must be rep $|k_i - l_i| = 1$. Of course, $l' = (l_1, ..., l_{n-1})$. It will be sufficie $\varphi'(x')$ p<sub>(k',k_p)(x) where $x \in \mathbb{R}_n$ and $k = (k', k_n)$. If $1 \le x_n \le 2$
($F^{-1}\psi_kFf$) $(x$ *i* $\ell_k F'((r, 0))(x') = \sum_{m=-\infty}^{\infty} (F^{-1} \varphi'_k \varphi_{(k',m)} F)$
dicates sums of the above type whe
dicates sums of the above type whe
 $-\ell_i] = 1$. Of course, $l' = (\ell_1, ..., \ell_n, \ell_n)$
the sequel. The above Proposition
 $\varphi_{(k',k_0)}(x)$ wh

$$
\psi_k(x) = \varphi'_{k'}(x') \varphi_{(k',k_0)}(x) \text{ where } x \in \mathbf{R}_n \text{ and } k = (k', k_n). \text{ If } 1 \le x_n \le 2 \text{ then we have}
$$
\n
$$
|(F^{-1}\psi_k Ff)(x', 0)|^p \le c(\psi_k f)_n^{*p}(x) \text{ where } x \in (x', x_n).
$$
\n(18)\nWe put (18) in (17). Let $0 < p \le 1$. Afterwards we apply the inequality

$$
\left(\sum_{j} \alpha_{j}\right)^{p} \leq \sum_{j} \alpha_{j}^{p} \quad \text{if} \quad \alpha_{j} \geq 0 \tag{19}
$$

and integrate over $\mathbf{R}_{n-1} \times [1, 2]$. Then we have

We put (15) in (17). Let
$$
0 < p \le 1
$$
. Afterwards we apply the inequality
\n
$$
\left(\sum_{j} \alpha_{j}\right)^{p} \le \sum_{j} \alpha_{j}^{p} \quad \text{if} \quad \alpha_{j} \ge 0
$$
\nand integrate over $\mathbb{R}_{n-1} \times [1, 2]$. Then we have
\n
$$
||h_{k'}(\cdot)| L_{p}(\mathbb{R}_{n-1})||^{p} \le c \sum_{m=-\infty}^{\infty} ||(\psi_{(k',m)}f)_{n}^{*}(\cdot)| L_{p}(\mathbb{R}_{n})||^{p}, \quad 0 < p \le 1.
$$
\n(20)\nIf $1 \le p \le \infty$, then we use the triangle inequality instead of (19) and obtain that

 ∞ . then we use the triangle inequality instead of (19) and obtain that

rate over
$$
\mathbf{R}_{n-1} \times [1, 2]
$$
. Then we have
\n
$$
||h_{k'}(\cdot) | L_p(\mathbf{R}_{n-1})||^p \leq c \sum_{m=-\infty}^{\infty} ||(\psi_{(k',m)}f)_n^*(\cdot) | L_p(\mathbf{R}_n)||^p, \qquad 0 < p \leq 1.
$$
\n(20)\n
$$
\leq \infty \text{ then we use the triangle inequality instead of (19) and obtain that}
$$
\n
$$
||h_{k'}(\cdot) | L_p(\mathbf{R}_{n-1})|| \leq c \sum_{m=-\infty}^{\infty} ||(\psi_{(k',m)}f)_n^*(\cdot) | L_p(\mathbf{R}_n)||, \qquad 1 \leq p \leq \infty.
$$
\n(21)\n
$$
a \leq n \leq 1 \text{ in particular we have } \overline{p} = p. \text{ Then (15) and (20) yield}
$$

Let $0 < q \leq p \leq 1$, in particular we have $\overline{p} = p$. Then (15) and (20) yield

grate over
$$
\mathbf{R}_{n-1} \times [1, 2]
$$
. Then we have
\n
$$
||h_{k'}(\cdot) | L_p(\mathbf{R}_{n-1})||^p \leq c \sum_{m=-\infty}^{\infty} ||(\psi_{(k',m)}f)_n^* (\cdot) | L_p(\mathbf{R}_n)||^p, \quad 0 < p \leq 1. \quad (20)
$$
\n
$$
\leq \infty. \text{ then we use the triangle inequality instead of (19) and obtain that}
$$
\n
$$
||h_{k'}(\cdot) | L_p(\mathbf{R}_{n-1})|| \leq c \sum_{m=-\infty}^{\infty} ||(\psi_{(k',m)}f)_n^* (\cdot) | L_p(\mathbf{R}_n)||, \quad 1 \leq p \leq \infty. \quad (21)
$$
\n
$$
q \leq p \leq 1, \text{ in particular we have } \overline{p} = p. \text{ Then (15) and (20) yield}
$$
\n
$$
\sum_{k' \in \mathbb{Z}_{n-1}} a_k'^q ||h_{k'} | L_p(\mathbf{R}_{n-1})||^q
$$
\n
$$
\leq c \sum_{m=-\infty}^{\infty} \sum_{k' \in \mathbb{Z}_{n-1}} a_k'^q ||(\psi_{(k',m)}f)_n^* | L_p(\mathbf{R}_n)||^q \leq c \sum_{l \in \mathbb{Z}_n} a_l^q ||(\psi_lf)_n^* | L_p(\mathbf{R}_n)||^q. \quad (22)
$$
\nsimilar estimates for the terms indicated in (17) by + ···. Now it follows from
\n) and the Proposition that
\n
$$
||f(\cdot, 0) | B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})|| \leq c ||f | B_{p,q}^{\alpha}(\mathbf{R}_n)||.
$$
\n
$$
||\overline{f}(\cdot, 0) | B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})|| \leq c ||f | B_{p,q}^{\alpha}(\mathbf{R}_n)||.
$$
\n
$$
||\overline{f}(\cdot, 0) | B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})|| \leq c ||f | B_{p,q}^{\alpha}(\mathbf{R}_n)||.
$$
\n(23)

We have similar estimates for the terms indicated in (17) by $+ \cdots$. Now it follows from (17), (22) and the Proposition that

$$
||f(\cdot, 0)||B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})|| \leq c||f||B_{p,q}^{\alpha}(\mathbf{R}_{n})||.
$$
 (23)

Hence, *R* is a bounded operator from $B_{p,q}^{\alpha}(\mathbf{R}_n)$ into $B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})$. Let $0 < p \leq 1$ and $p < q \leq \infty$. Again we have $p = \overline{p}$. Then we apply Hölder's inequality with respect to

 $+\frac{p}{q}=1$ to (20) and obtain that $\frac{p}{q} + \frac{p}{q} = 1$
 $\left\| h_k \right\|$
 $\leq \epsilon$

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\n
$$
\frac{p}{q} + \frac{p}{\sigma} = 1 \text{ to } (20) \text{ and obtain that}
$$
\n
$$
||h_k(\cdot) | L_p(\mathbf{R}_{n-1})||^p
$$
\n
$$
\leq c \left(\sum_{m=-\infty}^{\infty} a_{(k',m)}^{-\sigma} \right)^{p/\sigma} \left(\sum_{m=-\infty}^{\infty} a_{(k',m)}^q ||(\psi_{(k',m)}f)_n^* (\cdot) | L_p(\mathbf{R}_n)||^q \right)^{p/q}.
$$
\n(24)
\nWe use (16), take the $\frac{q}{p}$ -power of (24), sum over $k' \in \mathbb{Z}_{n-1}$ and apply again (17) and
\nthe Proposition. The result is (23). The case $1 \leq p \leq \infty$ follows the same line where
\none has to replace (20) by (21). Hence, (23) holds in all cases under consideration.
\nStep 3. We construct an extension operator T from $B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})$ into $B_{p,q}^{\alpha}(\mathbf{R}_n)$ which
\nsatisfies (14). Let $\chi(t) \in S(\mathbf{R}_1)$ be a function with
\n
$$
\sup p \chi = \left(-\frac{1}{4}, \frac{1}{4}\right) \text{ and } (F_1^{-1}\chi) (0) = 1,
$$
\n(25)
\nwhere F_1^{-1} is the inverse one-dimensional Fourier transform. Let $g(x') \in B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})$.
\nWe use the above notations and put

We use (16), take the $\frac{q}{n}$ -power of (24), sum over $k' \in \mathbf{Z}_{n-1}$ and apply again (17) and $\leq c \left(\sum_{m=-\infty} a_{(k',m)}^{-2} \right)$ $\left(\sum_{m=-\infty} a_{(k',m)}^{2} \| (\psi_{(k',m)}f)_n^* (\cdot) \| L_p(\mathbf{R}_n) \|^q \right)$ (24)

