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Singularly perturbed elliptic problems of second order with a singular line

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Für einige Klassen singulär gestörter elliptischer Probleme, wo die Charakteristiken des reduzierten Problems parallel oder senkrecht zur singulären Linie verlaufen, werden gleichmäßige asymptotische Approximationen konstruiert. Die Approximationsordnung ist abhängig vom Verhalten der Koeffizienten des reduzierten Problems in Umgebung der singulären Linie und von den Eigenschaften des betrachteten Gebietes.

Для некоторых классов сингулярно-возмущённых эллиптических задач, в которых характеристики вырожденной задачи параллельны или перпендикулярны к особой кривой, строятся равномерно-асимптотические аппроксимации. Порядок аппроксимации зависит от свойств коэффициентов вырожденной задачи в окрестности особой кривой и от характера рассматриваемой области.

For some classes of singularly perturbed elliptic problems, in which the characteristics of the reduced problem are parallel or perpendicular to the singular line, uniform asymptotic approximations are constructed. The order of approximation depends on the behaviour of the coefficients of the reduced problem in the neighbourhood of the singular line and on the properties of the considered domain.

1. Introduction

We consider boundary value problems of the form

$$\begin{aligned} L_\varepsilon u_\varepsilon &\equiv \varepsilon L_1 u_\varepsilon + L_0 u_\varepsilon = 0 \quad \text{in } \Omega \\ u_\varepsilon &= g \qquad\qquad\qquad\quad \text{in } \partial\Omega, \end{aligned} \tag{1.1}$$

where L_1 denotes a linear uniformly elliptic second order differential operator:

$$L_1 \equiv \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{2} a_i \frac{\partial}{\partial x_i} + a_0 \qquad (a_0 \geqq 0),$$

L_0 is a linear first order differential operator:

$$L_0 \equiv \sum_{i=1}^{2} b_i \frac{\partial}{\partial x_i},$$

and ε is a small positive parameter.

Φ_ε is called *uniform asymptotic approximation* of u_ε on the subdomain $\tilde{\Omega} \subseteq \bar{\Omega}$ if the inequality

$$\|u_\varepsilon - \Phi_\varepsilon\|_{\tilde{\Omega}} = \sup_{x \in \tilde{\Omega}} |u_\varepsilon(x) - \Phi_\varepsilon(x)| \leqq k_\varepsilon^{\sigma} \qquad (\sigma > 0)$$

is satisfied for a positive constant K independent of ε. In case of $\tilde{\Omega} = \bar{\Omega}$ we shall omit the subskript $\tilde{\Omega}$ and only write $\|\cdot\|$. The asymptotic behaviour of the solution u_ε

of (1.1) depends on the characteristics of L_0, in particular on the existence of singular (or turning) points. A point $x^* \in \bar{\Omega}$ is called a *singular point* if and only if $b_i(x^*) = 0$ for $i = 1, 2$. For isolated singular points the asymptotic behaviour of the solution u_ε of (1.1) have been investigated by many authors in detail (see, for instance [2, 4, 8, 10—13]). In the case of a singular line, up to now, only very special problems of the type described above have been considered [1, 3, 5—7, 16].

Our main objective is to construct uniform approximations of the solution of typical classes of (1.1) in which the characteristics of L_0 are parallel or perpendicular to the singular line. For this purpose let us assume that $x_2 = 0$ is the singular line and that Ω can be described by

$$\Omega = \{(x_1, x_2) \mid -1 < x_1 < +1, -f_1(x_1) < x_2 < f_2(x_1)\}$$

or

$$\Omega = \{(x_1, x_2) \mid -1 < x_1 < +1, 0 < x_2 < f_2(x_1)\}$$

by means of smooth functions f_1, f_2 with $f_{1,2}(\pm 1) = 0$ and $f_{1,2}(x_1) > 0$ on the open interval $(-1, +1)$. As a result Ω is an admissible domain in the sense of [15] and the existence of a uniquely determined classical solution u_ε of (1.1) can be guaranteed for sufficiently smooth coefficients of L_1 and L_0.

