(1.1)

Asymptotic conditions at the two first eigenvalues for the periodic solutions of Liénard differential equations and an inequality of E. Schmidt.

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ir untersuchen die periodische Randwertaufgabe

$$
x''(t) + f(x(t)) x'(t) + g(t, x(t)) = e(t),
$$

$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
$$

unter Nicht-Resonanz-Bedingungen auf $x^{-1}g(t, x)$ für $|x| \to \infty$.

Исследуется периодическая краевая задача

$$
x''(t) + f(x(t)) x'(t) + g(t, x(t)) = e(t),
$$

$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
$$

при нерезонансных условиях на $x^{-1}g(t, x)$ для $|x| \to \infty$.

We study the periodic boundary problem

$$
x''(t) + f(x(t)) x'(t) + g(t, x(t)) = e(t),
$$

$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
$$

under some non-resonance conditions on the asymptotic behavior of $x^{-1}g(t, x)$ for $|x| \to \infty$.

1. Introduction

This paper is devoted to the study of the periodic boundary value problem

$$
x''(t) + f(x(t)) x'(t) + g(t, x(t)) = e(t),
$$

$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
$$

when the asymptotic behavior of $x^{-1}g(t, x)$ is compared with the two first eigenvalues 0 and 1 of the linear problem

$$
x'' + \lambda x = 0,
$$

\n
$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.
$$
\n(1.2)

The results are in the line of the ones given by MAWHIN and WARD in [2] and [3] where a review of the preceding literature can be found. They essentially differ from [2] and [3] by generalizing the conditions on the function Γ which is such that

> $\limsup x^{-1}g(t, x) \leq \Gamma(t)$. $|\dot{x}| \rightarrow \infty$

Instead of assuming, like in [2] or [3] that $\Gamma(t) \leq 1$ with strict inequality on a subset of [0, 2 π] with positive measure, we write Γ in the form $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_{\infty}$

3 Analysis Bd. 3, Heft 1 (1984)

34 C. P. Guera and J. Mawmn

with Γ_0 satisfying the above condition on Γ , $\Gamma_1 \in L^1(0, 2\pi)$, $\Gamma_\infty \in L^\infty(0, 2\pi)$ and $| \Gamma_1 |_{L^1}$ and $| \Gamma_{\infty} |_{L^{\infty}}$ sufficiently small. Thus the expression

$$
\limsup_{|x|\to\infty}x^{-1}g(t,x)
$$

can now cross *any* number of eigenvalue *n2* of the problem (1.2) as far as those crossings take place in subsets of $[0, 2\pi]$ of sufficiently small measure. See GossEz [1] for similar results around the first eigenvalue.

As in *[2]* and [3], the results depend on lemmas giving a' priori inequalities and degree arguments. For Theorems 1 and 2, respectively in Sections 3 and 4, and which apply to the general case (1.1), those lemmas are slight improvements of the ones in [2] and [3]. For Theorem 3 in Section 5, which requires *f* to be constant, a rather different lemma is introduced which makes uses of an inequality of E. SCHMIDT [5] for periodic absolutely continuous functions. This lemma allows an improvement on the condition on *I'* when $\Gamma_0 = \Gamma_\infty = 0$ and $f \equiv 0$, but this condition is no more sharp when applied to the case of a constant Γ .

We end this introduction by mentioning that besides the classical spaces $C([0, 2\pi])$. $C^k([0, 2\pi])$ and $L^k(0, 2\pi)$ of continuous, k-times continuously differentiable or measurable real functions whose kth power of the absolute value is Lebesgue integrable, we shall make use in what follows of the Sobolev space $H¹(0, 2\pi)$ defined by

$$
H^1(0, 2\pi) = \{x : [0, 2\pi] \to \mathbb{R} \mid x \text{ is abs. cont. on } [0, 2\pi] \text{ and } x' \in L^2(0, 2\pi)\},
$$

with the inner product defined, by

$$
H^1(0, 2\pi) = \{x : [0, 2\pi] \to \mathbb{R} \mid x \text{ is abs. cont. on } [0, 2\pi] \text{ and } x \in
$$

\ninner product defined by
\n
$$
(x, y)_{H^1} = \left(\frac{1}{2\pi} \int_0^{2\pi} x(t) dt\right) \left(\frac{1}{2\pi} \int_0^{2\pi} y(t) dt\right) + \frac{1}{2\pi} \int_0^{2\pi} x'(t) y'(t) dt
$$

\ncorresponding norm $|\cdot|_{H^1}$. Notice also that we define for con
\n
$$
L^2(0, 2\pi)
$$
 by
\n
$$
|x|_{L^2} = \left(\frac{1}{2\pi} \int_0^{2\pi} |x(t)|^k dt\right)^{1/k}.
$$

and the corresponding norm $\|\cdot\|_{H^1}$. Notice also that we define for convenience the norm in $L^k(0, 2\pi)$ by

$$
|x|_{L^k}=\left(\frac{1}{2\pi}\int\limits_0^{2\pi}|x(t)|^k\ dt\right)^{1/k}.
$$

2. 'An inequality for some Liénard operators with periodic boundary conditions

For $x \in L^1(0, 2\pi)$, let us write

equality for some Liénard operators with period:
\n
$$
\bar{x}^{1}(0, 2\pi), \text{ let us write}
$$
\n
$$
\bar{x} = (2\pi)^{-1} \int_{0}^{2\pi} x(t) dt, \qquad \tilde{x}(t) = x(t) - \bar{x},
$$
\n
$$
\tilde{x}(t) dt = 0. \text{ Let } \tilde{H}^{1}(0, 2\pi) = \{x \in H^{1}(0, 2\pi) : \bar{x} = 0\}
$$
\n
$$
\text{Ilowing result is proved in MAWHIN-WARD [3].}
$$
\n
$$
\text{na 1:} \text{Let } \Gamma \in L^{1}(0, 2\pi) \text{ be such that, for a.e. } t \in [0, 1] \leq 1
$$

so that $\int \tilde{x}(t) dt = 0$. Let $\tilde{H}^1(0, 2\pi) = \{x \in H^1(0, 2\pi) : \tilde{x} = 0\}.$

The following result is proved in MAWHIN-WARD [3].

Lemma 1: Let $\Gamma \in L^1(0, 2\pi)$ be such that, for a.e. $t \in [0, 2\pi]$, one has

$$
T(t)\leq 1
$$

$$
(2.1)
$$

with the strict inequality on a subset of $[0, 2\pi]$ *of positive measure. Then there exists* $\delta = \delta(\Gamma) > 0$ such that for all $\tilde{x} \in \tilde{H}^1(0, 2\pi)$ one has
 $B_f(\tilde{x}) = (2\pi)^{-1} \int_0^{2\pi} [(\tilde{x}'(t))^2 - I'(t) \tilde{x}^2(t)] dt \geq \delta |\til$ $\delta = \delta(\Gamma) > 0$ such that for all $\tilde{x} \in \tilde{H}^1(0, 2\pi)$ one has

$$
B_{\Gamma}(\tilde{x}) = (2\pi)^{-1} \int_{0}^{2\pi} [(\tilde{x}'(t))^{2} - I'(t) \tilde{x}^{2}(t)] dt \geq \delta |\tilde{x}|_{H^{1}}^{2}.
$$