We use (16), take the $\frac{q}{p}$ -power of (24), sum over $k' \in \mathbf{Z}_{n-1}$ and apply again (17) and

the Proposition. The resu

Step 3. We construct an extension operator *T* from $B_{p,q}^a(\mathbf{R}_{n-1})$ into $B_{p,q}^a(\mathbf{R}_n)$ which satisfies (14). Let $\chi(t) \in S(R_1)$ be a function with

to replace (20) by (21). Hence, (23) holds in all cases under consideration.
\nWe construct an extension operator T from
$$
B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})
$$
 into $B_{p,q}^{\alpha}(\mathbf{R}_n)$ which
\n(14). Let $\chi(t) \in S(\mathbf{R}_1)$ be a function with
\n
$$
\sup p \chi \subset \left(-\frac{1}{4}, \frac{1}{4}\right) \text{ and } (F_1^{-1} \chi) (0) = 1,
$$
\n(25)
\n
$$
\chi^{-1}
$$
 is the inverse one-dimensional Fourier transform. Let $g(x') \in B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})$,
\nthe above notations and put
\n
$$
f(x) = (Tg) (x) = \sum_{k' \in \mathbb{Z}_{n-1}} (F_1^{-1} \chi) (x_n) (F'^{-1} \varphi'_{k'} F'g) (x'), \quad x = (x', x_n).
$$
\n(26)
\n2, T is linear. Let $\sum_{k' \in \mathbb{Z}_{n-1}} \varphi'_{k'}(x') \equiv 1$ in \mathbf{R}_{n-1} . Then we have $f(x', 0) = g$, i.e. (14)
\n
$$
\chi'(\mathbf{R}_{n-1})
$$
\n
$$
\chi''(\mathbf{R}_{n-1})
$$

where F_1^{-1} is the inverse one-dimensional Fourier transform. Let $g(x') \in B_{n,q}^{\alpha'}(\mathbf{R}_{n-1})$. We use the above notations and put

$$
f(x) = (Tg)(x) = \sum_{k' \in \mathbb{Z}_{n-1}} (F_1^{-1} \chi) (x_n) (F'^{-1} \varphi'_{k'} F'g) (x'), \qquad x = (x', x_n). \qquad (26)
$$

Of course, *T* is linear. Let $\sum_{x \in \mathcal{X}} \varphi'_k(x') = 1$ in \mathbb{R}_{n-1} . Then we have $f(x', 0) = g$, i.e. (14) holds. Let $\{\varphi_k(x)\}_{k\in \mathbb{Z}_n}$ be an admissible system in \mathbb{R}_n , where we may assume that supp $\chi \subset \left(-\frac{1}{4}, \frac{1}{4}\right)$

where F_1^{-1} is the inverse or

We use the above notations
 $f(x) = (Tg)(x) =$

Of course, T is linear. Let $\sum_{k' \in \mathbb{Z}}$

holds. Let $\{\varphi_k(x)\}_{k \in \mathbb{Z}_n}$ be an a
 $\varphi_k(x) = \varphi'_{k'}(x') \varrho_{k_n}($

$$
\varphi_k(x) = \varphi'_{k'}(x') \varrho_{k_n}(x_n), \qquad k = (k', k_n), \qquad x = (x', x_n),
$$

with appropriate functions $\varrho_m(x_n)$, $m \in \mathbb{Z}_1$. We may assume that $\varrho_m(t) \chi(t) = 0$ if $m \neq 0$. We have

$$
(F^{-1}\varphi_k F f) (x) = (F_1^{-1}\varrho_{k_n}\chi) (x_n) (F'^{-1}\varphi_k'^2 F' g) (x') + \cdots \qquad (27)
$$

 $\begin{aligned}\n\sup_{x \in \mathbb{R}} p \times \mathbb{C} \left(-\frac{1}{4}, \frac{1}{4} \right) \quad \text{and} \quad & (r_1 - \chi) \ (0) = 1, \\
\text{or} \quad \text{where } \text{ the above notations and put}\n\end{aligned}\n\begin{aligned}\n\lim_{\epsilon \to 0} \left(F_1^{-1} \chi \right) (x_n) \left(F'^{-1} \varphi_{k'} F' g \right) (x'), \quad x = (x', \\
\text{where } \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ where $+ \cdots$ must be understood similarly as in (17). Again we can restrict our attention to the first term on the right-hand side of (27). If $k_n \neq 0$ then $(F^{-1}\varphi_k Ff)$ (x) $= 0$ by our assumption. Then it follows that by our assumption. Let $\sum_{k' \in \mathbb{Z}_{n-1}} (F_1^{-1}z) (x_n) (F'^{-1}\varphi'_k F'g) (x'), \quad x = (x', x_n).$

Of course, T is linear. Let $\sum_{k' \in \mathbb{Z}_{n-1}} \varphi'_k(x') = 1$ in \mathbb{R}_{n-1} . Then we have $f(x', 0) = g$, i.e. tholds. Let $\{\varphi_k(x)\}_{k \in \mathbb{Z}_n}$ be have
 $(\mathbb{F}^{-1}\varphi_k Ff)(x) = (F_1^{-1}\varphi_{k,n}\chi)(x_n) (F'^{-1}\varphi_k'^2 F'g)(x') + \cdots$ (27)
 \cdots must be understood similarly as in (17). Again we can restrict our atten-

first term on the right-hand side of (27). If $k_n \neq 0$ then $(F^{-1$

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\n
$$
\frac{p}{q} + \frac{p}{\sigma} = 1
$$
 to (20) and obtain that
\n
$$
||h_{K}(\cdot)||_{L_{p}(\mathbf{R}_{n-1})}||^{p}
$$
\n
$$
\leq c \left(\sum_{m=-\infty}^{\infty} a_{\mathbf{q}} \overline{r}_{,m}\right)^{p/q} \left(\sum_{m=-\infty}^{\infty} a_{\mathbf{q}}^{\sigma} \overline{r}_{,m} \right) \left[(\psi_{\mathbf{R}',m}f)_{n}^{*}(\cdot)||_{L_{p}(\mathbf{R}_{n})}||^{p}\right)^{p/q}.
$$
\nWe use (16), take the $\frac{q}{p}$ -power of (24), sum over $k' \in \mathbb{Z}_{n-1}$ and apply again (17) and the Proposition 7. The result is (23). The case 1 \leq p \leq \infty 6 follows the same line where
\none has to replace (20) by (21). Hence, (23) holds in all cases under consideration.
\nStep 3. We construct an extension operator T from $B_{p,q}^{\omega}(\mathbf{R}_{n-1})$ into $B_{p,q}^{\omega}(\mathbf{R}_{n})$ which
\nsatisfies (14). Let $\chi(t) \in S(\mathbf{R}_{1})$ be a function with
\n
$$
\sup_{\mathbf{R}} \chi = \left(-\frac{1}{4}, \frac{1}{4}\right) \text{ and } (F_{1}^{-1}\chi)(0) = 1,
$$
\nwhere F_{1}^{-1} is the inverse one-dimensional Fourier transform. Let $g(\mathbf{z}') \in B_{p,q}^{\omega}(\mathbf{R}_{n-1})$.
\nWe use the above notations and put
\n
$$
f(\mathbf{z}) = (f'q) (\mathbf{z}) = \sum_{k} \sum_{k} f_{k}(x') = 1
$$
 in \mathbf{R}_{n-1} . Then we have $f(\mathbf{z}', 0) = g$, i.e. (14)
\nholds. Let $\{\varphi_{\mathbf{q}}(x) |_{\mathbf{R}(\mathbf{z}, \mathbf{z})} = k \sum_{k} f_{k}(x') = 1$ in \mathbf{R}_{n-1} . Then we have $f(\mathbf{z}', 0) = g$, i.e. (

$$
ca_{k}^{'q} \|F'^{-1}\varphi_{k}^{'2}F'g \| L_p(\mathbf{R}_{n-1})\|^q + \cdots \tag{29}
$$

We sum over $k' \in \mathbb{Z}_{n-1}$ and apply the Proposition. We obtain that

$$
||Tg \mid B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)|| \leq c ||g \mid B_{p,q}^{\mathfrak{a}'}(\mathbf{R}_{n-1})|| \tag{30}
$$

advantages, in particular if one wishes to apply interpolation methods, where couples of different spaces of type $B_{p,q}^{\alpha}(\mathbf{R}_n)$ and $F_{p,q}^{\alpha}(\mathbf{R}_n)$ come in. This will be done in Sub- $\leq c\alpha_{k',0} ||h' - \varphi_{k'} h' g + L_p(\mathbf{R}_{n-1}) ||^2 + \cdots$

(modification if $q = \infty$). By our hypotheses we have $\alpha_{(k',0)} \leq A^{-1} \alpha'_{k'}$. I

can be estimated from above by
 $c\alpha_{k}^{\prime 2} ||F'^{-1} \varphi_{k'}^{\prime 2} F' g + L_p(\mathbf{R}_{n-1}) ||^2 + \cdots$

We

section 4.2. The price for the independence of T on a, p , q is paid by the aditional assumption $a'_k \geq A a_{k',0}$. In the next subsection we shall show that this assumption can he omitted (at least for some *p* and q), but the extension operator *T* depends on the sequence a (and hence is not useful for interpolation purposes). 2. The price for the incomponal $a'_{k'} \geq A a_{(k',0)}$. In the
initted (at least for some
nome a (and hence is not
ifications
aim to remove the cond
ary: Let $0 < p \leq \infty$,
of positive numbers with
ly. If
either $0 < q \leq \overline{p}$

3.2. Modifications

It is our aim to remove the condition $a'_{k'} \geq A a_{(k',0)}$ from Theorem 1.

Corollary: Let $0 < p \leq \infty$, $\overline{p} = \min(1, p)$, and $0 < q \leq \infty$. Let $\{a_k\}_{k \in \mathbb{Z}_n}$ be a *sequence of positive numbers with* (5). Let $a' = \{a'_k\}_{k' \in \mathbb{Z}_{n-1}}$ be given by (15) and (16), *respectively. If*

> (31) *P*
OP $0 < q \leq \overline{p}$
OP $1 \leq p < q \leq \infty$

then R is a retraction from $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ *onto* $B_{p,q}^{\mathfrak{a}'}(\mathbf{R}_{n-1})$.