2. A singular line perpendicular to the characteristics

We consider the singularly perturbed problem

$$L_\varepsilon u_\varepsilon \equiv \varepsilon L_1 u_\varepsilon \pm x_2^l \frac{\partial u_\varepsilon}{\partial x_2} = 0 \quad \text{in } \Omega \qquad (2.1)$$
$$u_\varepsilon = g \quad \text{on } \partial\Omega.$$

A uniform asymptotic approximation will be constructed by applying the method of matched asymptotic expansion. In dependence on the behaviour of the characteristics near the boundary we use following notations for the smooth parts of the boundary:

$$\Gamma_\pm = \{x \in \partial\Omega \mid b(x) \cdot \nu(x) \gtrless 0\}, \quad \Gamma_0 = \{x \in \partial\Omega \mid b(x) \cdot \nu(x) = 0\},$$

where $b = (b_1, b_2)$ and ν denotes the outward directed unit normal.

First we assume that

$$\Omega = \{(x_1, x_2) \mid -1 < x_1 < +1, -f_1(x_1) < x_2 < f_2(x_1)\}.$$

Now, let $f_{1,2}$ vanish at $x_1 = \pm 1$ of the order $m_{1,2}^\pm$,

$$g|_{x_2=f_2(x_1)} = g_2(x_1), \qquad g|_{x_2=-f_1(x_1)} = g_1(x_1)$$

and l be even. Then, without restriction of generality, we can assume that $L_0 = -x_2^{2k} \dfrac{\partial}{\partial x_2}$ and it holds

$$\Gamma_- = \{(x_1, x_2) \mid x_2 = f_2(x_1)\}, \qquad \Gamma_+ = \{(x_1, x_2) \mid x_2 = -f_1(x_1)\}.$$

Consequently, the solution of the global problem becomes $g_2(x_1)$, such that the boundary condition on Γ_+ does not satisfied and local correctors are needed.

In the neighbourhood of Γ_+ let $\zeta = \dfrac{x_2 + f_1(x_1)}{\varepsilon}$ and

$$L_\varepsilon = \varepsilon^{-1}\left(-a(x_1)\frac{\partial^2}{\partial\zeta^2} - f_1'(x_1)\frac{\partial}{\partial\zeta}\right) + L_\varepsilon^*.$$

Constructing a local corrector $v = v_0 + \varepsilon v_1$ with $|L_\varepsilon v| \leqq K \cdot \varepsilon$ such that the boundary conditions on Γ_+ are satisfied by $v + g_2$ we obtain

$$v_0(x_1, \zeta) = [g_1(x_1) - g_2(x_1)] \exp\left(-\frac{f_1'(x_1)}{a(x_1)} \cdot \frac{f_1(x_1) + x_2}{\varepsilon}\right).$$

On Γ_- v_0 satisfies $v_0|_{\Gamma_-} = (g_1 - g_2) \exp\left(-\dfrac{f_1'}{a}\dfrac{f_1 + f_2}{\varepsilon}\right)$ and is exponentially small for $|x_1| \leqq x_0 < 1$. In the neighbourhood of $x_1 = 1$ $v_0|_{\Gamma_-}$ behaves as $\mu \exp\left(-\dfrac{\mu^{lm_1^+ + \min(m_1^+, m_2^+)}}{\varepsilon}\right)$. For arbitrary positive r it holds $\left|\mu \exp\left(-\dfrac{\mu^r}{\varepsilon}\right)\right| \leqq K\varepsilon^{\frac{1}{r}}$, such that $|v_0|_{\Gamma_-}| \leqq K\varepsilon^{\frac{1}{s_1}}$ with $s_1 = \max\left(2km_1^+ + \min(m_1^+, m_2^+), 2km_1^- + \min(m_1^-, m_2^-)\right)$. Summing up we obtain

