Zeitschrift für Analysis
und ihre Anwendungen
Bd. 3 (1) 1984, S. 33–42

Asymptotic conditions at the two first eigenvalues for the periodic solutions of Liénard differential equations and an inequality of E. Schmidt

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Wir untersuchen die periodische Randwertaufgabe

$$x''(t) + f(x(t))\,x'(t) + g(t, x(t)) = e(t),$$
$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

unter Nicht-Resonanz-Bedingungen auf $x^{-1}g(t, x)$ für $|x| \to \infty$.

Исследуется периодическая краевая задача

$$x''(t) + f(x(t))\,x'(t) + g(t, x(t)) = e(t),$$
$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

при нерезонансных условиях на $x^{-1}g(t, x)$ для $|x| \to \infty$.

We study the periodic boundary problem

$$x''(t) + f(x(t))\,x'(t) + g(t, x(t)) = e(t),$$
$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

under some non-resonance conditions on the asymptotic behavior of $x^{-1}g(t, x)$ for $|x| \to \infty$.

1. Introduction

This paper is devoted to the study of the periodic boundary value problem

$$\begin{aligned} &x''(t) + f(x(t))\,x'(t) + g(t, x(t)) = e(t), \\ &x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \end{aligned} \tag{1.1}$$

when the asymptotic behavior of $x^{-1}g(t, x)$ is compared with the two first eigenvalues 0 and 1 of the linear problem

$$\begin{aligned} &x'' + \lambda x = 0, \\ &x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0. \end{aligned} \tag{1.2}$$

The results are in the line of the ones given by Mawhin and Ward in [2] and [3] where a review of the preceding literature can be found. They essentially differ from [2] and [3] by generalizing the conditions on the function Γ which is such that

$$\limsup_{|x|\to\infty} x^{-1}g(t, x) \leq \Gamma(t).$$

Instead of assuming, like in [2] or [3] that $\Gamma(t) \leq 1$ with strict inequality on a subset of $[0, 2\pi]$ with positive measure, we write Γ in the form $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$

with Γ_0 satisfying the above condition on Γ, $\Gamma_1 \in L^1(0, 2\pi)$, $\Gamma_\infty \in L^\infty(0\ 2\pi)$ and $|\Gamma_1|_{L^1}$ and $|\Gamma_\infty|_{L^\infty}$ sufficiently small. Thus the expression

$$\limsup_{|x|\to\infty} x^{-1}g(t, x)$$

can now cross *any* number of eigenvalue n^2 of the problem (1.2) as far as those crossings take place in subsets of $[0, 2\pi]$ of sufficiently small measure. See GOSSEZ [1] for similar results around the first eigenvalue.

As in [2] and [3], the results depend on lemmas giving a priori inequalities and degree arguments. For Theorems 1 and 2, respectively in Sections 3 and 4, and which apply to the general case (1.1), those lemmas are slight improvements of the ones in [2] and [3]. For Theorem 3 in Section 5, which requires f to be constant, a rather different lemma is introduced which makes uses of an inequality of E. SCHMIDT [5] for periodic absolutely continuous functions. This lemma allows an improvement on the condition on Γ when $\Gamma_0 = \Gamma_\infty = 0$ and $f \equiv 0$, but this condition is no more sharp when applied to the case of a constant Γ.

We end this introduction by mentioning that besides the classical spaces $C([0, 2\pi])$, $C^k([0, 2\pi])$ and $L^k(0, 2\pi)$ of continuous, k-times continuously differentiable or measurable real functions whose k^{th} power of the absolute value is Lebesgue integrable, we shall make use in what follows of the Sobolev space $H^1(0, 2\pi)$ defined by

$$H^1(0, 2\pi) = \{x\colon [0, 2\pi] \to \mathbf{R} \mid x \text{ is abs. cont. on } [0, 2\pi] \text{ and } x' \in L^2(0, 2\pi)\},$$

with the inner product defined by

$$(x, y)_{H^1} = \left(\frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt\right)\left(\frac{1}{2\pi}\int_0^{2\pi} y(t)\,dt\right) + \frac{1}{2\pi}\int_0^{2\pi} x'(t)\,y'(t)\,dt$$

and the corresponding norm $|\cdot|_{H^1}$. Notice also that we define for convenience the norm in $L^k(0, 2\pi)$ by

$$|x|_{L^k} = \left(\frac{1}{2\pi}\int_0^{2\pi} |x(t)|^k\,dt\right)^{1/k}.$$

2. An inequality for some Liénard operators with periodic boundary conditions

For $x \in L^1(0, 2\pi)$, let us write

$$\bar{x} = (2\pi)^{-1}\int_0^{2\pi} x(t)\,dt, \qquad \tilde{x}(t) = x(t) - \bar{x},$$

so that $\int_0^{2\pi} \tilde{x}(t)\,dt = 0$. Let $\tilde{H}^1(0, 2\pi) = \{x \in H^1(0, 2\pi)\colon \bar{x} = 0\}$.