Corollary: Let $0 < p \le \infty$, $\overline{p} = \min(1, p)$, and $0 < q \le \infty$. Let $\{a_k\}_{k \in \mathbb{Z}_n}$ be a
 paecively. If
 and (16),
 pectively. If
 and (16),
 and (16),
 and (16),
 and (16),
 and (16),
 and (16),
 and **Proof:** We used the additional assumption $a'_{k'} \geq A a_{k',0}$ from Theorem 1 only in Step 3 of the proof of this theorem. In other words, only the construction of the extension operator T from (26) must be modified. Let 0 then R is a retraction from $B_{p,q}^{\alpha}(\mathbf{R}_n)$ onto $B_{p,q}^{\alpha'}(\mathbf{R}_{n-1})$.

Proof: We used the additional assumption $a'_{k'} \geq A a_{k',0}$ from Theorem 1 only

in Step 3 of the proof of this theorem. In other words, only t we choose *1* = i(k') E *^Z ¹*such that If *^x*is given by (25) then we pu t. *I*(*I*) *I*(

 $\gamma_i(t) = \gamma(t - l)$. We modify (26) by

$$
f(x) = (Tg)(x) = \sum_{k' \in \mathbb{Z}_{n-1}} (F_1^{-1} \chi_{l(k')}) (x_n) (F'^{-1} \varphi'_{k'} F'g) (x'), \qquad x = (x', x_n). \tag{32}
$$

Again we have $f(x', 0) = g(x')$. The modified estimate (28) with $a_{(k',0)}$ instead of $u_{(k',0)}$ and the arguments afterwards show that T has the desired properties. Let $1 \leq p < q \leq \infty$. Then $\overline{p} = 1$. If $k' \in \mathbb{Z}_{n-1}$ then we choose a natural number $N(k')$ and a number $C(k')$ with $1 < C(k') < 2$ such that $\begin{align*}\n\sum_{i=1}^{n} p < q \leq \infty \\
\text{on from } B_{p,q}^{\mathfrak{a}}(\mathbf{R})\n\end{align*}$
 $\begin{align*}\n\sum_{i=1}^{n} q_i(\mathbf{R}) \leq \mathbf{Z}_1 \text{ such that} \\
\sum_{i=1}^{n} q_i(\mathbf{R}) \leq \mathbf{Z}_2 \text{ such that} \\
\sum_{i=1}^{n} q_i(\mathbf{R}) = \sum_{i=1}^{n} q_i(\mathbf{R}) \leq \mathbf{Z}_1.\n\end{align*}$

$$
C(k') \sum_{|l| \le N(k')} \frac{a_{k'}^{\prime \prime}}{a_{(k',l)}^{\sigma}} = 1, \tag{33}
$$

cf. (16). Let

$$
C(k') \sum_{|l| \le N(k')} \frac{a_{k'}^{l'}}{a_{(k',l)}^{\sigma}} = 1,
$$
\n
$$
C(k') \sum_{|l| \le N(k')} \frac{a_{k'}^{l'}}{a_{(k',l)}^{\sigma}} = 1,
$$
\n
$$
f(x) = (Tg)(x) = \sum_{k' \in \mathbb{Z}_{n-1}} C(k') \sum_{|l| \le N(k')} \frac{a_{k'}^{l''}}{a_{(k',l)}^{\sigma}} (F_1^{-1} \chi_l) (x_n) (F'^{-1} \varphi'_{k'} F'g) (x)', \quad (34)
$$

 $x = (x', x_n)$. We have $f(x', 0) = g(x')$. We use (27). Then the counterpart of (28) reads as follows,

$$
I \leq p < q \leq \infty. \text{ Then } p = 1. \text{ If } k \in \mathbb{Z}_{n-1} \text{ then we choose a natural number } N(k)
$$
\nand a number $C(k')$ with $1 < C(k') < 2$ such that

\n
$$
C(k') \sum_{|l| \leq N(k')} \frac{a_{k'}^{\prime o}}{a_{k',l}} = 1,
$$
\ncf. (16). Let

\n
$$
f(x) = (Tg)(x) = \sum_{k' \in \mathbb{Z}_{n-1}} C(k') \sum_{|l| \leq N(k')} \frac{a_{k'}^{\prime o}}{a_{k',l}} (F_1^{-1} \chi_l) (x_n) (F_1^{V-1} \varphi_k F_2^{\prime o}) (x'),
$$
\n(34)

\n
$$
x = (x', x_n). \text{ We have } f(x', 0) = g(x'). \text{ We use (27). Then the counterpart of (28)}\nreads as follows,

\n
$$
\sum_{m=-\infty}^{\infty} a_{k',m}^q \|F^{-1} \varphi_{k',m} F_1^{\prime} \| L_p(\mathbf{R}_n) \|^q
$$
\n
$$
\leq c a_{k'}^{\prime o q} \|F_1^{V-1} \varphi_{k'}^{\prime} F_2^{\prime o} \| L_p(\mathbf{R}_{n-1}) \|^q \sum_{m=-\infty}^{\infty} a_{k',m}^{q(1-o)} + \cdots
$$
\n(35)

\n(modification if $q = \infty$). We have $\frac{1}{q} + \frac{1}{q} = 1$ and consequently $q(1 - \sigma) = -\sigma$.
$$

and $\sigma(q - 1) = q$. Then it follows that *If* $\frac{1}{m} = -\infty$ (i.m.)
 ∞). We have $\frac{1}{\sigma} + \frac{1}{q} = 1$ and consequently $q(1 - \sigma) = -\sigma$

en it follows that
 $|F^{-1}\varphi_{(k',m)}Ff | L_p(\mathbf{R}_n)||^q \leq c a_k'^q ||F'^{-1}\varphi_k'^2 F'g | L_p(\mathbf{R}_{n-1})||^q + \cdots$ (36)

$$
\sum_{m=-\infty}^{\infty} a_{(k',m)}^q \|F^{-1}\varphi_{(k',m)}Ff \| L_p(\mathbf{R}_n)\|^q \leq c a_{k'}^{'q} \|F'^{-1}\varphi_{k'}^{'2}F'g \| L_p(\mathbf{R}_{n-1})\|^q + \cdots \tag{36}
$$

which yields the desired result by the above arguments. **I**

Remark 7: in contrast to (26), the operator *T* from (32) and (34) depends on the sequence $a = \{a_k\}_{k \in \mathbb{Z}_n}$. Of course, (31) does not cover all cases. What remains is the case $0 < p < 1$, $p < q$. Probably the Corollary remains valid also in this case. One must modify (34) (replacement of F_1 ⁻¹ χ ¹ by more sophisticated functions). However we we have not checked this proposal in detail.

4. Traces of $F_{p,q}^{\alpha}(\mathbf{R}_n)$

4.1. Main Assertion

All notations have the same meaning as in Subsection 3.1.

Theorem 2: Let $0 < p < \infty$ and $0 < q \leq \infty$. Let $a = \{a_k\}_{k \in \mathbb{Z}_n}$ be a sequence of *positive numbers with* (5). If $k \in \mathbb{Z}_n$ then we put $k = (k', k_n)$ with $k' \in \mathbb{Z}_{n-1}$ and $k_n \in \mathbb{Z}_1$. $\begin{aligned} \text{tted function} \ = \{a_k\}_{k \in \mathbb{Z}_n} \ \text{for} \ k' \in \mathbb{Z}_n \ \end{aligned}$

By assumption there exists a positive number c such that
\n
$$
a_{(k',m)} \ge c |m|^x a_{(k',0)} \quad \text{with} \quad z > \max\left(1, \frac{1}{p}, \frac{1}{q}\right)
$$
\n(37)

holds for all $k' \in \mathbb{Z}_{n-1}$ *and* $m \in \mathbb{Z}_1$ *. Then* $F_{p,q}^{\alpha'}(\mathbf{R}_{n-1})$ *with*

$$
a' = \{a'_{k'} = a_{(k',0)}\}_{k' \in \mathbb{Z}_{n-1}}
$$

is a space in the sense of Definition 1 <i>(ii) and R is a retraction from $F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ *onto* $F_{p,q}^{\alpha'}(\mathbf{R}_{n-1})$. Furthermore, T from (26) *is a corresponding coretraction.*

Proof (Outline): *Step* 1. It is easy to see that $F^{\mathfrak{a}'}_{p,q}(\mathbf{R}_{n-1})$ is a space in the sense of Definition 1 (ii). Furthermore, it follows from the arguments in Step 3 of the proof of Theorem 1 that T from (26) is an extension operator with the desired properties. $\mathbf{A}^{H}(\mathbf{R}_{n-1}) = \{a'_{k'} = a_{(k',0)}\}_{k' \in \mathbb{Z}_{n-1}}$
 $\mathbf{A}^{H} = \{a'_{k'} = a_{(k',0)}\}_{k' \in \mathbb{Z}_{n-1}}$
 e in the sense of Definition 1 (*ii*) and *R* is a re
 A. *Furthermore*, *T from* (26) is a corresponding co:

(Out

Step 2. We must prove that *R* is a bounded mapping from F_n^a . $\frac{f_{p,q}^{a}(\mathbf{R}_{n})}{f}$ as in Step $_{a}(\mathbf{R}_{n})$ into Let $1 = b_0 < b_1 < b_2 < b_3 < \cdots$. We beginn in the same way as in Step 2 of the proof of Theorem 1. Instead of (18) we use (Outline): *Step* 1. It is easy to see that $F_{p,q}^*(\mathbf{R}_{n-1})$ is a space in the sense of

1 1 (ii). Furthermore, it follows from the arguments in Step 3 of the proof

2 in 1 that T from (26) is an extension operator wit

$$
|(F^{-1}\psi_k F f)(x', 0)|^2 \leq c b_{m+1}^{d\lambda} (\psi_k f)_n^{d\lambda} (x), \tag{38}
$$

where $x = (x', x_n)$, $b_m \le x_n \le b_{m+1}$, $k = (k', m)$ with $m \ge 0$ and $\lambda > 0$. Here *d* has the meaning of (9). Integration yields

$$
|(F^{-1}\psi_k F f)(x',0)|^2 \le c b_{m+1}^{d\lambda} (b_{m+1} - b_m)^{-1} \int_{b_m}^{b_{m+1}} (\psi_k f)_n^{d\lambda} (x',x_n) dx_n
$$
 (39)
with $k = (k', m)$, $m \ge 0$. A similar formula holds if $m < 0$. Let $0 < \lambda < \min(p, q)$.
Let e.g. $0 < q \le 1$. We use the abbreviation $h_{k'}(x')$ from (17). Then we have

with $k = (k', m), m \ge 0$. A similar formula holds if $m < 0$. Let $0 < \lambda < \min (p, q)$.