$$|L_\varepsilon(u_\varepsilon - (g_2 + v))| \leqq K\varepsilon \text{ in } \Omega, \quad |u_\varepsilon - (g_2 + v)| \leqq K\varepsilon^{\frac{1}{s_1}}$$

on $\partial\Omega$. The barrier function $K_1\varepsilon^{\frac{1}{s_1}} - K_2\varepsilon^{\frac{2}{2k+1}}v_\varepsilon(x_2)$ with

$$v_\varepsilon(x_2) = \int_{-\infty}^{x_2/\varepsilon^{\frac{1}{2k+1}}} \int_{-\infty}^{q} \exp\left(\frac{p^{2k+1} - q^{2k+1}}{K_3}\right) dp\, dq$$

yields the estimate $|u_\varepsilon - (g_2 + v)| \leqq K\varepsilon^{\frac{1}{s_1}} + K\varepsilon^{\frac{2}{2k+1}}$ [17]. Because of $s_1 \geqq 2k + 1$ we obtain

Theorem 2.1: *Let* $L_0 \equiv -x_2^{2k}\dfrac{\partial}{\partial x_2}$ *and let the singular line lie in the interior of* Ω. *Then, it holds that*

$$\|u_\varepsilon - (g_2 + v_0)\| \leqq K\varepsilon^{\frac{1}{s_1}},$$

with $s_1 = \max(\min(m_1^+, m_2^+) + 2km_1^+, \min(m_1^-, m_2^-) + 2km_1^-)$.

Considering the subdomains $x_2 > 0$ and $x_2 < 0$, respectively, we obtain analogous problems. Let

$$\Omega = \{(x_1, x_2) \mid -1 < x_1 < +1,\ 0 < x_2 < f_2(x_1)\}.$$

Again $g_2(x_1)$ is the solution of the global problem. Setting $\zeta = \dfrac{x_2}{\varepsilon^{\frac{1}{2k+1}}}$ it holds

$$L_\varepsilon = \varepsilon^{\frac{2k-1}{2k+1}}\left(-a(x_1)\frac{\partial^2}{\partial\zeta^2} - \zeta^{2k}\frac{\partial}{\partial\zeta}\right) + \varepsilon^{\frac{2k}{2k+1}}L_\varepsilon^*.$$

Requiring

$$a\frac{\partial^2 v}{\partial\zeta^2} + \zeta^{2k}\frac{\partial v}{\partial\zeta} = 0, \qquad v|_{\zeta=0} = g_1(x_1) - g_2(x_1)$$

we obtain

$$v = \frac{(g_1(x_1) - g_2(x_1))}{A} \int\limits_{x_2/\varepsilon^{\frac{1}{2k+1}}}^{\infty} \exp\left(-\frac{t^{2k+1}}{a(2k+1)}\right) dt$$

with $A = \int\limits_0^\infty \exp\left(-\frac{t^{2k+1}}{a(2k+1)}\right) dt$. Now $u_\varepsilon - (g_2 + v)$ satisfies

$$|L_\varepsilon(u_\varepsilon - (g_2 + v))| \leqq K\varepsilon^{\frac{2k}{2k+1}} \quad \text{in } \Omega,$$

$$u_\varepsilon - (g_2 + v)|_{x_2=0} = 0,$$

$$|u_\varepsilon - (g_2 + v)|_{x_2=f_2(x_1)}| \leqq K\varepsilon^{\frac{1}{s_2}}, \qquad s_2 = (2k+1) \max(m_2^+, m_2^-),$$

where the last follows from the estimate

$$\mu \int\limits_{\mu^m/\varepsilon^{\frac{1}{2k+1}}}^{\infty} \exp\left(-\frac{t^{2k+1}}{2k+1}\right) dt \leqq K^* \varepsilon^{\frac{1}{m(2k+1)}}.$$

Then, because of $s_2 \geqq 2k + 1$, the barrier function

$$K_1\varepsilon^{\frac{1}{s_2}} - K_2\varepsilon^{\frac{1}{2k+1}} v_\varepsilon(x_2)$$

yields

Theorem 2.2: *Let $L_0 \equiv -x_2^{2k} \frac{\partial}{\partial x_2}$ and let the singular line belong to the boundary of Ω. Then it holds that*

$$\|u_\varepsilon - (g_2 + v)\| \leqq K\varepsilon^{\frac{1}{s_2}}$$

with $s_2 = (2k+1) \max(m_2^+, m_2^-)$.

