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On the existence of the solution of an abstract optimization problem related to a quasi-variational inequality

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Es wird ein allgemeines Konzept zur Erlangung von Existenz- und Regularitätsresultaten für ein Optimierungsproblem angegeben, das in einem engen Zusammenhang zu einer Quasi-Variationsungleichung steht.

Дается общий подход получения результатов о существовании и регулярности для некоторой проблемы оптимизации, которая тесно связана с одним квази-вариационным неравенством.

A general concept is given to get existence and regularity results for an optimization problem that is closely connected to a quasi-variational inequality.

In this paper a general concept is given to get existence results for the problem

$$\|x - u\|_B = \inf_{\substack{y \in X_0, v \in U_0 \\ [y,v] \in G(S)}} \|y - v\|_B \tag{1}$$

where X_0 and U_0 are closed and convex subsets of a reflexive Banach space B_0 being continuously imbedded into the reflexive Banach space B, and $[y, v] \in G(S)$ means that y is a solution corresponding to the parameter v of a certain parametric problem in B.

If the mentioned parametric problem is a parametric variational inequality (and this will be assumed later on)

$$\left.\begin{aligned} &y \in C(v), \qquad v \in U \subset B \\ &(A(y, v) - y^*, z - y)_B \geqq h(y, v) - h(z, v) \qquad \forall\, z \in C(v) \end{aligned}\right\} \tag{2}$$

and $X_0 = U_0$ then (1) is closely connected to the quasi-variational inequality

$$\left.\begin{aligned} &u \in C(u), \qquad u \in U_0 \\ &(A(u, u) - y^*, z - u)_B \geqq h(u, u) - h(z, u) \qquad \forall\, z \in C(u) \end{aligned}\right\} \tag{3}$$

(cf. [1]).

The main reasons to investigate (1) instead of (3) are the following:

(i) (1) can be solved under milder conditions than (3); a solution of (1) can be considered as a generalized solution of (3).

(ii) If (3) is solvable then the solution sets of (1) and (3) coincide.

(iii) To solve (1) optimization techniques can be used (cf. [1] where approximation procedures are given).

Compared with the literature on existence for quasi-variational inequalities (cf. e.g. [3, 5]) here the condition

$$SU_0 \subset U_0 \tag{4}$$

(where S is the solution operator of (2)) is not needed. Further, in the sequel the uniform coercivity condition of [5] is weakened and other conditions are generalized to our case.

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Let B_0, B be reflexive Banach spaces, let B_0 be continuously imbedded into B, B^* the adjoint space of B, $(\cdot,\cdot)$ the pairing between B^* and B, $\|\cdot\|$ the norm in B and B^*, $\|\cdot\|_0$ the norm in B_0. Let further U be a w-closed subset of B, X_0 and U_0 w-closed subsets of B_0, $U_0 \subset U \cap B_0$. Let on U a multivalued mapping C be defined, $\emptyset \neq C(u) \subset B$, $C(u)$ closed and convex. Further, let $A(\cdot, u)$ be an operator from (the whole of) $C(u)$ into B^*, $h(\cdot, u)$ an admissible functional on $C(u)$, and $y^* \in B^*$.

We solve (1) with the help of the following theorem of Weierstrass:

In a reflexive Banach space a w-l.s.c. functional attaines its inf on a bounded, w-closed subset.

The functional

$$f(x, u) = \|x - u\|$$

is w-l.s.c. on $B_0 \times B_0$. Indeed, let be $x_i \rightharpoonup x$, $u_i \rightharpoonup u$ in B_0. As the imbedding into B is linear and continuous it is also w-continuous, i.e. $x_i \rightharpoonup x$, $u_i \rightharpoonup u$ in B. Hence $\|x - u\| \leqq \underline{\lim} \|x_i - u_i\|$ because of the w-l.s.continuity of the norm.

