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Uniform approximation by solutions of general boundary value problems for elliptic equations of arbitrary order I

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Es sei $\Omega \subset \mathbf{R}^n$ ein beschränktes, glattes Gebiet, Γ eine geschlossene, glatte, $(n-1)$-dimensionale Fläche im Innern von Ω und V eine offene Teilmenge des Randes $\partial\Omega$. In Ω werde ein eigentlich elliptischer Differentialoperator L beliebiger Ordnung mit glatten Koeffizienten betrachtet. $B_1, \ldots, B_m$ sei ein normales System von Randoperatoren auf $\partial\Omega$, welches der klassischen Wurzelbedingung genügt. $L_V(\Gamma)$ bezeichne den Raum der Einschränkungen der Funktionen des Raumes

$$L_V(\Omega) = \{u : Lu = 0 \text{ in } \Omega,\ B_1 u|_{\partial\Omega} = \cdots = B_m u|_{\partial\Omega} = 0 \quad \text{in } \partial\Omega \setminus V\}$$

auf Γ. Es wird unter anderem bewiesen, daß $L_V(\Gamma)$ im Raum $W^{m-1}(\Gamma)$ der Whitneyschen Taylorfelder der Ordnung $m-1$ dicht liegt, d. h., alle Ableitungen bis zur Ordnung $m-1$ lassen sich auf Γ gleichmäßig approximieren.

Пусть $\Omega \subset \mathbf{R}^n$ — органиченная, гладкая область, Γ — замкнутая, гладкая $(n-1)$-мерная площадь внутри области Ω и V — открытое подмножество края $\partial\Omega$. Рассматривается в Ω собственный эллиптический дифференциальный оператор L любого порядка с гладкими коэффициентами. Пусть $B_1, \ldots, B_m$ — нормальная система краевых операторов на $\partial\Omega$, удовлетворяющая классическому условию на корни. $L_V(\Gamma)$ обозначает пространство ограничений на Γ функций пространства

$$L_V(\Omega) = \{u : Lu = 0 \text{ в } \Omega,\ B_1 u|_{\partial\Omega} = \cdots = B_m u|_{\partial\Omega} = 0 \text{ в } \partial\Omega \setminus V\}.$$

Доказывается, между прочим, что $L_V(\Gamma)$ плотно в пространстве $W^{m-1}(\Gamma)$ Тейлоровых полей Витнея порядка $m-1$, т. е. что все производные до $(m-1)$-ого порядка допускают равномерную аппроксимацию на Γ.

Let $\Omega \subset \mathbf{R}^n$ be a bounded, smooth domain, Γ a closed, smooth, $(n-1)$-dimensional surface in the interior of Ω and V an open subset of the boundary $\partial\Omega$. In Ω we consider a properly elliptic differential operator L of arbitrary order with smooth coefficients. Let $B_1, \ldots, B_m$ be a normal system of boundary operators on $\partial\Omega$, which fulfils the classical roots condition. $L_V(\Gamma)$ denotes the space of the restrictions on Γ of the functions from

$$L_V(\Omega) = \{u : Lu = 0 \text{ in } \Omega,\ B_1 u|_{\partial\Omega} = \cdots = B_m u|_{\partial\Omega} = 0 \text{ in } \partial\Omega \setminus V\}.$$

Among other things it is proved, that the space $L_V(\Gamma)$ is dense in the space $W^{m-1}(\Gamma)$ of the Whitney-Taylorfields of the order $m-1$, i.e. all derivatives up to the order $m-1$ can be uniformly approximated on Γ.

1. In 1960 H. BECKERT [2] proved the following result: Let L be an elliptic differential operator of the second order with sufficiently smooth coefficients, $\Omega \subset \mathbf{R}^n$ a bounded smooth domain, $\Gamma \subset \Omega$ a smooth surface, such that $\Omega \setminus \Gamma$ is connected, V a given open subset of the boundary $\partial\Omega$ and

$$L_V(\Omega) = \{u \in C^2(\Omega) \cap C(\bar{\Omega}) : Lu = 0 \quad \text{in} \quad \Omega,\ u|_{\partial\Omega \setminus V} = 0\}.$$

Furthermore it is supposed, that the homogeneous Dirichlet problem for $Lu = 0$ does not have a non-trivial solution in Ω (Condition (U)). Then the space $L_V(\Gamma)$ of the restrictions on Γ of the space $L_V(\Omega)$ is dense in $L^2(\Gamma)$.

A. GÖPFERT [6, 7] generalized this result in the following directions:

- For the case, that the condition (U) for the domain Ω does not hold.
- For the case, that Γ is a closed surface in Ω, such that $\Omega \setminus \Gamma$ is not connected.
- For the second and the third boundary-value problem.
- For the elliptic system of the theory of elasticity.
- For parabolic equations of the second order.

In the case of the Laplace operator and if the boundary $\partial\Omega$ is regular and $\Omega \setminus \Gamma$ is connected, G. ANGER [1] proved the density of $L_V(\Gamma)$ in $C(\Gamma)$. G. WANKA [10] has given this result for general elliptic equations of the second order with sufficiently smooth coefficients — also for a closed surface Γ and without condition (U).