The following result is proved in MAWHIN-WARD [3].

Lemma 1: *Let $\Gamma \in L^1(0, 2\pi)$ be such that, for a.e. $t \in [0, 2\pi]$, one has*

$$\Gamma(t) \leq 1 \tag{2.1}$$

with the strict inequality on a subset of $[0, 2\pi]$ *of positive measure. Then there exists* $\delta = \delta(\Gamma) > 0$ *such that for all* $\tilde{x} \in \tilde{H}^1(0, 2\pi)$ *one has*

$$B_\Gamma(\tilde{x}) \equiv (2\pi)^{-1} \int_0^{2\pi} \left[(\tilde{x}'(t))^2 - \Gamma(t)\,\tilde{x}^2(t)\right] dt \geqq \delta\, |\tilde{x}|_{H^1}^2.$$

Lemma 2: *Let* $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ *where* $\Gamma_\infty \in L^\infty(0, 2\pi)$, $\Gamma_1 \in L^1(0, 2\pi)$, *and* $\Gamma_0 \in L^1(0, 2\pi)$ *is such that* $\Gamma_0(t) \leqq 1$ *for a.e.* $t \in [0, 2\pi]$ *with strict inequality on a subset of* $[0, 2\pi]$ *of positive measure. Let* $\delta(\Gamma_0) > 0$ *be given by Lemma* 1. *Then one has, for all* $\tilde{x} \in \tilde{H}^1(0, 2\pi)$,

$$B_\Gamma(\tilde{x}) \geqq \left[\delta(\Gamma_0) - \frac{\pi^2}{3}\,|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\right] |\tilde{x}|_{H^1}^2.$$

Proof: We have

$$B_\Gamma(\tilde{x}) = (2\pi)^{-1} \int_0^{2\pi} \left([\tilde{x}'(t)]^2 - \Gamma_0(t)\,\tilde{x}^2(t)\right) dt$$

$$- (2\pi)^{-1} \int_0^{2\pi} \Gamma_1(t)\,\tilde{x}^2(t)\, dt - (2\pi)^{-1} \int_0^{2\pi} \Gamma_\infty(t)\,\tilde{x}^2(t)\, dt.$$

Using the fact that $H^1(0, 2\pi) \subset C^0[0, 2\pi]$ and the well-known inequalities (see e.g. Rouche-Mawhin [4: p. 208])

$$|\tilde{x}|_{L^2} \leqq |\tilde{x}'|_{L^2} = |\tilde{x}|_{H^1}, \qquad |\tilde{x}|_{L^\infty} \leqq \frac{\pi}{\sqrt{3}}\,|\tilde{x}'|_{L^2} = \frac{\pi}{\sqrt{3}}\,|\tilde{x}|_{H^1},$$

as well as Lemma 1, we obtain

$$B_\Gamma(\tilde{x}) \geqq \delta(\Gamma_0)\,|\tilde{x}|_{H^1}^2 - |\Gamma_1|_{L^1}\,|\tilde{x}|_{L^\infty}^2 - |\Gamma_\infty|_{L^\infty}\,|\tilde{x}|_{L^2}^2$$

$$\geqq \left[\delta(\Gamma_0) - \frac{\pi^2}{3}\,|\Gamma_1|_{L^1} - |\Gamma|_{L^\infty}\right] |\tilde{x}|_{H^1}^2 \quad ∎$$

Remark 1: The best value for $\delta(0)$ is clearly 1, so that $B_{\Gamma_1}(\tilde{x}) \geqq \left(1 - \frac{\pi^2}{3}\,|\Gamma_1|_{L^1}\right) |\tilde{x}|_{H^1}^2$ for all $\tilde{x} \in \tilde{H}^1(0, 2\pi)$.