Let $G(S)$ be the graph of S, i.e.

$$G(S) = \{[y, v] \in B \times U \quad \text{such that } y \in Sv\}$$

and let us suppose that

$$\text{there is an } [y_0, v_0] \in G(S) \text{ with } y_0 \in X_0,\ v_0 \in U_0. \tag{5}$$

We consider the (non-empty) set $M_1 \subset B_0 \times B_0$,

$$M_1 = \{[y, v] \in G(S): y \in X_0, v \in U_0, \|y - v\|_0 \leqq \|y_0 - v_0\|_0\}.$$

If M_1 is

(i) bounded in $B_0 \times B_0$ and

(ii) w-closed in $B_0 \times B_0$

then by the Weierstrass theorem $f(x, u)$ will attain its inf on M_1. This inf then is clearly a solution of (1).

Sufficient for the boundedness of M_1 in $B_0 \times B_0$ is that

$$\left.\begin{array}{l} M = \{y \in X_0 : \exists\, v \in U_0 \text{ s.t. } [y, v] \in G(S) \text{ and } \|y - v\|_0 \leqq c\} \\ \text{is bounded in } B_0 \text{ for every } c \geqq 0. \end{array}\right\} \tag{6}$$

Indeed, let us take $c = \|y_0 - v_0\|_0$. If y is bounded then also v has to be bounded since $\|y - v\|_0 \leqq c$. This is (i).

Sufficient for the w-closedness of M_1 is that

$$G(S) \cap (X_0 \times U_0) \text{ is w-closed in } B_0 \times B_0. \tag{7}$$

Indeed, let be $[y_i, v_i] \in M_1$, $[y_i, v_i] \rightharpoonup [y, v]$ in $B_0 \times B_0$. Since $[y_i, v_i] \in G(S) \cap (X_0 \times U_0)$, (7) implies $[y, v] \in G(S) \cap (X_0 \times U_0)$. We have further $\|y - v\|_0 \leqq \underline{\lim} \|y_i - v_i\|_0 \leqq \|y_0 - v_0\|_0$. This is (ii).

If $G(S)$ is w-closed in $B \times B$ then clearly (7) holds.

In the sequel we will assume that $[y, v] \in G(S)$ means y is a solution corresponding to the parameter v of the parametric variational inequality (2) and give sufficient conditions for (6) and (7).

Let us begin with (6).

If one of X_0 and U_0 is bounded M is clearly bounded. In the case where both X_0 and U_0 are unbounded we have

Proposition 1: *Let for every $u \in U_0$ there be an $Nu \in C(u)$ s.t.*

$$\|Nu\| \leqq b\,\|u\| + c, \qquad b \geqq 0, \tag{8}$$

$$|h(Nu, u)| \leqq d\,\|u\| + c, \qquad d \geqq 0, \tag{9}$$

and let $y \in X_0$, $u \in U_0$ with $\|y - u\|_0 \leqq c_1$ and $\|y\|_0 \to \infty$ imply

$$\big((A(y, u), y - Nu) + h(y, u)\big)/\|y\|_0 \to +\infty. \tag{10}$$

Then M is bounded in B_0 for an arbitrary $y^* \in B^*$.

Proof: (Attention: Throughout this proof and for the rest of the paper the letters $c, c_1, c_2, \ldots$ will symbolize "a certain constant".)
For $y \in M$ there is a $v \in U_0$ s.t. $\|y - v\|_0 \leqq c$ and

$$\big(A(y, v), y - Nv\big) + h(y, v) \leqq (y^*, y - Nv) + h(Nv, v)$$
$$\leqq \|y^*\|\,\|y - Nv\| + |h(Nv, v)|. \tag{11}$$