In the present paper a corresponding theorem for elliptic equations of arbitrary order with smooth coefficients is given. We shall prove (Theorem 2), that the space $L_V(\Gamma)$ of the restrictions on Γ of the space

$$L_V(\Omega) = \{u : Lu = 0 \quad \text{in} \quad \Omega, B_1u|_{\partial\Omega} = \cdots = B_mu|_{\partial\Omega} = 0 \quad \text{in} \quad \partial\Omega \setminus V\} \tag{1.1}$$

is dense in the space $W^{m-1}(\Gamma)$ of the Whitney-Taylorfields on Γ ($2m$ order of the differential operator). This means, all the derivatives up to the order $m - 1$ can be approximated uniformly by the corresponding derivatives of $u \in L_V(\Omega)$. $B_1, B_2, \ldots, B_m$ is a given normal system of boundary operators with ord $B_j \leq 2m - 1$, which fulfils the classical roots condition (see [8, 9]). Here we suppose, that the condition (U) with respect to $L, B_1, \ldots, B_m$ and Ω is fulfiled. Furthermore it must be supposed, that there are not eigensolutions of the Dirichlet problem for $Lu = 0$ in the inner domain Ω_i, which is bounded by Γ. Otherwise the assertion of Theorem 1 and 2 fails (Theorem 5). Contrary to H. Beckert, A. Göpfert and G. Wanka in the proof we do not use directly the Cauchy problem. We consider an open set $G \subset \Omega_a := \Omega \setminus \bar{\Omega}_i$ and define

$$\begin{aligned} L_G(\Omega) = \{u : Lu = g \text{ in } \Omega, g \in C^\lambda(\Omega)\ (0 < \lambda < 1), \\ g \equiv 0 \text{ in } \Omega / G, \quad B_ju|_{\partial\Omega} = 0 \quad (j = 1, \ldots, m)\}. \end{aligned} \tag{1.2}$$

Under the so-called condition for uniqueness in the Cauchy problem in the small for the adjoint operator L^* we prove, that the space $L_G(\Gamma)$ of the restrictions on Γ of the space $L_G(\Omega)$ is dense in $W^{m-1}(\Gamma)$ (Theorem 1). The proof of Theorem 2 is then reduced to Theorem 1. Moreover we use results from the potential theory of eelliptic equations of higher order, developed in [11, 9].

Founding upon the same idea one can prove the density of $L_V(\Gamma)$ in the Sobolev space $W_2^{2m-1}(\Gamma)$, a generalization of a further result of H. BECKERT [2] to elliptic equations of higher order. This will be published in a forthcoming paper. For $\Omega \setminus \Gamma$ connected the approximation in $W^{m-1}(\Gamma)$ by potentials is studied in [9]. Theorems of an other type for the approximation of solutions of elliptic equations of arbitrary order are given by F. E. BROWDER [4, 5].

In Section 2 we summarize the basic facts from the general theory of elliptic boundary value problems and from the potential theory. In Section 3 we prove the Theorems 1 and 2. Section 4 contains some additional remarks.

2. Let

$$L = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$$

($m > 0$ an integer, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geq 0$ integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$) be a properly elliptic differential operator with real coefficients in $C^\infty(\mathbf{R}^n)$. We suppose, that for the adjoint operator

$$L^*u = \sum_{|\alpha| \leqq 2m} (-1)^{|\alpha|} D^\alpha\big(a_\alpha(x)\, u\big)$$

the "condition for uniqueness in the Cauchy problem in the small" holds. This means, if $u \in C^{2m}(\tilde{\Omega})$ is a solution of $L^*u = 0$ in a connected open set $\tilde{\Omega}$, vanishing on a non-vacuous open subset $\Omega_0 \subset \tilde{\Omega}$, then u must be identically zero in $\tilde{\Omega}$.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and Γ a $(n-1)$-dimensional, closed, smooth surface, which splits up the domain Ω in two parts Ω_i and Ω_a:

$$\Omega = \Omega_i \cup \Omega_a \cup \Gamma.$$

It is further supposed, that the Dirichlet problem

$$Lu = 0 \quad \text{in} \quad \Omega_i,$$

$$\left.\frac{\partial^{j-1}u}{\partial n^{j-1}}\right|_\Gamma = 0 \quad (j = 1, \ldots, m;\ n \text{ outer normal direction})$$

only has the trivial solution.

On the boundary $\partial\Omega$ we consider a normal system $B_1, \ldots, B_m$ of boundary operators with ord $B_j = m_j \leqq 2m - 1$ and smooth coefficients, which fulfils the classical roots condition (see [8, 9, 13]). Moreover we suppose, that the problem

$$Lu = 0 \quad \text{in} \quad \Omega,$$

$$B_j u|_{\partial\Omega} = 0 \quad (j = 1, \ldots, m).$$

only has the trivial solution.