Lemma 3: *Let* $\gamma \in L^1(0, 2\pi)$, $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ *be like in Lemma* 2 *and* $\delta(\Gamma_0)$ *be given by Lemma* 1. *Then, for all measurable real functions* p *on* $[0, 2\pi]$ *such that* $\bar{\gamma} \leqq \bar{p}$, $p(t) \leqq \Gamma(t)$ *a.e. on* $[0, 2\pi]$, *all continuous functions* $f\colon \mathbf{R} \to \mathbf{R}$ *and all* $x \in W^{2,1} \times (0, 2\pi)$ *such that*

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \tag{2.2}$$

one has

$$(2\pi)^{-1} \int_0^{2\omega} \left(\bar{x} - \tilde{x}(t)\right)\left(x''(t) + f(x(t))\,x'(t) + p(t)\,x(t)\right) dt$$

$$\geqq \bar{\gamma}\bar{x}^2 + \left[\delta(\Gamma_0) - \frac{\pi^2}{3}\,|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\right] |\tilde{x}|_{H^1}^2.$$

Proof: If $x \in W^{1,2}(0, 2\pi)$ and satisfies (2.2), we obtain easily, integrating by parts and using Lemma 2,

$$(2\pi)^{-1}\int_0^{2\pi}\left(\bar{x}-\tilde{x}(t)\right)\left(x''(t)+f(x(t))\,x'(t)+p(t)\,x(t)\right)dt$$
$$=\overline{p}\bar{x}^2+(2\pi)^{-1}\int_0^{2\pi}\left([\tilde{x}'(t)]^2-p(t)\,\tilde{x}^2(t)\right)dt$$
$$\geqq\overline{\gamma}\bar{x}^2+B_\Gamma(\tilde{x})\geqq\overline{\gamma}\bar{x}^2+\left[\delta(\Gamma_0)-\frac{\pi^2}{3}\,|\Gamma_1|_{L^1}-|\Gamma_\infty|_{L^\infty}\right]|\tilde{x}|^2_{H^1}\ \blacksquare$$

3. Nonresonance conditions for the existence of periodic solutions for some forced Liénard equations

Let $f\colon \mathbf{R}\to\mathbf{R}$ be continuous and let $g\colon [0,2\pi]\times\mathbf{R}\to\mathbf{R}$, $(t,x)\mapsto g(t,x)$ be such that $g(\cdot,x)$ is measurable on $[0,2\pi]$ for each $x\in\mathbf{R}$ and $g(t,\cdot)$ is continuous on $\mathbf{R}$ for almost each $t\in[0,2\pi]$. Assume moreover that for each $r>0$ there exists a $\gamma_r\in L^1(0,2\pi)$ such that $|g(t,x)|\leqq\gamma_r(t)$ for a.e. $t\in[0,2\pi]$ and all $x\in[-r,r]$. Such a g will be said to satisfy the Carathéodory conditions. Consider the following periodic boundary-value problem for the Liénard equation, with $e\in L^1(0,2\pi)$,

$$\begin{aligned}&x''(t)+f(x(t))\,x'(t)+g(t,x(t))=e(t),\qquad t\in[0,2\pi],\\&x(0)-x(2\pi)=x'(0)-x'(2\pi)=0.\end{aligned}\tag{3.1}$$

We prove the following existence result for (3.1).

Theorem 1: *Assume that the inequalities*

$$\gamma(t)\leqq\liminf_{|x|\to\infty}x^{-1}g(t,x)\leqq\limsup_{|x|\to\infty}x^{-1}g(t,x)\leqq\Gamma(t)\tag{3.2}$$

hold uniformly a.e. in $t\in[0,2\pi]$ *and that* γ *and* Γ *satisfy the following conditions*

a) $\gamma\in L^1(0,2\pi)$ *and* $\overline{\gamma}>0$,

b) $\Gamma=\Gamma_0+\Gamma_1+\Gamma_\infty$ *with* $\Gamma_1\in L^1(0,2\pi)$,

$\Gamma_\infty\in L^\infty(0,2\pi)$, Γ_0 *is measurable on* $[0,2\pi]$, $\Gamma_0(t)\leqq 1$ *a.e. on* $[0,2\pi]$ *with strict inequality on a subset of measure zero and* $\frac{\pi^2}{3}|\Gamma_1|_{L^1}+|\Gamma_\infty|_{L^\infty}<\delta(\Gamma_0)$, *where* $\delta(\Gamma_0)$ *is given by Lemma* 1.

Then problem (3.1) *has at least one solution for each* $e\in L^1(0,2\pi)$.