Let e.g.
$$
0 < q \le 1
$$
. We use the abbreviation $h_{k'}(x')$ from (17). Then we have
\n
$$
\sum_{k \in \mathbb{Z}_{n-1}} a_{k'}^{q} |h_{k'}(x')|^q
$$
\n
$$
\le \sum_{m=-\infty}^{\infty} \sum_{k' \in \mathbb{Z}_{n-1}} a_{k'}^{q} |(F^{-1}\psi_{(k',m)}Ff)(x',0)|^q
$$
\n
$$
\le c \sum_{m=0}^{\infty} \left\{ \sum_{k' \in \mathbb{Z}_{n-1}} \left[a_{k'}^{l'} (b_{m+1} - b_m)^{-1} b_{m+1}^{dl} \int_{b_m}^{b_{m+1}} (\psi_{(k',m)}f)_n^{*l'}(x',x_n) dx_n \right]^{q/l} \right\}^{(l/q)\cdot (q/l)} + \cdots,
$$
\n(40)

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where $+ \cdots$ indicates a corresponding term with $m = -1, -2, -3, \ldots$ If one uses the triangle inequality for l_q , then one obtains a corresponding formula with $q > 1$. We put temporarily *ak* = *a' .(b 1 — bm) - 'R b* if *^k*= (k', *m)* with *m* ^ 0. (41)

$$
\tilde{a}_k = a'_{k'}(\dot{b}_{m+1} - b_m)^{-1/2} b^d_{m+1} \quad \text{if} \quad k = (k', m) \quad \text{with} \quad m \ge 0. \tag{41}
$$

Because $\frac{q}{\lambda} > 1$, the first term on the right-hand side of (40) can be estimated from above by $a_k = a$

Because $\frac{q}{\lambda} > a$

above by
 $c \sum_{m=0}^{\infty}$

Because $\frac{p}{\lambda} > a$
 $\left\{\begin{array}{l} b_{m+1} \\ \int_{b_m}^{b_{m+1}} 0 \text{d} \lambda \end{array}\right.$

Let $p \leq q$ and
 $a_k = a_k$

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\nwhere + ... indicates a corresponding term with
$$
m = -1, -2, -3, ...
$$
 If one uses
\nthe triangle inequality for l_q , then one obtains a corresponding formula with $q > 1$.
\nWe put temporarily
\n
$$
\bar{a}_k = a'_k(b_{m+1} - b_m)^{-1/\lambda} b_{m+1}^d \quad \text{if} \quad k = (k', m) \quad \text{with} \quad m \ge 0. \tag{41}
$$
\nBecause $\frac{q}{\lambda} > 1$, the first term on the right-hand side of (40) can be estimated from
\nabove by
\n
$$
c \sum_{m=0}^{\infty} \left\{ \int_{b_m}^{b_{m+1}} \left[\sum_{k \in \mathbb{Z}_{n-1}} \bar{a}_{(k',m)}^q (\psi_{(k',m)} f)_n^{*q} (x', x_n) \right]^{1/q} dx_n \right\}^{q/\lambda}. \tag{42}
$$
\nBecause $\frac{p}{\lambda} > 1$, it follows from Hölder's inequality that
\n
$$
\left\{ \int_{b_m}^{b_{m+1}} [\ldots]^{1/q} dx_n \right\}^{q/\lambda} \le \left(\int_{b_m}^{b_{m+1}} [\ldots]^{p/q} dx_n \right)^{q/p} (b_{m+1} - b_m)^{\frac{p-\lambda}{p\lambda}}. \tag{43}
$$

Because $\frac{p}{\lambda} > 1$, it follows from Hölder's inequality that

comporarily

\n
$$
k = a'_k \cdot (b_{m+1} - b_m)^{-1/\lambda} b_{m+1}^d \quad \text{if} \quad k = (k', m) \quad \text{with} \quad m \ge 0.
$$
\n
$$
\frac{q}{\lambda} > 1, \text{ the first term on the right-hand side of (40) can be estimated from}
$$
\n
$$
c \sum_{m=0}^{\infty} \left\{ \int_{b_m}^{b_{m+1}} \left[\sum_{k \in \mathbb{Z}_{n-1}} \bar{a}_{(k',m)}^q (\psi_{(k',m)} f)_n * q (x', x_n) \right]^{1/q} dx_n \right\}^{q/\lambda}
$$
\n
$$
\frac{p}{\lambda} > 1, \text{ it follows from Hölder's inequality that}
$$
\n
$$
\left\{ \int_{b_m}^{b_{m+1}} \left[\ldots \right]^{1/q} dx_n \right\}^{q/\lambda} \le \left(\int_{b_m}^{b_{m+1}} \left[\ldots \right]^{p/q} dx_n \right)^{q/p} (b_{m+1} - b_m)^{\frac{p-\lambda}{p\lambda}}.
$$
\n
$$
q \text{ and}
$$
\n
$$
a_k = \bar{a}_k (b_{m+1} - b_m)^{\frac{1}{\lambda} - \frac{\lambda}{p}} = a'_k (b_{m+1} - b_m)^{-\frac{1}{p}} b_{m+1}^d, \quad d > \frac{1}{p},
$$
\n
$$
(k', m), m \ge 0. \text{ Then (40), (42) and (43) yield}
$$

•

• S

$$
a_k = \bar{a}_k (b_{m+1} - b_m)^{\frac{1}{4} - \frac{4}{p}} = a'_k (b_{m+1} - b_m)^{-\frac{1}{p}} b'_{m+1}, \quad d > \frac{1}{p}, \tag{44}
$$

with $k = (k', m), m \geq 0$. Then (40), (42) and (43) yield

$$
\begin{cases}\n\int_{b_m}^{b_{m+1}} \left[\ldots\right]^{i/q} dx_n\right|^{q/4} \leq \int_{b_m}^{b_{m+1}} \left[\ldots\right]^{p/q} dx_n\right)^{q/p} (b_{m+1} - b_m)^{\frac{p-2}{p}}.\n\end{cases}
$$
\n(43)
\nLet $p \leq q$ and
\n
$$
a_k = \bar{a}_k(b_{m+1} - b_m)^{\frac{1}{4} - \frac{1}{p}} = a'_k(b_{m+1} - b_m)^{-\frac{1}{p}} b'_{m+1}, \quad d > \frac{1}{p},
$$
\n(44)
\nwith $k = (k', m), m \geq 0$. Then (40), (42) and (43) yield
\n
$$
\int_{\mathbf{R}_{n-1}} \left(\sum_{k \in \mathbb{Z}_{n-1}} a'^{q}_{k} |h_k(x')|^q\right)^{p/q} dx' \leq c \sum_{m=0}^{\infty} \int_{\mathbf{R}_{n-1}} \left[\ldots\right]^{p/q} dx' + \ldots
$$
\n
$$
\leq c' \sum_{m=0}^{\infty} \int_{\mathbf{R}_{n-1}} \int_{b_m}^{b_{m+1}} \left[\sum_{k \in \mathbb{Z}_{n-1}} a^q_{k',m} |(\psi_{k',m})f\rangle^{*q} (x', x_n)\right]^{p/q} dx_n dx' + \ldots
$$
\n
$$
\leq c' \sum_{m=0}^{\infty} \int_{\mathbf{R}_{n-1}} \int_{b_m}^{b_m} \left[\sum_{k \in \mathbb{Z}_n} a^q_{k}(w_k)\right]^{k/q} (x', x_n)\right]^{p/q} dx_n dx' + \ldots
$$
\n
$$
\leq c' \int_{\mathbf{R}_{n}} \left[\sum_{k \in \mathbb{Z}_{n}} a^q_{k}(w_k)\right]^{k/q} dx.
$$
\nBecause $p \leq q$ and $d > \frac{1}{p}$ it follows from the Proposition that the right-hand side of (45) can be estimated from above by $c\|f| \sum_{p,q} \left(\mathbf{R}_p\right)\|^{p}$. Together with (17) this yields the desired assertion, provided that $0 < p \leq q \leq 1$ and that a_k and a