It is well-known (see [8, 13]), that the system $(B_j)_{j=1,\ldots,m}$ by a (not uniquely determined) normal system $(C_j)_{j=1,\ldots,m}$ (ord $C_j = l_j \leqq 2m - 1$) can be completed to a Dirichlet system $(B_1, \ldots, B_m, C_1, \ldots, C_m)$ of order $2m$ on $\partial\Omega$. This means, that the completed system is a normal system and the set of the orders of the operators is $\{0, 1, \ldots, 2m - 1\}$. If the operators $(C_j)_{j=1,\ldots,m}$ are fixed, then in an unique way one can find $2m$ boundary operators $(B_j')_{j=1,\ldots,m}$, $(C_j')_{j=1,\ldots,m}$ with smooth coefficients on $\partial\Omega$, such that the following properties hold:

(i) ord $B_j' = m_j' = 2m - 1 - l_j$, ord $C_j' = l_j' = 2m - 1 - m_j$

(ii) $(B_1', \ldots, B_m', C_1', \ldots, C_m')$ is a Dirichlet system of the order $2m$ on $\partial\Omega$ and for $u, v \in C^\infty(\Omega)$ the Green formula

$$\int_\Omega (Lu)\, v\, dx - \int_\Omega uL^*v\, dx = \sum_{j=1}^m \int_{\partial\Omega} C_j u B_j' v\, d\sigma - \sum_{j=1}^m \int_{\partial\Omega} B_j u C_j' v\, d\sigma \tag{2.1}$$

holds.

According to results of J. M. Berezanskij and J. A. Rojtberg [3] (see also [13]) the unique solution of the boundary value problem

$$Lu = g \quad \text{in} \quad \Omega,$$

$$B_j u|_{\partial\Omega} = \varphi_j \qquad (j = 1, \ldots, m)$$

under certain smoothness conditions for g and φ_j (which in the following always are fulfiled) can be represented by a Green function $G(x, y)$ in the form

$$u(x) = \int_\Omega g(y)\, G(x, y)\, dy + \sum_{j=1}^m \int_{\partial\Omega} \varphi_j(y)\, C_j'\, G(x, y)\, d\sigma(y). \tag{2.2}$$

The operators C_j' are the corresponding operators from the Green formula (2.1), and they are applied to the variable y. Under our conditions the function $G(x, y)$ for $x \neq y$ has derivatives of arbitrary order with respect to both variables. By application of the differential operators with respect to y we have

$$L^*G(x, y) = 0 \quad (x \neq y), \qquad (2.3)$$
$$B_j'G(x, y)|_{y \in \partial\Omega} = 0 \quad (j = 1, \ldots, m).$$

Furthermore we suppose the existence of a global fundamental solution $\Phi(x, y)$ for the operator L, i.e.

$$\varphi(z) = \int \Phi(x, z)\, L^*\varphi(x)\, dx, \; \varphi(x) = \int \Phi(x, z)\, L\, \varphi(z)\, dz \qquad (2.4)$$

for all $\varphi \in C_0^\infty(\mathbf{R}^n)$. Set $\Phi_\alpha^\beta(x, z) := D_x^\alpha D_z^\beta\, \Phi(x, z)$. For $|\alpha| + |\beta| \leqq 2m - 2$, $|x - z| \leqq$ q the estimates

$$|\Phi_\alpha^\beta(x, z)| \leqq \begin{cases} c_{\alpha\beta}\, |x - z|^{2-n} & \text{for} \quad n > 2 \\ c_{\alpha\beta}\, (1 + |\log|x - z||) & \text{for} \quad n = 2 \end{cases} \qquad (2.5)$$

hold (see [9]). The Green function $G(x, y)$ then can be written as a sum

$$G(x, y) = \Phi(x, y) + h(x, y), \qquad (2.6)$$

where $h(x, y)$ for fixed $x \in \Omega$ with respect to y is a regular solution of the following boundary value problem:

$$L^*h(x, y) = 0 \quad \text{in} \quad \Omega,$$
$$B_j'h(x, y)|_{y \in \partial\Omega} = -B_j'\Phi(x, y)|_{y \in \partial\Omega} \qquad (j = 1, \ldots, m) \qquad (2.7)$$

(The operators applied to y.) With $\Phi_\alpha(x, y) := D_x^\alpha \Phi(x, y)$ the function

$$\Phi^*\mu(x) = \sum_{|\alpha| \leq m-1} \int_\Gamma \Phi_\alpha(x, y)\, d\mu_\alpha(x) \qquad (2.8)$$

is called the *adjoint potential* with respect to the vector measure $(\mu_\alpha)_{|\alpha| \leq m-1}$ on Γ. In [9; p. 228—229] the potential is defined more general as a vector function. The function (2.8) is the first component of the potential in the sense of [9].