Proof: If $\eta=\frac{1}{2}\min\left[\overline{\gamma},\delta(\Gamma_0)-\frac{\pi^2}{3}|\Gamma_1|_{L^1}-|\Gamma_\infty|\right]>0$, then, by (3.2), we can find $r>0$ such that for a.e. $t\in[0,2\pi]$ and all x with $|x|\geqq r$ we have $\gamma(t)-\eta\leqq x^{-1}g(t,x)\leqq\Gamma(t)+\eta$. We then write, like in the proof of Theorem 1 of [2], the equation in (3.1) in the form

$$x''(t)+f(x(t))\,x'(t)+\tilde{\gamma}(t,x(t))\,x(t)+h(t,x(t))=e(t),$$

where

$$\gamma(t)-\eta\leqq\tilde{\gamma}(t,x)\leqq\Gamma(t)+\eta,\qquad|h(t,x)|\leqq\alpha(t)\tag{3.3}$$

for a.e. $t\in[0,2\pi]$, all $x\in\mathbf{R}$ and some $\alpha\in L^1(0,2\pi)$. By the same degree argument than in the proof of Theorem 1, our result will be proved if we show that the set of possible solutions, of the family of equations

$$\begin{aligned}&x''(t) + \lambda f(x(t))\,x'(t) + [(1-\lambda)\,\Gamma(t) + \lambda\tilde{\gamma}(t, x(t))]\,x(t)\\&\quad + \lambda h(t, x(t)) = \lambda e(t), \qquad \lambda \in [0, 1],\\&x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\end{aligned} \tag{3.4}$$

is a priori bounded in $C^1([0, 2\pi])$ independently of $\lambda \in [0, 1]$. If x is a solution of (3.4), then multiplying (3.4) by $\bar{x} - \tilde{x}$, integrating over $[0, 2\pi]$ and using (3.3) together with Lemma 3 with Γ_∞ replaced by $\Gamma_\infty + \eta$ and γ replaced by $\gamma - \eta$, we find

$$\begin{aligned}0 &= (2\pi)^{-1}\int_0^{2\pi} (\bar{x} - \tilde{x}(t))\,\{x''(t) + \lambda f(x(t)\,x'(t))\\&\quad + [(1-\lambda)\,\Gamma(t) + \lambda\tilde{\gamma}(t, x(t))]\,x(t) + \lambda h(t, x(t)) - \lambda e(t)\}\,dt\\&\geqq (\bar{\gamma} - \eta)\,\bar{x}^2 + \left[\delta(\Gamma_0) - \frac{\pi^2}{3}\,|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \eta\right]|\tilde{x}|_{H^1}^2\\&\quad - (|\alpha|_{L^1} + |e|_{L^1})\,|\bar{x} - \tilde{x}|_{L^\infty}\\&\geqq \frac{\bar{\gamma}}{2}\,\bar{x}^2 + \frac{1}{2}\left[\delta(\Gamma_0) - \frac{\pi^2}{3}\,|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\right]|\tilde{x}|_{H^1}^2 - \beta\,|x|_{H^1}\\&\geqq \eta\,|x|_{H^1}^2 - \beta\,|x|_{H^1},\end{aligned}$$

and hence $|x|_{H^1} \leqq \beta/\eta$. This implies then, like in the proof of Theorem 1 of [2] that $|x|_{C^1} < R$ for some R independent of $\lambda \in [0, 1]$ ∎

4. Periodic solutions for a Liénard equation at resonance

Let $f: \mathbf{R} \to \mathbf{R}$ be continuous and let $g: [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$, $(t, x) \mapsto g(t, x)$ be such that $g(\cdot, x)$ is measurable on $[0, 2\pi]$ for each $x \in \mathbf{R}$ and $g(t, \cdot)$ is continuous on $\mathbf{R}$ for a.e. $t \in [0, 2\pi]$. Assume moreover that for each $r > 0$ there exists $\gamma_r \in L^1(0, 2\pi)$ such that $|g(t, x)| \leqq \gamma_r(t)$ for a.e. $t \in [0, 2\pi]$ and all $x \in [-r, r]$. We consider the following periodic boundary-value problem for the Liénard equation

$$\begin{aligned}&x''(t) + f(x(t))\,x'(t) + g(t, x(t)) = e(t), \qquad t \in [0, 2\pi],\\&x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.\end{aligned} \tag{4.1}$$

We prove the following existence result for (4.1).