Because $p \leq q$ and $d > \frac{1}{p}$ it follows from the Proposition that the right-hand side of (45) can be estimated from above by $c \, ||f|| F^a_{p,q}(\mathbf{R}_n) ||^p$. Together with (17) this yields Because $p \leq q$ and $d > \frac{1}{p}$ it follows from the Pro
of (45) can be estimated from above by $c \parallel f \mid F_{p,q}^a(\mathbf{R}_n)$
the desired assertion, provided that $0 < p \leq q \leq$ $\prod_{k=1}^{n} a_k$ and a'_k are related Because $p \leq q$ and $d > \frac{1}{p}$ it follows from the Proposition that the right-hand side
of (45) can be estimated from above by $c \, ||f| \, F_{p,q}^{\mathfrak{a}}(\mathbf{R}_n) ||^p$. Together with (17) this yields
the desired assertion, pro $m=0$ $\leq c' \sum_{m=0}^{\infty} \int_{R_{n-1}}^{\infty} \int_{R}^{\infty} \left[\sum_{k \in \mathbb{Z}_n} a_k^q (\psi_k f)_n^{*q} (x', x_n) \right]^{p/q} dx_n dx' + \cdots$
 $\leq c' \int_{R_n}^{\infty} \left[\sum_{k \in \mathbb{Z}_n} a_k^q (\psi_k f)_n^{*q} (x) \right]^{p/q} dx.$

Because $p \leq q$ and $d > \frac{1}{p}$ it follows from the Propositi by (44). If $q \ge 1$ and
to $\sum_{m=0}^{\infty} \{... \}$ in (40). If
 $\sum_{m=0}^{\infty} \{... \} =$
if $\varrho \sigma > 1$, i.e. $\varrho >$ $p \leq q$ and $d > \frac{1}{p}$ it follows from the Proposition that the right-hand side

n be estimated from above by $c \, ||f| \, F^a_{p,q}(\mathbf{R}_n) ||^p$. Together with (17) this yields

ed assertion, provided that $0 < p \leq q \leq 1$ and t

$$
\sum_{n=0}^{\infty} \{\ldots\} = \sum_{m=0}^{\infty} (1+m)^e \{\ldots\} (1+m)^{-e} \leq c \left(\sum_{m=0}^{\infty} (1+m)^{e(p/q)} \{\ldots\}^{p/q} \right)^{q/p}
$$
\ni.e. $\varrho > \frac{p-q}{p}$. In that case (i.e. $q \leq 1$ and $p > q$) we replace (44) by

$$
a_k = a'_k (b_{m+1} - b_m)^{-1/p} b^d_{m+1} (1 + m)^{q/q}, \qquad d > \frac{1}{q}, \qquad (46)
$$

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with $k = (k', m), m \ge 0$. The rest is the same as in the case $0 < p \le q \le 1$. The case $q > 1$ can be treated in a similar way, where one has again two subcases, $p \le q$ and $p > q$. with $k = (k', m), m \ge 0$. The rest is the same as in the case $0 < p \le q \le 1$. The case $q > 1$ can be treated in a similar way, where one has again two subcases, $p \le q$ and $p > q$.
Step 3. Let again $q \le 1$. We discuss (44) and (4

 $b_{m+1} - b_m = (1 + m)^{-\delta}$ if $m = 0, 1, 2, ...$ and $0 < \delta < 1$. Then we have h $k = (k', m), m \ge 0$. The rest is the same as in

> 1 can be treated in a similar way, where one

> q.

Step 3. Let again $q \le 1$. We discuss (44) and (
 $1 - b_m = (1 + m)^{-\delta}$ if $m = 0, 1, 2, ...$ and 0 <
 $(1 + m)^{1-\delta}$. With δ nea *1=1* (*k'*, *m*), $m \ge 0$. The rest is the same as in the case $0 < p \le q \le 1$. The case

n be treated in a similar way, where one has again two subcases, $p \le q$ and

Let again $q \le 1$. We discuss (44) and (46). We recall that b Let again $q \leq 1$. We discuss (44) and (46). We recall that $b_0 = 1$ and put
 $a_k = (1 + m)^{-\delta}$ if $m = 0, 1, 2, ...$ and $0 < \delta < 1$. Then we have $b_m \sim \sum_{l=1}^{m} l^{-\delta}$
 $a_k \sim a'_k (1 + m)^{\frac{1}{p} + \epsilon}$, $\varepsilon > 0$, if $p \leq q \leq 1$ (47)

and

$$
u_k \sim a'_k (1+m)^{\frac{1}{p}+\epsilon}, \qquad \epsilon > 0, \quad \text{if} \quad p \leq q \leq 1 \tag{47}
$$

I

$$
\epsilon_k \sim a'_k (1+m)^{q^{n-k}}, \qquad \epsilon > 0, \quad \text{if} \quad q \leq 1 \quad \text{and} \quad p \geq q. \tag{48}
$$

The desired estimate is now a consequence of (37). Similarly one deals with the case $q>1$.

Remark 8: The question is whether our choice of the numbers b_m in Step 3 of the last proof is optimal. Our limiting exponents \varkappa from (37) are shown in Fig. 1. If one asks the same question, i.e.

for the spaces $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ then one has to examine (15) and (16). The corresponding limiting exponents x are shown in Fig. 2. Because $B_{p,\nu}^{\alpha}(\mathbf{R}_n) = F_{p,\nu}^{\alpha}(\mathbf{R}_n)$ one can try to interpolate these two figures and to improve the assertions for $\widetilde{F}_{n,q}^{\mathfrak{a}}(\mathbf{R}_n)$ on that way. We sketch this possibility in the following subsection.

N

4.2. Improvements

In [16: 2.4.9] we developed a complex interpolation method which can also be applied to our situation. (One has to replace the dyadic covering of \mathbf{R}_n by the congruent covering in the sense of the introduction.) Without any further explanations we use the notations introduced in [16: 2.4.9]. In particular if $F_{p,q_0}^{a^*}(\mathbf{R}_n)$ and $F_{p_1,q_1}^{a^*}(\mathbf{R}_n)$ are two spaces with the sequences $a^0 = {a_k^0}_{k\epsilon\mathbf{Z}_n}$ and $a^1 = {a_k^1}_{k\epsilon\mathbf{Z}_n}$ then the complex interpolation $(\cdot, \cdot)_{\theta}$ with $0 < \theta < 1$ yields *Provements*
 P. 4.9] we developed a comp

ituation. (One has to replaid

in the sense of the introduct

is introduced in [16: 2.4.9]

with the sequences $a^0 = \{a \in \Lambda : \lambda, b\}$
 $(\cdot, \cdot)_\theta$ with $0 < \theta < 1$ yiel
 $(F_{p,\mathcal{A}}^{$ $\rho_{p,q}$, $\rho_{p,q}$, *z* we developed a comption. (One has to replant the sense of the introduct troduced in [16: 2.4.9 the sequences $\mathfrak{a}^0 = \{ \partial_{\theta} \}$, with $0 < \theta < 1$ yie $\mathfrak{a}_s \mathfrak{a}_s (\mathbf{R}_n)$, $F_{p_1, q_1}^{a^1} (\mathbf{R}_n) \mathfrak{b}_0 =$
 $=$ **h** ped a complex interpolation method which can all

as to replace the dyadic covering of \mathbf{R}_n by the c

e introduction.) Without any further explanation
 $[16: 2.4.9]$. In particular if $F_{p_0, q_0}^a(\mathbf{R}_n)$ and and $a^1 = {n \choose a_k} {n \choose b_k}$ and $a^1 = {a_k} {n \choose b_k} {n \choose b_k}$ and $a^1 = {a_k} {n \choose b_k} {n \choose b_k}$ (49)
 R_n), (49)
 P_p = $\frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q_0} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. (50)
 a) and $r = 0, 1$ then $a_n^{\theta} \sim a_{n+1}^{\theta$

$$
(F_{p_o,q_o}^{a^o}(\mathbf{R}_n),\,F_{p_1,q_1}^{a^1}(\mathbf{R}_n))_{\theta} = F_{p_{\theta'}q_{\theta}}^{a^{\theta}}(\mathbf{R}_n),\qquad(49)
$$

\n The sequences \n
$$
\mathbf{a}^0 = \{a_k^0\}_{k \in \mathbb{Z}_n}
$$
\n and \n $\mathbf{a}^1 = \{a_k^1\}_{k \in \mathbb{Z}_n}$ \n then the complex interval \n $(\cdot, \cdot)_{\theta}$ \n with \n $0 < \theta < 1$ \n yields\n $\left(F_{p_o,q_o}^{\mathfrak{a}^0}(\mathbf{R}_n), F_{p_1,q_1}^{\mathfrak{a}^1}(\mathbf{R}_n)\right)_{\theta} = F_{p_{\theta},q_{\theta}}^{\mathfrak{a}^0}(\mathbf{R}_n),$ \n (49)\n $\mathbf{a}^0 = \{a_k^0 = (a_k^0)^{1-\theta} \ (a_k^1)^0\}_{k \in \mathbb{Z}_n}, \ \frac{1}{p_{\theta}} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ \frac{1}{q_{\theta}} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$ \n (50)\n