A system $g = (g_\alpha)_{|\alpha| \leq m-1}$ of continuous functions, defined on Γ, is called a Whitney-Taylorfield of order $m - 1$ on Γ, if there exists a function $\varphi \in C^{m-1}(\mathbf{R}^n)$ with $D^\alpha\varphi|_\Gamma = g_\alpha(|\alpha| \leqq m - 1)$. $W^{m-1}(\Gamma)$ denotes the vector space of all such Whitney-Taylorfields. From the smoothness of Γ follows, that $W^{m-1}(\Gamma)$ is a Banach space with respect to the norm

$$\|g\|_{W^{m-1}(\Gamma)} = \sum_{|\alpha| \leqq m-1} \sup_{x \in \Gamma} |g_\alpha(x)| \qquad (2.9)$$

($(m - 1)$ — regularity of Γ). Every continuous linear functional l on $W^{m-1}(\Gamma)$ can be represented with the help of a vector measure $(\mu_\alpha)_{|\alpha| \leq m-1}$ (supp $\mu_\alpha \subset \Gamma$) in the form

$$l(g) = \sum_{|\alpha| \leqq m-1} \int_\Gamma g_\alpha(x)\, d\mu_\alpha(x) \qquad (2.10)$$

(see [9]).

For the proof of Theorem 1 we need a special result of the general balayage theory (developed in [11, 9]) for the domain Ω_i. At first we consider the Dirichlet problem for L^* and Ω_i in the formulation with Whitney-Taylorfields: Given a Taylorfield $g = (g_\alpha) \in W^{m-1}(\Gamma)$ we are looking for a solution of the equation $L^*w = 0$ in Ω_i with $D^\beta w|_\Gamma = (g_\beta)_{|\beta| \leq m-1}$ for $|\beta| \leqq m - 1$. In [13: p. 77, remark 7.2] it is shown, that under the preceding conditions there exist so-called harmonic measures $\tau_z^\beta(|\beta| \leqq m - 1,\ z \in \Omega_i)$, which can be represented by smooth densities on Γ, such that the unique solution of the Dirichlet problem for $z \in \Omega_i$ is given by

$$w(z) = \sum_{|\beta| \leq m-1} \int_\Gamma g_\beta(y) \, d\tau_z^\beta(y). \tag{2.11}$$

Moreover, from the general balayage theory in [9] follows.

Lemma 1: *For $z \in \Omega_i$ fixed we have*

$$\Phi_\alpha(x, z) = \sum_{|\beta| \leq m-1} \int_\Gamma \Phi_\alpha^\beta(x, y) \, d\tau_z^\beta(y)$$

for any $x \in \Gamma$ and $|\alpha| \leq m - 1$.

It is well-known, that the fundamental solution of the Laplace operator is given by

$$\Phi_N(x, y) = \begin{cases} \dfrac{1}{(n-2)\,\omega_n} \dfrac{1}{|n - y|^{n-2}} & \text{for } n > 2 \\ \dfrac{1}{2\pi} \log \dfrac{1}{|x - y|} & \text{for } n = 2 \end{cases}$$

(ω_n area of the n-dimensional unit sphere). Let $\mathfrak{M}_0^+$ be the set of all nonnegative Radon measures with compact support and $\mathfrak{F}^+ \subset \mathfrak{M}_0^+$ the subset of all measures λ, for which the Newton potentials

$$\Phi_N\lambda(x) = \int \Phi_N(x, y) \, d\lambda(y)$$

are continuous in the whole space. $\mathscr{I}_\lambda$ denotes the class of the universal measurable sets A with $\lambda(A) = 0$ and

$$\mathscr{I}_N = \bigcap_{\lambda \in \mathfrak{F}^+} \mathscr{I}_\lambda .$$

A Borel set B is called a *set of capacity zero*, if $B \in \mathscr{I}_N$. If an assertion holds with the exception of a set of capacity zero, then we shall say, that the assertion holds $\mathscr{I}_N$-almost everywhere ($\mathscr{I}_N$-a.e.). For instance we have (see [9])

Lemma 2: *For an arbitrary measure $\mu \in \mathfrak{M}_0^+$*

$$\Phi_N\mu(x) < \infty \ \mathscr{I}_N\text{-a.e. holds.}$$

The measure μ is called *$\mathscr{I}_N$-absolutely continuous,* if $\mathscr{I}_N \subseteq \mathscr{I}_\mu$. For instance the harmonic measures in (2.11) are $\mathscr{I}_N$-absolutely continuous.