Lemma 2: Let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_{\infty}$ where $\Gamma_{\infty} \in L^{\infty}(0, 2\pi)$, $\Gamma_1 \in L^1(0, 2\pi)$, and $\mathbf{I}_0 \in L^1(0, 2\pi)$ is such that $\Gamma_0(t) \leqq 1$ for a.e. $t \in [0, 2\pi]$ with strict inequality on a subset *of* $[0, 2\pi]$ *of positive measure. Let* $\delta(\Gamma_0) > 0$ *be given by Lemma 1. Then one has, for all* $\tilde{x} \in \tilde{H}^1(0, 2\pi)$, $B_{\Gamma}(\tilde{x}) \equiv (2\pi)^{-1}$

1a 2: Let $\Gamma =$

1a 2: Let $\Gamma =$

10, 2 π) is such that

10, 2 π),
 $B_{\Gamma}(\tilde{x}) \geq \left[\delta(\Gamma_0) \right]$
 \vdots We have $f_1 + \sum_{\infty}$ wh
 $f(t) \leq 1$ for a.e. t
 f_2 . Let $\delta(\Gamma_0) > 0$
 $\frac{\pi^2}{3} | \Gamma_1 |_{L^1} - | \Gamma_{\infty} |$ *on a subset*
 or all $\tilde{x} \in \tilde{H}$
 $\int_{0}^{2\pi} |(\tilde{x}'(t))^{2} - \Gamma_{0} + \Gamma_{1} + \Gamma_{0}(t) \leq 1$ for
 i $\Gamma_{0}(t) \leq 1$ for
 i $\text{Var}e$. Let $\delta(I - \frac{\pi^{2}}{3} | \Gamma_{1}|_{L^{1}})$

$$
B_{\Gamma}(\tilde{x}) \geq \left[\delta(\Gamma_0) - \frac{\pi^2}{3} | \Gamma_1|_{L^1} - | \Gamma_{\infty}|_{L_{\infty}} \right] |\tilde{x}|_{H^1}^2.
$$

Proof: We have

 $\frac{1}{2}$

$$
B_{\Gamma}(\tilde{x}) = (2\pi)^{-1} \int_{0}^{2\pi} \left([\tilde{x}'(t)]^{2} - \Gamma_{0}(t) \tilde{x}^{2}(t) \right) dt
$$

\n
$$
- (2\pi)^{-1} \int_{0}^{2\pi} \Gamma_{1}(t) \tilde{x}^{2}(t) dt - (2\pi)^{-1} \int_{0}^{2\pi} \Gamma_{\infty}(t) \tilde{x}^{2}(t) dt.
$$

\ne fact that $H^{1}(0, 2\pi) \subset C^{0}[0, 2\pi]$ and the well-known inec
\nMAWHIN [4: p. 208])
\n
$$
|\tilde{x}|_{L^{1}} \leq |\tilde{x}'|_{L^{1}} = |\tilde{x}|_{H^{1}}, \qquad |\tilde{x}|_{L^{\infty}} \leq \frac{\pi}{\sqrt{3}} |\tilde{x}'|_{L^{1}} = \frac{\pi}{\sqrt{3}} |\tilde{x}|_{H^{1}},
$$

\nis Lemma 1, we obtain

Using the fact that $H^1(0, 2\pi) \subset \mathbb{C}^0[0, 2\pi]$ and the well-known inequalities (see e.g. ROUCHE-MAWHIN $[4: p. 208]$).

$$
|\tilde{x}|_{L^{\mathbf{x}}}\leq |\tilde{x}'|_{L^{\mathbf{x}}} = |\tilde{x}|_{H^1}, \qquad |\tilde{x}|_{L^\infty}\leq \frac{\pi}{\sqrt{3}} |\tilde{x}'|_{L^{\mathbf{x}}} = \frac{\pi}{\sqrt{3}} |\tilde{x}|_{H^1}
$$

as well as Lemma 1, we obtain

e fact that
$$
H^1(0, 2\pi) \subset C^0[0, 2\pi]
$$
 and the well-k
\nMaximum [4: p. 208])
\n $|\tilde{x}|_{L^1} \leq |\tilde{x}'|_{L^1} = |\tilde{x}|_{H^1}, \qquad |\tilde{x}|_{L^\infty} \leq \frac{\pi}{\sqrt{3}} |\tilde{x}'|_{L^1} = \frac{\pi}{\sqrt{3}}$
\ns Lemma 1, we obtain
\n $B_\Gamma(\tilde{x}) \geq \delta(\Gamma_0) |\tilde{x}|_{H^1}^2 - |\Gamma_1|_{L^1} |\tilde{x}|_{L^\infty}^2 - |\Gamma_\infty|_{L^\infty} |\tilde{x}|_{L^1}^2$
\n $\geq \left[\delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^1} - |\Gamma|_{L^\infty} \right] |\tilde{x}|_{H^1}^2$
\nrk 1: The best value for $\delta(0)$ is clearly 1, so that B_Γ

Remark 1: The best value for $\delta(0)$ is clearly 1, so *that* $B_{\Gamma_1}(\tilde{x}) \ge \left(1 - \frac{\pi^2}{3} |I_1|_{L^1}\right) |\tilde{x}|_H^2$
 Γ all $\tilde{x} \in \tilde{H}^1(0, 2\pi)$.

Lemma 3: *Let* $\gamma \in L^1(0, 2\pi)$, $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_{\infty}$ be like in Le Using the fact that $H^1(0, 2\pi) \subset C^0[0, 2\pi]$ and the well-known inequalities (see

ROUCHE-MAWHIN [4: p. 208])
 $|\tilde{x}|_{L^1} \leq |\tilde{x}'|_{L^1} = |\tilde{x}|_{H^1}, \qquad |\tilde{x}|_{L^\infty} \leq \frac{\pi}{\sqrt{3}} |\tilde{x}'|_{L^1} = \frac{\pi}{\sqrt{3}} |\tilde{x}|_{H^1},$

as well as *be given by Lemma 1. Then, for all measurable real functions p on* $[0, 2\pi]$ *such that* $\overline{\gamma} \leq \overline{p}$, $p(t) \leq \Gamma(t)$ *a.e. on* $[0, 2\pi]$, *all continuous functions* $f: \mathbb{R} \to \mathbb{R}$ *and all* $x \in W^{2,1}$ *Sor all* $\tilde{x} \in \tilde{H}^1(0, 2\pi)$ *.*
Lemma 3: Let $\gamma \in L^1(0, 2\pi)$, $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_{\infty}$ be like in Lemma 2 and $\delta(\Gamma_0)$ \times (0, 2π) such that s Lemma 1, we obtain
 $\hat{B}_\Gamma(\tilde{x}) \geq \delta(\Gamma_0) |\tilde{x}|_{H^1}^2 - |\Gamma_1|_{L^1} |\tilde{x}|_{L^\infty}^2 - |\Gamma_\infty|_{L^\infty} |\tilde{x}|_{L^1}^2$
 $\geq \left[\delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^1} - |\Gamma|_{L^\infty} \right] |\tilde{x}|_{H^1}^2$
 Γ is 1; The best value for $\delta(0)$ is clearly

$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \tag{2.2}
$$

one has

*3**

$$
(t) \leq \Gamma(t) \text{ a.e. on } [0, 2\pi], \text{ all continuous functions } f: \mathbf{H}
$$