Theorem 2: *Assume that there exists $\Gamma \in L^1(0, 2\pi)$ such that*

$$\limsup_{|x|\to\infty} \frac{g(t, x)}{x} \leqq \Gamma(t) \tag{4.2}$$

uniformly a.e. in $t \in [0, 2\pi]$ and such that $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ where $\Gamma_\infty \in L^\infty(0, 2\pi)$, $\Gamma_1 \in L^1(0, 2\pi)$ and $\Gamma_0 \in L^1(0, 2\pi)$ are such that $\Gamma_0(t) \leqq 1$ for a.e. $t \in [0, 2\pi]$, with strict inequality on a subset of $[0, 2\pi]$ of positive measure and $|\Gamma_\infty|_{L^\infty} + \frac{\pi^2}{3}\,|\Gamma_1|_{L^1} < \delta(\Gamma_0)$. Assume moreover that there exists real numbers a, A, r, and R with $a \leqq A$ and $r < 0 < R$ such that

$$g(t, x) \geqq A \tag{4.3}$$

for a.e. $t \in [0, 2\pi]$ *and all* $x \geqq R$ *and*

$$g(t, x) \leqq a \tag{4.4}$$

for a.e. $t \in [0, 2\pi]$ *and all* $x \leqq r$. *Then the problem* (3.1) *has at least one solution for each* $e \in L^1(0, 2\pi)$ *such that*

$$a \leqq \bar{e} \leqq A. \tag{4.5}$$

Proof: Define g_1 on $[0, 2\pi] \times \mathbf{R}$ by $g_1(t, x) = g(t, x) - (1/2)(a + A)$ and e_1 on $[0, 2\pi]$ by $e_1(t) = e(t) - (1/2)(a + A)$, so that, for a.e. $t \in [0, 2\pi]$, using (4.3) to (4.5), we have

$$g_1(t, x) \geqq (1/2)(A - a) \geqq 0 \quad \text{if} \quad x \geqq R, \tag{4.6}$$

$$g_1(t, x) \leqq (1/2)(a - A) \leqq 0 \quad \text{if} \quad x \leqq r, \tag{4.7}$$

and

$$(1/2)(a - A) \leqq \bar{e}_1 \leqq (1/2)(A - a). \tag{4.8}$$

Clearly, the equation in (4.1) is equivalent to

$$x''(t) + f(x(t))\, x'(t) + g_1(t, x(t)) = e_1(t). \tag{4.9}$$

Moreover, we have

$$\limsup_{|x| \to \infty} x^{-1} g_1(t, x) \leqq \Gamma(t)$$

uniformly a.e. in $t \in [0, 2\pi]$ and if $|x| \geqq \max(R, -r)$, then for a.e. $t \in [0, 2\pi]$ we have also $x^{-1} g_1(t, x) \geqq 0$. So that $\Gamma(t) \geqq 0$ a.e. on $[0, 2\pi]$.

Let $\eta = \frac{1}{2}\left[\delta(\Gamma_0) - |\Gamma_\infty|_{L^\infty} - \frac{\pi^2}{3}|\Gamma_1|_{L^1}\right]$. Then there exists $r_1 > 0$ such that for a.e. $t \in [0, 2\pi]$ and for all x with $|x| \geqq r_1$, one has

$$0 \leqq x^{-1} g_1(t, x) \leqq \Gamma(t) + \eta. \tag{4.10}$$

Proceeding like in the proof of Theorem 1 of [3] we can write the equation in (4.9) in the equivalent form

$$x''(t) + f(x(t))\, x'(t) + \gamma_1(t, x(t))\, x(t) + h(t, x(t)) = e_1(t), \tag{4.11}$$

where $0 \leqq \gamma_1(t, x) \leqq \Gamma(t) + \eta$, $|h(t, x)| \leqq \alpha(t)$ for a.e. $t \in [0, 2\pi]$, all $x \in \mathbf{R}$ and some $\alpha \in L^1(0, 2\pi)$. Again, degree arguments will imply the existence of a solution for (4.1) if the set of possible solutions of the family of equations

$$\begin{aligned} &x''(t) + \lambda f(x(t))\, x'(t) + \left[(1 - \lambda)\left(\Gamma(t) + \eta\right) + \lambda\gamma_1(t, x(t))\right] x(t) \\ &+ \lambda h(t, x(t)) = \lambda e_1(t), \qquad \lambda \in [0, 1], \\ &x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \end{aligned} \tag{4.12}$$

is a priori bounded independently of $\lambda \in [0, 1]$. If x is a possible solution of (4.12) for some $\lambda \in [0, 1]$, then, integrating (4.12) over $[0, 2\pi]$ after multiplication by