3. Theorem 1: *Let be fulfiled the suppositions of Section 2 with respect to L, Ω and Γ. Then the space $L_G(\Gamma)$ is dense in $W^{m-1}(\Gamma)$ with respect to the norm (2.9).*

Proof: a) $\overline{L_G(\Gamma)} = W^{m-1}(\Gamma)$ holds iff every $l \in \left(W^{m-1}(\Gamma)\right)'$ with $l(u) = 0$ for all $u \in L_G(\Gamma)$ vanishes identically. Therefore we consider $l \in \left(W^{m-1}(\Gamma)\right)'$ and suppose

$$l(u) = 0 \quad \text{for all} \quad u \in L_G(\Gamma). \tag{3.1}$$

By (2.10) l can be represented with the help of a vector measure $(\mu_\alpha)_{|\alpha| \leq m-1}$ (supp $\mu_\alpha \subseteq \Gamma$) and we get

$$l(u) = \sum_{|\alpha| \leq m-1} \int_\Gamma D^\alpha u(x) \, d\mu_\alpha(x). \tag{3.2}$$

By (2.2) for $u \in L_C(\Gamma)$ and $x \in \Gamma$ we have

$$u(x) = \int_G g(y) \, G(x, y) \, dy. \tag{3.3}$$

Because $G \cap \Gamma = \varphi$ the derivatives $D^\alpha u$ on Γ can be calculated by differentiation of (3.3) under the integral:

$$D^\alpha u(x) = \int_G g(y) \cdot D_x{}^\alpha G(x, y)\, dy \tag{3.4}$$

$x \in \Gamma$, $|\alpha| \leqq m - 1$). Putting (3.4) in (3.2) it follows

$$l(u) = \sum_{|\alpha| \leqq m-1} \int_\Gamma \left(\int_G g(y)\, D_x{}^\alpha G(x, y)\, dy \right) d\mu_\alpha(x)$$

$$= \int_G g(y) \left\{ \sum_{|\alpha| \leqq m-1} \int_\Gamma D_x{}^\alpha G(x, y)\, d\mu_\alpha(x) \right\} dy.$$

We define $G^*\mu(y) := \sum_{|\alpha| \leqq m-1} \int_\Gamma D_x{}^\alpha G(x, y)\, d\mu_\alpha(x)$. Since g in G can be choosen arbitrary and since $G^*\mu$ is smooth on G (supp $\mu_\alpha \subset \Gamma$!), from (3.1) we get $G^*\mu(y) \equiv 0$ in G. From (2.3) follows ,that $G^*\mu$ is a solution of the equation $L^*u = 0$ in $\Omega \setminus \Gamma$. Therefore the condition for uniqueness in the Cauchy problem in the small implies $G^*\mu(y) \equiv 0$ in $\Omega \setminus \Gamma$.

b) In the next step we prove for $|\beta| \leqq m - 1$

$$D^\beta G^*\mu(y) := \sum_{|\alpha| \leqq m-1} \int_\Gamma D_y{}^\beta D_x{}^\alpha\, G(x, y)\, d\mu_\alpha(x) = 0 \qquad \mathcal{I}_N - \text{a.e. on } \Gamma.$$

Using (2.6) and (2.7), we have

$$G^*\mu(y) = \sum_{|\alpha| \leqq m-1} \int_\Gamma \Phi_\alpha(x, y)\, d\mu_\alpha(x) + R(y) = \Phi^*\mu(y) + R(y),$$

where $R(y)$ is smooth in a neighbourhood of Γ. Moreover

$$D^\beta G^*\mu(y) = \sum_{|\alpha| \leqq m-1} \int_\Gamma \Phi_\alpha{}^\beta(x, y)\, d\mu_\alpha(x) + D^\beta R(y).$$

Lemma 2 implies

$$\int_\Gamma \Phi_N(x, y)\, d\, |\mu_\alpha|\, (x) < \infty \qquad \mathcal{I}_N - \text{a.e. on } \Gamma.$$

We choose a point $y_1 \in \Gamma$ with

$$\int_\Gamma \Phi_N(x, y_1)\, d\, |\mu_\alpha|\, (x) < \infty \quad \text{for} \quad |\alpha| \leqq m - 1 \tag{3.5}$$

and consider $y \in \Omega_a$ on the normal direction through y_1. Since $G^*\mu \equiv 0$ in Ω_a, we have $D^\beta G^*\mu(y) = 0$, such that

$$|D^\beta G^*\mu(y_1)| = |D^\beta G^*\mu(y_1) - D^\beta G^*\mu(y)|$$

$$\leqq \left| \sum_{|\alpha| \leqq m-1} \int_\Gamma \Phi_\alpha{}^\beta(x, y_1)\, d\mu_\alpha(x) - \sum_{|\alpha| \leqq m-1} \int_\Gamma \Phi_\alpha{}^\beta(x, y)\, d\mu_\alpha(x) \right|$$

$$+ |D^\beta R(y_1) - D^\beta R(y)|.$$

Obviously $|D^\beta R(y_1) - D^\beta R(y)| < \frac{\varepsilon}{2}$ for $|y - y_1| < \delta_1(\varepsilon)$ holds. Moreover we have

$$\begin{aligned}
&\left|\sum_{|\alpha|\leq m-1} \int_\Gamma \Phi_\alpha{}^\beta(x, y_1)\, d\mu_\alpha(x) - \sum_{|\alpha|\leq m-1} \int_\Gamma \Phi_\alpha{}^\beta(x, y)\, d\mu_\alpha(x)\right| \\
&\leq \sum_{|\alpha|\leq m-1} \int_\Gamma |\Phi_\alpha{}^\beta(x, y_1) - \Phi_\alpha{}^\beta(x, y)|\, d|\mu_\alpha|\,(x) \\
&= \sum_{|\alpha|\leq m-1} \int_{\Gamma\setminus K_\delta(y_1)} |\Phi_\alpha{}^\beta(x, y_1) - \Phi_\alpha{}^\beta(x, y)|\, d|\mu_\alpha|\,(x) \\
&\quad + \sum_{|\alpha|\leq m-1} \int_{\Gamma\cap K_\delta(y_1)} |\Phi_\alpha{}^\beta(x, y_1) - \Phi_\alpha{}^\beta(x, y)|\, d|\mu_\alpha|\,(x) \\
&= I_1 + I_2 .
\end{aligned}$$