\n
$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
$$

\n
$$
(2\pi)^{-1} \int_{0}^{2\omega} (\bar{x} - \bar{x}(t)) \left(x''(t) + f(x(t)) \, x'(t) + p(t) \, x(t)\right) dt
$$

\n
$$
\geq \overline{\gamma} \bar{x}^{2} + \left[\delta(\Gamma_{0}) - \frac{\pi^{2}}{3} | \Gamma_{1} |_{L^{1}} - | \Gamma_{\infty} |_{L^{\infty}} \right] |\bar{x}|_{H^{1}}^{2},
$$

\nIf $x \in W^{1,2}(0, 2\pi)$ and satisfies (2.2), we obtain easily

Proof: If $x \in W^{1,2}(0, 2\pi)$ and satisfies (2.2), we obtain easily, integrating by parts

and using Lemma 2,

C. P. Gupra and J. MawHIN
\ng Lemma 2,
\n
$$
(2\pi)^{-1} \int_{0}^{2\pi} (\bar{x} - \bar{x}(t)) (x''(t) + f(x(t)) x'(t) + p(t) x(t)) dt
$$
\n
$$
= \bar{p}\bar{x}^{2} + (2\pi)^{-1} \int_{0}^{2\pi} ([\bar{x}'(t)]^{2} - p(t) \bar{x}^{2}(t)) dt
$$
\n
$$
\geq \bar{p}\bar{x}^{2} + B_{\Gamma}(\bar{x}) \geq \bar{p}\bar{x}^{2} + \left[\delta(\Gamma_{0}) - \frac{\pi^{2}}{3} |[\Gamma_{1}]_{L^{1}} - |\Gamma_{\infty}|_{L^{\infty}}\right] |\bar{x}|_{H^{1}}^{2} \blacksquare
$$
\nSuppose conditions for the existence of periodic solutions

3. Nonresonance conditions for the existence **of** periodic solutions for some forced Liénard equations

Let $f: \mathbb{R} \to \mathbb{R}$ be continuous and let $g: [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$, $(t, x) \mapsto g(t, x)$ be such that $g(\cdot, x)$ is measurable on $[0, 2\pi]$ for each $x \in \mathbb{R}$ and $g(t, \cdot)$ is continuous on \mathbb{R} for almost each $t \in [0, 2\pi]$. Assume moreover that for each $r > 0$ there exists a $\gamma_r \in$ $L^1(0, 2\pi)$ such that $|g(t, x)| \leq \gamma_r(t)$ for a.e. $t \in [0, 2\pi]$ and all $x \in [-r, r]$. Such a g will be said to satisfy the Carathéodory conditions. Consider the following periodic boundary-value problem for the Liénard equation, with $e \in L^1(0, 2\pi)$, such that $|g(t, x)| \leq \gamma_r(t)$ for a.e. $t \in [0, 2\pi]$ and also id to satisfy the Carathéodory conditions. Consider ν -value problem for the Liénard equation, with $e \in L$
 $x''(t) + f(x(t)) x'(t) + g(t, x(t)) = e(t), \quad t \in [0, 2\pi],$ be continuous and let $g: [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$, $(t, x) \mapsto g(t, x)$
rable on $[0, 2\pi]$ for each $x \in \mathbb{R}$ and $g(t, \cdot)$ is continue
 $[0, 2\pi]$. Assume moreover that for each $r > 0$ there
that $|g(t, x)| \leq \gamma_r(t)$ for a.e. t

$$
x''(t) + f(x(t)) x'(t) + g(t, x(t)) = e(t), \qquad t \in [0, 2\pi],
$$

\n
$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.
$$
\n(3.1)

Theorem 1: Assume that the inequalities

We prove the following existence result for (3.1).
\n
$$
\begin{aligned}\n\text{Theorem 1:} \text{ Assume that the inequalities} \\
\gamma(t) &\leq \lim_{|x| \to \infty} \inf x^{-1} g(t, x) \leq \lim_{|x| \to \infty} \sup x^{-1} g(t, x) \leq \Gamma(t) \quad (3.2)\n\end{aligned}
$$

hold uniformly a.e. in t \in [0, 2 π] *and that* γ *and* Γ *satisfy the following conditions*

a) $\gamma \in L^1(0, 2\pi)$ and $\overline{\gamma} > 0$,

b) $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ with $\Gamma_1 \in L^1(0, 2\pi)$,

 $\Gamma_{\infty} \in L^{\infty}(0, 2\pi)$, Γ_0 is measurable on $[0, 2\pi]$, $\Gamma_0(t) \leq 1$ a.e. on $[0, 2\pi]$ with strict in*equality on a subset of measure zero and* $\frac{\pi^2}{3} | \Gamma_1|_{L^1} + | \Gamma_{\infty}|_{L^{\infty}} < \delta(\Gamma_0)$, where $\delta(\Gamma_0)$ is *given by Lemma* **1.** a) $\gamma \in L^1(0, 2\pi)$ and $\overline{\gamma} > 0$,

b) $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ with $\Gamma_1 \in L^1(0, 2\pi)$,
 $\delta \in L^\infty(0, 2\pi)$, Γ_0 is measurable on $[0, 2\pi]$, $\Gamma_0(t) \le 1$ and μ

is multity on a subset of measure zero and $\frac{\pi^2}{3$

Then problem (3.1) has at least one solution for each $e \in L^1(0, 2\pi)$.

 > 0 , then, by (3.2), we
 $\ln |x| \ge r$ we have $y(t) - \eta$ can find $r > 0$ such that for a.e. $t \in [0, 2\pi]$ and all x with $|x| \geq r$ we have $y(t) - \eta$ *xg(t, x) 1(1) + 77.* We then write, like in the proof of Theorem 1 of [2], the Ih(t, x)I 5 *a(t)* equation in (3.1) in the form $\leq x^{-1}g(t, x) \leq \Gamma(t) + \eta$. We then write, like in the proof of Theorem 1 of [2], the

$$
x''(t) + f(x(t)) x'(t) + \tilde{\gamma}(t, x(t)) x(t) + h(t, x(t)) = e(t),
$$

where

$$
\gamma(t) - \eta \leq \tilde{\gamma}(t, x) \leq \Gamma(t) + \eta, \qquad |h(t, x)| \leq \alpha(t) \tag{3.3}
$$

for a.e. $t \in [0, 2\pi]$, all $x \in \mathbb{R}$ and some $\alpha \in L^1(0, 2\pi)$. By the same degree argument than in the proof of Theorem 1, our result will be proved if we show that the set Asymptotic cof possible solutions, of the family of equations
 $x''(t) + \lambda f(x(t)) x'(t) + [(1 - \lambda) T(t) + \lambda \tilde{y}(t, x))]$