The case

$$\Omega = \{(x_1, x_2) \mid -1 < x_1 < +1, -f_1(x_1) < x_2 < 0\}$$

is analogous to that which has been considered first. Setting $g|_{x_2=0} = g_0(x_1)$ we obtain $\|u_\varepsilon - (g_0 + v)\| \leqq K\varepsilon^{\frac{1}{s_3}}$ with $s_3 = (2k+1) \max(m_1^+, m_1^-)$.

Now let l be odd, $l = 2k - 1$. In this case the sign of $\pm x_2^{2k-1} \frac{\partial}{\partial x_2}$ is of decisive importance. First we consider

$$L_\varepsilon u_\varepsilon \equiv \varepsilon L_1 u_\varepsilon - x_2^{2k-1} \frac{\partial u_\varepsilon}{\partial x_2} = 0 \quad \text{in } \Omega$$

$$u_\varepsilon = g \quad \text{on } \partial\Omega$$

with

$$\Omega = \{(x_1, x_2) \mid -1 < x_1 < +1, -f_1(x_1) < x_2 < f_2(x_1)\}.$$

Because of $\Gamma_- = \partial\Omega$ the solution of the global problem becomes

$$u = \begin{cases} g_2(x_1), & x_2 > 0 \\ g_1(x_1), & x_2 < 0 \end{cases}.$$

and at $x_2 = 0$ a free boundary layer originates. By setting $\zeta = \dfrac{x_2}{\varepsilon^{\frac{1}{2k}}}$ in the neighbourhood of $x_2 = 0$ we obtain

$$L_\varepsilon = \varepsilon^{\frac{2k-1}{2k}} \left\{ -a(x_1) \frac{\partial^2}{\partial \zeta^2} - \zeta^{2k-1} \frac{\partial}{\partial \zeta} \right\} + \varepsilon^{\frac{2k-1}{2k}} L_\varepsilon^*.$$

The requirements

$$a \frac{\partial^2 v}{\partial \zeta^2} + \zeta^{2k-1} \frac{\partial v}{\partial \zeta} = 0, \quad v|_{\zeta=-\infty} = \frac{g_1 - g_2}{2}, \quad v|_{\zeta=+\infty} = \frac{g_2 - g_1}{2}$$

yield

$$v = \frac{g_2 - g_1}{2A^*} \int_0^\zeta \exp\left(-\frac{1}{a\,2k} t^{2k}\right) dt \quad \text{with} \quad A^* = \int_0^\infty \exp\left(-\frac{1}{a\,2k} t^{2k}\right) dt.$$

Theorem 2.3: *Let $L_0 = -x_2^{2k-1} \dfrac{\partial}{\partial x_2}$ and let the singular line lie in the interior of Ω. Then it holds that*

$$\left\| u_\varepsilon - \left(\frac{g_2 + g_1}{2} + v \right) \right\| \leqq K \begin{cases} \varepsilon^{\frac{1}{2}} |\ln \varepsilon| & \text{if } k = 1 \text{ and } s_4 = 2 \\ \varepsilon^{\frac{1}{s_4}} & \text{otherwise} \end{cases}$$

with $s_4 = 2k \max(m_2^+, m_2^-, m_1^+, m_1^-)$.