$\bar{x} - \tilde{x}$, we obtain, using Lemma 3 with $\gamma = 0$ and Γ_∞ replaced by $\Gamma_\infty + \eta$,

$$0 = (2\pi)^{-1} \int_0^{2\pi} \Big\{\big(\bar{x} - \tilde{x}(t)\big) \big[x''(t) + \lambda f\big(x(t)\big) x'(t)$$
$$+ \big((1 - \lambda)(\Gamma(t) + \eta) + \lambda\gamma_1(t, x(t))\big) x(t) + \lambda h\big(t, x(t)\big) - \lambda e_1(t)\big]\Big\}\, dt$$
$$\geqq \left[\delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \eta\right] |\tilde{x}|^2_{H^1}$$
$$- (|\alpha|_{L^1} + |e_1|_{L^1}) |\bar{x} - \tilde{x}|_{L^\infty} \geqq \eta\, |\tilde{x}|^2_{H^1} - \beta(|\bar{x}| + |\tilde{x}|_{H^1}).$$

Consequently,

$$|\tilde{x}|^2_{H^1} \leqq (\beta/\eta)\,(|\bar{x}| + |\tilde{x}|_{H^1}).$$

Integrating the differential equation in (4.12) over $[0, 2\pi]$, we obtain

$$(1 - \lambda)(2\pi)^{-1} \int_0^{2\pi} \big(\Gamma(t) + \eta\big) x(t)\, dt + \lambda(2\pi)^{-1} \int_0^{2\pi} \big[g_1\big(t, x(t)\big) - e_1(t)\big]\, dt = 0. \tag{4.13}$$

If $x(t) \geqq R$ for all $t \in [0, 2\pi]$, then (3.6) and (3.8) imply that $(1 - \lambda)(\bar{\Gamma} + \eta) R \leqq 0$, a contradiction with $\bar{\Gamma} \geqq 0$. Similarly we cannot have $x(t) \leqq r$ for all $t \in [0, 2\pi]$. Consequently there exists $\tau \in [0, 2\pi]$ such that $r < x(\tau) < R$ and we can achieve the proof like in Theorem 1 of [3] ∎

5. An inequality for some linear second order operators with periodic boundary conditions and periodic solutions of some Duffing equations

We shall show in this section that a partial extension of Theorem 2 can be obtained when f is constant and $\Gamma_0 = \Gamma_\infty = 0$. It depends upon an inequality given by the following Lemma.

Lemma 4: *Let $c \in \mathbb{R}$, $e \in L^1(0, 2\pi)$, $\Gamma \in L^1(0, 2\pi)$ with $\bar{\Gamma} \geqq 0$. Then every possible solution x of the problem*

$$\begin{aligned} &x''(t) + cx'(t) + p(t)\, x(t) = e(t), \\ &x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \end{aligned} \tag{5.1}$$

with $p \in L^1(0, 2\pi)$ such that

$$\bar{p} \leqq \bar{\Gamma}, \qquad 0 \leqq p(t) \tag{5.2}$$

a.e. on $[0, 2\pi]$ satisfies the inequality

$$\left(1 - \frac{\pi^2}{4}\bar{\Gamma}\right) |x'' + cx'|^2_{L^1} \leqq 2\, |e|_{L^1} |x'' + cx'|_{L^1} + \bar{\Gamma}\, |e|_{L^1} |x|_{L^\infty}. \tag{5.3}$$

Proof: Let p be like above and let x be a possible solution of (5.1). Then, multiplying the equation by x and integrating over $[0, 2\pi]$ we obtain

$$-\frac{1}{2\pi} \int_0^{2\pi} x'^2(t)\, dt + \frac{1}{2\pi} \int_0^{2\pi} p(t)\, x^2(t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} e(t)\, x(t)\, dt. \tag{5.4}$$

Now, by Schwarz inequality and (5.2) we have, as $p^{1/2}x$ and $p^{1/2}$ belong to $L^2(0, 2\pi)$,

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |p(t)\,x(t)|\,dt\right)^2 \leqq \left(\frac{1}{2\pi}\int_0^{2\pi} p(t)\,dt\right)\left(\frac{1}{2\pi}\int_0^{2\pi} p(t)\,x^2(t)\,dt\right)$$

$$\leqq \bar{\Gamma}\left(\frac{1}{2\pi}\int_0^{2\pi} p(t)\,x^2(t)\,dt\right), \tag{5.5}$$

and hence, using (5.1),

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |e(t) - x''(t) - cx'(t)|\,dt\right)^2 \leqq \bar{\Gamma}\left(\frac{1}{2\pi}\int_0^{2\pi} p(t)\,x^2(t)\,dt\right). \tag{5.6}$$