Asymptotic conditions at eigenvalues
\nof possible solutions, of the family of equations
\n
$$
x''(t) + \lambda f(x(t)) = x'(t) + [(1 - \lambda) \Gamma(t) + \lambda \tilde{\gamma}(t, x(t))]x(t)
$$
\n
$$
+ \lambda h(t, x(t)) = \lambda e(t), \quad \lambda \in [0, 1],
$$
\n(3.4)
\n
$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
$$
\nas a priori bounded in $C^1([0, 2\pi])$ independently of $\lambda \in [0, 1]$. If x is a solution of

is a priori bounded in $C^1([0, 2\pi])$ independently of $\lambda \in [0, 1]$. If x is a solution of (3.4), then multiplying (3.4) by $\bar{x} - \bar{x}$, integrating over [0, 2π] and using (3.3) together with Lemma 3 with Γ_{∞} replaced by $\Gamma_{\infty} + \eta$ and γ replaced by $\gamma - \eta$, we find

en multiplying (3.4) by
$$
\bar{x} - \tilde{x}
$$
, integrating over [0, 2 π] and u
\nwith Lemma 3 with Γ_{∞} replaced by $\Gamma_{\infty} + \eta$ and γ replaced
\n
$$
0 = (2\pi)^{-1} \int_{0}^{2\pi} (\bar{x} - \tilde{x}(t)) \{x''(t) + \lambda f(x(t) x'(t))
$$
\n
$$
+ [(1 - \lambda) \Gamma(t) + \lambda \tilde{\nu}(t, x(t))] x(t) + \lambda h(t, x(t)) - \lambda e(t) \} dt
$$
\n
$$
\geq (\bar{\gamma} - \eta) \bar{x}^{2} + \left[\delta(\Gamma_{0}) - \frac{\pi^{2}}{3} | \Gamma_{1} |_{L^{1}} - | \Gamma_{\infty} |_{L^{\infty}} - \eta \right] |\tilde{x}|_{H^{1}}^{2}
$$
\n
$$
- (|\alpha|_{L^{1}} + i e|_{L^{1}}) |\bar{x} - \tilde{x}|_{L^{\infty}}
$$
\n
$$
\geq \frac{\bar{\gamma}}{2} \bar{x}^{2} + \frac{1}{2} \left[\delta(\Gamma_{0}) - \frac{\pi^{2}}{3} | \Gamma_{1} |_{L^{1}} - | \Gamma_{\infty} |_{L^{\infty}} \right] |\tilde{x}|_{H^{1}}^{2} - \beta |x|_{H^{1}}
$$
\n
$$
\geq \eta |x|_{H^{1}}^{2} - \beta |x|_{H^{1}},
$$
\n
$$
\text{se } |x|_{H^{1}} \leq \beta/\eta. \text{ This implies then, like in the proof of Theorem 8 for some } R \text{ independent of } \lambda \in [0, 1] \blacksquare
$$

and hence $|x|_{H} \leq \beta/\eta$. This implies then, like in the proof of Theorem 1 of [2] that $|x|_{C} < R$ for some *R* independent of $\lambda \in [0, 1]$ **I**

4. Periodic solutions for a Liénard equation at resonance

Let $f: \mathbb{R} \to \mathbb{R}$ be continuous and let $g: [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$, $(t, x) \mapsto g(t, x)$ be such that $g(\cdot, x)$ is measurable on $[0, 2\pi]$ for each $x \in \mathbb{R}$ and $g(t, \cdot)$ is continuous on \mathbb{R} for a.e. $t \in [0, 2\pi]$. Assume moreover that for each $r > 0$ there exists $\gamma_r \in L^1(0, 2\pi)$ such that $|g(t, x)| \leq \gamma_r(t)$ for a.e. $t \in [0, 2\pi]$ and all $x \in [-r, r]$. We consider the following periodic boundary-value problem for the Liénard equation $\geq \eta |x|_{H^1}^2 - \beta |x|_{H^1}$,
 $\geq \eta |x|_{H^1} \leq \beta/\eta$. This implies then, like in the proof of Theorem is
 $\alpha \in \{x|_{H^1} \leq \beta/\eta\}$. This implies then, like in the proof of Theorem is
 α for some R independent of λ ous and let $g: [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$, $(t, x) \mapsto g(t, x)$ *b* e such

on [0, 2π] for each $x \in \mathbf{R}$ and $g(t, \cdot)$ is continuous on **R**
 r moreover that for each $r > 0$ there exists γ , $\in L^1(0, 2\pi)$

or a.e. $t \in [$

$$
x''(t) + f(x(t)) x'(t) + g(t, x(t)) = e(t), \qquad t \in [0, 2\pi],
$$

\n
$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.
$$
\n(4.1)

We prove the following existence result for (4.1)

Theorem 2: Assume that there exists $\Gamma \in L^1(0, 2\pi)$ *such that*

$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.
$$

ove the following existence result for (4.1).
rem 2: Assume that there exists $\Gamma \in L^1(0, 2\pi)$ such that

$$
\limsup_{|z| \to \infty} \frac{g(t, z)}{z} \le \Gamma(t)
$$
(4.2)

uniformly a.e. in t \in [0, 2π] *and such that* $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ where $\Gamma_\infty \in L^\infty(0, 2\pi)$, $P_1 \in L^1(0, 2\pi)$ and $P_0 \in L^1(0, 2\pi)$ are such that $\Gamma_0(t) \leq 1$ for a.e. $t \in [0, 2\pi]$, with *strict inequality on a subset of* $[0, 2\pi]$ *of positive measure and* $|\Gamma_{\infty}|_{L^{\infty}} + \frac{\pi}{3} | \Gamma_{1}|_{L} < \delta(\Gamma_{0}).$ Assume moreover that there exists real numbers a, A, r, and R with $a \leq A$ and $r < 0 < R$ *such that* $x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.$

ove the following existence result for (4.1).

cem 2: *Assume that there exists* $\Gamma \in L^1(0, 2\pi)$ such that
 $\limsup_{|x| \to \infty} \frac{g(t, x)}{x} \le \Gamma(t)$ (4.2)
 y *a.e.* in $t \in [0, 2\pi]$ and such tha

$$
g(t,x)\geq A
$$

$$
f_{\rm{max}}
$$

for a.e. $t \in [0, 2\pi]$ and all $x \ge R$ and

$$
g(t,x)\leq a\tag{4.4}
$$

for a.e. $t \in [0, 2\pi]$ and all $x \leq r$. Then the problem (3.1) has at least one solution for each $e \in L^1(0, 2\pi)$ such that

$$
a \leq \bar{e} \leq A. \tag{4.5}
$$

Proof: Define g_1 on $[0, 2\pi] \times R$ by $g_1(t, x) = g(t, x) - (1/2) (a + A)$ and e_1 on $[0, 2\pi]$ by $e_1(t) = e(t) - (1/2)(a + A)$, so that, for a.e. $t \in [0, 2\pi]$, using (4.3) to (4.5) , we have

$$
g_1(t,x) \ge (1/2) (A - a) \ge 0 \quad \text{if} \quad x \ge R,
$$
\n
$$
(4.6)
$$

$$
g_1(t,x) \leq (1/2) (a-A) \leq 0 \quad \text{if} \quad x \leq r, \tag{4.7}
$$

and

$$
(1/2) (a - A) \leq \bar{e}_1 \leq (1/2) (A - a).
$$
 (4.8)

Clearly, the equation in (4.1) is equivalent to

$$
x''(t) + f(x(t)) x'(t) + g_1(t, x(t)) = e_1(t).
$$
\n(4.9)

Moreover, we have

 $\limsup x^{-1}g_1(t, x) \leq \Gamma(t)$ $|x| \rightarrow \infty$

uniformly a.e. in $t \in [0, 2\pi]$ and if $|x| \ge \max(R, -r)$, then for a.e. $t \in [0, 2\pi]$ we have also $x^{-1}g_1(t, x) \ge 0$. So that $\Gamma(t) \ge 0$ a.e. on [0, 2π].

