On the existence of the solution of an abstract optimization problem related to a quasi-variational inequality

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Es wird ein allgemeines Konzept zur Erlangung von Existenz- und Regularitätsresultaten für ein Optimierungsproblem angegeben, das in einem engen Zusammenhang zu einer Quasi-Variationsungleichung steht.

Дается общий подход получения результатов о существовании и регулярности для некоторой проблемы оптимизации, которая тесно связана с одним квази-вариационным неравенством.

A general concept is given to get existence and regularity results for an optimization problem that is closely connected to a quasi-variational inequality.

In this paper a general concept is given to get existence results for the problem

$$
||x - u||_B = \inf_{\substack{y \in X_0, v \in u_0 \\ [y, v] \in G(S)}} ||y - v||_B
$$

where X_0 and U_0 are closed and convex subsets of a reflexive Banach space B_0 being continuously imbedded into the reflexive Banach space B, and $[y, v] \in G(S)$ means that y is a solution corresponding to the parameter v of a certain parametric problem in B .

If the mentioned parametric problem is a parametric variational inequality (and this will be assumed later on)

$$
y \in C(v), \qquad v \in U \subset B
$$

$$
(A(y, v) - y^*, z - y)_B \ge h(y, v) - h(z, v) \qquad \forall z \in C(v)
$$
 (2)

and $X_0 = U_0$ then (1) is closely connected to the quasi-variational inequality

$$
u \in C(u), \qquad u \in U_0
$$

$$
(A(u, u) - y^*, z - u)_B \ge h(u, u) - h(z, u) \qquad \forall z \in C(u)
$$
 (3)

 $(cf. [1]).$

The main reasons to investigate (1) instead of (3) are the following:

(i) (1) can be solved under milder conditions than (3) ; a solution of (1) can be considered as a generalized solution of (3) .

(ii) If (3) is solvable then the solution sets of (1) and (3) coincide.

(iii) To solve (1) optimization techniques can be used (cf. [1] where approximation procedures are given).

Compared with the literature on existence for quasi-variational inequalities (cf. e.g. $[3, 5]$ here the condition

$$
{SU_{\mathfrak a}}\!\subset U_{\mathfrak a}
$$

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 (4)

 (1)

(where S is the solution operator of (2)) is not needed. Further, in the sequel the uniform coercivity condition of [51 is weakened and other conditions are generalized to our case.

The author is obliged to R. **KLUGE** for hints and discussions.

Let B_0 , *B* be reflexive Banach spaces, let B_0 be continuously imbedded into *B*, B^* the adjoint space of *B*, (\cdot, \cdot) the pairing between B^* and *B*, $\|\cdot\|$ the norm in *B* and B^* , $\|\cdot\|_0$ the norm in B_0 . Let further U be a w-closed subset of B, X_0 and U_0 w-closed subsets of B_0 , $U_0 \subset U \cap B_0$. Let on *U* a multivalued mapping *C* be defined, $\emptyset + C(u) \subset B$, $C(u)$ closed and convex. Further, let $A(\cdot, u)$ be an operator from (the whole of) $C(u)$ into B^* , $h(\cdot, u)$ an admissible functional on $C(u)$, and $y^* \in B^*$. We solve (1) with the help of the following theorem of Weierstrass:

In a re/lexive Bavach space aw-l.s.c. functional aitaines its ml on a bounded, w-closed subset.

The functional

$$
f(x, u) = ||x - u||
$$

is w-l.s.c. on $B_0 \times B_0$. Indeed, let be $x_i \to x$, $u_i \to u$ in B_0 . As the imbedding into *B* is linear and continuous it is also w-continuous, i.e. $x_i \rightharpoonup x$, $u_i \rightharpoonup u$ in *B*. Hence $||x - u|| \leq \lim_{x \to \infty} ||x_i - u_i||$ because of the w-l.s.continuity of the norm. The functional
 $f(x, u) = ||x - u||$

s w-1.s.c. on $B_0 \times B_0$. Indeed, let be $x_i \to x$, $u_i \to u$ in B_0 . As the im

s linear and continuous it is also w-continuous, i.e. $x_i \to x$, $u_i \to$
 $|x - u|| \le \lim_{t \to \infty} ||x_i - u||$ because of the volet, B^X , $n(\cdot, u)$ an admission entertormal on $\infty(u)$, x_0 , $y_0 \in B$ is help of the following theorem of Weierstrass:

pace $a w$ -*l.s.c. functional attaines its inf on a bounded, w-closed*
 $||u||$

indeed, let be x

Let $G(S)$ be the graph of S, i.e.