Proof: According to the construction we have

$$\left| L_\varepsilon \left(u_\varepsilon - \left(\frac{g_2 + g_1}{2} + v \right) \right) \right| \leqq K \varepsilon^{\frac{2k-1}{2k}} \quad \text{in } \Omega.$$

Since v is a function of boundary layer type, it follows that

$$\frac{g_2 + g_1}{2} + v|_{\Gamma_-} = g + r \quad \text{with} \quad |r| \leqq K\varepsilon^N \quad (N \text{ arbitrary})$$

in the domain $|x| \leqq x_0 < 1$. However, analogous to the above, we obtain only

$$\left\| u_\varepsilon - \left(\frac{g_2 + g_1}{2} + v \right) \right\|_{\Gamma_-} \leqq K \varepsilon^{\frac{1}{s_4}}$$

with $s_4 = 2k \max(m_2^+, m_2^-, m_1^+, m_1^-)$. Now, we can conclude that $K_1 \varepsilon^{\frac{1}{s_4}} \pm K_2 \varepsilon^{\frac{1}{2k}} \times w_\varepsilon(x_2)$ and $K_1 \varepsilon^{\frac{1}{2}} |\ln \varepsilon| - K_2 \varepsilon^{\frac{1}{2}} w_\varepsilon(x_2)$ if $k = 1$ and $s_4 = 2$, respectively, with

$$w_\varepsilon(x_2) = \int_0^{x_2/\varepsilon^{\frac{1}{2k}}} \int_0^q \exp\left(\frac{1}{K_3} \left(p^{2k} - q^{2k} \right) \right) dp\, dq,$$

are barrier functions for $u_\varepsilon - \left(\frac{g_2 + g_1}{2} + v\right)$. Because of

$$|w_\varepsilon(x_2)| \leqq K\,|\ln \varepsilon| \quad \text{for} \quad k = 1 \quad \text{and} \quad |w_\varepsilon(x_2)| \leqq K \quad \text{for} \quad k > 1$$

we obtain the assertion of Theorem 2.3 ∎

In the case $\Omega = \{(x_1, x_2) \mid -1 < x_1 < +1, 0 < x_2 < f_2(x_1)\}$ a boundary layer of the above type appears and it does not change so much.

Now we consider an other type of a singularly perturbed problem, namely

$$L_\varepsilon u_\varepsilon \equiv \varepsilon L_1 u_\varepsilon + x_2^{2k-1} \frac{\partial u_\varepsilon}{\partial x_2} = 0 \quad \text{in } \Omega$$

$$u_\varepsilon = g \quad \text{on } \partial\Omega$$

where $\Omega = \{(x_1, x_2) \mid -1 < x_1 < +1, -f_1(x_1) < x_2 < f_2(x_1)\}$. Here we have

$$\Gamma_+ = \partial\Omega \setminus \{(1, 0) \cup (-1, 0)\}.$$

Therefore we cannot impose conditions on the solution of the global problem $U(x_1)$. Let $v^{1,2}$ be the local correctors in the neighbourhood of the boundary, such that $|L_\varepsilon v^{1,2}| \leqq K_\varepsilon^2$ and $v^{1,2} + U$ satisfy the boundary conditions for $x_2 = f_2(x_1)$ and $x_2 = -f_1(x_1)$, respectively. Then, it holds

$$v^{1,2} = v_0^{1,2} + \varepsilon v_1^{1,2} + \varepsilon^2 v_2^{1,2}$$

with

$$v_0^1 = (g_1 - U) \exp\left(-\frac{f_1^{2k-1}}{a} \frac{f_1(x_1) + x_2}{\varepsilon}\right),$$

$$v_0^2 = (g_2 - U) \exp\left(-\frac{f_2^{2k-1}}{a} \frac{f_2(x_1) - x_2}{\varepsilon}\right).$$

If a_{11}, a_1 and a_0 do not depend on x_2, then we can improve the accuracy of the approximation by requiring

$$a_{11} \frac{\partial^2 U}{\partial x_1^2} + a_1 \frac{\partial U}{\partial x_1} + a_0 U = 0.$$

If follows $|L_\varepsilon(u_\varepsilon - (U + v^1 + v^2))| \leqq K\varepsilon^2$.