$K_\delta(y_1)$ denotes the ball with center y_1 and radius δ. For $\varepsilon > 0$ we can find $\delta_2(\varepsilon) > 0$, such that

$$|\Phi_\alpha{}^\beta(x, y_1) - \Phi_\alpha{}^\beta(x, y)| < \frac{\varepsilon}{4|\mu|_{\max}(\Gamma)\, k}$$

for $|\alpha| \leq m - 1$, $|y - y_1| < \delta_2$ and uniformly with respect to $x \in \Gamma\setminus K_\delta(y_1)$. Here k is the number of the terms in I_1 and

$$|\mu|_{\max}(\Gamma) = \max_{|\alpha|\leq m-1} \int_\Gamma d\,|\mu_\alpha|\,(x).$$

Thus

$$|I_1| < \frac{\varepsilon}{4\,|\mu|_{\max}(\Gamma)\cdot k}\, k|\mu|_{\max}\big(\Gamma\setminus K_\delta(y_1)\big) < \frac{\varepsilon}{4}$$

for $|y - y_1| < \delta_2$.

From the smoothness condition for Γ follows, that there exists a cone in Ω_a with vertex in y_1 and axis in the normal direction. From this we obtain the existence of a constant $c_1 > 0$ with $|x - y| \geqq c_1\,|x - y_1|$ for $x \in \Gamma \cap K_\delta(y_1)$ and therefore

$$\Phi_N(x, y) \leqq c_2\Phi_N(x, y_1).$$

Using this inequality, (3.5) and the estimates (2.5), which hold since we have $|\alpha| + |\beta| \leqq 2m - 2$, we get

$$\begin{aligned}
|I_2| &\leqq \sum_{|\alpha|\leq m-1} \int_{\Gamma\cap K_\delta(y_1)} |\Phi_\alpha{}^\beta(x, y_1)|\, d\,|\mu_\alpha|\,(x) + \sum_{|\alpha|\leq m-1} \int_{\Gamma\cap K_\delta(y_1)} |\Phi_\alpha{}^\beta(x, y)|\, d\,|\mu_\alpha|\,(x) \\
&\leqq \sum_{|\alpha|\leq m-1} c_{\alpha\beta} \int_{\Gamma\cap K_\delta(y_1)} \Phi_N(x, y_1)\, d|\mu_\alpha|(x) + \sum_{|\alpha|\leq m-1} c_{\alpha\beta} \int_{\Gamma\cap K_\delta(y_1)} \Phi_N(x, y)\, d|\mu_\alpha|\,(x) \\
&\leqq \sum_{|\alpha|\leq m-1} \bar{c}_{\alpha\beta} \int_{\Gamma\cap K_\delta(y_1)} \Phi_N(x, y_1)\, d\,|\mu_\alpha|\,(x) < \frac{\varepsilon}{4}
\end{aligned}$$

for sufficiently small δ $\big(\bar{c}_{\alpha\beta} = c_{\alpha\beta}\,(1 + c_2)\big)$.

We now have $|D^\beta G^*\mu(y_1)| < \varepsilon$ for any $\varepsilon > 0$, i.e. $D^\beta G^*\mu(y_1) = 0$. By Lemma 2 this holds for every $\beta(|\beta| \leqq m - 1)$ and $\mathscr{I}_N$ — a.e. on Γ.

c) Next, we prove $G^*\mu(z) = 0$ for all $z \in \Omega_i$. We write down the right-hand side of (2.11), replacing $g_\beta(y)$ by the expressions $D^\beta G^*\mu(y)$, which vanishes $\mathscr{I}_N$-almost everywhere on Γ. Because the harmonic measures $\tau_z{}^\beta$ are $\mathscr{I}_N$-absolutely continuous and using (2.6) we get

$$0 = \sum_{|\beta| \leq m-1} \int_\Gamma \left\{ \sum_{\alpha \leq m-1} \int_\Gamma D_x^\alpha D_y^\beta G(x, y) \, d\mu_\alpha(x) \right\} d\tau_z^\beta(y)$$

$$= \sum_{|\alpha| \leq m-1} \int_\Gamma \left\{ \sum_{|\beta| \leq m-1} \int_\Gamma D_x^\alpha D_y^\beta G(x, y) \, d\tau_z^\beta(y) \right\} d\mu_\alpha(x)$$

$$= \sum_{|\alpha| \leq m-1} \int_\Gamma \left\{ \sum_{|\beta| \leq m-1} \int_\Gamma \Phi_\alpha^\beta(x, y) \, d\tau_z^\beta(y) \right\} d\mu_\alpha(x)$$

$$+ \sum_{|\alpha| \leq m-1} \int_\Gamma \left\{ \sum_{|\beta| \leq m-1} \int_\Gamma D_x^\alpha D_y^\beta h(x, y) \, d\tau_z^\beta(y) \right\} d\mu_\alpha(x). \qquad (3.6)$$