 $G(S) = \{ [y, v] \in B \times U \text{ such that } y \in Sv \}$

and let us suppose that

there is an $[y_0,$

We consider the (non-empty) set $M_1 \subset B_0 \times B_0$,

there is an
$$
[y_0, v_0] \in G(S)
$$
 with $y_0 \in X_0$, $v_0 \in U_0$.
\nider the (non-empty) set $M_1 \subset B_0 \times B_0$,
\n $M_1 = \{[y, v] \in G(S) : y \in X_0, v \in U_0, ||y - v||_0 \le ||y_0 - v_0||_0\}$.

If M_1 is

(i) bounded in $B_0 \times B_0$ and

(ii) w-closed in $B_0 \times B_0$

then by the Weierstrass theorem $f(x, u)$ will attain its inf on M_1 . This inf then is clearly a solution of (1) .

Sufficient for the boundedness of M_1 in $B_0 \times B_0$ is that

Using
$$
v_1 < v_2
$$
 and $[y_0, v_0] \in G(S)$ with $y_0 \in X_0$, $v_0 \in U_0$.

\nUnder the (non-empty) set $M_1 \subset B_0 \times B_0$.

\n $M_1 = \{(y, v) \in G(S) : y \in X_0, v \in U_0, \|y - v\|_0 \le \|y_0 - v_0\|_0\}$.

\nundefined in $B_0 \times B_0$ and closed in $B_0 \times B_0$.

\nthe Weierstrass theorem $f(x, u)$ will attain its inf on M_1 . This inf then is solution of (1).

\nisolution of (1).

\nif $M = \{y \in X_0 : \exists v \in U_0 \text{ s.t. } [y, v] \in G(S) \text{ and } ||y - v||_0 \leq c\}$ is bounded in B_0 for every $c \geq 0$.

\nLet us take $c = \|y_0 - v_0\|_0$. If y is bounded then also v has to be bounded.

\nwhere v is the following property.

Indeed, let us take $c = \|y_0 - v_0\|_0$. If y is bounded then also v has to be bounded since $||y - v||_0 \leq c$. This is (i).

Sufficient for the w-closedness of M_1 is that V

$$
G(S) \cap (X_0 \times U_0)
$$
 is w-closed in $B_0 \times B_0$.

 $M_1 = \{ [y, v] \in G(S) : y \in \Delta_0, v \in \partial_0, ||y - v||_0 \leq ||y - v||_0 \}$

unded in $B_0 \times B_0$ and

the Weierstrass theorem $f(x, u)$ will attain its inf on M_1 . This inf then is

solution of (1).

Int for the boundedness of M_1 in $B_0 \times$ $\text{Indeed, let be } [y_i, v_i] \in M_1, [y_i, v_i] \rightarrow [y, v] \text{ in } B_0 \times B_0. \text{ Since } [y_i, v_i] \in G(S) \cap (X_0 \times U_0),$
 $\text{lim } ||y_i - v_i|| \leq \lim_{n \to \infty} ||y - v_i|| \leq \lim_{n \to \infty} ||y - v_i||$ Sufficient for the w-closedness of M_1 is that
 $G(S) \cap (X_0 \times U_0)$ is w-closed in $B_0 \times B_0$.

Indeed, let be $[y_i, v_i] \in M_1$, $[y_i, v_i] \rightarrow [y, v]$ in $B_0 \times B_0$. Since $[y_i, v_i] \in G(S)$

(7) implies $[y, v] \in G(S) \cap (X_0 \times U_0)$. We hav (7) implies $[y, v] \in G(S) \cap (X_0 \times U_0)$. We have further $||y - v||_0 \leq \underline{\lim} ||y_i - v_i||_0$ \leq $||y_0 - v_0||_0$. This is (ii). Let us begin with (6).

Let us begin with (6).

If $G(S)$ is w-closed in $B \times B$ then clearly (7) holds.

In the sequel we will assume that $[y, v] \in G(S)$ means y is a solution corresponding to the parameter *v* of the parametric variational inequality (2) and give sufficient conditions for (6) and (7).