Let
$$
\eta = \frac{1}{2} \left[\delta(\Gamma_0) - |\Gamma_{\infty}|_{L^{\infty}} - \frac{\pi^2}{3} |\Gamma_1|_{L^1} \right]
$$
. Then there exists $r_1 > 0$ such that for

a.e. $t \in [0, 2\pi]$ and for all x with $|x| \ge r_1$, one has

$$
0 \leq x^{-1}g_1(t, x) \leq \Gamma(t) + \eta. \tag{4.10}
$$

Proceeding like in the proof of Theorem 1 of [3] we can write the equation in (4.9) in the equivalent form

$$
x''(t) + f(x(t)) x'(t) + \gamma_1(t, x(t)) x(t) + h(t, x(t)) = e_1(t), \qquad (4.11)
$$

where $0 \leq \gamma_1(t, x) \leq \Gamma(t) + \eta$, $|h(t, x)| \leq \alpha(t)$ for a.e. $t \in [0, 2\pi]$, all $x \in \mathbb{R}$ and some $\alpha \in L^1(0, 2\pi)$. Again, degree arguments will imply the existence of a solution for (4.1) if the set of possible solutions of the family of equations

$$
x''(t) + \lambda f(x(t)) x'(t) + [(1 - \lambda) (T(t) + \eta) + \lambda \gamma_1(t, x(t))] x(t)
$$

+ $\lambda h(t, x(t)) = \lambda e_1(t), \qquad \lambda \in [0, 1],$

$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
$$
 (4.12)

is a priori bounded independently of $\lambda \in [0, 1]$. If x is a possible solution of (4.12) for some $\lambda \in [0, 1]$, then, integrating (4.12) over $[0, 2\pi]$ after multiplication by

38

 \tilde{x} , we obtain, using Lemma 3 with $\gamma = 0$ and Γ_{∞} replaced by $\Gamma_{\infty} \div \eta$,

/

$$
0 = (2\pi)^{-1} \int_{0}^{2\pi} \{(\bar{x} - \bar{x}(t)) [x''(t) + \lambda f(x(t)) x'(t) + (1-\lambda) (T(t) + \eta) + \lambda \gamma_1(t, x(t))] x(t) + \lambda h(t, x(t)) - \lambda e_1(t)]\} dt
$$

\n
$$
\geq \left[\delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^1} - |\Gamma_{\infty}|_{L^{\infty}} - \eta \right] |\bar{x}|_H^2,
$$

\n
$$
- (|\alpha|_{L^1} + |e_1|_{L^1}) |\bar{x} - \bar{x}|_{L^{\infty}} \geq \eta |\bar{x}|_H^2 - \beta(|\bar{x}| + |\bar{x}|_{H^1}).
$$

\nently,

Consequently,

 $|\tilde{x}|_{H^1}^2 \leq (\beta/\eta) (|\tilde{x}| + |\tilde{x}|_{H^1}).$

Integrating the differential equation in (4.12) over [0, 2π], we obtain

$$
(1 - \lambda) (2\pi)^{-1} \int_{0}^{2\pi} \left(\Gamma(t) + \eta \right) x(t) dt + \lambda (2\pi)^{-1} \int_{0}^{2\pi} \left[g_1(t, x(t)) - e_1(t) \right] dt = 0. \tag{4.13}
$$

If $x(t) \ge R$ for all $t \in [0, 2\pi]$, then (3.6) and (3.8) imply that $(1 - \lambda) (\overline{P} + \eta) R \le 0$, a contradiction with $\overline{\overline{P}} \geq 0$. Similarly we cannot have $x(t) \leq r$ for all $t \in [0, 2\pi]$. Consequently there exists $\tau \in [0, 2\pi]$ such that $r < x(\tau) < R$ and we can achieve the proof like in Theorem 1 of **[3]I** If $x(t) \ge R$ for all $t \in [0, 2\pi]$, then (3.6) and (3.

a contradiction with $\overline{\Gamma} \ge 0$. Similarly we call

Consequently there exists $\tau \in [0, 2\pi]$ such the

the proof like in Theorem 1 of [3] \blacksquare

5. An inequality f R for all $t \in [0, 2\pi]$, then (3.6) and (3.8) imply that $(1 -$
diction with $\overline{\Gamma} \ge 0$. Similarly we cannot have $x(t) \le$
ently there exists $\tau \in [0, 2\pi]$ such that $r < x(\tau) < R$ a
like in Theorem 1 of $[3] \blacksquare$
equality f (4.13)
 $- 2$) ($\overline{\Gamma} + \eta$) $R \le 0$,

r for all $t \in [0, 2\pi]$.

nd we can achieve

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 α 2 can be obtained

uality given by the
 Then every possible

(5.1)

(5

r

5. An inequality for some linear second order operators with periodic boundary conditions and periodic solutions of some Dulling equations

We shall show in this section that a partial extension of Theorem 2 can be obtained when *f* is constant and $\Gamma_0 = \Gamma_\infty = 0$. It depends upon an inequality given by the following Lemma. uality for some linear second order operators with periodic boundary

ns and periodic solutions of some Duffing equations

now in this section that a partial extension of Theorem 2 can be obtained

constant and $\Gamma_0 = \Gamma_\in$ *a.e. on* [0, 2 π] *satisfies the inequality*
 $\vec{P} = \vec{P}$ $\infty = 0$. It depends upon an inequality gives following Lemma.

Lemma 4: Let $c \in \mathbb{R}$, $e \in L^1(0, 2\pi)$, $\Gamma \in L^1(0, 2\pi)$ with $\overline{\Gamma} \geq 0$. Then every s

Lemma 4: Let $c \in \mathbb{R}$, $e \in L^1(0, 2\pi)$, $\Gamma \in L^1(0, 2\pi)$ *with* $\overline{\Gamma} \geq 0$. *Then every possible solution x of the problem*

$$
x''(t) + cx'(t) + p(t) x(t) = e(t),
$$

\n
$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
$$
\n(5.1)

with $p \in L^1(0, 2\pi)$ *such that*

$$
\overline{p} \leq \overline{\Gamma}, \qquad 0 \leq p(t) \tag{5.2}
$$

$$
m a 4: Let c ∈ R, e ∈ L1(0, 2π), Γ ∈ L1(0, 2π) with $\overline{\Gamma} \ge 0$. Then every possible
\nx of the problem
\nx''(t) + cx'(t) + p(t) x(t) = e(t),
\nx(0) - x(2π) = x'(0) - x'(2π) = 0
\nL¹(0, 2π) such that
\n $\overline{p} \le \overline{\Gamma}$, $0 \le p(t)$
\n0, 2π] satisfies the inequality
\n
$$
\left(1 - \frac{\pi^2}{4} \overline{\Gamma}\right) |x'' + cx'|_{L^1}^2 \le 2 |e|_{L^1} |x'' + cx'|_{L^1} + \overline{\Gamma} |e|_{L^1} |x|_{L^\infty}.
$$
\n(5.3)
\nf: Let p be like above and let x be a possible solution of (5.1). Then, multi-
\nthe equation by x and integrating over [0, 2π] we obtain
$$