On the other hand, by an inequality of E. Schmidt [5] we have, for every absolutely continuous function y on $[0, 2\pi]$ such that $y(0) = y(2\pi)$, $\bar{y} = 0$, the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} y^2(t)\,dt \leqq \frac{\pi^2}{4}\left(\frac{1}{2\pi}\int_0^{2\pi} |y'(t)|\,dt\right)^2 - \left(\frac{M+m}{2}\right)^2,$$

where $M = \max\limits_{[0,2\pi]} y$, $m = \min\limits_{[0,2\pi]} y$ and $\frac{\pi^2}{4}$ is the best possible constant. Applying this inequality to $x' + c\tilde{x}$, we find

$$\frac{1}{2\pi}\int_0^{2\pi} [x'(t) + c\tilde{x}(t)]^2\,dt = \frac{1}{2\pi}\int_0^{2\pi} [x'(t)]^2\,dt + \frac{c^2}{2\pi}\int_0^{2\pi} \tilde{x}^2(t)\,dt$$

$$\leqq \frac{\pi^2}{4}\left(\frac{1}{2\pi}\int_0^{2\pi} |x''(t) + cx'(t)|\,dt\right)^2. \tag{5.7}$$

Introducing (5.6) and (5.7) in (5.4), we obtain

$$-\frac{\pi^2}{4}|x'' + cx'|_{L^1}^2 + \bar{\Gamma}^{-1}|e - x'' - cx'|_{L^1}^2 \leqq |e|_{L^1}\,|x|_{L^\infty},$$

and hence (5.3) by elementary computations ∎

Let now $c \in \mathbf{R}$ and $g\colon [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ be like in the first paragraph of Section 4, $e \in L^1(0, 2\pi)$ and consider the following periodic boundary value problem for the Duffing equation

$$\begin{aligned} &x''(t) + cx'(t) + g(t, x(t)) = e(t),\\ &x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0. \end{aligned} \tag{5.8}$$

Theorem 3: *Assume that there exists $\Gamma \in L^1(0, 2\pi)$ such that*

$$\limsup_{|x|\to\infty} x^{-1}g(t, x) \leqq \Gamma(t)$$

uniformly a.e. on $[0, 2\pi]$ and such that $\bar{\Gamma} < 4/\pi^2$. Assume moreover that there exists real numbers a, A, r and R with $a \leqq A$ and $r < 0 < R$ such that, for a.e. $t \in [0, 2\pi]$,

$g(t, x) \geqq A$ when $x \geqq R$ and $g(t, x) \leqq a$ when $x \leqq r$. Then the problem (5.8) *has at least one solution for each $e \in L^1(0, 2\pi)$ verifying the relation $a \leqq \bar{e} \leqq A$.*

Proof: We first define g_1 and e_1 like in the proof of Theorem 2 so that the equation can be written

$$x''(t) + cx'(t) + g_1(t, x(t)) = e_1(t) \tag{5.9}$$

with $g_1(t, x) \geqq 0$ when $x \geqq R$ and $g_1(t, x) \leqq 0$ when $x \leqq r$, and $\limsup_{|x| \to \infty} x^{-1} g_1(t, x) \leqq \Gamma(t)$ uniformly a.e. on $[0, 2\pi]$. Consequently,

$$\Gamma(t) \geqq 0 \tag{5.10}$$

a.e. on $[0, 2\pi]$. Let $\eta = (1/2)(4/\pi^2 - \bar{\Gamma}) > 0$ so that $\bar{\Gamma} + \eta < 4/\pi^2$ and let $r_1 > 0$ be such that $0 \leqq x^{-1} g_1(t, x) \leqq \Gamma(t) + \eta$ for all x with $|x| \geqq r_1$ and a.e. $t \in [0, 2\pi]$. Proceeding like in the proof of Theorem 1 of [3] we can write the equation in (5.8) in the form

$$x''(t) + cx'(t) + \gamma_1(t, x(t))\, x(t) + h(t, x(t)) = e_1(t) \tag{5.11}$$

where $0 \leqq \gamma_1(t, x) \leqq \Gamma(t) + \eta$, $|h(t, x)| \leqq \alpha(t)$ for a.e. $t \in [0, 2\pi]$, all $x \in \mathbf{R}$ and some $\alpha \in L^1(0, 2\pi)$. The same degree arguments will imply the existence of a solution for (5.8) if the set of possible solutions of the family of equations