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If one of X_0 **and** U_0 **is bounded** M **is clearly bounded. In the case where both** X_0 **

d** U_0 **are unbounded we have

Proposition 1:** Let for every $u \in U_0$ there be an $Nu \in C(u)$ s.t.
 $||Nu|| \leq b ||u|| + c$, $b \geq 0$, (8)
 and *U0* are unbounded we have

Proposition 1: Let for every $u \in U_0$ there be an $Nu \in C(u)$ s.t.

or
$$
\Delta_0
$$
 and U_0 is bounded in is clearly bounded. In the case where both X_0 are unbounded we have
position 1: Let for every $u \in U_0$ there be an $Nu \in C(u)$ s.t.
 $||Nu|| \le b ||u|| + c$, $b \ge 0$, (8)

$$
|h(Nu,u)| \le d ||u|| + c, \qquad d \ge 0,
$$
\n
$$
(9)
$$

and let $y \in X_0$ *,* $u \in U_0$ *with* $||y - u||_0 \leq c_1$ *and* $||y||_0 \to \infty$ *imply*

$$
((A(y,u),y-Nu)+h(y,u))/||y||_0\rightarrow+\infty.
$$
 (10)

Then M is bounded in B₀ for an arbitrary $y^* \in B^*$ *.*

Proof: (Attention: Throughout this proof and for the rest of the paper the letters c, c_1, c_2, \ldots will symbolize "a certain constant".)

For $y \in M$ there is a $v \in U_0$ s.t. $||y - v||_0 \leq c$ and

$$
(A(y, v), y - Nv) + h(y, v) \le (y^*, y - Nv) + h(Nv, v)
$$

\n
$$
\le ||y^*|| ||y - Nv|| + |h(Nv, v)|.
$$

\n
$$
||y - Nv|| \le ||y|| + ||Nv|| \le ||y|| + b ||v|| + c
$$
\n(11)

We have

 $\ell \in M$ there is a $v \in U_0$ s.t. $||y - v||_0 \leq c$ and
 $\big(A(y, v), y - Nv) + h(y, v) \leq (y^*, y - Nv) + h(Nv, v)$
 $\leq ||y^*|| \, ||y - Nv|| + |h(Nv, v)|$
 $||y - Nv|| \leq ||y|| + ||Nv|| \leq ||y|| + b ||v|| + c$
 $||v|| = ||v - y + y|| \leq ||y - v|| + ||y|| \leq ||v|| + k ||y - v||_0 \leq$
 $- Nv|| \leq (1 + b) ||y|| + c$. $||y|| + c$, and
 $||v|| = ||v - v||$

hence $||y - Nv|| \le$

we get $(1 + b) ||y|| + c$. Setting this into (11) and having regard to (9) we get (b) $||v|| = ||v - y + y|| \le ||y - v|| + ||y|| \le ||v|| + k ||y - v||_0 \le ||y|| + c$,

hence $||y - Nv|| \le (1 + b) ||y|| + c$. Setting this into (11) and having regard to

we get

(A(y, v), y - Nv) + h(y, v) $\le (||y^*|| (1 + b) + d) ||y|| + c \le c_1 ||y||_0 + c$,

i.e.

and

$$
(A(y, v), y - Nv) + h(y, v) \leq (||y*|| (1 + b) + d) ||y|| + c \leq c
$$

$$
((A(y, v), y - Nv) + h(y, v)) / ||y||_0 \leq c \quad \text{if } ||y||_0 \to \infty.
$$

The contradiction means that $||y||_0$ has to be bounded **I**

Remark 1: In the case $B_0 = B$ instead of (10) anothe

coercivity condition is
 $((A(y, u), y - Nu) + h(y, u))/||y - Nu|| \rightarrow +\infty$

which is used in the second part of [1].

Remark 2 Remark 1: In the case $B_0 = B$ instead of (10) another possibility for a uniform coercivity condition is

$$
((A(y, u), y - Nu) + h(y, u)) / ||y - Nu|| \rightarrow +\infty \text{ if } ||y - Nu|| \rightarrow \infty,
$$

which is used in the second part of [1].

Remark 2 (cf. [4: p. 211]): If

$$
||\langle \mathbf{1}(y, u), y - \mathbf{1}(u) + h(y, u)||y - \mathbf{1}(u)|| \rightarrow +\infty \text{ if } ||y - \mathbf{1}(u)|| \rightarrow \infty,
$$

s used in the second part of [1].

$$
\text{ark } 2 (\text{cf. } [4: p. 211]): \text{ If}
$$

$$
h(y, v) \ge -c_1 ||y|| - c_2 ||v|| - c_3
$$
 (12)

6*

The contradiction meantle contradiction is
 $\text{Remark 1: In the}\n\text{coercivity condition is}\n\quad (\text{(}A(y, u), y - \text{which is used in the set}\n\text{Remark 2 (cf. [4:]}\n)
$$
h(y, v) \ge -c_1
$$
\n\text{the condition}\n
$$
(A(y, v), y - \text{is sufficient for (10). I})
$$$ y||₀ has to
= *B* instead
 $u(y, u)$ |||y
t of [1].
f
 $||v|| - c_3$ **(***A*(*y*, *u*), *y* - *Nu*) + *h*(*y*, *u*))/||*y* - *Nu*|| \rightarrow + ∞ if

used in the second part of [1].