If $a_k \sim a_{(k',0)}^r (1 + |m|)^{k_r}$ with $k = (k', m)$ and $r = 0, 1$ then $a_k \circ \sim a_{(k',0)}^{\theta} (1 + |m|)^{k_{0}}$ polation $(\cdot, \cdot)_\theta$ with $0 < \theta < 1$ yields
 $(F_{p_s,q_s}^{a^s}(\mathbf{R}_n), F_{p_i,q_s}^{a^t}(\mathbf{R}_n))_\theta = F_{p_g,q_g}^{a^g}(\mathbf{R}_n),$ (49)
 $\mathfrak{a}^0 = \{a_k^{\theta} = (a_k^{\theta})^{1-\theta} (a_k^{\theta})^{\theta}\}_{k \in \mathbb{Z}_n}, \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q_\theta} = \frac{1-\theta}{q_0$ Theorem 1 for $F_{q_0,q_0}^{a^o}(\mathbf{R}_n) = B_{q_0,q_0}^{a^o}(\mathbf{R}_n)$ and the hypotheses of Theorem 2 for are satisfied then we have the well-known standard situation of interpolation theory: The restriction operator $(Rf)(x') = f(x', 0)$ and the extension operator *T* from (26) are the same for both spaces. Then the interpolation property yields that R is also a retraction from $F^{\mathfrak{a}^{\theta}}_{p_{\theta},q_{\theta}}(\mathbf{R}_n)$ onto $F^{\mathfrak{a}'\mathfrak{b}}_{p_{\theta},q_{\theta}}(\mathbf{R}_{n-1})$ with $\mathfrak{a}'^{\theta} = \{a'^{\theta}_{\mathbf{k}'} = a^{\theta}_{(\mathbf{k}',0)}\}_{\mathbf{k}' \in \mathbf{Z}_{n-1}}$ and that T $p_0(p)$ ^x with $k = (k, 0)$
 $p_1 + z_1 \theta$ and $a_{(k',0)}^{\theta}$
 $(\mathbf{R}_n) = B_{\theta_0, q_0}^{\theta}(\mathbf{R}_n)$ and $\theta_{(k',0)}^{\theta}$
 θ have the well-kno
 $\mathbf{r}_0 \cdot \theta_0$ (\mathbf{R}_n) onto $F_{\theta_0, q_0}^{\theta}$
 $\mathbf{r}_0 \cdot \theta_0$ (\mathbf{R}_n) ont from (26) is a corresponding coretraction. Now one can apply this procedure in order to improve the limiting exponents for these spaces $F_{n,a}^{\mathfrak{a}}(\mathbf{R}_n)$ from Fig. 1. We are the same for both spaces. Then the interpolation property yields that *R* is also
a retraction from $F_{p_{\theta},q_{\theta}}^{a^{\theta}}(\mathbf{R}_n)$ onto $F_{p_{\theta},q_{\theta}}^{a^{\theta}}(\mathbf{R}_{n-1})$ with $a^{\prime\theta} = \{a^{\prime\theta}_{k'} = a^{\theta}_{(k',0)}\}_{k' \in \mathbf{Z}_{n-1}}$ $p < p_{\theta}$ and determine θ such that $\frac{1}{p_{\theta}} = \frac{1 - \theta}{q} + \frac{1}{p_{\theta}}$ $F_{q,q}^{\mathfrak{a}^{\bullet}}(\mathbf{R}_n) = B_{q,q}^{\mathfrak{a}^{\bullet}}(\mathbf{R}_n)$ and $F_{p,q}^{\mathfrak{a}^{\bullet}}(\mathbf{R}_n)$ via (49) with $\frac{1}{p_q} = \frac{1-\theta}{q} + \frac{\theta}{p}$ and $q_\theta q$ (this corresponds to the heavy line in Fig. 3). From $p \rightarrow 0$ follows $\theta \rightarrow 0$. Now we identify x_0 and x_1 with the limiting exponents from Fig. 2 and Fig. 1, respectively, i.e. $= 1 - \frac{1}{q}$ and $\varkappa_1 = \frac{1}{n}$ ve the well-known standard situation of interport
 $r(Rf)(x') = f(x', 0)$ and the extension operator

spaces. Then the interpolation property yields (\mathbf{R}_n) onto $F_{\rho_g}^{\rho_g} q_g(\mathbf{R}_{n-1})$ with $a'^{\rho} = \{a'^{\rho}_{k'} = a^{\rho}_{(k',0)}\}_{k' \$ *q* example, cf. Fig. 3. Let
 *B*_{q,q}(**R**_n) and $F_{p,q}^{a^*}(\mathbf{R}_n)$
 s to the heavy line in

and \varkappa_1 with the limiting $\frac{1}{q}$

and $\varkappa_1 = \frac{1}{p}$. Then we
 $= (1 - \theta) \varkappa_0 + \theta \varkappa_1 = ($
 $= \frac{1}{p_{\theta}} + (1 - \theta) (1 - \$ $x_{\theta} = (1 - \theta) x_0 + \theta z_1 = (1 - \theta) \left(1 - \frac{1}{q}\right) + \frac{\theta}{p}$ *q* $\frac{1}{p_{\theta}} = \frac{1}{q} + \frac{1}{p}$. We
 *q*ia (49) with $\frac{1}{p_{\theta}} = \frac{1}{q}$
 Fig. 3). From $p \to 0$ for
 x ponents from Fig. 2 and

have
 $1 - \theta \left(1 - \frac{1}{q}\right) + \frac{\theta}{p}$
 $\frac{2}{q}$
 $\Rightarrow \frac{1}{p_{\theta}} - \frac{1}{q} + 1 - \frac{1}{q}$
 $\frac{2$

$$
= \frac{1}{p_{\theta}} + (1 - \theta) \left(1 - \frac{2}{q} \right) \rightarrow \frac{1}{p_{\theta}} - \frac{1}{q} + 1 - \frac{1}{q}.
$$

In other words: R is a retraction from $F^{\mathfrak{a}}_{p_{\theta},q}(\mathbf{R}_n)$ onto $F^{\mathfrak{a}'}_{p_{\theta},q}(\mathbf{R}_{n-1})$ where p_{θ} and q have In other words: *R* is a retraction from $F_{p_\theta, q}^{\mathfrak{a}}(\mathbf{R}_n)$ onto $F_{p_\theta, q}^{\mathfrak{a}'}(\mathbf{R}_{n-1})$ where p_θ and *q* have
the above meaning if $\mathfrak{a} = \{a_k\}_{k \in \mathbb{Z}_n}$ with $a_{(k',m)} \ge c |m|^k a_{(k',0)}, c > 0, \varkappa > \varkappa_\theta$ with Theorem 2. If we calculate the best possible limiting exponents x which can be obtained on that way then we arrive at Fig. 4, which is (at least partly) an improvement *of* the limiting exponents from Fig. 1. However we are not sure whether these limiting exponents are natural (in contrast to the limiting exponents for the spaces $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$ from Fig. 2).

5. **Continuous** Version

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<<<<<**
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5. Continuous Version

Let $\varphi(x) \geq 0$ be a compactly supported infinitely differentiable function on **R**_n with, say, 99. Continuous Version

1. Let $\varphi(x) \ge 0$ be a compactly supported infinitely differentiable function on \mathbf{R}_n with,

say,
 $\varphi(x) = 1$ if $x = (x_1, ..., x_n) \in \mathbf{R}_n$ and $|x_j| \le 1$, where $j = 1, ..., n$. (51)

1. Let $\varphi_p(x) = \var$

$$
\varphi(x) = 1
$$
 if $x = (x_1, ..., x_n) \in \mathbb{R}_n$ and $|x_j| \le 1$, where $j = 1, ..., n$. (51)

system in the sense of Section 2 after the immaterial replacement of condition (i) from Section 2 by the assumption that supp φ is compact. Roughly speaking we shall $\;$ -try to replace the discrete sequence ${\varphi_k(x)}_{k\in\mathbb{Z}_n}$ in Definition 1 of the spaces $B_{p,q}^{\alpha}(\mathbf{R}_n)$ and $F_{p,q}^{\alpha}(\mathbf{R}_n)$ by its continuous counterpart $\{\varphi_{\nu}(x)\}_{\nu \in \mathbf{R}_n}$. We extend the definition of the maximal function from (8) by uous Version
 ≥ 0 be a compactly supported infinitely differentiable function on \mathbf{R}_n wit

if $x = (x_1, ..., x_n) \in \mathbf{R}_n$ and $|x_j| \leq 1$, where $j = 1, ..., n$. (5
 $= \varphi(x - y)$ if $x \in \mathbf{R}_n$ and $y \in \mathbf{R}_n$. Of course, $\$ of Section 2 after the interpretation of the sequence $\{\varphi_k(x)\}_{k\in\mathbb{N}}$
 secrete sequence $\{\varphi_k(x)\}_{k\in\mathbb{N}}$
 m (8) by
 $\sup_{z\in\mathbf{R}_n} \frac{|(F^{-1}\varphi_y Ff)(x-z)|}{1+|z|^b}$, *X* differentiable function on \mathbf{R}_n with,
 ≤ 1 , where $j = 1, ..., n$. (51)
 Course, $\{\varphi_k(x)\}_{k \in \mathbb{Z}_n}$ is an admissible
 X and *X* and *E* admissible
 X and *Z* = 1, ..., *n*. (51)
 $\epsilon \mathbf{z}_n$ is an admissible

nent of condition (i)

lly speaking we shall

f the spaces $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$

the definition of the
 R_{*n*}, (52)

1, (53)

4nd *z*). Then $\mathfrak{a} = \{a_k\}$ *i p p*_{*p_{<i>a*}}(**R**_{*n*}) *by* its continuous counterpart $\{\varphi_k(x)\}_{k \in \mathbb{Z}_n}$ in $F_{p,q}^a(\mathbf{R}_n)$ by its continuous counterpart $\{\varphi_k(x)\}_{k \in \mathbb{Z}_n}$
maximal function from (8) by
 $(\varphi_p f)^* (x) = \sup_{z \in \mathbf{R}_n} \frac{|(F^{-1}\$

$$
(\varphi_y f)^* (x) = \sup_{z \in \mathbf{R}_n} \frac{|(F^{-1} \varphi_y F f)(x - z)|}{1 + |z|^b}, \quad x \in \mathbf{R}_n, \quad y \in \mathbf{R}_n,
$$
 (52)
\n) and $b > 0$.
\n
$$
e \text{ m 3: Let } a(y) > 0 \text{ be a continuous function on } \mathbf{R}_n \text{ with}
$$

\n
$$
a(y) \le ca(z) \quad \text{if} \quad y \in \mathbf{R}_n, \quad z \in \mathbf{R}_n \quad \text{and} \quad |y - z| \le 1,
$$
 (53)
\n
$$
e \text{ a is an annormalize number (which is independent of } y \text{ and } z). \text{ Then } a = \{a_n\}
$$

. $f \in S^c(\mathbf{R}_n)$ and $b>0$.