$$\begin{aligned} &x''(t) + cx'(t) + \left[(1 - \lambda)\left(\Gamma(t) + \eta\right) + \lambda \gamma_1(t, x(t))\right] x(t) \\ &= \lambda e_1(t) - \lambda h(t, x(t)), \qquad \lambda \in [0, 1], \\ &x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \end{aligned} \tag{5.12}$$

is a priori bounded independently of $\lambda \in [0, 1]$ in the uniform norm on $[0, 2\pi]$. As

$$0 \leqq (1 - \lambda)\left(\Gamma(t) + \eta\right) + \lambda \gamma_1(t, x(t)) \leqq \Gamma(t) + \eta$$

ofr a.e. $t \in [0, 2\pi]$, with $\bar{\Gamma} + \eta < 4/\pi^2$, and as

$$|e_1 + h(\cdot, x(\cdot))|_{L^1} \leqq |e_1|_{L^1} + |\alpha|_{L^1},$$

it follows from Lemma 4 that the inequality

$$\begin{aligned} \left[1 - \frac{\pi^2}{4}(\bar{\Gamma} + \eta)\right] |x'' + cx'|_{L^1}^2 &\leqq 2(|e_1|_{L^1} + |\alpha|_{L^1})\, |x'' + cx'|_{L^1} \\ &\quad + (\bar{\Gamma} + \eta)(|e_1|_{L^1} + |\alpha|_{L^1})\, |x|_{L^\infty} \end{aligned} \tag{5.13}$$

holds. We can now proceed like in the proof of Theorem 2 to obtain the existence of $\tau \in [0, 2\pi]$ such that

$$r < x(\tau) < R. \tag{5.14}$$

It is easy to write explicitly the unique periodic solution having mean value zero of the problem $x''(t) + cx'(t) = y(t)$ where $y \in L^1(0, 2\pi)$ and has mean value zero and to deduce from those formulas the existence of $\delta_1 = \delta_1(c) > 0$ and $\delta_2 = \delta_2(c) > 0$ such that

$$|\tilde{x}|_{L^\infty} \leqq \delta_1 |\tilde{x}'' + c\tilde{x}'|_{L^1} = \delta_1 |x'' + cx'|_{L^1}, \tag{5.15}$$

$$|x'|_{L^\infty} \leqq \delta_2 |\tilde{x}'' + c\tilde{x}'|_{L^1} = \delta_2 |x'' + cx'|_{L^1} \tag{5.16}$$

for every $x \in C^1([0, 2\pi])$ with x' absolutely continuous and satisfying the periodic boundary conditions. Inserting (5.15) in (5.13) we get

$$\left[1 - \frac{\pi^2}{4}(\bar{\Gamma} + \eta)\right] |x'' + cx'|_{L^1}^2$$
$$\leqq (|e_1|_{L^1} + |\alpha|_{L^1}) [2 + \delta_1(\bar{\Gamma} + \eta)] |x'' + cx'|_{L^1} + (\bar{\Gamma} + \eta)(|e_1|_{L^1} + |\alpha|_{L^1}) |\bar{x}|. \quad (5.17)$$

Now, by (5.14) we have, for all $t \in [0, 2\pi]$,

$$|x(t)| = \left| x(\tau) + \int_\tau^t x'(s)\, ds \right| < \max(-r, R) + 2\pi |x'|_{L^\infty}$$
$$\leq \max(-r, R) + 2\pi\delta_2 |x'' + cx'|_{L^1},$$

so that

$$|\bar{x}| \leqq \frac{1}{2\pi} \int_0^{2\pi} |x(t)|\, dt < \max(-r, R) + 2\pi\delta_2 |x'' + cx'|_{L^1}. \quad (5.18)$$

Inserting (5.18) in (5.17), we easily deduce the existence of $\varrho_1 = \varrho_1(\Gamma, e_1, \eta, c, r, R) > 0$ such that $|x'' + cx'|_{L^1} < \varrho_1$ which by (5.15) and (5.18) implies the existence of $\varrho > 0$ depending on the same quantities only and such that $|x|_{L^\infty} < \varrho$, which completes the proof. ∎

Remark 2: With respect to Remark 1, we see that when Γ_0 and $\Gamma_\infty = 0$, the condition on Γ is improved from $\bar{\Gamma} < 3/\pi^2$ into $\bar{\Gamma} < 4/\pi^2$ but the existence result requires that f is constant. Notice that, in contrast with Theorem 2, Theorem 3 is not sharp when applied to the case of a constant Γ.

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Manuskripteingang: 04. 01. 1983

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