rk 2 (cf. [4: p. 211]): If
 h(*y*, *v*) \ge -*c*₁ ||*y*|| - *c*₂ ||*v*|| - *c*₃

ition

(*A*(*y*, *v* $\begin{aligned} \text{the condition} \qquad & \langle \ \begin{array}{c} (A(y,v),\,y\,-\,Nv) / \|y\|_0\to\infty & \text{ for } &\|y\|_0\to\infty \end{array} \end{aligned}$ is sufficient for (10). Indeed,

The contradiction means that
$$
||y||_0
$$
 has to be bounded \blacksquare
\nRemark 1: In the case $B_0 = B$ instead of (10) another possibility for a uni
\ncoercivity condition is
\n
$$
((A(y, u), y - Nu) + h(y, u))/||y - Nu|| \rightarrow +\infty \text{ if } ||y - Nu|| \rightarrow \infty,
$$

\nwhich is used in the second part of [1].
\nRemark 2 (cf. [4: p. 211]): If
\n
$$
h(y, v) \ge -c_1 ||y|| - c_2 ||v|| - c_3
$$

\nthe condition
\n
$$
(A(y, v), y - Nv)/||y||_0 \rightarrow \infty \text{ for } ||y||_0 \rightarrow \infty
$$

\nis sufficient for (10). Indeed,
\n
$$
(A(y, v), y - Nv) + h(y, v) \ge (A(y, v), y - Nv) - c_1 ||y|| - c_2 ||v|| - c_3
$$

\nand from $||v|| \le ||y|| + c$ and $||y|| \le k ||y||_0$ we obtain
\n
$$
((A(y, v), y - Nv) + h(y, v))/||y||_0 \ge (A(y, v), y - Nv)/||y||_0 - c
$$

\nfor $||y||_0 \rightarrow \infty$.

$$
((A(y, v), y - Nv) + h(y, v))/||y||_0 \geq (A(y, v), y - Nv)/||y||_0 - c
$$

for $||y||_0 \rightarrow \infty$.

(13)

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It should be mentioned that (12) is clearly satisfied if e.g. $h \ge 0$ holds. Further, *h*(., *u*) convex and l.s.c. imply $h(y, v) \ge -c_1(v) \|y\| + h(0, v)$ [4: p. 136]. *If* the mentioned that (12) is clearly satisf
 If the mentioned that (12) is clearly satisf
 If $h(y, v) \ge -c_1(v)$ ||y||
 If $A(\cdot, u)$, $y - Nu$ ||/||y||₀ $\le c$
 $\Rightarrow \infty$, $||y - u||_0 \le c$, $u \in U_0$, $y \in X_0$ the $h(y, u) / ||y||_0$ *h* the mentioned that (12) is clear
 hvex and l.s.c. imply $h(y, v) \ge -\alpha$
 rk 3 (cf. [4: p. 211]): If $A(\cdot, u)$ is
 $|(A(Nu, u), y - Nu)|/||y||_0 \le c$
 $\Rightarrow \infty$, $||y - u||_0 \le c$, $u \in U_0$, $y \in$
 $h(y, u)/||y||_0 \Rightarrow +\infty$

ifficient for (10).

Remark 3 (cf. $[4: p. 211]$): If $A(\cdot, u)$ is monotone and

$$
||A(Nu, u), y - Nu||/||y||_0 \leq c \qquad (14)
$$

for $||y||_0 \to \infty$, $||y - u||_0 \leq c$, $u \in U_0$, $y \in X_0$ then the condition

$$
h(u, u)/||u||_{\alpha} \to +\infty
$$

will be sufficient for (10). Indeed,

$$
h(y, u)/||y||_0 \rightarrow +\infty
$$

will be sufficient for (10). Indeed,

$$
(h(y, u) + (A(y, u), y - Nu))/||y||_0 \geq h(y, u)/||y||_0 + (A(Nu, u), y - Nu)/||y||_0.
$$