Proof: Let p be like above and let x be a possible solution of (5.1). Then, multiplying the equation by x and integrating over $[0, 2\pi]$ we obtain

$$
\overline{p} \leq \overline{\Gamma}, \qquad 0 \leq p(t) \tag{5.2}
$$
\n
$$
0, 2\pi] satisfies the inequality
$$
\n
$$
\left(1 - \frac{\pi^2}{4} \overline{\Gamma}\right) |x'' + cx'|_{L^1}^2 \leq 2 |e|_{L^1} |x'' + cx'|_{L^1} + \overline{\Gamma} |e|_{L^1} |x|_{L^\infty}. \tag{5.3}
$$
\n
$$
\therefore \text{Let } p \text{ be like above and let } x \text{ be a possible solution of (5.1). Then, multi-\nine equation by } x \text{ and integrating over } [0, 2\pi] \text{ we obtain}
$$
\n
$$
-\frac{1}{2\pi} \int_{0}^{2\pi} x'^2(t) dt + \frac{1}{2\pi} \int_{0}^{2\pi} p(t) x^2(t) dt = \frac{1}{2\pi} \int_{0}^{2\pi} e(t) x(t) dt. \tag{5.4}
$$

40 C. P. GUPTA and J. MAWHIN

Now, by Schwarz inequality and (5.2) we have, as $p^{1/2}x$ and $p^{1/2}$ belong to $L^2(0,\,2n)_\bullet$

y Schwarz inequality and (5.2) we have, as
$$
p^{1/2}x
$$
 and $p^{1/2}$ belong to $L^2(0, 2\pi)$,
\n
$$
\left(\frac{1}{2\pi}\int_{0}^{2\pi}|p(t)x(t)| dt\right)^2 \le \left(\frac{1}{2\pi}\int_{0}^{2\pi}p(t) dt\right)\left(\frac{1}{2\pi}\int_{0}^{2\pi}p(t)x^2(t) dt\right).
$$
\n
$$
\le \overline{\Gamma}\left(\frac{1}{2\pi}\int_{0}^{2\pi}p(t)x^2(t) dt\right),
$$
\ne, using (5.1),
\ne, using (5.1),
\n
$$
\left(\frac{1}{2\pi}\int_{0}^{2\pi}|e(t) - x''(t) - cx'(t)| dt\right)^2 \le \overline{\Gamma}\left(\frac{1}{2\pi}\int_{0}^{2\pi}p(t)x^2(t) dt\right).
$$
\n(5.6)
\nother hand, by an inequality of E. SCHMDT [5] we have, for every absolutely
\nus function y on [0, 2\pi] such that $y(0) = y(2\pi)$, $\overline{y} = 0$, the inequality
\n
$$
\frac{1}{2\pi}\int_{0}^{2\pi}y^2(t) dt \le \frac{\pi^2}{4}\left(\frac{1}{2\pi}\int_{0}^{2\pi}|y'(t)| dt\right)^2 - \left(\frac{M+m}{2}\right)^2,
$$
\n
$$
I = \max_{[0,2\pi]} y, m = \min_{[0,2\pi]} y
$$
 and $\frac{\pi^2}{4}$ is the best possible constant. Applying this
\nby to $x' + c\tilde{x}$, we find
\n
$$
\frac{1}{2\pi}\int_{0}^{2\pi}[x'(t) + c\tilde{x}(t)]^2 dt = \frac{1}{2\pi}\int_{0}^{2\pi}[x'(t)]^2 dt + \frac{c^2}{2\pi}\int_{0}^{2\pi} \tilde{x}^2(t) dt
$$
\n
$$
\le \frac{\pi^2}{2\pi}\left(\frac{1}{\pi}\int_{0}^{2\pi}|x''(t) + cx'(t)| dt\right)^2.
$$
\n(5.7)

and hence, using (5.1),

$$
\leq \Gamma\left(\frac{1}{2\pi}\int_{0}^{2\pi} p(t) x^{2}(t) dt\right),
$$
\n(5.3)

\n(5.4)

\n
$$
\left(\frac{1}{2\pi}\int_{0}^{2\pi} |e(t) - x''(t) - cx'(t)| dt\right)^{2} \leq \overline{\Gamma}\left(\frac{1}{2\pi}\int_{0}^{2\pi} p(t) x^{2}(t) dt\right).
$$
\n(5.6)

\n(5.6)

\n(5.6)

\n(5.7)

\n(5.8)

\n(5.9)

\n(5.1)

continuous function y on $[0, 2\pi]$ such that $y(0) = y(2\pi)$, $\bar{y} = 0$, the inequality

$$
\left(\frac{1}{2\pi}\int_{0}^{1} |e(t) - x''(t) - cx'(t)| dt\right) \leq \overline{\Gamma}\left(\frac{1}{2\pi}\int_{0}^{1} p(t) dt\right)
$$

other hand, by an inequality of E. SCHMIDT [5] we have
us function y on [0, 2 π] such that $y(0) = y(2\pi)$, $\overline{y} =$

$$
\frac{1}{2\pi}\int_{0}^{2\pi} y^{2}(t) dt \leq \frac{\pi^{2}}{4}\left(\frac{1}{2\pi}\int_{0}^{2\pi} |y'(t)| dt\right)^{2} - \left(\frac{M + m}{2}\right)^{2},
$$

$$
= \max_{\begin{subarray}{l} [0, 2\pi] \\ y \text{ to } x' + c\overline{x}, \text{ we find} \end{subarray}} \frac{\pi^{2}}{4} \text{ is the best possible of }
$$

where $M = \max_{n \ge 1} y$, $m = \min_{n \ge 1} y$ and $\frac{\pi}{4}$ is the best possible constant. Applying this inequality to $x' + c\bar{x}$, we find

$$
\lim_{\Delta z \to 0} \frac{2\pi}{2} \int_{0}^{2\pi} y^{2}(t) dt \leq \frac{\pi^{2}}{4} \left(\frac{1}{2\pi} \int_{0}^{2\pi} |y'(t)| dt \right)^{2} - \left(\frac{M + m}{2} \right)^{2},
$$
\n
$$
= \max_{0,2\pi 1} y, m = \min_{0,2\pi 1} y \text{ and } \frac{\pi^{2}}{4} \text{ is the best possible constant. Applying this}
$$
\n
$$
y \text{ to } x' + c\bar{x}, \text{ we find}
$$
\n
$$
\frac{1}{2\pi} \int_{0}^{2\pi} [x'(t) + c\bar{x}(t)]^{2} dt = \frac{1}{2\pi} \int_{0}^{2\pi} [x'(t)]^{2} dt + \frac{c^{2}}{2\pi} \int_{0}^{2\pi} \bar{x}^{2}(t) dt
$$
\n
$$
\leq \frac{\pi^{2}}{4} \left(\frac{1}{2\pi} \int_{0}^{2\pi} |x''(t) + cx'(t)| dt \right)^{2}.
$$
\n
$$
\lim_{\Delta z \to 0} (5.6) \text{ and } (5.7) \text{ in (5.4), we obtain}
$$
\n
$$
= \frac{\pi^{2}}{4} |x'' + cx'|_{L^{1}}^{2} + \overline{T}^{-1} |e - x'' - cx'|_{L^{2}}^{2} \leq |e|_{L^{1}} |x|_{L^{\infty}},
$$
\n
$$
= \frac{\pi^{3}}{4} |x'' + cx'|_{L^{1}}^{2} + \overline{T}^{-1} |e - x'' - cx'|_{L^{2}}^{2} \leq |e|_{L^{1}} |x|_{L^{\infty}},
$$