 Theorem 3: *Let* $a(y) > 0$ *be a continuous function on* \mathbf{R}_n *with*

$$
a(y) \leq ca(z) \quad \text{if} \quad y \in \mathbf{R}_n, \qquad z \in \mathbf{R}_n \quad \text{and} \quad |y - z| \leq 1, \tag{53}
$$

where c > 0 is an appropriate number (which is independent of y and z). Then $a = \{a_k\}$ $= a(k)$ _{$\mathbf{k} \in \mathbf{Z}_n$ *satisfies* (5).}

$$
= a(k) \log z_n \text{ satisfies (5)}.
$$

(i) Let $0 < p \le \infty$ and $0 < q \le \infty$. Let $b > \frac{n}{p}$ in (52). Then

$$
(\varphi_y f)^* (x) = \sup_{z \in \mathbf{R}_n} \frac{|F \cdot \varphi_y F f|(x - z)|}{1 + |z|^b}, \quad x \in \mathbf{R}_n, \quad y \in \mathbf{R}_n,
$$
 (52)
\n
$$
(\mathbf{R}_n, \mathbf{R}_n) \quad \text{and} \quad b > 0.
$$
\n
$$
\text{From 3: Let } a(y) > 0 \text{ be a continuous function on } \mathbf{R}_n \text{ with}
$$
\n
$$
a(y) \leq ca(z) \quad \text{if} \quad y \in \mathbf{R}_n, \quad z \in \mathbf{R}_n \quad \text{and} \quad |y - z| \leq 1,
$$
 (53)
\n
$$
0 \text{ is an appropriate number (which is independent of } y \text{ and } z). \text{ Then } a = \{a_k\}
$$
\n
$$
\mathbf{z}_n \text{ satisfies (5)}.
$$
\n
$$
0 < p \leq \infty \text{ and } 0 < q \leq \infty. \text{ Let } b > \frac{n}{p} \text{ in (52). Then}
$$
\n
$$
\left(\int_{\mathbf{R}_n} a^q(y) \, ||F^{-1}\varphi_y F f | L_p(\mathbf{R}_n) ||^q \, dy\right)^{1/q}
$$
 (54)
\n
$$
\left(\int_{\mathbf{R}_n} a^q(y) \, ||(\varphi_y f)^* | L_p(\mathbf{R}_n) ||^q \, dy\right)^{1/q}
$$
 (55)
\n
$$
\text{aient quasi-norms on } B_{p,q}^a(\mathbf{R}_n). \text{ (Usually modification if } q = \infty).
$$

and

$$
\left(\int\limits_{\mathbf{R}_n} a^q(y) \ ||(\varphi_y f)^* || L_p(\mathbf{R}_n) ||^q \ dy\right)^{1/q} \tag{55}
$$

are equivalent quasi-norms on $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$. (Usual modification if $q = \infty$).

$$
a(y) \leq ca(z) \quad \text{if} \quad y \in \mathbb{R}_n, \quad z \in \mathbb{R}_n \quad \text{and} \quad |y - z| \leq 1, \tag{53}
$$
\nwhere $c > 0$ is an appropriate number (which is independent of y and z). Then $a = \{a_k\}$
\n $= a(k)\}_{k\in\mathbb{Z}_n}$ satisfies (5).\n(i) Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $b > \frac{n}{p}$ in (52). Then\n
$$
\left(\int_R a^q(y) \|F^{-1}p_y Ff \mid L_p(\mathbf{R}_n)\|^q dy\right)^{1/q} \tag{54}
$$
\nand\n
$$
\left(\int_R a^q(y) \|(\varphi_p f)^* \mid L_p(\mathbf{R}_n)\|^q dy\right)^{1/q} \tag{55}
$$
\nare equivalent quasi-norms on $B_{p,q}^{\alpha}(\mathbf{R}_n)$. (Usually modification if $q = \infty$).\n(ii) Let $0 < p < \infty$ and $0 < q \leq \infty$. Let $b > \frac{n}{\min(p,q)}$ in (52). Then\n
$$
\left\|\left(\int_R a^q(y) |(F^{-1}p_y Ff) \cdot (\cdot)|^q dy\right)^{1/q} |L_p(\mathbf{R}_n)\right\| \tag{56}
$$
\nand\n
$$
\left\|\left(\int_R a^q(y) |(F^{-1}p_y Ff) \cdot (\cdot)|^q dy\right)^{1/q} |L_p(\mathbf{R}_n)\right\| \tag{57}
$$
\nare equivalent quasi-norms on $F_{p,q}^{\alpha}(\mathbf{R}_n)$. Usually modification if $q = \infty$).\nProof. Step 1. It is obvious that the sequence a satisfies (5). We prove (i). By the multiplier theorem from [16: 1.5.2 or 1.6.3] it follows that there exists a positive number c such that\n
$$
\|F^{-1}p_y Ff \mid L_p(\mathbf{R}_n)\| \leq c \|F^{-1}p_k Ff \mid L_p(\mathbf{R}_n)\| + \cdots
$$
\nfor all $y \in \mathbf{R}_n$ and all $f \in S^c(\mathbf{R}_n)$, where $k \in \mathbf{Z}_n$ is the nearest lattice point to y,

$$
\left\| \left(\int_{\mathbf{R}_n} a^q(y) \, |(\varphi_y f)^* (\cdot) |^q \, dy \right)^{1/q} \right| \, L_p(\mathbf{R}_n) \, \right\| \tag{57}
$$

are equivalent quasi-norms on $F^a_{p,q}(\mathbf{R}_n)$ *. Usual modification if* $q = \infty$ *).*

Proof: $Step 1$. It is obvious that the sequence a satisfies (5). We prove (i). By the multiplier theorem from [16: 1.5.2 or 1.6.3] it follows that there exists a positive number *c* such that

$$
||F^{-1}\varphi_y F f \mid L_p(\mathbf{R}_n)|| \le c \, ||F^{-1}\varphi_k F f \mid L_p(\mathbf{R}_n)|| + \cdots \tag{58}
$$

for all $y \in \mathbf{R}_n$ and all $f \in S^c(\mathbf{R}_n)$, where $k \in \mathbf{Z}_n$ is the nearest lattice point to *y*, and $+ \cdots$ indicates terms with $F^{-1}\varphi_l Ff$ where $l \in \mathbb{Z}_n$ and $|l - k| \leq c'$ (the constant *c'* depends only on φ). Conversely, if $k \in \mathbb{Z}_n$ and $y \in \mathbb{R}_n$ with $|k - y| \leq 1$, then we have by the same multiplier theorem that there exists an appropriate positive number *c*

such that

$$
||F^{-1}\varphi_k Ff \mid L_p(\mathbf{R}_n)|| \le c \, ||F^{-1}\varphi_\nu Ff \mid L_p(\mathbf{R}_n)|| + \cdots \tag{59}
$$

-

 \mathbf{v} -

 $|L_p(\mathbf{R}_n)| \le c ||F^{-1} \varphi_y F f | L_p(\mathbf{R}_n) || + \cdots$
Here $+ \cdots$ indicates terms with φ_{y+e} instead on c' depends only on φ). Integration over y with $a(y)$, resp. $a(k)$, q-power summation over For all $f \in S^c(\mathbf{R}_n)$. Here $\|\mathbf{F}^{-1}\varphi_k F f \mid L_p(\mathbf{R}_n)\| \leq c \|F^{-1}\varphi_\nu F f \mid L_p(\mathbf{R}_n)\| + \cdots$ (59)
for all $f \in S^c(\mathbf{R}_n)$. Here $+ \cdots$ indicates terms with $\varphi_{\nu+\varrho}$ instead of φ_ν , where $\varrho \in \mathbf{Z}_n$
and $[\varrho]$ (59), multiplication with $a(y)$, resp. $a(k)$, q-power summation over k and a similar procedure starting with (58) show that (54) is an equivalent quasi-norm on $B^{\mathfrak{a}}_{p,q}(\mathbf{R}_n)$. for all $f \in S^c(\mathbf{R}_n)$. Here $+ \cdots$ indicates terms with φ_{y+e} instead of φ_y , where $g \in \mathbf{Z}_n$
and $|g| \le c'$ (again c' depends only on φ). Integration over y with $|k - y| \le 1$ in
(59), multiplication with Under the same hypotheses as in (58) and with the same interpretation of $+\cdots$ we have (*X*) $\|F^{-1}\varphi_k Ff \mid L_p(\mathbf{R}_n)\| \leq c \|F^{-1}\varphi_\nu Ff \mid L_p(\mathbf{R}_n)\| + \cdots$
 $\in S^c(\mathbf{R}_n)$. Here $+ \cdots$ indicates terms with $\varphi_{\nu+e}$ ins
 $\in c'$ (again c' depends only on φ). Integration over

tiplication with $a(y)$, res $p_{y}Ff \mid L_{p}(\mathbf{R}_{n})|| + \cdots$
es terms with φ_{y+e}
on φ). Integration ($u(k)$, q -power summ
at (54) is an equival)
and with the same
 $x \in \mathbf{R}_{n}$.
 \downarrow [16]. Similarly we
 $x \in \mathbf{R}_{n}$,
(and with the same
he abov and $|e| \leq c'$ (again c' depends only on φ). Integration over y with $|k - y| \leq 1$ in