If one is only interested in some special $y^* \in B^*$ then a sufficient condition for the boundedness of *M* is given by $\begin{aligned} &\text{rk 3 (cf. [4: p. 211]): If }~ A\ &\left|\left(A(Nu, u), y - Nu)\right|\right|\left|\left|y\right|\right|_{0}\right| \\ &\Rightarrow \infty,~\left|\left|y - u\right|\right|_{0} \leqq c,~u \in h(y, u)/\left|\left|y\right|\right|_{0} \rightarrow +\infty \\ &\text{efficient for (10). Indeed,}\\ &\cdot~\left(A(y, u), y - Nu)\right)\right|\left|\left|y\right|\right|_{0}\text{only interested in some s} \\ &\text{ness of }M\text{ is given by} \\ &\text{rk 4 (cf. [3]): Let for eve} \\ &\left.h(Nu, u$ (A(y, *u*), *y* - *Nu*))/||*y*||₀ $\geq h(y, u)/||y||_0 + (A(Nu, u))$

only interested in some special $y^* \in B^*$ then a suffices of *M* is given by

rk 4 (cf. [3]): Let for every $u \in U_0$ there be an *Nu*
 $h(Nu, u) \leq c$,
 $\in X_0, u \$ *•* sufficient for (10). Indeed,
 $\rho + (A(y, u), y - Nu))/||y||_0 \ge h(y, u)/||y||_0 + (A(Nu,$
 is only interested in some special $y^* \in B^*$ then a su

edness of *M* is given by
 $n \text{ a rk } 4$ (cf. [3]): Let for every $u \in U_0$ there be an *N*

Remark 4 (cf. [3]): Let for every
$$
u \in U_0
$$
 there be an $Nu \in C(u)$ s.t.
\n
$$
h(Nu, u) \leq c,
$$
\nand let $y \in X_0$, $u \in U_0$ s.t. $||y||_0 \to \infty$ and $||y - u||_0 \leq c$ imply

and let $y \in X_0$, $u \in U_0$ s.t. $||y||_0 \to \infty$ and $||y - u||_0 \le c$ imply

$$
(A(y, u) - y^*, y - Nu) + h(y, u) \rightarrow +\infty.
$$
 (17)

Then M is bounded. Indeed, for $y \in M$ we have

$$
(A(y,v)-y^*,y-Nv)+h(y,v)\leq h(Nv,v)\leq c.
$$

This contradicts (17) for $||y||_0 \rightarrow \infty$.

We give some easy examples in the case $B_0 = B$.

Example 1 (cf. [2]): Let be $A: B \rightarrow B^*, N: U_0 \rightarrow B$ with

$$
(A(y, v) - y^*, y - Nv) + h(y, v) \ge h(Nv, v) \ge c.
$$

tradicts (17) for $||y||_0 \rightarrow \infty$.
ve some easy examples in the case $B_0 = B$.
uple 1 (cf. [2]): Let be $A : B \rightarrow B^*, N : U_0 \rightarrow B$ with
 $(Ax - Ay, x - y) \ge \delta(||x - y||), \quad \delta(1) > 0,$ (18)
 $||Ay|| \le L ||y|| + c.$ (19)

rk 4 (cf. [3]): Let for every
$$
u \in U_0
$$
 there be an $Nu \in C(u)$ s.t.
\n $h(Nu, u) \leq c$,
\n $\in X_0, u \in U_0$ s.t. $||y||_0 \rightarrow \infty$ and $||y - u||_0 \leq c$ imply
\n $(A(y, u) - y^*, y - Nu) + h(y, u) \rightarrow +\infty$.
\nis bounded. Indeed, for $y \in M$ we have
\n $(A(y, v) - y^*, y - Nv) + h(y, v) \leq h(Nv, v) \leq c$.
\ntradicts (17) for $||y||_0 \rightarrow \infty$.
\nwe some easy examples in the case $B_0 = B$.
\n $||y||_0 = 1$ (cf. [2]): Let be $A : B \rightarrow B^*, N : U_0 \rightarrow B$ with
\n $(Ax - Ay, x - y) \geq \delta(||x - y||)$, $\delta(1) > 0$, (18)
\n $||Ay|| \leq L ||y|| + c$, $b < \delta(1)/2L$ $(b < m/L$ if $\delta(r) = mr^2)$ (20)