We have

$$\|y - Nv\| \leqq \|y\| + \|Nv\| \leqq \|y\| + b\,\|v\| + c$$

and

$$\|v\| = \|v - y + y\| \leqq \|y - v\| + \|y\| \leqq \|v\| + k\,\|y - v\|_0 \leqq \|y\| + c,$$

hence $\|y - Nv\| \leqq (1 + b)\,\|y\| + c$. Setting this into (11) and having regard to (9) we get

$$\big(A(y, v), y - Nv\big) + h(y, v) \leqq \big(\|y^*\|\,(1 + b) + d\big)\,\|y\| + c \leqq c_1\,\|y\|_0 + c,$$

i.e.

$$\big((A(y, v), y - Nv) + h(y, v)\big)/\|y\|_0 \leqq c \quad \text{if } \|y\|_0 \to \infty.$$

The contradiction means that $\|y\|_0$ has to be bounded ∎

Remark 1: In the case $B_0 = B$ instead of (10) another possibility for a uniform coercivity condition is

$$\big((A(y, u), y - Nu) + h(y, u)\big)/\|y - Nu\| \to +\infty \quad \text{if } \|y - Nu\| \to \infty,$$

which is used in the second part of [1].

Remark 2 (cf. [4: p. 211]): If

$$h(y, v) \geqq -c_1\,\|y\| - c_2\,\|v\| - c_3 \tag{12}$$

the condition

$$\big(A(y, v), y - Nv\big)/\|y\|_0 \to \infty \quad \text{for } \|y\|_0 \to \infty \tag{13}$$

is sufficient for (10). Indeed,

$$\big(A(y, v), y - Nv\big) + h(y, v) \geqq \big(A(y, v), y - Nv\big) - c_1\,\|y\| - c_2\,\|v\| - c_3$$

and from $\|v\| \leqq \|y\| + c$ and $\|y\| \leqq k\,\|y\|_0$ we obtain

$$\big((A(y, v), y - Nv) + h(y, v)\big)/\|y\|_0 \geqq \big(A(y, v), y - Nv\big)/\|y\|_0 - c$$

for $\|y\|_0 \to \infty$.

It should be mentioned that (12) is clearly satisfied if e.g. $h \geqq 0$ holds. Further, $h(\cdot, u)$ convex and l.s.c. imply $h(y, v) \geqq -c_1(v) \|y\| + h(0, v)$ [4: p. 136].

Remark 3 (cf. [4: p. 211]): If $A(\cdot, u)$ is monotone and

$$|(A(Nu, u), y - Nu)|/\|y\|_0 \leqq c \tag{14}$$

for $\|y\|_0 \to \infty$, $\|y - u\|_0 \leqq c$, $u \in U_0$, $y \in X_0$ then the condition

$$h(y, u)/\|y\|_0 \to +\infty \tag{15}$$

will be sufficient for (10). Indeed,

$$\big(h(y, u) + (A(y, u), y - Nu)\big)/\|y\|_0 \geqq h(y, u)/\|y\|_0 + \big(A(Nu, u), y - Nu\big)/\|y\|_0.$$

If one is only interested in some special $y^* \in B^*$ then a sufficient condition for the boundedness of M is given by

Remark 4 (cf. [3]): Let for every $u \in U_0$ there be an $Nu \in C(u)$ s.t.

$$h(Nu, u) \leqq c, \tag{16}$$

and let $y \in X_0$, $u \in U_0$ s.t. $\|y\|_0 \to \infty$ and $\|y - u\|_0 \leqq c$ imply

$$\big(A(y, u) - y^*, y - Nu\big) + h(y, u) \to +\infty. \tag{17}$$

Then M is bounded. Indeed, for $y \in M$ we have

$$\big(A(y, v) - y^*, y - Nv\big) + h(y, v) \leqq h(Nv, v) \leqq c.$$

This contradicts (17) for $\|y\|_0 \to \infty$.

We give some easy examples in the case $B_0 = B$.