For fixed $x \in \Gamma$ the function $h(x, y)$ with respect to z is a smooth solution of the equation $L^*w = 0$ in Ω_i with boundary values $D_y^\beta h(x, y)$. Consequently by (2.11) we have

$$h(x, z) = \sum_{|\beta| \leq m-1} \int_\Gamma D_y^\beta h(x, y) \, d\tau_z^\beta(y).$$

Differentiation with respect to x implies

$$D_x^\alpha h(x, z) = \sum_{|\beta| \leq m-1} \int_\Gamma D_x^\alpha D_y^\beta h(x, y) \, d\tau_z^\beta(x)$$

for $|\alpha| \leq m - 1$. Using Lemma 1, from (3.6) then follows

$$0 = \sum_{|\alpha| \leq m-1} \int_\Gamma \Phi_\alpha(x, z) \, d\mu_\alpha(x) + \sum_{|\alpha| \leq m-1} \int_\Gamma D_x^\alpha h(x, z) \, d\mu_\alpha(x)$$

$$= \sum_{|\alpha| \leq m-1} \int_\Gamma D_x^\alpha G(x, z) \, d\mu_\alpha(x) = G^*\mu(z)$$

for any $z \in \Omega_i$.

d) Using $G^*\mu(y) = 0$ $\mathscr{I}_N$-almost everywhere in Ω, we shall conclude $l = 0$. For that reason we consider the set

$$D(\Gamma) = \{\varphi|_\Gamma : \varphi \in C_0^\infty(\Omega)\},$$

which is dense in $W^{m-1}(\Gamma)$, and show

$$l(\varphi) = \sum_{|\alpha| \leq m-1} \int_\Gamma D^\alpha \varphi(x) \, d\mu_\alpha(x) = 0 \quad \text{for all} \quad \varphi \in D(\Gamma).$$

Since supp $\mu_\alpha \subset \Gamma(|\alpha| \leq m - 1)$, then $l = 0$ follows. By (2.4) for every $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega G(x, y) \, L\varphi(y) \, dy = \int_\Omega \Phi(x, y) \, L\varphi(y) \, dy + \int_\Omega h(x, y) \, L\varphi(y) \, dy$$

$$= \varphi(x) + \int_\Omega L_y^* h(x, y) \, \varphi(y) \, dy = \varphi(x)$$

holds. Because the Lebesgue measure is $\mathscr{I}_N$-absolutely continuous, from $G^*\mu(y) = 0$ $\mathscr{I}_N$ — a.e. in Ω then follows

$$l(\varphi) = \sum_{|\alpha| \leq m-1} \int_\Gamma \left\{ \int_\Omega D_x^\alpha G(x, y) \, L\varphi(y) \, dy \right\} d\mu_\alpha(x)$$

$$= \int_\Omega \left\{ \sum_{|\alpha| \leq m-1} \int_\Gamma D_x^\alpha G(x, y) \, d\mu_\alpha(x) \right\} L\varphi(y) \, dy = \int_\Omega G^*\mu(y) \, L\varphi(y) \, dy = 0.$$

The proof of $\overline{L_G(\Gamma)} = W^{m-1}(\Gamma)$ now is complete ∎

Besides the suppositions of Theorem 1 now we presume, that the domain Ω can be enlarged to a smooth domain $\Omega_1 \supset \Omega$, such that the following properties hold:

(i) $\partial\Omega \setminus V \subset \partial\Omega_1$

(ii) The coefficients of B_j can be extended to $\partial\Omega_1 \setminus \partial\Omega$ in such a way, that for the new system of boundary operators on $\partial\Omega_1$ all the suppositions, formulated in Section 2, are satisfied.

If the coefficients of B_j are constant (for instance in the case of the Dirichlet problem) condition (ii) is always fulfiled.

Theorem 2: *Under the preceding suppositions the space $L_V(\Gamma)$ of the restrictions on Γ of the space $L_V(\Omega)$, defined in* (1.1), *is dense in $W^{m-1}(\Gamma)$ with respect to the norm* (2.9).