$$
h(Nu, u) \leq c,
$$
\n
$$
t y \in X_0, u \in U_0 \text{ s.t. } ||y||_0 \to \infty \text{ and } ||y - u||_0 \leq c \text{ imply}
$$
\n
$$
(A(y, u) - y^*, y - Nu) + h(y, u) \to +\infty.
$$
\n
$$
M \text{ is bounded. Indeed, for } y \in M \text{ we have}
$$
\n
$$
(A(y, v) - y^*, y - Nv) + h(y, v) \leq h(Nv, v) \leq c.
$$
\n
$$
\text{ontradicts (17) for } ||y||_0 \to \infty.
$$
\ngive some easy examples in the case $B_0 = B$.\n
$$
\text{ample 1 (cf. [2]): Let be } A : B \to B^*, N : U_0 \to B \text{ with}
$$
\n
$$
(Ax - Ay, x - y) \geq \delta(||x - y||), \quad \delta(1) > 0,
$$
\n
$$
||Ay|| \leq L ||y|| + c,
$$
\n
$$
||Nu|| \leq b ||u|| + c, \quad b < \delta(1)/2L \quad (b < m/L \text{ if } \delta(r) = mr^2)
$$
\n
$$
\text{then the conditions of Proposition 1 are fulfilled.}
$$
\n
$$
\text{If } \delta(x) = \frac{|x - y|}{|x - y|} \quad \text{(20)}
$$
\n
$$
\text{If } \delta(x) = \frac{|x - y|}{|x - y|} \quad \text{(21)}
$$
\n
$$
\text{If } \delta(x) = \frac{|x - y|}{|x - y|} \quad \text{(22)}
$$

and $h \equiv 0$. Then the conditions of Proposition 1 are fulfilled

Proof: Taking into account Lemma 1 of the second part of [1] and $||y - v|| \leq c$ we find for great $\left\| y \right\|$ and $h =$
Proof
we find for
i.e.

$$
(Ay, y - Nv) = (Ay - A0, y) - (Ay, Nv) + (A0, y) \ge \frac{o(1)}{2} ||y||^2
$$

- bL $||y||^2 - c_1 ||y|| - c$,

$$
(Ay, y - Nv) / ||y|| \ge \left(\frac{\delta(1)}{2} - bL\right) ||y|| - c \to \infty \quad \text{if } ||y|| \to \infty.
$$

$$
(Ay, y - Nv)/||y|| \geq \left(\frac{\delta(1)}{2} - bL\right)||y|| - c \to \infty \quad \text{if } ||y|| \to \infty.
$$

(Ay, y - Nv) = $(Ay - A0, y) - (Ay, Nv) + (A0, y) \ge \frac{\delta(1)}{2} ||y||^2$
 bL $||y||^2 - c_1 ||y|| - c$,
 (Ay, y - Nv)/||y|| $\ge \left(\frac{\delta(1)}{2} - bL\right) ||y|| - c \rightarrow \infty$ *if* $||y|| \rightarrow \infty$ *.

Example 2: Let* $A = J_{\varphi}$ *be the duality mapping corresponding to th* $h = 0$, and let *N* fulfil (8) with $b < 1$. Then the conditions of Proposition 1 are satisfied.

Proof: We use $(Ay, y) \geq \varphi(||y||)$ $||y||$, $||Ay|| = \varphi(||y||)$. Then

$$
(A y, y = N v) = (A y, y) - (A y, N v) \geq \varphi(||y||) ||y|| - \varphi(||y||) ||N v||,
$$

i.e.

$$
(Ay, y - Nv)/||y|| \geq \varphi(||y||) (1 - b) - c \rightarrow \infty \quad \text{if} \quad ||y|| \rightarrow \infty.
$$

(15)

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We now give sufficient conditions for (7) i.e. $G(S) \cap (X_0 \times U_0)$ is w-closed in $B_0 \times B_0$. Generalizing the definition of the "weak *A*-continuity of C" in [3] we prove the An abstract optimization problem
 After a abstract optimization problem
 A(*x u*) is w-closed in
 A(*x u*) monotone and hemicontinuous $\forall u \in U_0$,
 A(*x*, *u*) monotone and hemicontinuous $\forall u \in U_0$,
 A(*y*, *(A)* we give sufficient conditions for (7) i.e. $G(S) \cap (X(G))$
 Generalizing the definition of the "weak A-continuous
 A(*v, u) monotone and hemicontinuous* $\forall u \in U_0$,
 $(A(y, \cdot), \cdot)$ *w-continuous on* $U_0 \times B \forall y \in X_0$ **An abstract optimization problem** 85
 After abstract optimization problem 85
 A(y, iii) is w-closed in
 A(*y, iii) At be*
 A(, u) monotone and hemicontinuous $\forall u \in U_0$, (21)
 $(A(y, \cdot), \cdot)$ *w-continuous on* U **h** abstract optimization problem
 Herefore in a abstract optimization problem
 *Herefore in a set of the intime of the investal a-continuity of C' in [3] we

e

<i>A*(*, u*) *monotone and hemicontinuous* $\forall u \in U_0$,
 $(A(y$

Proposition-2: *Let be*

Let further the following implication be true:

If $A(\cdot, u)$ monotone and hemicontinuous $\forall u \in U_0$,
 $\{A(y, \cdot), \cdot\}$ *w*-continuous on $U_0 \times B \forall y \in X_0$ (e.g., if
 $A(y, \cdot)$ is increasing continuous on U_0),
 h w-l.s.c. on $X_0 \times U_0$ and $h(\cdot, u)$ convex $\forall u \in U_0$ $y_k \in S u_k$, *i.e. (A(y_t, u_t)* $\leq X_0 \times U_0$ *and f*
(A(y_t, u_t) $\leq X_0 \times U_0$,
 $(A(y_t, u_t) - y^*, z - y_t) \geq$
and $[y_t, u_t] \to [y, u]$ *in* $h(x, u)$ convex $\forall u \in U_0$.
 $h(x, u)$ convex $\forall u \in U_0$.
 $h(x_k, u_k) - h(z, u_k) \quad \forall z \in C(u_k),$
 $B_0 \times B_0$ *and* $[y_k, u_k] \rightarrow [y, u]$ *in* $B_0 \times B_0$ *theoryon and hemicontinuous* $\forall u \in A(\cdot, u)$ monotone and hemicontinuous $\forall u \in (A(y, \cdot), \cdot)$ w-continuous on $U_0 \times B \times y \in X$,
 $A(y, \cdot)$ is increasing continuous on U_0),
 h w-l.s.c. on $X_0 \times U_0$ and $h(\cdot, u)$ convex $\forall u$ (24) *(i)yEC(u) and* (ii) $\forall w \in C(u) \exists w_k \in C(u_k)$ *s.t.* $(A(y_k, u_k) - y^*, z - y_k) \ge h(y_k, u_k) - h(z, u_k) \quad \forall z \in C(u_k),$

and $[y_k, u_k] \to [y, u]$ in $B_0 \times B_0$

then

(i) $y \in C(u)$ and

(ii) $\forall w \in C(u) \exists w_k \in C(u_k) \quad s.t.$
 $\lim_{h \to 0} ((A(y_k, u_k) - y^*, w - w_k) - h(w_k, u_k)) \ge -h(w, u).$

Then $G(S) \cap (X_0 \times U_0)$ *is w-closed in* $B_0 \times B_0$ *.*

Proof: We take $[y_k, u_k] \in X_0 \times U_0$, $y_k \in Su_k$, $[y_k, u_k] \to [y, u]$ in $B_0 \times B_0$. We show that $y \in S$ u. Let be $w \in C(u)$ then $\exists w_k \in C(u_k)$ with property (24) (ii). We have

$$
0 \geq (A(y_k, u_k) - y^*, y_k - w_k) + h(y_k, u_k) - h(w_k, u_k)
$$

= $(A(y_k, u_k) - y^*, w - w_k) + (A(y_k, u_k) - y^*, y_k - w) + h(y_k, u_k)$
- $h(w_k, u_k)$.

As $(A(y_k, u_k) - y^*, y_k - w) \ge (A(w, u_k) - y^*, y_k - w)$ we have further

$$
(A(w, u_k) - y^*, w - y_k) - h(y_k, u_k) \geq (A(y_k, u_k) - y^*, w - w_k) - h(w_k, u_k)
$$

over to $\lim_{k \to \infty} \text{ we get}$

Going over to lim we get

$$
u_k(u_k) - y^*, y_k - w \geq (A(w, u_k) - y^*, y_k - w)
$$
 we have further
\n
$$
(A(w, u_k) - y^*, w - y_k) - h(y_k, u_k) \geq (A(y_k, u_k) - y^*, w - w_k)
$$
\n
$$
= \text{ker to } \overline{\lim} \text{ we get}
$$
\n
$$
-h(w, u) \leq \overline{\lim} \left\{ (A(w, u_k) - y^*, w - y_k) - h(y_k, u_k) \right\}
$$
\n
$$
\leq (A(w, u) - y^*, w - y) - h(y, u).
$$
\n
$$
\text{quivalent to (2), i.e. } y \in S u
$$
\n
$$
\text{icient condition for (24) of Proposition 2 is given by}
$$
\n
$$
\text{rk 1 (cf. [1, 3, 4]): If}
$$
\n
$$
A \text{ is bounded as a mapping from } B \times U \text{ into } B^*,
$$
\n
$$
h \text{ is } [s, w] \text{-}u.s.c. \text{ on } B \times U,
$$
\n
$$
u_k \in U_0, u_k \to u \text{ in } B_0 \Rightarrow w_{B_k} \text{-Lim } (C(u_k) \cap X_0) \subseteq C(u),
$$
\n
$$
u_k \in U_0, u_k \to u \text{ in } B_0 \Rightarrow C(u) \subseteq s \text{-Lim } C(u_k),
$$
\n
$$
\text{dition (24) holds.}
$$
\n
$$
\text{Let the assumption of the implication (24) be true. Then } y_k \in C
$$

This is equivalent to (2), i.e. $y \in S_u$

A sufficient condition for (24) of Proposition 2 is given by Remark I (cf. [1; 3,4]): If

(25)

(26)

(27)

$$
u_k \in U_0, u_k \to u \text{ in } B_0 \Rightarrow C(u) \subseteq \text{s-Lim } C(u_k), \tag{28}
$$

then condition (24) holds.