Introducing (5.6) **and (5.7) in (5.4), we** obtain

$$
-\frac{\pi^2}{4}|x''+cx'|_{L^1}^2+\bar{F}^{-1}|e-x''-cx'|_{L^1}^2 \leq |e|_{L^1}|x|_{L^\infty},
$$

and hence (5.3) by elementary computations **^I**

Let now $c \in \mathbf{R}$ and $g: [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ be like^t in the first paragraph of Section 4. Let now $e \in \mathbf{R}$ and $g \cdot [0, 2x] \wedge \mathbf{R}$ be the following periodic boundary value problem for the Duffing equation
 $x''(t) + cx'(t) + g(t, x(t)) = e(t)$, (5.8) Duffing equation

$$
x''(t) + cx'(t) + g(t, x(t)) = e(t),
$$

\n
$$
x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.
$$
\n(5.8)

Theorem 3: Assume that there exists $\Gamma \in L^1(0, 2\pi)$ such that

$$
\limsup_{|x|\to\infty}x^{-1}g(t,x)\leq \Gamma(t)
$$

uniformly a.e. on $[0, 2\pi]$ *and such that* $\overline{\Gamma} < 4/\pi^2$. Assume moreover that there exists *real numbers a, A, r and R with a* $\leq A$ *and r* $< 0 < R$ *such that, for a.e. t* \in $[0, 2\pi]$.

Asymptotic conditions at eigenvalues 41
 $g(t, x) \ge A$ when $x \ge R$ and $g(t, x) \le a$ when $x \le r$. Then the problem (5.8) has at least one solution for each $e \in L^1(0, 2\pi)$ verifying the relation $a \le \bar{e} \le A$. *least one solutions at eigenv*
g(t, x) \geq *A when* $x \geq R$ *and g(t, x)* \leq *a when* $x \leq r$. Then the problem
least one solution for each e \in *L*¹(0, 2*x*) *verifying the relation a* \leq $\bar{e} \leq$ *A*. Asymptotic
 A when $x \ge R$ and $g(t, x) \le a$ when $x \le r$

solution for each $e \in L^1(0, 2\pi)$ verifying the re

: We first define g_1 and e_1 like in the proof of

ritten
 $x''(t) + cx'(t) + g_1(t, x(t)) = e_1(t)$
 $(x) \ge 0$ when $x \ge R$ Asymptotic condition
 $x) \ge A$ when $x \ge R$ and $g(t, x) \le a$ when $x \le r$. Then

st one solution for each $e \in L^1(0, 2\pi)$ verifying the relation

Proof: We first define g_1 and e_1 like in the proof of Theor

i be written

Proof: We first define g₁ and e_1 like in the proof of Theorem 2 so that the equation can be written

$$
x''(t) + cx'(t) + g_1(t, x(t)) = e_1(t)
$$
\n(5.9)

with $g_1(t, x) \ge 0$ when $x \ge R$ and $g_1(t, x) \le 0$ when $x \le r$, and $\limsup_{|x| \to \infty} x^{-1}g_1(t, x) \le \Gamma(t)$ uniformly a.e. on $[0, 2\pi]$. Consequently, $|x| \rightarrow \infty$

$$
\Gamma(t) \geq 0 \tag{5.10}
$$

A when $x \ge R$ and
solution for each $e \in$
: We first define g_1 a
ritten
 $x''(t) + cx'(t) + g_1(t)$
 $x' \ge 0$ when $x \ge 0$
informly a.e. on $[0, 1]$
 $\Gamma(t) \ge 0$
 $\Gamma(t) \ge 0$
 $\Gamma(t) \le x^{-1}g_1(t, x)$
 $\Gamma(t) \le x^{-1}g_1(t, x)$ a.e. on [0, 2π]. Let $\eta = (1/2) (4/\pi^2 - \bar{T}) > 0$ so that $\bar{T} + \eta < 4/\pi^2$ and let $r_1 > 0$ be such that $0 \le x^{-1}g_1(t, x) \le \Gamma(t) + \eta$ for all x with $|x| \ge r_1$ and a.e. $t \in [0, 2\pi]$. Proceeding like in the proof of Theorem 1 of $[3]$ we can write the equation in (5.8) in the form x *C* irst define g_1 and e_1 like in the proof of Theorem 2 so that the equation
 $x''(t) + cx'(t) + g_1(t, x(t)) = e_1(t)$ (5.9)
 $x' = 0$ when $x \ge R$ and $g_1(t, x) \le 0$ when $x \le r$, and $\limsup_{|x| \to \infty} x^{-1}g_1(t, x)$

informly a.e. o

$$
x''(t) + cx'(t) + \gamma_1(t, x(t)) x(t) + h(t, x(t)) = e_1(t)
$$
\n(5.11)

where $0 \leq \gamma_1(t, x) \leq \Gamma(t) + \eta$, $|h(t, x)| \leq \alpha(t)$ for a.e. $t \in [0, 2\pi]$, all $x \in \mathbb{R}$ and some $\alpha \in L^1(0, 2\pi)$. The same degree arguments will imply the existence of a solution for (5.8) if the set of possible solutions of the family of equations

$$
\Gamma(t) \geq 0
$$
\n
$$
(5.10)
$$
\n
$$
2\pi
$$
\n<math display="block</math>

is a priori bounded independently of $\lambda \in [0, 1]$ in the uniform norm on $[0, 2\pi]$. As

$$
0\leq (1-\lambda)\bigl(\Gamma(t)+\eta\bigr)+\lambda\gamma_1(t,x(t))\leq \Gamma(t)+\eta
$$

ofr a.e. $t \in [0, 2\pi]$, with $\overline{\Gamma} + \eta < 4/\pi^2$, and as

 $|e_1 + h(\cdot, x(\cdot))|_{L^1} \leq |e_1|_{L^1} + |\alpha|_{L^1}$

it follows from Lemma 4 that the inequality

$$
0 \leq (1 - \lambda) \left\{ \Gamma(t) + \eta \right\} + \lambda \gamma_1(t, x(t)) \leq \Gamma(t) + \eta
$$

\n
$$
\in [0, 2\pi], \text{ with } \overline{\Gamma} + \eta < 4/\pi^2, \text{ and as}
$$