In the domain $|x| \leqq x_0 < 1$ v_0^1 and v_0^2 are again exponentially small for $x_2 = f_2(x_1)$ and $x_2 = -f_1(x_1)$, respectively. In the neighbourhood of $|x_1| = 1$ we obtain the conditions

$$U(1) = g(1) \quad \text{and} \quad U(-1) = g(-1)$$

by requiring

$$|v_0^1|_{x_2 = f_2(x_1)}| + |v_0^2|_{x_2 = -f_1(x_1)}| \leqq K\varepsilon^{\frac{1}{s}} \quad (s > 0).$$

Since $a_0 \geqq 0$ we can determine the solution $U(x_1)$ of the global problem as the unique solution of the boundary value problem

$$a_{11} \frac{\partial^2 U}{\partial x_1^2} + a_1 \frac{\partial U}{\partial x_1} + a_0 U = 0 \tag{2.2}$$

$$U(1) = g(1), \qquad U(-1) = g(-1).$$

For s we obtain, as above, that

$$s = s_5 = \max\big(\min(m_1^+, m_2^+) + lm_1^+, \min(m_1^+, m_2^+) + lm_2^+, \min(m_1^-, m_2^-) + lm_1^-, \min(m_1^-, m_2^-) + lm_2^-\big).$$

By means of the barrier function

$$K_1\varepsilon^{\frac{1}{s_5}} - K_2\varepsilon \exp(-K_3x_1)$$

we immediately obtain

Theorem 2.4: *Let $L_0 \equiv x_2^{2k-1}\frac{\partial}{\partial x_2}$ and let the singular line lie in the interior of the domain Ω. Furthermore, let a_{11}, a_1, a_0 be independent of x_2. Then it holds*

$$\|u_\varepsilon - (U + v_0^1 + v_0^2)\| \leqq K\varepsilon^{\frac{1}{s_5}}$$

with the above defined s_5, where $U(x_1)$ denotes the solution of the boundary value problem (2.2).

Remark: In the case $k = 1$ a_{11}, a_1 and a_0 can also depend on x_2. Then, we require

$$a_{11}(x_1, 0)\frac{\partial^2 U}{\partial x_1^2} + a_1(x_1, 0)\frac{\partial U}{\partial x_1} + a_0(x_1, 0)\, U = 0,$$

set $\Phi_\varepsilon = U(x_1) + \varepsilon U_1(x_1, x_2) + v^1 + v^2$ and determine U_1 from

$$x_2\frac{\partial U_1}{\partial x_2} = [a_{11}(x_1, x_2) - a_{11}(x_1, 0)]\frac{\partial^2 U}{\partial x_1^2} + [a_1(x_1, x_2) - a_1(x_1, 0)]\frac{\partial U}{\partial x_1} + [a_0(x_1, x_2) - a_0(x_1, 0)]\, U.$$

It always hold $s \gtreqqless 2$. $s = 2$ holds if and only if $m_1^+ = m_2^+ = m_1^- = m_2^- = 1$ and $k = 1$. The assertion of GRASMAN [6], that $s = 1$ for $k = 1$, is not valid.

The cases

$$\Omega = \{(x_1, x_2) \mid 0 < x_2 < f_2(x_1)\} \quad \text{and} \quad \Omega = \{(x_1, x_2) \mid -f_1(x_1) < x_2 < 0\},$$

respectively, are less interesting since the solutions of the global problem, namely $g_1(x_1)$ and $g_2(x_1)$, respectively are uniquely determined.

3. A singular line parallel to the characteristics

We now consider

$$L_\varepsilon u_\varepsilon = \varepsilon L_1 u_\varepsilon + x_2^l\frac{\partial u_\varepsilon}{\partial x_1} = 0 \quad \text{in } \Omega$$
$$u_\varepsilon = g \quad \text{on } \partial\Omega$$

with $\Omega = \{(x_1, x_2) \mid -1 < x_1 < +1, 0 < x_2 < f(x_1)\}$ and assume that the lines $x_2 = const.$ intersect $\partial\Omega$ at exactly two points (except $x_2 = 0$ and $x_2 = \sup f(x_1)$). Furthermore, let us assume that $f(x_1) < f(0)$ for all $x_1 \neq 0$, such that

$$\Gamma_- = \{(x_1, x_2) \mid -1 < x_1 < 0, x_2 = f(x_1)\},$$
$$\Gamma_+ = \{(x_1, x_2) \mid 0 < x_1 < +1, x_2 = f(x_1)\}.$$