Example 1 (cf. [2]): Let be $A: B \to B^*$, $N: U_0 \to B$ with

$$(Ax - Ay, x - y) \geqq \delta(\|x - y\|), \qquad \delta(1) > 0, \tag{18}$$

$$\|Ay\| \leqq L \|y\| + c, \tag{19}$$

$$\|Nu\| \leqq b \|u\| + c, \qquad b < \delta(1)/2L \qquad (b < m/L \text{ if } \delta(r) = mr^2) \tag{20}$$

and $h \equiv 0$. Then the conditions of Proposition 1 are fulfilled.

Proof: Taking into account Lemma 1 of the second part of [1] and $\|y - v\| \leqq c$ we find for great $\|y\|$

$$(Ay, y - Nv) = (Ay - A0, y) - (Ay, Nv) + (A0, y) \geqq \frac{\delta(1)}{2} \|y\|^2 - bL \|y\|^2 - c_1 \|y\| - c,$$

i.e.

$$(Ay, y - Nv)/\|y\| \geqq \left(\frac{\delta(1)}{2} - bL\right) \|y\| - c \to \infty \quad \text{if } \|y\| \to \infty.$$

Example 2: Let $A = J_\varphi$ be the duality mapping corresponding to the function φ, $h \equiv 0$, and let N fulfil (8) with $b < 1$. Then the conditions of Proposition 1 are satisfied.

Proof: We use $(Ay, y) \geqq \varphi(\|y\|) \|y\|$, $\|Ay\| = \varphi(\|y\|)$. Then

$$(Ay, y - Nv) = (Ay, y) - (Ay, Nv) \geqq \varphi(\|y\|) \|y\| - \varphi(\|y\|) \|Nv\|,$$

i.e.

$$(Ay, y - Nv)/\|y\| \geqq \varphi(\|y\|) (1 - b) - c \to \infty \quad \text{if } \|y\| \to \infty.$$

We now give sufficient conditions for (7) i.e. $G(S) \cap (X_0 \times U_0)$ is w-closed in $B_0 \times B_0$. Generalizing the definition of the "weak A-continuity of C" in [3] we prove the

Proposition 2: *Let be*

$$A(\cdot, u) \quad \text{monotone and hemicontinuous } \forall u \in U_0, \tag{21}$$

$$\big(A(y, \cdot), \cdot\big) \quad w\text{-continuous on } U_0 \times B \; \forall y \in X_0 \text{ (e.g., if } A(y, \cdot) \text{ is increasing continuous on } U_0), \tag{22}$$

$$h \; w\text{-l.s.c. on } X_0 \times U_0 \text{ and } h(\cdot, u) \text{ convex } \forall u \in U_0. \tag{23}$$

Let further the following implication be true:

$$\left.\begin{array}{l} \text{If } \; [y_k, u_k] \in X_0 \times U_0, \quad y_k \in Su_k, \quad \text{i.e.} \\ \big(A(y_k, u_k) - y^*, z - y_k\big) \geqq h(y_k, u_k) - h(z, u_k) \qquad \forall z \in C(u_k), \\ \text{and } \; [y_k, u_k] \rightharpoonup [y, u] \; \text{ in } \; B_0 \times B_0 \\ \text{then} \\ \text{(i) } y \in C(u) \; \text{ and} \\ \text{(ii) } \forall w \in C(u) \; \exists w_k \in C(u_k) \quad \text{s.t.} \\ \overline{\lim} \Big(\big(A(y_k, u_k) - y^*, w - w_k\big) - h(w_k, u_k)\Big) \geqq -h(w, u). \end{array}\right\} \tag{24}$$

Then $G(S) \cap (X_0 \times U_0)$ is w-closed in $B_0 \times B_0$.