\n
$$
|e_1 + h(\cdot, x(\cdot))|_{L^1} \leq |e_1|_{L^1} + |\alpha|_{L^1},
$$

\nfrom Lemma 4 that the inequality
\n
$$
\left[1 - \frac{\pi^2}{4} \left(\overline{\Gamma} + \eta\right) |x'' + cx'|_{L^1}^2 \leq 2(|e_1|_{L^1} + |\alpha|_{L^1}) |x'' + cx'|_{L^1} + (\overline{\Gamma} + \eta) (|e_1|_{L^1} + |\alpha|_{L^1}) |x|_{L^{\infty}}
$$
(5.13)
\ne can now proceed like in the proof of Theorem 2 to obtain the existence
\n 2π such that
\n $r < x(\tau) < R.$
\n y to write explicitly the unique periodic solution having mean value zero
\n $\text{coblem } x''(t) + cx'(t) = y(t) \text{ where } y \in L^1(0, 2\pi) \text{ and has mean value zero}$
\n $\text{duce from those formulas the existence of } \delta_1 = \delta_1(c) > 0 \text{ and } \delta_2 = \delta_2(c) > 0$
\nt
\n $|\tilde{x}|_{L^{\infty}} \leq \delta_1 |\tilde{x}'' + c\tilde{x}'|_{L^1} = \delta_1 |x'' + cx'|_{L^1},$
\n $|x'|_{L^{\infty}} \leq \delta_2 |\tilde{x}'' + c\tilde{x}'|_{L^1} = \delta_2 |x'' + cx'|_{L^1}$ (5.15)
\n $|x'|_{L^{\infty}} \leq \delta_2 |\tilde{x}'' + c\tilde{x}'|_{L^1} = \delta_2 |x'' + cx'|_{L^1}$

holds. We can now proceed like in the proof of Theorem 2 to obtain the existence of $\tau \in [0, 2\pi]$ such that

$$
r < x(\tau) < R. \tag{5.14}
$$

It' is easy to write explicitly the unique periodic solution having mean value zero of the problem $x''(t) + cx'(t) = y(t)$ where $y \in L^1(0, 2\pi)$ and has mean value zero and to deduce from those formulas the existence of $\delta_1 = \delta_1(c) > 0$ and $\delta_2 = \delta_2(c) > 0$
such that
 $|\tilde{x}|_{L^\infty} \leq \delta_1 |\tilde{z}'' + c\tilde{x}'|_{L^1} = \delta_1 |z'' + cx'|_{L^1}$, (5.15) it follows from Lemma 4 that the inequality
 $\left[1 - \frac{\pi^2}{4} (\overline{F} + \eta)\right] |x'' + cx'|_{L^1}^2 \leq 2(|c_1|_{L^1} + |\alpha|_{L^1}) |x'' + cx'|_{L^1} + (\overline{F} + \eta)(|c_1|_{L^1} + |\alpha|_{L^1}) |x|_{L^{\infty}}\right]$

holds. We can now proceed like in the proof of Theorem (5.13)

orem 2 to obtain the existence

(5.14)

ution having mean value zero
 2π) and has mean value zero
 $\delta_1(c) > 0$ and $\delta_2 = \delta_2(c) > 0$

(5.15)

(5.16)

$$
|\tilde{x}|_{I,\infty} \leq \delta_1 |\tilde{x}'' + c\tilde{x}'|_{L^1} = \delta_1 |x'' + cx'|_{L^1}, \tag{5.15}
$$

$$
|x'|_{L^{\infty}} \leq \delta_2 |\tilde{x}'' + c\tilde{x}'|_{L^1} = \delta_2 |x'' + cx'|_{L^1}
$$
\n(5.16)

for every $x \in C^1([0, 2\pi])$ with x' absolutely continuous and satisfying the periodic boundary conditions. Inserting (5.15) in (5.13) we get

y conditions. Inserting (5.15) in (5.13) we get
\n
$$
\left[1 - \frac{\pi^2}{4} (\overline{F} + \eta)\right] |x'' + cx'|_{L^1}^2
$$
\n
$$
\leq (|e_1|_{L^1} + |\alpha|_{L^1}) [2 + \delta_1(\overline{F} + \eta)] |x''| + cx'|_{L^1} + (\overline{F} + \eta) (|e_1|_{L^1} + |\alpha|_{L^1}) |\overline{x}|.
$$
\n(5.14) we have, for all $t \in [0, 2\pi]$,
\n
$$
|x(t)| = \left|x(\tau) + \int x'(s) ds\right| < \max(-r, R) + 2\pi |x'|_{L^\infty}
$$
\n
$$
\leq \max(-r, R) + 2\pi\delta_2 |x'' + cx'|_{L^1},
$$
\n
$$
|\overline{x}| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |x(t)| dt < \max(-r, R) + 2\pi\delta_2 |x'' + cx'|_{L^1}.
$$
\n(5.18)
\n
$$
g(5.18) \text{ in (5.17), we easily deduce the existence of } \rho_1 = \rho_1(\Gamma, \rho_1, \eta, c, r, R) > 0
$$
\nt $|x'' + cx'|_{L^1} < \rho_1$ which by (5.15) and (5.18) implies the existence of $\rho > 0$

Now, by (5.14) we have, for all $t \in [0, 2\pi]$,

$$
\leq (|e_1|_{L^1} + |\alpha|_{L^1}) [2 + \delta_1(\bar{T} + \eta)] |x'' \perp | cx'|_{L^1} + (\bar{T} + \delta_1(\bar{T} + \eta)) |x'' \perp | cx'|_{L^1} + (\bar{T} + \delta_1(\bar{T} + \eta)) |x'' \perp | cx'|_{L^1} + (\bar{T} + \delta_1(\bar{T} + \eta)) |x'' \perp | cx'|_{L^1}
$$
\n
$$
\leq |x(t)| = |x(t) + \int_0^t x'(s) ds| < \max (-r, R) + 2\pi |x'|_{L^1}
$$
\n
$$
\leq \max (-r, R) + 2\pi \delta_2 |x'' + cx'|_{L^1},
$$

so that

$$
|\bar{x}| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |x(t)| dt < \max(-\tau, R) + 2\pi \delta_2 |x'' + cx'|_{L^1}.
$$
 (5.18)

Inserting (5.18) in (5.17), we easily deduce the existence of $\varrho_1 = \varrho_1(\Gamma, e_1, \eta, c, r, R) > 0$ such that $|x'' + cx'|_{L^1} < \rho_1$ which by (5.15) and (5.18) implies the existence of $\rho > 0$ depending on the same quantities only and such that $|x|_{L^\infty} < \rho$, which completes the proof. I

Remark 2: With respect to Remark 1, we see that when Γ_0 and $\Gamma_\infty = 0$, the condition on *I'* is improved from $\overline{I} \leq 3/\pi^2$ into $\overline{I} \leq 4/\pi^2$ but the existence result requires that *f* is constant. Notice that, in contrast with Theorem 2, Theorem 3 is not sharp when applied to the case of a constant *r.*

REFERENCES

- [1] GossEz, J. P.: Some nonlinear differential equations with resonance at the first eigenvalue In: Atti **30** Seminario di Analisi Funzionale ed Applicazioni (SAFA III), Confer. Semin. Mat. Univ. Bari 163-168 (1979), 355-389.
- [2] MAWBTN, 3., and J. R. **WARD:** Nonuniform nonresonance conditions at the two first eigenvalues for periodic solutions of forced Liénard and Duffing equations. Rocky Mountain J. Math. 12 (1982), 643-654.
- [3] MAwHIN, J., and J. R. WARD: Periodic solutions of some forced Liénard differential equations at resonance (to appear).
- [4] ROUCHE, N., and J. MAWHIN: Ordinary Differential Equations. Stability and Periodic Solutions. Pitman: Boston 1980.
- [5] Scnaunv, E.: Uber die Ungleichung, weiche die Integrals fiber eine Potenz einer Funktion und fiber eine andere Potenz ihrer Ableitung verbindet. Math. Ann. 117 (1940), 301-326.

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