Let u be the solution of the global problem

$$\frac{\partial u}{\partial x_1} = 0, \qquad u|_{\Gamma_-} = g.$$

u is a smooth function in $\bar{\Omega}$ except a neighbourhood of $(0, f(0))$. Near the singular line $x_2 = 0$ let $\zeta = \frac{x_2}{\varepsilon^{\frac{1}{2+l}}}$ and let v satisfy

$$a_{22}(x_1, 0) \frac{\partial^2 v}{\partial \zeta^2} - \zeta^l \frac{\partial v}{\partial x_1} = 0, \qquad v|_{\zeta=0} = g - u|_{x_2=0}.$$

This boundary layer problem has been studied in [16], because of the singularity of $\frac{\partial^2 v}{\partial x_1 \partial \zeta}, \frac{\partial^2 v}{\partial x_1^2}$ at $(-1,0)$ a regularization method has been applied.

Theorem 3.1: *Let $\Omega_\gamma = \Omega \cap \{x \mid \operatorname{dist}(x, \Gamma_+) > \gamma\}$. Then it holds*

$$\|u_\varepsilon - (u + v)\|_{\bar{\Omega}_\gamma} \leqq K\varepsilon^{\frac{1}{2(2+l)}}.$$

Proof: In $\Omega_{\frac{\gamma}{2}}$, for corresponding regularization, we have

$$|L_\varepsilon(u_\varepsilon - (u + v))| \leqq K\varepsilon^{1-\frac{3}{2(2+l)}},$$

$$|u_\varepsilon - (u + v)|_{x_2=0} \leqq K\varepsilon^{\frac{1}{2(2+l)}}, \quad |u_\varepsilon - (u + v)|_{\Gamma_-} = 0, \quad |u_\varepsilon - (u + v)| \leqq K$$

according to the construction. Now, let $\psi = \psi(x_1, x_2)$ be a smooth function with $\psi \equiv 0$ in Ω_γ, $\psi \equiv 1$ in $\Omega \setminus \Omega_{\frac{\gamma}{2}}$ and $\frac{\partial \psi}{\partial x_1} \geqq 0$. We construct a barrier function S_ε by setting

$$S_\varepsilon(x_1, x_2) = K_1\psi(x_1, x_2) + K_2\varepsilon^{\frac{1}{2(2+l)}} \exp(K_3 x_1) \left(K_4 + V_\varepsilon(x_2)\right)$$

with

$$V_\varepsilon(x_2) = \varepsilon^{-\frac{1}{2(2+l)}} x_2^{\frac{1}{2}} S_{\frac{2-l}{2(2+l)}, \frac{1}{2+l}} \left(\frac{2}{2+l} x_2^{\frac{2+l}{2}} \varepsilon^{-\frac{1}{2}}\right)$$

and appropriately chosen constants K_1, K_2, K_3, K_4. $S_{p,q}(\cdot)$ denotes the Lommel function. As a result we obtain that S_ε majorizes $u_\varepsilon - (u + v)$ in $\Omega_{\frac{\gamma}{2}}$. The restriction to Ω_γ yields the assertion ∎

Remark: Of course, it is possible to construct a local corrector in the neighbourhood of Γ_+. For the study of the asymptotic behaviour in the neighbourhood of the points $(0, f(0))$ and $(1,0)$ we refer to [14, 9].

The case $\Omega = \{(x_1, x_2) \mid -1 < x_1 < +1, -f_1(x_1) < x_2 < f_2(x_1)\}$ and l even is not interesting, since nothing happens along the singular line. However, if $l =$ odd, the problem becomes more difficult. For $l = 1$ this case have been considered by Barton [1] and Gorkov [5].

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