Proof: We take $[y_k, u_k] \in X_0 \times U_0$, $y_k \in Su_k$, $[y_k, u_k] \rightharpoonup [y, u]$ in $B_0 \times B_0$. We show that $y \in Su$. Let be $w \in C(u)$ then $\exists\, w_k \in C(u_k)$ with property (24) (ii). We have

$$\begin{aligned} 0 &\geqq \big(A(y_k, u_k) - y^*, y_k - w_k\big) + h(y_k, u_k) - h(w_k, u_k) \\ &= \big(A(y_k, u_k) - y^*, w - w_k\big) + \big(A(y_k, u_k) - y^*, y_k - w\big) + h(y_k, u_k) \\ &\quad - h(w_k, u_k). \end{aligned}$$

As $\big(A(y_k, u_k) - y^*, y_k - w\big) \geqq \big(A(w, u_k) - y^*, y_k - w\big)$ we have further

$$\big(A(w, u_k) - y^*, w - y_k\big) - h(y_k, u_k) \geqq \big(A(y_k, u_k) - y^*, w - w_k\big) - h(w_k, u_k)$$

Going over to $\overline{\lim}$ we get

$$\begin{aligned} -h(w, u) &\leqq \overline{\lim} \Big\{\big(A(w, u_k) - y^*, w - y_k\big) - h(y_k, u_k)\Big\} \\ &\leqq \big(A(w, u) - y^*, w - y\big) - h(y, u). \end{aligned}$$

This is equivalent to (2), i.e. $y \in Su$ ∎

A sufficient condition for (24) of Proposition 2 is given by

Remark 1 (cf. [1, 3, 4]): If

$$A \text{ is bounded as a mapping from } B \times U \text{ into } B^*, \tag{25}$$

$$h \text{ is } [s, w]\text{-u.s.c. on } B \times U, \tag{26}$$

$$u_k \in U_0, \; u_k \rightharpoonup u \text{ in } B_0 \Rightarrow \mathrm{w}_{B_0}\text{-Lim}\,\big(C(u_k) \cap X_0\big) \subseteqq C(u), \tag{27}$$

$$u_k \in U_0, \; u_k \rightharpoonup u \text{ in } B_0 \Rightarrow C(u) \subseteqq \text{s-Lim}\, C(u_k), \tag{28}$$

then condition (24) holds.

Proof: Let the assumption of the implication (24) be true. Then $y_k \in C(u_k) \cap X_0$

and (27) gives $y \in C(u)$. Further, let be $w_k \in C(u_k)$ with $w_k \to w$ in B, $w \in C(u)$ arbitrary. As $[y_k, u_k]$ is bounded also $A(y_k, u_k)$ is bounded, consequently $\left(A(y_k, u_k) - y, w - w_k\right) \to 0$. Then

$$\overline{\lim} \left\{\left(A(y_k, u_k) - y^*, w - w_k\right) - h(w_k, u_k)\right\} = -\underline{\lim}\, h(w_k, u_k) \geqq -h(w, u)$$

as $h(w, u) \geqq \overline{\lim}\, h(w_k, u_k) \geqq \underline{\lim}\, h(w_k, u_k)$, because of (26).

In the case where the imbedding of B_0 into B is compact a sufficient condition for (24) is given by

Remark 2 (cf. [3]): If

$$A \text{ is continuous as a mapping from } B \times U \text{ into } B^*, \tag{29}$$

$$h \text{ is } [w, s]\text{-u.s.c. on } B \times U, \tag{30}$$

$$u_k, u \in U_0,\ u_k \rightharpoonup u \text{ in } B \Rightarrow s_B\text{-Lim}\left(C(u_k) \cap X_0\right) \subseteqq C(u), \tag{31}$$

$$u_k, u \in U_0,\ u_k \rightharpoonup u \text{ in } B \Rightarrow C(u) \subseteqq w_B\text{-Lim}\, C(u_k), \tag{32}$$

then (24) holds.

The proof goes like the proof of Remark 1 if we use the fact that $[y_k, u_k] \rightharpoonup [y, u]$ in $B_0 \times B_0$ implies $[y_k, u_k] \to [y, u]$ in $B \times B$ and that $A(y_k, u_k)$ is converging strongly in B^*.

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