(59), multiplication with $a(y)$, resp. $a(k)$, q -power summation over k and a similar

procedure starting with (58) show that (54)

$$
(\varphi_y f)^* (x) \le c(\varphi_t f)^* (x) + \cdots, \qquad x \in \mathbb{R}_n.
$$

\n(60)
\nows from formula (1.6.3/2) in [16]. Similarly we have
\n
$$
(\varphi_t f)^* (x) \le c(\varphi_y f)^* (x) + \cdots, \qquad x \in \mathbb{R}_n,
$$

\na sawo brancheses as in (50) (and with the same interpretation of + ...) By

This follows from formula $(1.6.3/2)$ in [16]. Similarly we have

$$
(\varphi_k f)^* (x) \leq c(\varphi_y f)^* (x) + \cdots, \qquad x \in \mathbf{R}_n,\tag{61}
$$

under the same hypotheses as in (59) (and with the same interpretation of $+ \cdots$). By is an equivalent quasi-norms on $B_{p,q}^{\mathfrak{a}}(\mathbf{R}_n)$.

Step 2. We outline the proof of (ii). By the above procedure it follows from (60), (61) and the Proposition from Section 2 that (57) is an equivalent quasi-norm on $F_{p,q}^{\alpha}(\mathbf{R}_n)$. Of course, the quasi-norm in (56) can be estimated from above by the quasistant c such that *• Step 2.* We outline the proof of (61) and the Proposition from Sec $F^a_{p,q}(\mathbf{R}_n)$. Of course, the quasi-norm norm in (57), and hence by $c \, ||f|| F^a_{p,q}$ stant c such that $||\left(\sum_{k \in \mathbb{Z}_n} a^q(k) \, |(F^{-1}\varphi_k Ff) \, (\cdot$ *contracted in the Proposition from Section*
 c (*v*) *c* (*v*) *c* (*course, the quasi*
 cof course, the quasi
 cof course, the quasi
 cof that
 $\left\| \left(\sum_{k \in \mathbb{Z}_n} a^q(k) \right| (F^{-1} \varphi_k F) \right\|$
 $c \left\| \left(\int_a a^q(y) \right$

$$
F_{p,q}^{\alpha}(\mathbf{R}_n)
$$
 Of course, the quasi-norm in (56) can be estimated from above by the quasi-
norm in (57), and hence by c ||f | $F_{p,q}^{\alpha}(\mathbf{R}_n)$ ||. The proof is complete if we can find a con-
stant c such that

$$
\left\| \left(\sum_{k \in \mathbb{Z}_n} a^q(k) | (F^{-1}\varphi_k F f) | \cdot |)^q \right\}^{1/q} \right\| L_p(\mathbf{R}_n) \right\|
$$

$$
c \left\| \left(\int_{\mathbf{R}_n} a^q(y) | (F^{-1}\varphi_k F f) | \cdot |)^q dy \right\}^{1/q} \left| L_p(\mathbf{R}_n) \right|
$$

$$
c \left\| \left(\int_{\mathbf{R}_n} a^q(y) | (F^{-1}\varphi_\nu F f) | \cdot |)^q dy \right\}^{1/q} \left| L_p(\mathbf{R}_n) \right| \right\|
$$

$$
bolds for all $f \in S^c(\mathbf{R}_n)$. For the purpose we introduce

$$
\psi_k(x) = \int_{|y-k| \le 1} \varphi_y(x) dy, \qquad k \in \mathbf{Z}_n.
$$
$$

holds for all $f \in S^{\rm c}({\bf R}_n)$. For the purpose we introduce

$$
\psi_k(x) = \int\limits_{|y-k| \leq 1} \varphi_y(x) \, dy, \qquad k \in \mathbb{Z}_n.
$$

Then $\{\psi_k(x)\}_{k\in \mathbb{Z}_n}$ is an admissible system in the sense of Section 2 (again with an immaterial modification of condition (i), from Section 2). In particular, the left-hand side of (62) can be estimated from above by a corresponding quasi-norm with ψ_k from in (91), and hence

stant c such that
 $\|\left(\sum_{k \in \mathbb{Z}_n} a^q(k) \right) (F \|_{\mathbf{R}_n}$
 $c \|\left(\int_{\mathbf{R}_n} a^q(y) \right) (F \|_{\mathbf{R}_n})$
 $\psi_k(x) = \int_{\|y - k\| \le 1} q$

Then $\{\psi_k(x)\}_{k \in \mathbb{Z}_n}$ is an a

material modification

side of (62) $c \left\| \left(\int_{\mathbf{R}_n} a^q(y) \right) (F \right\|_{\mathbf{R}_n}$

holds for all $f \in S^c(\mathbf{R}_n)$.
 $\psi_k(x) = \int_{|y-k| \leq 1} \varphi$

Then $\{\psi_k(x)\}_{k \in \mathbf{Z}_n}$ is an a

material modification

side of (62) can be est

instead of φ_k . We have
 $(F^{-1}\$ Fig. $\int f(x) \, dx$ $\int f(x) \, dx$ $\int f(x) \, dx$. For the purpose we introduce
 $\int f(x) \, dy$, $k \in \mathbb{Z}_n$.

dmissible system in the sense of Section 2 (again with an im-

of condition (i) from Section 2). In particular, the left-hand instead of φ_k . We have ain with an
 r , the left-h:
 i -norm with
 i -norm with
 (60) that
 $\frac{1}{2}$, $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$

$$
(F^{-1}\psi_k F f) (x) = \int_{|y-k| \le 1} (F^{-1}\varphi_y F f) (x) dy.
$$

then Hölder's inequality yields

$$
|(F^{-1}\psi_k F f) (x)|^q \le c \int_{|y-k| \le 1} |(F^{-1}\varphi_k F f) (x)|^q dy
$$

If $q \geq 1$ then Hölder's inequality yields

• •

• •

 $|y-k|\leq 1$

• and (62) follows easely. If $0 < q < 1$ then we obtain from (63) and (60) that

$$
|(F^{-1}\psi_k Ff)(x)|^q \leq c \int_{|y-k| \leq 1} |(F^{-1}\psi_k Ff)(x)|^q dy
$$

and (62) follows easily. If $0 < q < 1$ then we obtain from (63) and (60) that

$$
|(F^{-1}\psi_k Ff)(x)| \leq c \int_{|y-k| \leq 1} |(F^{-1}\psi_k Ff)(x)|^q dy \big((\varphi_k f)^{1-q}(x) + \cdots \big), \tag{64}
$$

where $+ \cdots$ has the same meaning as in (60). By Hölder's inequality with $q + (1 - q)$ $=1$ we have

then Hölder's inequality yields
\n
$$
|(F^{-1}\psi_k Ff)(x)|^q \leq c \int \frac{|(F^{-1}\psi_k Ff)(x)|^q dy}{|y-k|\leq 1}
$$
\nfollows easily. If $0 < q < 1$ then we obtain from (63) and (60) th
\n
$$
|(F^{-1}\psi_k Ff)(x)| \leq c \int \frac{|(F^{-1}\psi_k Ff)(x)|^q dy((\varphi_k f)^{*1-q}(x) + \cdots),}{|y-k|\leq 1}
$$
\n
$$
\cdots
$$
 has the same meaning as in (60). By Hölder's inequality with q -
\nhave
\n
$$
\sum_{k \in \mathbb{Z}_n} a^q(k) |(F^{-1}\psi_k Ff)(x)|^q
$$
\n
$$
\leq \left(\sum_{k \in \mathbb{Z}_n} a^q(k) \int \frac{|(F^{-1}\psi_k Ff)(x)|^q dy}{|y-k|\leq 1} d\psi\right)^q \left(\sum_{k \in \mathbb{Z}_n} a^q(k) (\varphi_k f)^{*q}(x)\right)^{1-q} + \cdots
$$

/

Now it follows from (62),.a second application of Holder's inequality with.respect to $pq + p(1 - q) = p$ and the Proposition from Section 2 that

Modulation Sps
\nallows from (62), a second application of Hölder's inequality wit
\n
$$
1-q) = p
$$
 and the Proposition from Section 2 that
\n
$$
||f| F_{p,q}^a(\mathbf{R}_n) ||
$$
\n
$$
\leq c \left| \left| \left(\int_{\mathbf{R}_n} a^q(y) \mid (F^{-1} \varphi_y F f) (\cdot) |^q dy \right)^{1/q} \right| L_p(\mathbf{R}_n) \right|^{q} ||f|| F_{p,q}^a(\mathbf{R}_n) ||^{1-q},
$$

which yields the desired inequality. \blacksquare

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