Proof: Let the assumption of the implication (24) be true. Then $y_k \in C(u_k)$ of X_0

and (27) gives $y \in C(u)$. Further, let be $w_k \in C(u_k)$ with $w_k \to w$ in *B*, $w \in C(u)$ arbitrary. As $[y_k, u_k]$ is bounded also $A(y_k, u_k)$ is bounded, consequently $(A(y_k, u_k))$ $(y, w - w_k) \rightarrow 0$. Then gives $y \in C(u)$. Further, let be $w_k \in C(u_k)$ with $w_k \to w$ in *B*, $w \in C(u_k)$. As $[y_k, u_k]$ is bounded also $A(y_k, u_k)$ is bounded, consequently $(A(y_k, u_k) - w_k) \to 0$. Then
 $\lim_{h \to 0} \{(A(y_k, u_k) - y^*, w - w_k) - h(w_k, u_k)\} = -\lim_{h \to 0} h(w_k, u_k) \geq -$ Further, let be $w_k \,\epsilon\, C(u_k)$ with $w_k \to w$ in *B*, $w \epsilon\, C(u)$

sounded also $A(y_k, u_k)$ is bounded, consequently $\big(A(y_k, u_k)\big)$
 $-y^*, w - w_k \big) - h(w_k, u_k) \big\} = -\lim_{k \to \infty} h(w_k, u_k) \ge -h(w, u)$
 $\ge \lim_{k \to \infty} h(w_k, u_k)$, because of (26).

i gives $y \in C(u)$. Further, let be $w_k \in C(u_k)$ with $w_k \rightarrow w$ in *B*, $w \in C(u)$.

As $[y_k, u_k]$ is bounded also $A(y_k, u_k)$ is bounded, consequently $(A(y_k, u_k)$
 $w_k) \rightarrow 0$. Then
 $\lim_{h \to \infty} \{(A(y_k, u_k) - y^*, w - w_k) - h(w_k, u_k)\} = -\lim_{h \to \infty} h(w_k, u$ gives $y \in C(u)$. Further, let be $w_k \in C(u_k)$ with $w_k \rightarrow w$ in B , $w \in C(u)$.

As $[y_k, u_k]$ is bounded also $A(y_k, u_k)$ is bounded, consequently $(A(y_k, u_k)$
 $w_k) \rightarrow 0$. Then
 $\lim_{v \to \infty} \{(A(y_k, u_k) - y^*, w - w_k) - h(w_k, u_k)\} = -\lim_{v \to \infty} h(w_k, u$

$$
\overline{\lim} \left\{ (A(y_k, u_k) - y^*, w - w_k) - h(w_k, u_k) \right\} = -\underline{\lim} h(w_k, u_k) \geq -h(w, u)
$$

as $h(w, u) \geq \lim h(w_k, u_k) \geq \lim h(w_k, u_k)$, because of (26).

In the case where the imbedding of B_0 into B is compact a sufficient condition for (24) **i8** given by $\langle u_k, u_k \rangle \geq -h(w, u)$
sufficient condition
sufficient condition
(29)
(30)
(31)
converging strongly

 R emark 2 (cf. $[3]$): If

A is continuous as a mapping from $B \times U$ into B^* ,

h is $\{w, s\}$ -u.s.c. on $B \times U$,
 $u_k, u \in U_0, u_k \to u$ in $B \Rightarrow s_B$ -Lim $(C(u_k) \cap X_0) \subseteq C(u)$,

$$
u_k, u \in U_0, u_k \to u \text{ in } B \Rightarrow C(u) \subseteq w_B \text{-Lim } C(u_k),
$$
\n
$$
(32)
$$

then (24) holds.

The proof goes like the proof of Remark 1 if we use the fact that $[y_k, y_k] \rightarrow [y, u]$ in $B_0 \times B_0$ implies $[y_k, u_k] \rightarrow [y, u]$ in $B \times B$ and that $A(y_k, u_k)$ is converging strongly *in B*.*

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