Proof: We choose an open subset $G \subset \Omega_1 \setminus \Omega$ and consider the space $L_G(\Omega_1)$, defined in correspondence with (1.2). By Theorem 1 we have $\overline{L_G(\Omega_1)|_\Gamma} = W^{m-1}(\Gamma)$. Since obviously the inclusion $\overline{L_G(\Omega_1)}|_\Omega \subset L_V(\Omega)$ holds, it follows $\overline{L_V(\Gamma)} = W^{m-1}(\Gamma)$ ∎

Remark: Because the extension of Ω can be taken in such a manner, that the boundary value problem

$$\begin{aligned} Lu &= 0 \quad \text{in} \quad \Omega_1, \\ B_j u|_{\partial\Omega_1} &= 0 \qquad (j = 1, \ldots, m) \end{aligned} \tag{3.7}$$

only has the trivial solution, we see, that Theorem 2 also holds, if the corresponding homogeneous problem in Ω has non trivial solutions.

4. In the following we shall give some complementary results

Theorem 3: *We suppose, that for L and Ω the assumptions from Theorem* 1 *and Theorem* 2 *are fulfiled. Let $\Gamma \subset \Omega$ be a compact set, which satisfies the following conditions:*

(i) *For every $y \in \Gamma$ there exists a cone in $\Omega \setminus \Gamma$ with vertex in y.*
(ii) *$\Omega \setminus \Gamma$ is connected.*
(iii) *Γ is $(m-1)$-regular in the sense, that $W^{m-1}(\Gamma)$ is complete with respect to the norm* (2.9).

Then the statements of Theorem 1 *and* 2 *and of the preceding remark hold.*

Proof: From the proofs of Theorem 1 and 2 we see, that the step c in this case is omitted and that the properties (i)—(iii) for the other parts of the proof are sufficient ∎

The next Theorem is a result of the Browder-type (see [5]). It is a simple conclusion from Theorem 2.

Theorem 4: *We suppose, that for L, Ω, V and Γ the assumptions from Theorem* 2 *are fulfiled. We do not suppose, that the problem* (3.7) (*for* Ω) *only has the trivial solution. Let $u_0 \in C^{2m}(\Omega_i) \cap C^{m-1}(\bar{\Omega}_i)$ be a given solution of $Lu_0 = 0$ in Ω_i. Then for $\varepsilon > 0$ there exists a solution $u \in L_V(\Omega)$, such that*

$$\|u - u_0\|_{C^{m-1}(\bar{\Omega}_i)} < \varepsilon.$$

Proof: For the domain Ω_i the Agmon-Miranda-inequality (see [9]) holds, i.e. there is a constant $c > 0$, independent from u, such that for the solutions u of the equation $Lu = 0$ in Ω_i with $u \in C^{2m}(\Omega_i) \cap C^{m-1}(\bar{\Omega}_i)$ we have

$$\|u\|_{C^{m-1}(\bar{\Omega}_i)} \leq c \sum_{|\alpha| \leq m-1} \sup_{y \in \partial\Omega_i} |D^\alpha u(y)|.$$

The solution u_0 defines on $\Gamma = \partial\Omega_i$ an element $v \in W^{m-1}(\Gamma)$ (see [12]). By Theorem 2 there exists a sequence (u_n), $u_n \in L_V(\Omega)$ with $\|u_n|_\Gamma - v\|_{W^{m-1}(\Gamma)} \to 0$. Applying the Agmon-Miranda-inequality for $u_n - u_0$ and Ω_i, we get $\|u_n - u_0\|_{C^{m-1}(\bar{\Omega}_i)} \to 0$ ∎

Theorem 5: *The statements of Theorem 1 and 2 do not hold, if the homogeneous Dirichlet problem for the domain Ω_i has non-trivial solutions.*

Proof: Let $u \in L_G(\Omega)$ resp. $u \in L_V(\Omega)$, and let $v \in C^{2m}(\Omega_i) \cap C^{m-1}(\bar{\Omega}_i)$ be a solution of

$$L^*v = 0 \quad \text{in} \quad \Omega_i,$$

$$\left.\frac{\partial^{j-1}v}{\partial n^{j-1}}\right|_\Gamma = 0, \quad (j = 1, \ldots, m)$$

(Moreover, we have $v \in C^\infty(\bar{\Omega}_i)$, see [13: Lemma 4.6]). Using the Green formula (2.1) for u, v and Ω_i, $B_j = B_j' = \dfrac{\partial^{j-1}}{\partial n^{j-1}}$ and $Lu = 0$, $L^*v = 0$ in Ω_i, it follows

$$\sum_{j=1}^{m} \int_\Gamma \frac{\partial^{j-1}u}{\partial n^{j-1}}(y)\, C_j'v(y)\, d\sigma(y) = 0.$$

After rewriting the normal derivatives into partial derivatives the last equation can be interpreted as follows: There exists a vector measure $\mu = (\mu_\alpha)_{|\alpha| \le m-1} \neq 0$ (representable by smooth densities) with the property $\mu(u) = 0$ for all $u \in L_G(\Gamma)$ resp. $u \in L_V(\Gamma)$. This means $\overline{L_G(\Gamma)} \neq W^{m-1}(\Gamma)$ resp. $\overline{L_V(\Gamma)} \neq W^{m-1}(\Gamma)$ ∎

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